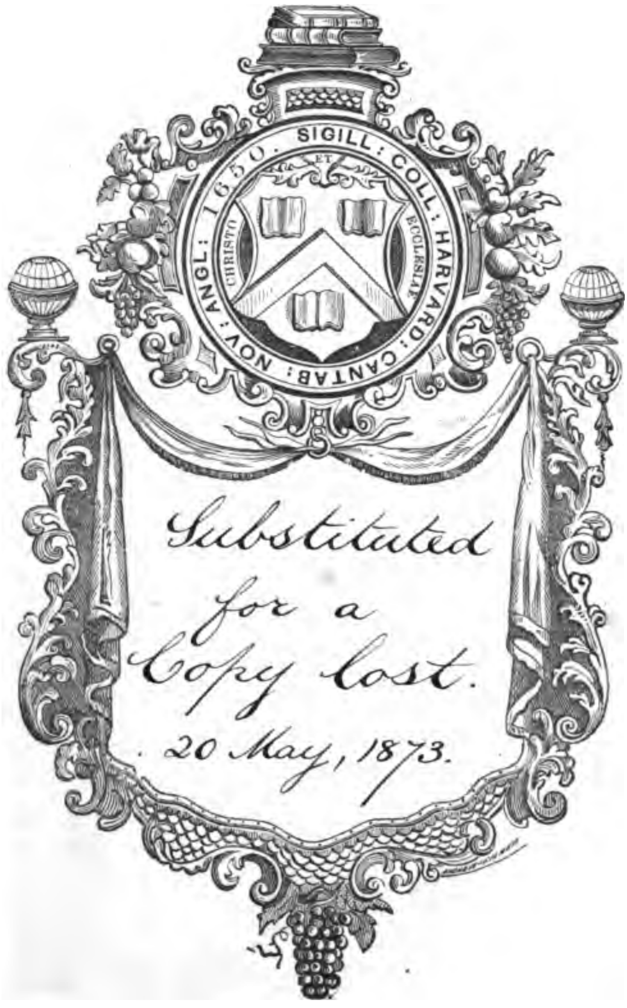

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AN

ELEMENTARY TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY,

WITH THEIR APPLICATIONS TO

NAVIGATION, SURVEYING, HEIGHTS AND DISTANCES, AND SPHERICAL ASTRONOMY,

AND PARTICULARLY ADAPTED TO EXPLAINING

THE CONSTRUCTION OF BOWDITCH'S NAVIGATOR AND THE NAUTICAL ALMANAC.

BY

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ADVERTISEMENT.

THE first three parts of this treatise, comprising Plane Trigonometry, Navigation and Surveying, and Spherical Trigonometry, have been revised for the present edition, under the direction of the author. Some sections have been rewritten, and some additions have been made, but in all essential respects the book remains unaltered. The original numbering of the formulas has been preserved; and in the few cases in which the numbering of the sections has been changed, that of the former editions has been speedily restored.

In the Spherical Astronomy, the notation of the chapter on Eclipses has been conformed to that used in the American Ephemeris and Nautical Almanac.

The tables of the Navigator referred to in the first three parts of the work may be found, with a very few exceptions of slight importance, in *Bowditch's Useful Tables*, a convenient selection from the Navigator, published by Messrs. E. and G. W. Blunt, New York. The only necessary tables, indeed, in this portion of the book, are those of logarithms of numbers, of logarithmic sines &c., of meridional parts, and of the correction for the middle latitude.

1861, March.

CONTENTS.

PLANE TRIGONOMETRY.

NAVIGATION AND SURVEYING.

SPHERICAL TRIGONOMETRY.

SPHERICAL ASTRONOMY.

NOTE ON MIDDLE LATITUDE SAILING.

It may be well to remark, in addition to what is said in § 39 of Navigation and Surveying, that the error of using the formulas of middle latitude sailing without correction diminishes with the diminution of the middle latitude, for any given difference of latitude, and is least when the middle latitude is zero; the increase in the error of the middle latitude being more than counterbalanced by the approximate parallelism of the meridians. The method of § 39 is, however, more accurate than that of middle latitude sailing, when the places are in opposite latitudes, and is obviously founded on the same principle.

ERRATA.

PAGE	47	LINE — 5	FOR	$\frac{AP}{2c}$	READ	$\frac{AP}{c}$
	96	4	AFTER	increased	INSERT	in
		19	FOR	furnish	READ	furnishes
	129	11, 28		*quadrantal*		*equilateral-quadrantal*

PAGE 171 INSTEAD OF § 20, INSERT :—Of the wandering stars, some are called *planets.* These are Mercury (☿), Venus (♀), the Earth (⊕), Mars (♂), the Asteroids, of which sixty-three are now known, (①, ②, &c.), Jupiter (♃), Saturn (♄), Uranus (♅), and Neptune (♆).

PAGE	186	LINE 19	FOR	houses	READ	hours
	187	17		14.596		1.4956
	195	12		$A_1 - 1$		$A_1 - A$
	202	9		$90° - ZS$		$90° - ZS'$
		20		triangles		triangle
				CS		ZS
	204	17		1339		1839
	205	— 6		$A_1 A -$		$A_1 - A$
	206	— 4		sum		sun
	215	—12		observation		observer
	218	13		from the ecliptic		from the equinoxes
	228	— 7		$17^h\ 51^m\ 17^s$		$17^h\ 51^m\ 14^s$
		— 1		$23°\ 47'\ 44''.7$		$23°\ 27'\ 44''.7$
	242	17		B		R
	250	16		$\delta h -$		$\delta h =$
	258	FORMULA (579)		$\frac{-T(43200 - T)}{2\,(43200)^2}$		$\frac{-T(43200 - T)}{2\,(43200)^2} B$
	260	(587)		$\frac{\cos.\,(a'-b') - \cos.\,E'}{\cos.\,a\,\cos.\,b}$		$\frac{\cos.\,(a'-b') - \cos.\,E'}{\cos.\,a'\,\cos.\,b'}$
	263	(604)		$P \cos.\,b = \delta b$		$P \cos.\,b - \delta b$

GEOGRAPHICAL POSITIONS

OF PLACES MENTIONED IN THIS WORK.

Authorities: the U. S. Coast Survey Reports, the American Ephemeris, the Connaissance des Temps, and Bowditch's Navigator.

Places.	Latitudes.	Longitudes from Greenwich.
Albany	42° 39′ N.	73° 45′ W.
Athens (Parthenon)	37 58 N.	23 44 E.
Barcelona (Cathedral)	41 23 N.	2 10 E.
Boston (* Light)	42 20 N.	70 53 W.
" (State House)	42 21 N.	71 03 W.
Botany Bay (Cape Banks)	34 02 S.	151 13 E.
Canton	23 08 N.	113 17 E.
Charleston (Light)	32 42 N.	79 52 W.
Disappointment I., N. Pacific Ocean	27 16 N.	140 51 E.
Disappointment Is., S. Pacific Ocean	14 10 S.	141 18 W.
Gibraltar	36 07 N.	5 21 W.
Good Hope, Cape of, (Point)	34 22 S.	18 29 E.
Georgetown, Bermudas	32 22 N.	64 38 W.
Greenwich (Observatory)	51 29 N.	0 00
Halifax (Dockyard)	44 40 N.	63 35 W.
Horn, Cape, (Point)	55 59 S.	67 16 W.
Java Head	6 48 S.	105 13 E.
Land's End	50 04 N.	5 42 W.
Lima (S. J. de Dios)	12 03 S.	77 08 W.
Liverpool (Observatory)	53 25 N.	3 00 W.
London (St. Paul's)	51 31 N.	0 06 W.
Melbourne, Australia	37 48 S.	144 59 E.
Moscow (Observatory)	55 45 N.	37 34 E.
Nantucket (Gr. Point Light)	41 23 N.	70 02 W.
Newfoundland (S. Pt. Great Bank)	42 56 N.	50 00 W.
New Orleans (City Hall)	29 58 N.	90 07 W.
New York (Battery)	40 42 N.	74 01 W.
" " (* Navesink Light)	40 24 N.	73 59 W.
Paris (Observatory)	48 50 N.	2 20 E.
Portland (Light)	43 37 N.	70 12 W.
St. Helena (Observatory)	15 55 S.	5 43 W.
St. Roque, Cape	5 28 S.	35 17 W.
St. Thomas, Cape	22 03 S.	41 00 W.
St. Vincent, Cape, (Convent)	37 03 N.	9 00 W.
San Francisco (Pt. Boneta Light)	37 49 N.	122 31 W.
Santa Cruz, C. Verde Islands	17 02 N.	25 15 W.
Smeerenburg Harbor, Spitzbergen	79 44 N.	11 11 E.
Stockholm (Observatory)	59 21 N.	18 03 E.
Verde, Cape	14 43 N.	17 31 W.
Washington (* Capitol)	38 53 N.	77 00 W.
" (Observatory)	38 54 N.	77 03 W.
Yankee Straits, New South Shetland	62 30 S.	60 22 W.

* Used in the examples in Navigation.

N. B. A few other places are referred to, in the Astronomy; but their geographical positions are given, where they occur.

PLANE TRIGONOMETRY.

PLANE TRIGONOMETRY.

CHAPTER I.

GENERAL PRINCIPLES OF PLANE TRIGONOMETRY.

1. *Trigonometry* is the science which treats of angles and triangles.

2. *Plane Trigonometry* treats of plane angles and plane triangles. [B., p. 36.*]

3. The sides and angles of a triangle are called its *parts*.

A triangle is said to be *known*, when all its parts are known.

To solve a triangle is to calculate the values of its unknown parts.

It has been proved in Geometry that, when three of the six parts of a triangle are given, the triangle can be constructed, provided one at least of the given parts is a side. In these cases, then, the unknown parts of the triangle can be determined geometrically, and it may readily be inferred that they can also be determined algebraically; that is, that it is possible to find equations which express the relation of the unknown parts to the known, and by which the unknown parts can be computed numerically.

But a great difficulty is met with on the very threshold of the attempt to apply the calculus to triangles. It arises from the circumstance that two kinds of quantities are to be introduced into the same formulas, — sides and angles. These quantities are not only of an entirely different species, but the law of their relative increase and decrease is so complicated that they cannot be determined from each other by any of the common operations of Algebra.

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* References between the brackets, preceded by the letter B., refer to the pages in the stereotype edition of Bowditch's Navigator.

4. To diminish the difficulty of solving triangles as much as possible, every method has been taken to compare triangles with each other, and the solution of all triangles has been reduced to that of a *Limited Series of Right Triangles.*

a. It is easily seen that the solution of all triangles is reducible to that of right triangles. For every oblique triangle is either the sum or the difference of two right triangles; and the sides and angles of the oblique triangle are the same with those of the right triangles, or may be obtained from them by addition or by subtraction. Thus the triangle ABC is the sum (fig. 2), or the difference (fig. 3), of the two right triangles ABP and BPC. In both figures the sides AB, BC, and the angle A belong at once to the oblique and the right triangles, and so does the angle BCA (fig. 2), or its supplement (fig. 3); while the angle ABC is the sum (fig. 2), or the difference (fig. 3), of ABP and PBC; and the side AC is the sum (fig. 2), or the difference (fig. 3), of AP and PC.

b. It follows from the well known propositions of Geometry concerning the similarity of right triangles [B., pp. 8, 12] that any assumed value of one of the acute angles of a right triangle determines the value of the other acute angle and the values of the various ratios between its sides, and that any assumed value of one of these ratios determines the values of the other ratios and of the acute angles. If, then, in the triangle ABC (fig. 4), right angled at C, we denote the hypothenuse by h and the legs opposite to the angles A and B by a and b respectively, and if we arrange in one column of a table all the possible values of the angle A, from 0° to 90°, and then calculate and arrange in other columns the corresponding values of the six ratios $\frac{a}{h}$, $\frac{h}{a}$, $\frac{a}{b}$, $\frac{b}{a}$, $\frac{h}{b}$, and $\frac{b}{h}$, and of the angle B, we shall have *a series of right triangles*, in which every possible case of the right triangle will be represented, and by reference to which, provided a sufficient number of parts are given, it can be solved. Suppose, for instance, that the angle A and the adjacent leg b are given. We are to look through the series of calculated triangles till we find one which has the angle A equal to the given angle; and this triangle is similar to that which we seek to solve. Then, to find the leg a, we have only to multiply the value which we have found of the ratio $\frac{a}{b}$ by b, and to find the hypothenuse, we have only to multiply $\frac{h}{b}$ by b, and the value of the angle B is given directly in the table.

Moreover, since one of the acute angles of every right triangle is included between 0° and 45°, it is evident that the last half of the above series is essentially a repetition of the first half, and is therefore unnecessary.

c. But as there is an infinite variety of values which an angle may assume, between any given limits, a perfect series of right triangles could never be constructed or calculated. Fortunately, such a series is not required; and it is sufficient for all practical purposes to calculate a series in which the successive angles differ only by a minute, or at least, by a second. Intermediate triangles can be obtained, when needed, by that simple principle of interpolation which is made use of to obtain the intermediate logarithms from those given in the tables.

5. Plane Trigonometry then embraces the methods of calculating the series, or table, above described, and of applying it to the solution of all kinds of plane triangles, together with such investigations as naturally grow out of the general theory of the science, though they may not be directly connected with the solution of triangles.

CHAPTER II.

SINES, TANGENTS, AND SECANTS.

6. The six ratios between the three sides of the right triangle, which ratios, as it has been seen, are fully determined by the value of either of the acute angles, are called the *trigonometric functions* of either acute angle; or, sometimes, the *trigonometric ratios.*

7. Each trigonometric function has its distinctive name.

The *Sine* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle by the hypothenuse.

Thus, if we denote (fig. 4) the legs BC and AC by the letters a and b, and the hypothenuse AB by the letter h, we have

$$\sin. A = \frac{a}{h}, \sin. B = \frac{b}{h}. \qquad (1)$$

The *Tangent* of an angle is the quotient obtained by dividing the leg opposite it in a right triangle by the adjacent leg.

Thus, (fig. 4),

$$\text{tang.}\, A = \frac{a}{b}, \text{tang.}\, B = \frac{b}{a}. \qquad (2)$$

The *Secant* of an angle is the quotient obtained by dividing the hypothenuse by the leg adjacent to the angle.

Thus, (fig. 4),

$$\sec. A = \frac{h}{b}, \sec. B = \frac{h}{a}. \qquad (3)$$

The *Cosine*, *Cotangent*, and *Cosecant* of an angle are respectively the sine, tangent, and secant of its complement.

8. *Corollary.* Since the two acute angles of a right triangle are complements of each other, the sine, tangent, and secant of the one must be the cosine, cotangent, and cosecant of the other.

Thus, (fig. 4),

$$\left.\begin{aligned}
\text{sin. } A &= \text{cos. } B = \frac{a}{h}\\
\text{cos. } A &= \text{sin. } B = \frac{b}{h}\\
\text{tang. } A &= \text{cotan. } B = \frac{a}{b}\\
\text{cotan. } A &= \text{tang. } B = \frac{b}{a}\\
\text{sec. } A &= \text{cosec. } B = \frac{h}{b}\\
\text{cosec. } A &= \text{sec. } B = \frac{h}{a}
\end{aligned}\right\} \qquad (4)$$

9. *Corollary.* It is evident that the sine and cosine of every angle are less than unity, that the secant and cosecant of every angle are greater than unity, and that the tangent and cotangent may have any value, the tangent being greater than unity when the cotangent is less, and less when the cotangent is greater.

10. *Corollary.* By inspecting the preceding equations (4), we perceive that the sine and cosecant of an angle are reciprocals of each other; as are also the cosine and secant, and also the tangent and cotangent.

So that

$$\left.\begin{aligned}
\text{cosec. } A \times \text{sin. } A &= \frac{h}{a} \times \frac{a}{h} = \frac{ah}{ah} = 1\\
\text{sec. } A \times \text{cos. } A &= \frac{h}{b} \times \frac{b}{h} = \frac{bh}{bh} = 1\\
\text{tang. } A \times \text{cotan. } A &= \frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1
\end{aligned}\right\} \qquad (5)$$

whence

$$\left.\begin{aligned}
\text{cosec. } A &= \frac{1}{\text{sin. } A}, \text{ or sin. } A = \frac{1}{\text{cosec. } A}\\
\text{sec. } A &= \frac{1}{\text{cos. } A}, \text{ or cos. } A = \frac{1}{\text{sec. } A}\\
\text{cotan. } A &= \frac{1}{\text{tan. } A}, \text{ or tang. } A = \frac{1}{\text{cotan. } A}
\end{aligned}\right\} \qquad (6)$$

As soon, then, as the sine, cosine, and tangent of an angle are known, their reciprocals, the cosecant, secant, and cotangent, may easily be obtained.

11. *Problem. To find the tangent of an angle when its sine and cosine are known.*

Solution. The quotient of sin. A divided by cos. A is, by equations (4),

$$\frac{\sin. A}{\cos. A} = \frac{a}{h} \div \frac{b}{h} = \frac{ah}{bh} = \frac{a}{b}.$$

But by (4),

$$\text{tang. } A = \frac{a}{b};$$

hence

$$\text{tang. } A = \frac{\sin. A}{\cos. A}. \qquad (7)$$

12. *Corollary.* Since the cotangent is the reciprocal of the tangent, we have

$$\text{cotan. } A = \frac{\cos. A}{\sin. A}. \qquad (8)$$

13. *Problem. To find the cosine of an angle when its sine is known.*

Solution. We have, by the Pythagorean proposition, in the right triangle ABC (fig. 4),

$$a^2 + b^2 = h^2.$$

But by (4),

$$(\sin. A)^2 + (\cos. A)^2 = \frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2 + b^2}{h^2} = \frac{h^2}{h^2} = 1,$$

or

$$(\sin. A)^2 + (\cos. A)^2 = 1; \qquad (9)$$

that is, *the sum of the squares of the sine and cosine of any angle is equal to unity.*

Hence

$$(\cos. A)^2 = 1 - (\sin. A)^2,$$
$$\cos. A = \sqrt{1 - (\sin. A)^2}. \qquad (10)$$

14. *Corollary.* Since

$$h^2 - a^2 = b^2,$$

we have by (4),

$$(\text{sec.}\ A)^2 - (\text{tan.}\ A)^2 = \frac{h^2}{b^2} - \frac{a^2}{b^2} = \frac{h^2 - a^2}{b^2} = \frac{b^2}{b^2} = 1,$$

or $$(\text{sec.}\ A)^2 - (\text{tang.}\ A)^2 = 1; \qquad (11)$$

whence $$(\text{sec.}\ A)^2 = 1 + (\text{tang.}\ A)^2.$$

15. *Corollary.* Since

$$h^2 - b^2 = a^2,$$

we have by (4),

$$(\text{cosec.}\ A)^2 - (\text{cotan.}\ A)^2 = \frac{h^2}{a^2} - \frac{b^2}{a^2} = \frac{h^2 - b^2}{a^2} = 1,$$

or $$(\text{cosec.}\ A)^2 - (\text{cotan.}\ A)^2 = 1; \qquad (12)$$

whence $$(\text{cosec.}\ A)^2 = 1 + (\text{cotan.}\ A)^2.$$

16. *Scholium.* The whole difficulty of calculating the trigonometric tables of sines and cosines, tangents and cotangents, secants and cosecants is, by the preceding propositions, reduced to that of calculating the sines alone. This agrees with the statement of § 4 *b*, that any one ratio determines the others.

17. Examples.

1. Given the sine of the angle A, equal to 0.4568, calculate its cosine, tangent, cotangent, secant, and cosecant.

Solution. By equation (10)

$$\cos.\ A = \sqrt{1 - (\sin.\ A)^2} = \sqrt{(1 + \sin.\ A)(1 - \sin.\ A)}.$$

$1 + \sin.\ A = 1.4568$	0.16340
$1 - \sin.\ A = 0.5432$	9.73496
$(\cos.\ A)^2$	2|9.89836
$\cos.\ A = 0.8896$	9.94918

By (7) and (8),

$$\text{tang.}\ A = \frac{\sin.\ A}{\cos.\ A}, \quad \text{cotan.}\ A = \frac{\cos.\ A}{\sin.\ A}$$

sin. $A = 0.4568$		9.65973 (ar. co.)	10.34027
cos. $A = 0.8896$ (ar. co.)	10.05082		9.94918
	9.71055	(ar. co.)	10.28945
tang. $A = 0.5135$	cotan. $A = 1.9474.$		

By (6),

$$\text{sec. } A = \frac{1}{\text{cos. } A} \qquad \text{cosec. } A = \frac{1}{\text{sin. } A}$$

$$\text{log. sec. } A = -\text{log. cos. } A = 0.05082,$$

$$\text{sec. } A = 1.1241,$$

$$\text{log. cosec. } A = -\text{log. sin. } A = 0.34027,$$

$$\text{cosec. } A = 2.1891.$$

2. Given sin. $A = 0.1111$; find the cosine, tangent, cotangent, secant, and cosecant of A.

Ans. cos. $A = 0.9938$
tang. $A = 0.1118$
cotan. $A = 8.9452$
sec. $A = 1.0062$
cosec. $A = 9.0010$

3. Given sin. $A = 0.9891$; find the cosine, tangent, cotangent, secant, and cosecant of A.

Ans. cos. $A = 0.1472$
tang. $A = 6.$[illegible]173
cotan. $A = 0.1489$
sec. $A = 6.79$[illegible]
cosec. $A = 1.0110$

18. *Theorem. The sine of an angle is equal to the perpendicular let fall from one extremity of the arc which measures it, in the circle whose radius is unity, upon the radius passing through the other extremity.*

The cosine is equal to so much of the radius drawn perpendicular to the sine as is included between the sine and the centre.

Proof. Let ACB (fig. 5) be the angle, and let the radius of the circle $ABA'A$ be the unit of length. Let fall on the radius CA

the perpendicular BP, and we have by § 7, in the right triangle BPC,

$$\text{sin. } ACB = \frac{PB}{CB} = \frac{PB}{1} = PB$$

$$\text{cos. } ACB = \frac{CP}{CB} = \frac{CP}{1} = CP.$$

19. *Theorem. In the circle of which the radius is unity, the secant is equal to the length of the radius which is drawn through one extremity of the arc which measures the angle and produced till it meets the tangent drawn through the other extremity.*

The trigonometric tangent is equal to so much of the tangent drawn through one extremity of the arc as is intercepted between the two radii which terminate the arc.

Proof. If CB (fig. 5) is produced to meet the tangent AT at T, we have, by (2) and (3), in the right triangle ACT,

$$\text{sec. } ACB = \frac{CT}{CA} = \frac{CT}{1} = CT$$

$$\text{tang. } ACB = \frac{AT}{CA} = \frac{AT}{1} = AT.$$

20. *Corollary.* If, in fig. 5, a radius CA'' be drawn perpendicular to CA, the angle $A''CB$ will be the complement of ACB. Hence, if a tangent $A''T'$ be drawn to meet the produced radius CT, the lengths $A''T'$ and CT' will be equal respectively to the tangent and the secant of $A''CB$; that is, to the *cotangent* and the *cosecant* of ACB.

21. *Scholium.* On account of their relation to the unit-circle, the trigonometric functions are often called *circular functions;* and most writers upon trigonometry have *defined* the sine, cosine, &c., as lines drawn in the manner described in §§ 18 – 20, but without limiting the radius of the circle to unity. [B., p. 6.]

If any radius is taken at pleasure, the values of the sine, &c., of any given angle are not fixed, but vary with the value of the radius; whereas, if the unit of length is always taken as radius, though any line may be made the unit, so that the *actual lengths* of the lines which represent the sine, &c., may vary, yet the *numerical values* of these lines remain the same, being their fixed ratios to the radius, which is always the unit. Hence, if R represents the value of the radius adopted in the common system, we have

sin., cos., &c., in the common system $= R \times$ sin., cos., &c., in this system.

22. *Corollary.* If the angle is very small, as C (fig. 6), the arc AB will be sensibly a straight line, perpendicular to the two radii, CA and CB, drawn to its extremities, and will sensibly coincide with the sine and tangent; while the cosine will sensibly coincide with the radius CA, and the secant with the radius CB.

Hence, the sine and tangent of a very small angle are nearly equal to the arc which measures the angle in the circle of which the radius is unity; and its cosine and secant are nearly equal to unity.

23. *Problem. To find the sine of a very small angle.*

Solution. Let the angle C (fig. 6) be the given angle, and suppose it to be exactly one minute. The arc AB must in this case be $\frac{1}{10800}$ of the semicircumference of which unity or CA is radius. But the value of the semicircumference of which unity is radius has been found in Geometry to be 3.1415926. Therefore, by § 22,

$$\sin. 1' = AB = \frac{3.1415926}{10800} = 0.000290888. \qquad (13)$$

In the same way we might find the sine of any other small angle, or we might, in preference, find it by the following proposition.

24. *Theorem. The sines of very small angles are proportional to the angles themselves.*

Proof. Let there be the two small angles, ACB and ACB' (fig. 7). Draw the arc ABB' with the centre C and the radius unity. Then, as angles are proportional to the arcs which measure them,

$$ACB : ACB' = AB : AB'.$$

But, by § 22,

$$\sin. ACB = AB, \quad \sin. ACB' = AB',$$

whence

$$ACB : ACB' = \sin. ACB : \sin. ACB'.$$

Scholium. §§ 22 – 24 are only approximately true of any angle; but the smaller the angle, the less is the error of these propositions. It is found that, for angles less than two degrees, the values of the sine found by this method are accurate in the first five places of decimals. Consequently, in calculating the sines of angles to five places of decimals, this method may be applied to angles less than two degrees; the investigation of the sines of larger angles requires the introduction of some new formulas. If more than five places are desired, the more accurate formulas must be introduced at an earlier point.

25. *Corollary.* It follows from the preceding theorem that if x is a very small angle,

$$\sin. x = x \sin. 1', \qquad (14)$$

provided that x in the second member denotes *the number of minutes* in the angle. But if x is expressed in *seconds*, we have

$$\sin. x = x \sin. 1''; \qquad (15)$$

and if x represents the angle by denoting *the length of the arc which measures it in the circle of which the radius is unity,*

$$\sin. x = x; \qquad (16)$$

and either of these different notations may be used at pleasure.

26. EXAMPLES.

1. Find the sine of 12′ 13″, knowing that

$$\sin. 1' = 0.0002909.$$

Solution. By (14),

$$1' : 12'\,13'' = \sin. 1' : \sin. 12'\,13'',$$

or

$$60'' : 733'' = 0.0002909 : \sin. 12'\,13''.$$

Hence

$$\sin. 12'\,13'' = \frac{733 \times 0.0002909}{60} = 0.00355. \quad \textit{Ans.}$$

2. Find the sine of 7′ 15″, knowing that

$$\sin. 1' = 0.0002909.$$

Ans. $\sin. 7'\,15'' = 0.00211.$

3. Find the sine of 1° 2′ 32″, knowing that

$$\sin. 1' = 0.0002909.$$

Ans. $\sin. 1^\circ\,2'\,32'' = 0.01819.$

27. *Problem. Given the sine of any angle, to find the sine of another angle which exceeds it by a very small quantity.*

Solution. Let the given angle be ACB (fig. 8), which we will denote by the letter M; and let the angle whose sine is required be ACB', exceeding the former by the angle BCB', which is supposed to be so small that the arc BB' may be considered as a straight line, as in § 22, and which we will denote by the letter m; so that

$$M = ACB, \qquad m = BCB',$$
$$M + m = ACB'.$$

From the vertex C as a centre, with the radius unity, describe the arc ABB'. From the points B and B', let fall BP and $B'P'$ perpendicular to AC.

We have, by § 18,

$$\sin. M = PB, \quad \cos. M = CP,$$
$$\sin. ACB' = \sin. (M + m) = P'B'.$$

Draw BR perpendicular to $B'P'$; and

$$P'B' = PB + RB',$$

or

$$\sin (M + m) = \sin. M + RB'.$$

The right triangles BCP and $BB'R$, having their sides perpendicular each to each, are similar, and give the proportion

$$CB : BB' = CP : RB'.$$

But, by § 22,

$$BB' = \sin. m.$$

Hence

$$1 : \sin. m = \cos. M : RB';$$
$$\text{and } RB' = \sin. m . \cos. M,$$

which gives, by substitution,

$$\sin. (M + m) = \sin. M + \sin. m . \cos. M. \qquad (17)$$

If m is $1'$, (17) becomes, by (13),

$$\sin. (M + 1') = \sin. M + \sin. 1' . \cos. M,$$
$$= \sin. M + 0.00029 \cos. M. \qquad (18)$$

We may, by this formula, find the sine of $2'$ from that of $1'$, thence that of $3'$, then of $4'$, of $5'$, &c., to the sine of an angle of any number of degrees and minutes.

28. *Corollary.* We can, in a similar way, deduce the value of cos. $(M+m)$.

For, by § 18,

$$\cos.\ (M+m) = CP' = CP - P'P$$
$$= \cos.\ M - RB.$$

But the similar triangles $BB'R$ and BCP give the proportion

$$CB : BB' = PB : RB,$$

or

$$1 : \sin.\ m = \sin.\ M : RB.$$

Hence

$$RB = \sin.\ m\ .\ \sin.\ M,$$

whence

$$\cos.\ (M+m) = \cos.\ M - \sin.\ m\ .\ \sin.\ M; \qquad (19)$$

and, if we make $m = 1'$, this equation becomes

$$\cos.\ (M+1') = \cos.\ M - \sin.\ 1'\ .\ \sin.\ M,$$
$$= \cos.\ M - 0.00029 \sin.\ M. \qquad (20)$$

29. Examples.

1. Given the sine of 23° 28′ equal to 0.39822, to find the sine of 23° 29′.

Solution. We find the cosine of 23° 28′ by (10) to be

$$\cos.\ 23°\ 28' = 0.91729.$$

Hence, by (18), making $M = 23°\ 28'$

$$\sin.\ 23°\ 29' = \sin.\ 23°\ 28' + 0.00029 \cos.\ 23°\ 28',$$
$$= 0.39822 + 0.00026,$$
$$= 0.39848.$$

Ans. sin. 23° 29′ = 0.39848.

2. Given the sine and cosine of 46° 58′ as follows,

sin. 46° 58′ = 0.73096, cos. 46° 58′ = 0.68242,

find the sine and cosine of 46° 59′.

Ans. sin. 46° 59′ = 0.73116,
cos. 46° 59′ = 0.68221.

3. Given the sine and cosine of 11° 10′ as follows,

sin. 11° 10′ = 0.19366, cos. 11° 10′ = 0.98107,

find the sine and cosine of 11° 11′.

Ans. sin. 11° 11′ = 0.19395.
cos. 11° 11′ = 0.98101.

30. By the formulas here given, a complete table of sines, cosines, &c., may be calculated. Such tables have been actually calculated, but generally by methods more convenient in practice than that explained in this chapter. Table XXIV of the Navigator is a table of sines and cosines calculated to five places of decimals; and Table XXVII gives the five-place logarithms of all the trigonometric functions.

The trigonometric functions themselves are called *natural*, as in Table XXIV, to distinguish them from their logarithms, which are more often used, and which are sometimes called the *artificial* sines, &c.

Table XXIV is constructed on the system of § 21, the radius being

$$10^5 = 100,000;$$

so that this table is reduced to the present system by dividing each number by this radius; that is, by putting the decimal point five places back, or prefixing it to each number as it is given in the table.

The radius of Table XXVII is

$$10^{10} = 10,000,000,000;$$

so that this table is reduced to the present system by subtracting from each number the logarithm of this radius, which is 10; that is, by subtracting 10 from each characteristic.

These values of the radius are taken in order to avoid printing the decimal point in the first case, and to avoid negative characteristics in the second case.

The method of using these two tables is fully explained in pp. 33 – 35 (given at the end of the Useful Tables) and pp. 391, 392 of the Navigator. It is supposed to be understood in the remainder of this book.

If we disregard the "Hour" columns in Table XXVII, with which at present we have nothing to do, and the insertion of angles greater than 90°, which will be explained in a future chapter, this table corresponds precisely to the series of right triangles described in § 4, the two opposite angles being always complements of each other, and the six principal columns giving the values of the six trigonometric ratios, each of which, as in § 8, bears complementary relations to the two angles.

CHAPTER III.

RIGHT TRIANGLES.

31. The general formulas which are obtained for the solution of triangles should in each case, as far as possible, express the unknown parts in terms only of those which are *given at the outset;* but it is occasionally better, for practical reasons, in the working of a numerical example, to compute certain of the unknown parts first, and then use these in finding the others.

Two classes of problems in the solution of right triangles may be distinguished; — the first class including those in which *an acute angle* and *a side* are given; the second, those in which *two sides* are given.

In problems of the first class, the general method of finding either unknown side is to *see what trigonometric function of the known angle is represented by the ratio of the side sought to the given side, find its value in the table, and multiply it by the given side.* The unknown angle is the complement of the known angle.

In problems of the second class, the general method of finding either acute angle is to *see what function of this angle is represented by the ratio of the given sides, find the value of this ratio, and look out the corresponding value of the angle in the table.* The unknown side may be found by the Pythagorean Proposition.

32. *Problem. To solve a right triangle, when the hypothenuse and one of the angles are known.* [B., p. 38.]

Solution. Given (fig. 4) the hypothenuse h and the angle A, to solve the triangle.

First. To find the other acute angle B, subtract the given angle from 90°.

Secondly. To find the opposite side a, we have, by (1),

$$\sin. A = \frac{a}{h},$$

which, multiplied by h, gives

$$a = h \sin. A; \tag{21}$$

or, by logarithms,

$$\log. a = \log. h + \log. \sin. A.$$

Thirdly. To find the side b, we have, by (4),

$$\cos. A = \frac{b}{h},$$

which, multiplied by h, gives

$$b = h \cos. A; \tag{22}$$

or, by logarithms,

$$\log. b = \log. h + \log. \cos. A.$$

33. *Problem. To solve a right triangle, when a leg and the opposite angle are known.* [B., p. 39.]

Solution. Given (fig. 4) the leg a and the opposite angle A, to solve the triangle.

First. The angle B is the complement of A.

Secondly. To find the hypothenuse h, we have, by (4),

$$\text{cosec}. A = \frac{h}{a},$$

which, multiplied by a, gives, by (6),

$$h = a \text{ cosec}. A = \frac{a}{\sin. A}; \tag{23}$$

or, by logarithms,

$$\log. h = \log. a + \log. \text{cosec}. A = \log. a + (\text{ar. co.}) \log. \sin. A.$$

Thirdly. To find the other leg b, we have, by (4),

$$\text{cotan}. A = \frac{b}{a},$$

$$b = a \text{ cotan}. A; \tag{24}$$

$$\log. b = \log. a + \log. \text{cotan } A.$$

34. *Problem. To solve a right triangle, when a leg and the adjacent angle are known.* [B., p. 39.]

Solution. Given (fig. 4) the leg a and the angle B, to solve the triangle.

First. The angle A is the complement of B.

Secondly. We have for h and b, from (4) and (6),

$$h = a \text{ sec. } B = \frac{a}{\text{cos. } B}, \qquad (25)$$

$$b = a \text{ tang. } B; \qquad (26)$$

or, by logarithms,

$$\text{log. } h = \text{log. } a + \text{log. sec. } B,$$

$$\text{log. } b = \text{log. } a + \text{log. tang. } B.$$

35. *Problem. To solve a right triangle, when the hypothenuse and a leg are known.* [B., p. 40.]

Solution. Given (fig. 4) the hypothenuse h and the leg a, to solve the triangle.

First. The angles A and B are obtained from equation (4),

$$\text{sin. } A = \text{cos. } B = \frac{a}{h}; \qquad (27)$$

or, by logarithms,

$$\text{log. sin. } A = \text{log. cos. } B = \text{log. } a + (\text{ar. co.}) \text{ log. } h.$$

Secondly. The leg b is deduced from the Pythagorean property of the right triangle, which gives

$$a^2 + b^2 = h^2, \qquad (28)$$

$$b^2 = h^2 - a^2 = (h + a)(h - a),$$

$$b = \sqrt{(h^2 - a^2)} = \sqrt{[(h + a)(h - a)]}; \qquad (29)$$

$$\text{log. } b = \tfrac{1}{2} \text{ log. } (h^2 - a^2) = \tfrac{1}{2} [\text{log. } (h + a) + \text{log. } (h - a)].$$

36. *Problem. To solve a right triangle, when the two legs are known.* [B., p. 40.]

Solution. Given (fig. 4) the legs a and b, to solve the triangle.

First. The angles are obtained from (4),

$$\text{tang. } A = \text{cotan. } B = \frac{a}{b}; \qquad (30)$$

$$\text{log. tang. } A = \text{log. cotan. } B = \text{log. } a + (\text{ar. co.}) \text{ log. } b.$$

Secondly. To find the hypothenuse, we have, by (28),

$$h = \sqrt{(a^2 + b^2)}. \qquad (31)$$

Thirdly. A practically better way of finding the hypothenuse is to make use of (23) or (25),

$$h = a \text{ cosec. } A = a \text{ sec. } B; \qquad (32)$$

$$\text{log. } h = \text{log. } a + \text{log. cosec. } A = \text{log. } a + \text{log. sec. } B.$$

37. Examples.

1. Given the hypothenuse of a right triangle equal to 49.58, and one of the acute angles equal to 54° 44′; to solve the triangle.

Solution. The other angle $= 90° - 54°\ 44' = 35°\ 16'$. Then making $h = 49.58$, and $A = 54°\ 44'$; we have, by (21) and (22),

$h = 49.58$	1.69531		1.69531
$A = 54°\ 44'$	* sin. 9.91194		cos. 9.76146
$a = 40.481$	1.60725;	$b = 28.627$	1.45677

Ans. The other angle $= 35°\ 16'$;

The legs $= \begin{cases} 40.481 \\ 28.627 \end{cases}$

2. Given the hypothenuse of a right triangle equal to 54.571, and one of the legs equal to 23.479; to solve the triangle.

Solution. Making $h = 54.571$, $a = 23.479$; we have, by (27),

$a = 23.479$		1.37068
$h = 54.571$	(ar. co.)	8.26304
$A = 25°\ 29'$ sin. / $B = 64°\ 31'$ cos.		9.63372
By (29),		
$h + a = 78.050$		1.89237
$h - a = 31.092$		1.49265
b^2	2	3.38502
$b = 49.262$		1.69251

Ans. The other leg $= 49.262$

The angles $= \begin{cases} 25°\ 29' \\ 64°\ 31' \end{cases}$

3. Given the two legs of a right triangle equal to 44.375, and 22.165; to solve the triangle.

* To avoid negative characteristics, the logarithms are retained as in the tables, according to the usual practice with the logarithms of decimals, as in B., p. 29.

Solution. Making $a = 44.375$, $b = 22.165$; we have

$a = 44.375$		1.64714		1.64714
$b = 22.165$	(ar. co.)	8.65433		
$A = 63° 27' 28''$ tang. / $B = 26° 32' 32''$ cotan.		10.30147;	cosec. / sec.	10.04837
		$h = 49.603$		1.69551

Ans. The hypothenuse = 49.603

The angles = { 63° 27′ 28″, 26° 32′ 32″ }

4. Given the hypothenuse of a right triangle equal to 37.364, and one of the acute angles equal to 12° 30′; to solve the triangle.

Ans. The other angle = 77° 30′

The legs = { 8.087, 36.478 }

5. Given one of the legs of a right triangle equal to 14.548, and the opposite angle equal to 54° 24′; to solve the triangle.

Ans. The hypothenuse = 17.892
The other leg = 10.415
The other angle = 35° 36′

6. Given one of the legs of a right triangle equal to 11.111, and the adjacent angle equal to 11° 11′; to solve the triangle.

Ans. The hypothenuse = 11.326
The other leg = 2.197
The other angle = 78° 49′

7. Given the hypothenuse of a right triangle equal to 100, and one of the legs equal to 1; to solve the triangle.

Ans. The other leg = 99.995

The angles = { 0° 34′ 23″, 89° 25′ 37″ }

8. Given the two legs of a right triangle equal to 8.148, and 10.864; to solve the triangle.

Ans. The hypothenuse = 13.58

The angles = { 36° 52′ 11″, 53° 7′ 49′ }

CHAPTER IV.

GENERAL FORMULAS.

38. Certain general problems, concerning the mutual relations of the trigonometric functions of angles which have simple relations to each other, are perpetually recurring in the applications of trigonometry; and as some of them arise in the solution of oblique triangles, it is convenient to bring them together and investigate them at this point.

39. *Problem. To find the sine of an angle equal to the sum of two other angles, in terms of the trigonometric functions of the two latter angles;* or, more briefly,

To find the sine of the sum of two angles.

Solution. Let the two angles be CAB and $B'AC$ (fig. 9), represented by the letters M and N. At any point C in the line AC, erect the perpendicular BB'. From B let fall on AB' the perpendicular BP. Then represent the several lines as follows,

$$a = CB, \quad a' = B'C, \quad b = AC$$
$$h = AB, \quad h' = AB', \quad x = PB$$
$$M = CAB, \quad N = B'AC.$$

Then, by (4),

$$\sin.\ CAB = \sin.\ M = \frac{a}{h}, \qquad \sin.\ N = \frac{a'}{h'}$$
$$\cos.\ M = \frac{b}{h}, \qquad \cos.\ M = \frac{b}{h'} \quad N$$
$$\sin.\ B'AB = \sin.\ (M + N) = \frac{PB}{AB} = \frac{x}{h}.$$

Now the triangles BPB' and $B'AC$, being right-angled, and having the angle B' common, are equiangular and similar.

Whence we derive the proportion

$$AB' : B'B = AC : PB,$$

or

$$h' : a + a' = b : x;$$

whence

$$x = \frac{a\,b + a'\,b}{h'},$$

and

$$\sin.\,(M + N) = \frac{x}{h} = \frac{a\,b + a'\,b}{h\,h'}.$$

The second member of this equation may be separated into factors, as follows,

$$\sin.\,(M + N) = \frac{a\,b}{h\,h'} + \frac{b\,a'}{h\,h'}$$
$$= \frac{a}{h}\cdot\frac{b}{h'} + \frac{b}{h}\cdot\frac{a'}{h'};$$

whence, by substitution, we obtain

$$\sin.\,(M + N) = \sin.\,M \cos.\,N + \cos.\,M \sin.\,N. \qquad (33)$$

40. *Problem. To find the sine of the difference of two angles.*

Solution. Let the two angles be CAB and CAB' (fig. 10), represented by M and N. At any point C in the line AC erect the perpendicular $CB'B$. From B let fall on AB' the perpendicular BP. Then, applying to fig. 10 the notation of § 39, we have

$$\sin.\,B'AB = \sin.\,(M - N) = \frac{PB}{AB} = \frac{x}{h}.$$

The triangles $B'AC$ and $BB'P$ are similar, because they are right-angled and the angles at B' are vertical and equal.

Whence

$$AB' : B'B = AC : PB,$$

or

$$h' : a - a' = b : x;$$

whence

$$x = \frac{ab - a'b}{h'},$$

and

$$\sin.\,(M - N) = \frac{x}{h} = \frac{a\,b - b\,a'}{h\,h'}$$
$$= \frac{a\,b}{h\,h'} - \frac{b\,a'}{h\,h'}$$
$$= \frac{a}{h}\cdot\frac{b}{h'} - \frac{b}{h}\cdot\frac{a'}{h'};$$

and by substitution,

$$\sin. (M - N) = \sin. M \cos. N - \cos. M \sin. N. \qquad (34)$$

41. *Problem. To find the cosine of the sum of two angles.*

Solution. Making use of fig. 9, with the notation of § 39 and also the following,

$$y = AP,\ z = PB';$$

we have

$$\cos. (M + N) = \frac{AP}{AB} = \frac{y}{h}.$$

But

$$y = AB' - PB' = h' - z.$$

The similar triangles BPB' and $B'AC$ give the proportion

$$AB' : B'B = B'C : PB',$$

or

$$h' : a + a' = a' : z;$$

whence

$$z = \frac{a\,a' + a'^2}{h'},$$

and

$$y = h' - z = h' - \frac{a\,a' + a'^2}{h'}$$

$$= \frac{h'^2 - a'^2 - a\,a'}{h'}.$$

But, from the right triangle $AB'C$,

$$h'^2 - a'^2 = (AB')^2 - (B'C)^2 = (AC)^2 = b^2;$$

whence

$$y = \frac{b^2 - a\,a'}{h'}$$

and

$$\cos. (M + N) = \frac{y}{h} = \frac{b^2 - a\,a'}{h\,h'},$$

$$= \frac{b^2}{h\,h'} - \frac{a\,a'}{h\,h'},$$

$$= \frac{b}{h} \cdot \frac{b}{h'} - \frac{a}{h} \cdot \frac{a'}{h'};$$

whence, by substitution,

$$\cos. (M + N) = \cos. M \cos. N - \sin. M \sin. N. \qquad (35)$$

42. *Problem.* *To find the cosine of the difference of two angles.*

Solution. Making use of fig. 10, with the notation of the preceding section, we have

$$\cos. B'AB = \cos. (M - N) = \frac{AP}{AB} = \frac{y}{h}.$$

But $$y = AB' + B'P = h' + z.$$

The similar triangles $BB'P$ and $B'AC$ give the proportion

$$AB' : B'B = CB' : B'P,$$

or $$h' : a - a' = a' : z;$$

whence $$z = \frac{a\,a' - a'^2}{h'},$$

and $$y = h' + z = h' + \frac{a\,a' - a'^2}{h'}$$

$$= \frac{h'^2 - a'^2 + a\,a'}{h'}.$$

But $$h'^2 - a'^2 = b^2.$$

Hence $$y = \frac{b^2 + a\,a'}{h'},$$

and $$\cos. (M - N) = \frac{y}{h} = \frac{b^2 + a\,a'}{h\,h'}$$

$$= \frac{b^2}{h\,h'} + \frac{a\,a'}{h\,h'}$$

$$= \frac{b}{h} \cdot \frac{b}{h'} + \frac{a}{h} \cdot \frac{a'}{h'};$$

or, by substitution,

$$\cos. (M - N) = \cos. M \cos. N + \sin. M \sin. N. \qquad (36)$$

43. *Corollary.* The similarity, in all but the signs, of the formulas (33) and (34) is such that they may both be written in the same form, as follows,

$$\sin. (M \pm N) = \sin. M \cos. N \pm \cos. M \sin. N, \qquad (37)$$

in which the upper signs correspond with each other, and also the lower ones.

In the same way, by the comparison of (35) and (36), we are led to the form

$$\cos. (M \pm N) = \cos. M \cos. N \mp \sin. M \sin. N. \qquad (38)$$

in which the upper signs correspond with each other, and also the lower ones.

44. *Corollary.* The sum of the equations (33) and (34) is

$$\sin. (M + N) + \sin. (M - N) = 2 \sin. M \cos. N. \qquad (39)$$

Their difference is

$$\sin. (M + N) - \sin. (M - N) = 2 \cos. M \sin. N. \qquad (40)$$

45. *Corollary.* The sum of (35) and (36) is

$$\cos. (M + N) + \cos. (M - N) = 2 \cos. M \cos. N. \qquad (41)$$

Their difference is

$$\cos. (M - N) - \cos. (M + N) = 2 \sin. M \sin. N. \qquad (42)$$

Formulas (39 - 42), like (37) and (38), may obviously be applied to any values of the angles M and N; and they are often found useful in trigonometric investigations.

46. *Corollary.* If, in (39 - 42), we make

$$M + N = A, \text{ and } M - N = B;$$

that is,

$$M = \tfrac{1}{2}(A + B), \qquad N = \tfrac{1}{2}(A - B);$$

they become, as follows,

$$\sin. A + \sin. B = 2 \sin. \tfrac{1}{2}(A + B) \cos. \tfrac{1}{2}(A - B) \qquad (43)$$

$$\sin. A - \sin. B = 2 \cos. \tfrac{1}{2}(A + B) \sin. \tfrac{1}{2}(A - B) \qquad (44)$$

$$\cos. A + \cos. B = 2 \cos. \tfrac{1}{2}(A + B) \cos. \tfrac{1}{2}(A - B) \qquad (45)$$

$$\cos. B - \cos. A = 2 \sin. \tfrac{1}{2}(A + B) \sin. \tfrac{1}{2}(A - B); \qquad (46)$$

and, in (43 - 46), A and B represent any two angles, because it is always possible to find two angles, M and N, of which the sum is equal to A and the difference to B.

47. *Corollary.* The quotient obtained by dividing (43) by (44) is

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin. \tfrac{1}{2}(A + B) \cos. \tfrac{1}{2}(A - B)}{\cos. \tfrac{1}{2}(A + B) \sin. \tfrac{1}{2}(A - B)}.$$

Reducing the second member by means of equations (6), (7), (8), we have the general formula: —

$$\frac{\sin. A+\sin. B}{\sin. A-\sin. B}=\text{tang.}\ \tfrac{1}{2}(A+B)\ \text{cotan.}\ \tfrac{1}{2}(A-B)$$

$$=\frac{\text{tang.}\ \frac{1}{2}(A+B)}{\text{tang.}\ \frac{1}{2}(A-B)}=\frac{\text{cotan.}\ \frac{1}{2}(A-B)}{\text{cotan.}\ \frac{1}{2}(A+B)}. \qquad (47)$$

48. *Corollary.* Dividing (46) by (45), and reducing, we have the general formula: —

$$\frac{\cos. B-\cos. A}{\cos. B+\cos. A}=\text{tang.}\ \tfrac{1}{2}(A+B)\ \text{tang.}\ \tfrac{1}{2}(A-B)$$

$$=\frac{\text{tang.}\ \frac{1}{2}(A+B)}{\text{cotan.}\ \frac{1}{2}(A-B)}=\frac{\text{tang.}\ \frac{1}{2}(A-B)}{\text{cotan.}\ \frac{1}{2}(A+B)}. \qquad (48)$$

49. *Corollary.* If, in (33) and (35), we suppose M and N equal to each other and represent their common value by A, we obtain, for *the sine and cosine of the double of any angle,*

$$\sin. 2A=\sin. A\cos. A+\sin. A\cos. A=2\sin. A\cos. A \qquad (49)$$

$$\cos. 2A=\cos. A\cos. A-\sin. A\sin. A$$

$$=(\cos. A)^2-(\sin. A)^2. \qquad (50)$$

50. *Corollary.* Comparing equation (50) with the following equation, which is the same as (9),

$$1=(\cos. A)^2+(\sin. A)^2,$$

we obtain, by addition and by subtraction,

$$1+\cos. 2A=2(\cos. A)^2 \qquad (51)$$

$$1-\cos. 2A=2(\sin. A)^2. \qquad (52)$$

51. *Corollary.* Making $2A=C$ and $A=\frac{1}{2}C$, in (49-52), we obtain

$$\sin. C=2\sin. \tfrac{1}{2}C\cos. \tfrac{1}{2}C \qquad (53)$$

$$\cos. C=(\cos. \tfrac{1}{2}C)^2-(\sin. \tfrac{1}{2}C)^2 \qquad (54)$$

$$1+\cos. C=2(\cos. \tfrac{1}{2}C)^2 \qquad (55)$$

$$1-\cos. C=2(\sin. \tfrac{1}{2}C)^2; \qquad (56)$$

and the equations (55) and (56) give, for *the sine, cosine, and tangent of the half of any angle,*

$$\cos.\ \tfrac{1}{2}\,C = \surd\,[\tfrac{1}{2}\,(1+\cos.\ C)] \qquad (57)$$

$$\sin.\ \tfrac{1}{2}\,C = \surd\,[\tfrac{1}{2}\,(1-\cos.\ C)] \qquad (58)$$

$$\text{tang.}\ \tfrac{1}{2}\,C = \frac{\sin.\ \frac{1}{2}\,C}{\cos.\ \frac{1}{2}\,C} = \surd\left(\frac{1-\cos.\ C}{1+\cos.\ C}\right). \qquad (59)$$

52. ***Problem.*** *To find the tangent and cotangent of the sum and of the difference of two angles.*

Solution. First. To find the tangent of the sum of two angles, which we will suppose to be M and N, we have, from (7),

$$\text{tang.}\ (M+N) = \frac{\sin.\ (M+N)}{\cos.\ (M+N)}.$$

Substituting (33) and (35),

$$\text{tang.}\ (M+N) = \frac{\sin.\ M\cos.\ N+\cos.\ M\sin.\ N}{\cos.\ M\cos.\ N-\sin.\ M\sin.\ N}.$$

Divide every term of both numerator and denominator of the second member by cos. M cos. N;

$$\text{tang.}\ (M+N) = \frac{\dfrac{\sin.\ M\cos.\ N}{\cos.\ M\cos.\ N}+\dfrac{\cos.\ M\sin.\ N}{\cos.\ M\cos.\ N}}{\dfrac{\cos.\ M\cos.\ N}{\cos.\ M\cos.\ N}-\dfrac{\sin.\ M\sin.\ N}{\cos.\ M\cos.\ N}},$$

$$= \frac{\dfrac{\sin.\ M}{\cos.\ M}+\dfrac{\sin.\ N}{\cos.\ N}}{1-\dfrac{\sin.\ M}{\cos.\ M}\cdot\dfrac{\sin.\ N}{\cos.\ N}},$$

which, reduced by means of (7), becomes

$$\text{tang.}\ (M+N) = \frac{\text{tang.}\ M+\text{tang.}\ N}{1-\text{tang.}\ M\ \text{tang.}\ N}. \qquad (60)$$

Secondly. To find the tangent of the difference of M and N. Since by (7)

$$\text{tang.}\ (M-N) = \frac{\sin.\ (M-N)}{\cos.\ (M-N)},$$

a bare inspection of (37) and (38) shows that we have only to change the signs which connect the terms in the value of tang. $(M + N)$ to obtain that of tang. $(M - N)$. This change, being made in (60), produces

$$\text{tang. } (M - N) = \frac{\text{tang. } M - \text{tang. } N}{1 + \text{tang. } M \text{ tang. } N}. \qquad (61)$$

Thirdly. As the cotangent is merely the reciprocal of the tangent, we have, by inverting the fractions, from (60) and (61),

$$\text{cotan. } (M + N) = \frac{1 - \text{tang. } M \text{ tang. } N}{\text{tang. } M + \text{tang. } N}, \qquad (62)$$

$$\text{cotan. } (M - N) = \frac{1 + \text{tang. } M \text{ tang. } N}{\text{tang. } M - \text{tang. } N}. \qquad (63)$$

53. *Corollary.* Make $M = N = A$, in (60) and (62). They give, for the double of any angle,

$$\text{tang. } 2\,A = \frac{2 \text{ tang. } A}{1 - (\text{tang. } A)^2}, \qquad (64)$$

$$\text{cotan. } 2\,A = \frac{1 - (\text{tang. } A)^2}{2 \text{ tang. } A}. \qquad (65)$$

CHAPTER V.

VALUES OF THE SINES, COSINES, TANGENTS, COTANGENTS, SECANTS, AND COSECANTS OF CERTAIN ANGLES.

54. The definitions of sine, cosine, &c. given in § 7 can be applied directly to acute angles only; but the general formulas which have been deduced from the definitions can be used to find values of these functions for obtuse angles, and, indeed, for angles of any magnitude.

We shall, therefore, by a natural enlargement of our previous conceptions, henceforth regard the sine, &c., as functions which belong to every angle, always having such values as to satisfy the general formulas which have been established.

An angle may be regarded as the measure of *the rotation of a line* which turns in a plane about one of its own points. When the line has made more than half a revolution, the angle of rotation is greater than 180°; when it has made more than a whole revolution, the angle is greater than 360°; and we thus arrive at the conception of *angles of all magnitudes*, up to infinity. We may even conceive of *negative angles;* for if the line, after having made a certain rotation, turn back towards its first position, the angle is diminished, and it is, therefore, proper to consider this backward rotation as negative; so that if rotation in one direction is positive, rotation in the opposite direction is negative, and the angle which measures it is negative, and this angle may be of any magnitude.

55. *Problem. To find the sine, &c. of* 0° *and of* 90°.

Solution. Since 0° and 90° are complements of each other, the sine of the one is the cosine of the other. It is evident, moreover, that § 22 is applicable strictly, and not merely approximately, to an angle of 0°. Hence

$$\sin. 0° = \cos. 90° = 0. \qquad (66)$$

$$\cos. 0° = \sin. 90° = 1. \qquad (67)$$

By (6) and (7), we have

$$\text{tang. } 0° = \text{cotan. } 90° = \frac{\text{sin. } 0°}{\text{cos. } 0°} = \frac{0}{1} = 0 \quad (68)$$

$$\text{cotan. } 0° = \text{tang. } 90° = \frac{1}{\text{tang.} 0°} = \frac{1}{0} = \pm \infty^{*} \quad (69)$$

$$\text{sec. } 0° = \text{cosec. } 90° = \frac{1}{\text{cos. } 0°} = \frac{1}{1} = 1 \quad (70)$$

$$\text{cosec. } 0° = \text{sec. } 90° = \frac{1}{\text{sin. } 0°} = \frac{1}{0} = \pm \infty. \quad (71)$$

56. *Problem. To find the sine, &c. of* 180°.

Solution. Make $A = 90°$ in (49) and (50); they become, by means of (66) and (67),

$$\text{sin. } 180° = 2 \text{ sin. } 90° \text{ cos. } 90° = 2 \times 1 \times 0 = 0 \quad (72)$$

$$\text{cos. } 180° = (\text{cos. } 90°)^2 - (\text{sin. } 90°)^2 = 0 - 1 = -1. \quad (73)$$

Hence, by (6) and (7),

$$\text{tang. } 180° = \frac{\text{sin. } 180°}{\text{cos. } 180°} = \frac{0}{-1} = 0 \quad (74)$$

$$\text{cotan. } 180° = \frac{\text{cos. } 180°}{\text{sin. } 180°} = \frac{-1}{0} = \pm \infty \quad (75)$$

$$\text{sec. } 180° = \frac{1}{\text{cos. } 180°} = \frac{1}{-1} = -1 \quad (76)$$

$$\text{cosec. } 180° = \frac{1}{\text{sin. } 180°} = \frac{1}{0} = \pm \infty. \quad (77)$$

57. *Problem. To find the sine, &c. of* 270°.

Solution. Make $M = 180°$ and $N = 90°$ in (33) and (35). They become, by means of (66, 67, 72, 73),

$$\text{sin. } 270° = \text{sin. } 180° \text{ cos. } 90° + \text{cos. } 180° \text{ sin. } 90° = -1 \quad (78)$$

$$\text{cos. } 270° = \text{cos. } 180° \text{ cos. } 90° - \text{sin. } 180° \text{ sin. } 90° = 0. \quad (79)$$

Hence, by (6) and (7),

$$\text{tang. } 270° = \frac{\text{sin. } 270°}{\text{cos. } 270°} = \frac{-1}{0} = \pm \infty \quad (80)$$

---

* If the denominator of $\frac{1}{a}$ is reduced to *zero*, the fraction itself becomes *infinite*; and since $+0 = -0$, we have $\frac{1}{0} = \frac{1}{+0} = +\infty$ or $\frac{1}{0} = \frac{1}{-0} = -\infty$.

$$\text{cotan.}\,270^\circ = \frac{\cos.\,270^\circ}{\sin.\,270^\circ} = \frac{0}{-1} = 0 \tag{81}$$

$$\text{sec.}\quad 270^\circ = \frac{1}{\cos.\,270^\circ} = \frac{1}{0} = \pm\,\infty \tag{82}$$

$$\text{cosec.}\,270^\circ = \frac{1}{\sin.\,270^\circ} = \frac{1}{-1} = -1. \tag{83}$$

58. *Problem. To find the sine, &c. of* 360°.

Solution. Make $A = 180^\circ$ in (49) and (50); and they become by (72, 73, 66, 67)

$$\sin.\,360^\circ = 0 = \sin.\,0^\circ \tag{84}$$

$$\cos.\,360^\circ = 1 = \cos.\,0^\circ. \tag{85}$$

Hence all the trigonometric functions of 360° *are the same as those of* 0°.

59. *Problem. To find the sine, &c. of* 45°.

Solution. Make $C = 90^\circ$ in (57) and (58). They become, by means of (66),

$$\cos.\,45^\circ = \sqrt{[\tfrac{1}{2}\,(1 + \cos.\,90^\circ)]} = \sqrt{\tfrac{1}{2}} \tag{86}$$

$$\sin.\,45^\circ = \sqrt{[\tfrac{1}{2}\,(1 - \cos.\,90^\circ)]} = \sqrt{\tfrac{1}{2}} = \cos.\,45^\circ. \tag{87}$$

Hence, by (6) and (7),

$$\text{tang.}\;\;45^\circ = \frac{\sin.\,45^\circ}{\cos.\,45^\circ} = 1 \tag{88}$$

$$\text{cotan.}\,45^\circ = \frac{1}{\text{tang.}\,45^\circ} = 1 = \text{tang.}\,45^\circ \tag{89}$$

$$\text{sec.}\quad 45^\circ = \frac{1}{\cos.\,45^\circ} = \frac{1}{\sqrt{\tfrac{1}{2}}} = \sqrt{2} \tag{90}$$

$$\text{cosec.}\,45^\circ = \frac{1}{\sin.\,45^\circ} = \frac{1}{\sqrt{\tfrac{1}{2}}} = \sqrt{2} = \text{sec.}\,45^\circ. \tag{91}$$

60. *Problem. To find the sine, &c. of* 30° *and* 60°.

Solution. Make $A = 30^\circ$ in (49). It becomes, from the consideration that 30° and 60° are complements of each other,

$$\sin.\,60^\circ = \cos.\,30^\circ = 2\,\sin.\,30^\circ\,\cos.\,30^\circ.$$

Dividing by cos. 30°, we have

$$1 = 2\,\sin.\,30^\circ,$$

or $$\sin. 30^\circ = \cos. 60^\circ = \tfrac{1}{2}; \qquad (92)$$

whence, from (6), (7), and (10),

$$\cos.\ 30^\circ = \sin.\ 60^\circ = \sqrt{(1-\tfrac{1}{4})} = \tfrac{1}{2}\sqrt{3} \qquad (93)$$

$$\text{tang. } 30^\circ = \text{cotan. } 60^\circ = \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \sqrt{\tfrac{1}{3}} \qquad (94)$$

$$\text{cotan. } 30^\circ = \text{tang. } 60^\circ = \frac{1}{\sqrt{\tfrac{1}{3}}} = \sqrt{3} = 3\sqrt{\tfrac{1}{3}} \qquad (95)$$

$$\text{sec. } 30^\circ = \text{cosec. } 60^\circ = \frac{1}{\tfrac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}} = 2\sqrt{\tfrac{1}{3}} \qquad (96)$$

$$\text{cosec. } 30^\circ = \text{sec. } 60^\circ = \frac{1}{\tfrac{1}{2}} = 2. \qquad (97)$$

61. *Problem. To find the sine, &c. of the supplement of an angle.*

Solution. Make $M = 180^\circ$ in (34) and (36). They become, by means of (72) and (73),

$$\sin. (180^\circ - N) = \sin. 180^\circ \cos. N - \cos. 180^\circ \sin. N = \sin. N \qquad (98)$$

$$\cos. (180^\circ - N) = \cos. 180^\circ \cos. N + \sin. 180^\circ \sin. N = -\cos. N, \qquad (99)$$

whence, by (6) and (7),

$$\text{tang. } (180^\circ - N) = \frac{\sin. N}{-\cos. N} = -\text{tang. } N \qquad (100)$$

$$\text{cotan. } (180^\circ - N) = \frac{1}{-\text{tang. } N} = -\text{cotan. } N \qquad (101)$$

$$\text{sec. } (180^\circ - N) = \frac{1}{-\cos. N} = -\text{sec. } N \qquad (102)$$

$$\text{cosec. } (180^\circ - N) = \frac{1}{\sin. N} = \text{cosec. } N; \qquad (103)$$

that is, *the sine and cosecant of the supplement of an angle are the same with those of the angle itself; and the cosine, tangent, cotangent, and secant of the supplement are the negatives of those of the angle.*

62. *Corollary.* Since every obtuse angle is the supplement of an acute angle, it follows, from the preceding proposition, that *the sine and cosecant of an obtuse angle are positive, while its cosine, tangent, cotangent, and secant are negative.*

In the application of logarithmic calculation to negative numbers, the *absolute* values of these numbers are used (that is, their values taken *without regard to their signs*), and the effect of the signs on the result is considered separately. B., Table XXVII is extended, on this principle, to angles between 90° and 180°; and each line of the table corresponds to two angles, supplements of each other, which are given as having the same log. sin., &c. When the logarithm of the cos., tang., cotan., or sec. of an obtuse angle is taken, the letter *n* may be written after the log., to show that it corresponds to a negative number. On the other hand, when we have to find an angle by this table, from its *log. cos., log. tang., log. cotan., or log. cosec.*, we should take the *acute* value of the angle or the *obtuse* value, according as the function is known to be *positive* or *negative;* but *when only the log. sin. or log. cosec. is given, either value of the angle may be taken, if the function is positive, and neither value, if the function is negative.* When both the values are admissible, the geometrical conditions of the problem which may be under consideration will either enable us to discriminate between these values or else show that the problem admits of two solutions.

63. *Corollary.* The preceding corollary may also be obtained, and the relation between the two angles which are found on the same side of the page, in B., Table XXVII, may be illustrated, by making $M = 90°$ in (33) and (35). For we have, by (66) and (67),

$$\sin. (90° + N) = \cos. N \qquad (104)$$

$$\cos. (90° + N) = -\sin. N; \qquad (105)$$

whence, by (6) and (7),

$$\text{tang.} (90° + N) = -\text{cotan.} N \qquad (106)$$

$$\text{cotan.} (90° + N) = -\text{tang.} N \qquad (107)$$

$$\sec. (90° + N) = -\text{cosec.} N \qquad (108)$$

$$\text{cosec.} (90° + N) = \sec. N; \qquad (109)$$

that is, *the sine and cosecant of an angle which exceeds* 90° *are equal to the cosine and secant of its excess above* 90°, *while its cosine, tangent, cotangent, and secant are equal to the negatives of the sine, cotangent, tangent, and cosecant of this excess.*

64. *Problem. To find the sine, &c. of a negative angle.*

Solution. Make $M = 0°$ in (34) and (36). They become, by means of (66) and (67),

$$\sin.\ (-N) = -\sin.\ N \tag{110}$$
$$\cos.\ (-N) = \cos.\ N; \tag{111}$$

whence, by (6) and (7),

$$\text{tang.}\ (-N) = -\text{tang.}\ N \tag{112}$$
$$\text{cotan.}\ (-N) = -\text{cotan.}\ N \tag{113}$$
$$\sec.\ (-N) = \sec.\ N \tag{114}$$
$$\text{cosec.}\ (-N) = -\text{cosec.}\ N; \tag{115}$$

so that *the cosine and secant of the negative of an angle are the same with those of the angle itself; and the sine, tangent, cotangent, and cosecant of the negative of the angle are the negatives of those of the angle.*

65. *Problem. To find the sine, &c. of an angle which exceeds* 180°.

Solution. Make $M = 180°$ in (33) and (35). They become, by means of (72) and (73),

$$\sin.\ (180° + N) = -\sin.\ N \tag{116}$$
$$\cos.\ (180° + N) = -\cos.\ N; \tag{117}$$

whence, by (6) and (7),

$$\text{tang.}\ (180° + N) = \text{tang.}\ N \tag{118}$$
$$\text{cotan.}\ (180° + N) = \text{cotan.}\ N \tag{119}$$
$$\sec.\ (180° + N) = -\sec.\ N \tag{120}$$
$$\text{cosec.}\ (180° + N) = -\text{cosec.}\ N; \tag{121}$$

that is, *the tangent and cotangent of an angle which exceeds* 180° *are equal to those of its excess above* 180°; *and the sine cosine, secant, and cosecant of this angle are the negatives of those of its excess.*

66. *Corollary.* If the excess of the angle above 180° is less than 90°, the angle is contained between 180° and 270°; so that *the tangent and cotangent of an angle which exceeds* 180° *and is less than* 270° *are positive; while its sine, cosine, secant, and cosecant are negative.*

If the excess of the angle above 180° is greater than 90° and less than 180°, the angle is contained between 270° and 360°; so that, by §§ 65 and 62, *the cosine and secant of an angle which exceeds* 270° *and is less than* 360° *are positive; while its sine, tangent, cotangent, and cosecant are negative.*

By the help of § 65, B., Table XXVII can be used to find the sine, &c. of an angle which exceeds 180°.

67. *Corollary.* The results of the preceding corollary may also be obtained from (34) and (36). For by making $M = 360°$, we have, by § 58,

$$\sin. (360° - N) = -\sin. N = \sin. (-N) \qquad (122)$$

$$\cos. (360° - N) = \cos. N = \cos. (-N); \qquad (123)$$

whence, by (6) and (7),

$$\text{tang.} (360° - N) = -\text{tang.} N = \text{tang.} (-N) \qquad (124)$$

$$\text{cotan.} (360° - N) = -\text{cotan.} N = \text{cotan.} (-N) \qquad (125)$$

$$\sec. (360° - N) = \sec. N = \sec. (-N) \qquad (126)$$

$$\text{cosec.} (360° - N) = -\text{cosec.} N = \text{cosec.} (-N); \qquad (127)$$

that is, *the cosine and secant of an angle are the same with those of the remainder after subtracting the angle from* 360° *while its sine, tangent, cotangent, and cosecant are the negatives of those of this remainder.*

68. *Problem. To find the sine, &c. of an angle which exceeds* 360°.

Solution. Make $M = 360°$ in (33) and (35). They become, by means of (84) and (85),

$$\sin. (360° + N) = \sin. N \qquad (128)$$

$$\cos. (360° + N) = \cos. N; \qquad (129)$$

that is, *all the trigonometric functions of an angle which exceeds* 360° *are equal to those of its excess above* 360°.

69. *Theorem. The sine, tangent, and secant of an acute angle increase with the increase of the angle; the cosine, cotangent, and cosecant decrease.*

Proof. I. It appears from (17) that sin. $(M + m)$ exceeds sin. M

by sin. m cos. M, which is a positive quantity when M is acute. If, therefore, an acute angle is increased by a very small amount, its sine is increased.

II. It appears from (19) that cos. M exceeds cos. $(M + m)$ by sin. m sin. M, which is a positive quantity; and, therefore, the cosine of the acute angle decreases with the increase of the angle.

III. The tangent of an angle is, by (7), the quotient of its sine divided by its cosine. It is, therefore, a fraction whose numerator increases with the increase of the angle, while its denominator decreases. Either of these changes in the terms of the fraction would increase its value; and, therefore, the tangent of an acute angle increases with the increase of the angle.

IV. The cosecant, secant, and cotangent of an angle are, by (6), the respective reciprocals of the sine, cosine, and tangent. But the reciprocal of a quantity increases with the decrease of the quantity, and the reverse. It follows, then, from the preceding demonstrations, that its secant increases with the increase of the acute angle, while its cosecant and cotangent decrease.

70. *Theorem. The absolute values* (*that is, the values taken without regard to their signs*) *of the sine, tangent, and secant of an obtuse angle decrease with the increase of the angle; while those of the cosine, cotangent, and cosecant increase.*

Proof. The supplement of an obtuse angle is an acute angle of which the sine, &c. are, in absolute value, by § 61, the same as those of the obtuse angle. But this acute angle decreases with the increase of the obtuse angle, and at the same time its sine, tangent, and secant decrease, while its cosine, cotangent, and cosecant increase.

71. *Scholium. The trigonometric functions of any angle can be represented geometrically*, by lines drawn according to the conditions prescribed in §§ 18 – 20; provided we adopt the principle, which has been already applied to angular magnitude, of using the *opposite signs*, *plus* and *minus*, to denote *opposite directions.*

Thus if $ABCD$ (figs. 64, 65, 66, and 67) is a circle, described with the radius unity, the trigonometric functions of the angle AOE can be represented as follows: —

sin. $AOE = PE = OR$, tang. $AOE = AT$, sec. $AOE = OT$,
cos. $AOE = RE = OP$, cotan. $AOE = BS$, cosec. $AOE = OS$.

To prove this, it is only necessary to show that these equations can always be made to conform with the results deduced above from the general formulas, if we assume those directions as positive which will make the functions of an acute angle positive.

Let it be agreed to measure the angle AOE from the line OA, calling the circular direction $ABCD$ positive, and the circular direction $ADCB$ negative; to call upward direction positive, for the sine and tangent, and downward, negative; direction towards the right positive, for the cosine and cotangent, and that towards the left negative; the direction of the radius OE positive, for the secant and cosecant, and the opposite direction negative.

Then, first, the trigonometric functions are represented, in the figures, with *the true signs.* All the functions of the acute angle (fig. 64) are made positive; if the angle is obtuse (fig. 65), the sin. (PE) and cosec. (OS) are made positive, while the cos. (RE), tang. (AT), cotan. (BS), and sec. (OT) are made negative, as in § 62; and, in like manner, § 66 is sustained, for angles in the third and fourth quadrants (figs. 66, 67).

Secondly, *the true absolute values* are given in the figures. Thus, the functions of the obtuse angle AOE (fig. 65) are, by the figure, the same, absolutely, as those of its supplement EOC, as in § 61. For, in absolute value,

sin. $EOC = PE$, tang. $EOC = CQ = AT$, sec. $EOC = OQ = OT$,
cos. $EOC = RE$, cotan. $EOC = BS$, cosec. $EOC = OS$.

So it may be shown that figs. 66 and 67 give the same results as § 65.

Again, the functions are the same, in the figures, whether we regard AOB as measured by the arc AB or by a whole circumference *plus* that arc; which agrees with § 68. And they are the same, whether we measure AOB in the direction $ABCD$ or in the direction $ADCB$; which agrees with § 67.

Moreover, as the angle increases, its functions increase or decrease, in the figures, conformably to §§ 69 and 70.

Lastly, the results of §§ 55 - 58 are easily obtained from the figures. The geometrical interpretation of $\pm\infty$ may be illustrated in the case of tang. 90°; — the radius OB, being then parallel to the tangent drawn at A, may be conceived to intersect it, if produced, *at an infinite distance from A, either above or below.*

CHAPTER VI.

OBLIQUE TRIANGLES.

72. *Theorem. The sides of a triangle are directly proportional to the sines of the opposite angles.* [B., p. 13.]

Proof. In the triangle ABC (figs. 2 and 3) denote the sides opposite the angles A, B, C, respectively, by the letters a, b, c. We are to prove that

$$\sin. A : \sin. B : \sin. C = a : b : c. \tag{130}$$

From the vertex B, let fall on the opposite side the perpendicular BP; and let

$$p = PB.$$

The right triangle BAP gives, by (1),

$$\sin. A = \frac{PB}{AB} = \frac{p}{c};$$

or

$$p = c \sin. A. \tag{131}$$

Also, the right triangle BCP gives

$$\sin. C = \frac{PB}{CB} = \frac{p}{a};$$

for if C is acute (as in fig. 2), this follows directly from (1); and if C is obtuse (as in fig. 3), it has, by § 61, the same sine as its supplement PCB. Hence we have

$$p = a \sin. C. \tag{132}$$

Comparing (131) and (132), we have

$$c \sin. A = a \sin. C,$$

which may be converted into the following proportion,

$$\sin. A : \sin. C = a : c.$$

In the same way, it may be proved that

$$\sin. A : \sin. B = a : b;$$

and these two proportions may be written in one, as follows: —

$$\sin. A : a = \sin. B : b = \sin. C : c;$$

or as in (130).

73. *Problem. To solve a triangle, when one of its sides and two of its angles are known.* [B., p. 41.]

Solution. First. The third angle may be found by subtracting the sum of the two given angles from 180°.

Secondly. To find either of the other sides, we have only to make use of a proportion, derived from § 72. As the sine of the angle opposite the given side is to the sine of the angle opposite the required side, so is the given side to the required side. Thus, if a (fig. 1) were the given and b the required side, we should have the proportion

$$\sin. A : \sin. B = a : b;$$

whence by (6)

$$b = \frac{a \sin. B}{\sin. A} = a \sin. B \text{ cosec. } A. \quad (133)$$

74. Examples.

1. Given one side of a triangle equal to 22.791, and the adjacent angles equal to 32° 41′ and 47° 54′; to solve the triangle.

Solution. Making

$$a = 22.791,\ B = 32° \ 41',\ C = 47° \ 54'$$

we have

$$A = 180° - (32° \ 41' + 47° \ 54') = 99° \ 25'.$$

Then, by (133),

$A = 99° \ 25'$ cosec.	10.00589		10.00589
$B = 32° \ 41'$ sin.	9.73239	$C = 47° \ 54'$ sin.	9.87039
$a = 22.791$	1.35776		1.35776
$b = 12.475$	*1.09604;	$c = 17.141$	*1.23404

Ans. The other angle = 99° 25′

The other sides = { 12.475, 17.141 }

2. Given one side of a triangle equal to 327.06, the opposite angle

* 20 is subtracted from each of these characteristics, because the two sines and the cosecant are taken from the table without the diminution which is required by § 30.

equal to 8° 3′, and one of the adjacent angles equal to 154° 22′; to solve the triangle.

Ans. The other angle = 17° 35′

$$\text{The other sides} = \begin{cases} 1010.4 \\ 705.5 \end{cases}$$

75. *Problem. To solve a triangle, when two of its sides and an angle opposite one of the given sides are known.* [B., p. 42.]

Solution. First. The angle opposite the other given side is found by the proportion of § 72. As the side opposite the given angle is to the other given side, so is the sine of the given angle to the sine of the required angle. Thus, if (fig. 1) a and b are the given sides and A the given angle, the angle B is found by the proportion

$$a : b = \sin. A : \sin. B;$$

whence

$$\sin. B = \frac{b \sin. A}{a}. \qquad (134)$$

Secondly. The third angle is found by subtracting the sum of the two known angles from 180°.

Thirdly. The third side is found by the proportion. As the sine of the given angle is to the sine of the angle opposite the required side, so is the side opposite the given angle to the required side. That is, in the present case,

$$\sin. A : \sin. C = a : c;$$

whence

$$c = \frac{a \sin. C}{\sin. A} = a \sin. C \operatorname{cosec.} A. \qquad (135)$$

76. *Scholium.* Since the angle B is found by means of its sine, and since the value of sin. B obtained from (134) is necessarily positive, we must, by § 62, have recourse to the geometrical conditions of the problem in order to determine which of the two supplementary angles given in the tables for the same sine ought to be taken as the value of B. The triangle is constructed geometrically from the given data as follows: — Draw an angle A (fig. 68) equal to the given angle, on one of its legs lay off AC equal to the adjacent side b, and from C as a centre, with a radius equal to a, describe an arc cutting the other leg of the angle A at B; draw AB, and

ABC is the triangle required. It is evident that if A is acute and if the radius a is less than b and greater than the perpendicular PC, the arc will cut AP in two points, B' and B'', giving the two triangles $AB'C$ and $AB''C$; and, since $B'CB''$ is isosceles, it is further evident that the two values of B, $CB''A$ and $CB'A$, are supplements of each other. The two values of B found by (134) correspond, therefore, to *two solutions* of the problem; but in some cases, one of these solutions is impossible, and, in some cases, both are impossible.

77. *Scholium.* If the given value of A is obtuse, the obtuse value of B and the corresponding solution of the problem must be rejected, because a triangle can have only one obtuse angle. In this case, the point B'' (fig. 69) falls on the wrong side of A, so that the triangle $AB''C$ does not contain the given angle.

If A is obtuse, and $a = b$ or $a < b$, neither solution is possible, for the obtuse angle of a triangle must be opposite the greatest side. In these cases, the geometrical construction also fails (fig. 69). If, however, $a > b$, that solution is always possible in which B is acute.

78. *Scholium.* If A is acute, and $a = b$ or $a > b$, the obtuse value of B cannot be taken, because the obtuse angle must be opposite the greatest side; and this is also evident (fig. 70) from the geometrical construction. But, in these cases, that solution is always possible in which B is acute.

If a is so much less than b as to be equal to the perpendicular PC, which is, by (1), equal to b sin. A, the points B' and B'' coincide, and there is only one solution, the right triangle APC.

If $a < b$ sin. A, the circle will not cut AP at all, and neither solution of the problem is possible.

79. *Scholium.* The above results may be also obtained from the trigonometric solution of the problem. For we must have $A + B <$ 180°, since $A + B + C = 180°$. Now, if $a > b$, (134) gives

$$\sin. B < \sin. A;$$

so that, by § 69, if A is acute, denoting the acute value of B by B' and the obtuse by B'', we have

$$B' < A, \qquad B'' > 180° - A;$$

but, if A is obtuse, by § 70,

$$B'' > A, \qquad B' < 180° - A;$$

and, in either case,

$$A + B' < 180°, \qquad A + B'' > 180°;$$

and B'' must be rejected.

It may be shown in like manner that if $a = b$, and if A is acute, B'' must be rejected, but if A is obtuse, both B' and B'' must be rejected; also that if $a < b$, both values of B are admissible when A is acute, and inadmissible when A is obtuse.

If $a = b$ sin. A, we have by (67),

$$\text{sin. } B = \frac{b \text{ sin. } A}{a} = 1,$$

$$B' = B'' = 90°;$$

and if $a < b$ sin. A,

$$\sin B > 1,$$

which is impossible for any real value of B.

80. Examples.

1. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the first side equal to 55° 28′ 12″; to solve the triangle.

Solution. Making

$$b = 92.341, \quad a = 77.245, \quad A = 55° \, 28' \, 12'',$$

we have, by (134),

$a = 77.245$		(ar. co.)	8.11213
$b = 92.341$			1.96540
$A = 55° \, 28' \, 12''$		sin.	9.91584
$B = 80° \, 1'$	or $= 99° \, 59'$	sin.	9.99337
$A + B = 135° \, 29' \, 12''$	or $= 155° \, 27' \, 12''$		
$C = 44° \, 30' \, 48''$	or $= 24° \, 32' \, 48''$		

Then, by (135),

$a = 77.245$		1.88787		1.88787
$C = 44° \, 30' \, 48''$	sin.	9.84576	or $= 24° \, 32' \, 48''$ sin.	9.61850
$A = 55° \, 28' \, 12''$	cosec.	10.08416		10.08416
$c = 65.734$		1.81779	or $= 38.952$	1.59053

Ans. The third side $= 65.734$ or $= 38.952$

$$\text{The other angles} = \begin{cases} 80° \, 1' \\ 44° \, 30' \, 48'' \end{cases} \text{or} = \begin{cases} 99° \, 59' \\ 24° \, 32' \, 48'' \end{cases}$$

2. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the second side equal to 55° 28′ 12″; to solve the triangle.

Ans. The third side $= 110.7$

The other angles $= \begin{cases} 43^\circ\ 33'\ 44'' \\ 80^\circ\ 58'\ 4'' \end{cases}$

3. Given two sides of a triangle equal to 40 and 50, and the angle opposite the first side equal to 45°; to solve the triangle.

Ans. The third side $\doteq 54.061$ or $= 16.65$

The other angles $= \begin{cases} 62^\circ\ 7' \\ 72^\circ\ 53' \end{cases}$ or $= \begin{cases} 117^\circ\ 53' \\ 17^\circ\ 7' \end{cases}$

4. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the second side equal to 124° 31′ 48″; to solve the triangle.

Ans. The third side $= 23.129$

The other angles $= \begin{cases} 43^\circ\ 33'\ 44'' \\ 11^\circ\ 54'\ 28'' \end{cases}$

5. Given two sides of a triangle equal to 77.245 and 92.341, and the angle opposite the first side equal to 124° 31′ 48″; to solve the triangle.

Ans. The question is impossible.

6. Given two sides of a triangle equal to 75.486 and 92.341, and the angle opposite the first side equal to 55° 28′ 12″; to solve the triangle.

Ans. The question is impossible.

81. *Theorem. The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.* [B., p. 13.]

Proof. We have (fig. 1)

$$a : b = \sin. A : \sin. B;$$

whence, by the theory of proportions,

$$a + b : a - b = \sin. A + \sin. B : \sin. A - \sin. B,$$

which, expressed fractionally, is

$$\frac{a+b}{a-b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B}.$$

But the general formula (47) gives, for any two angles A and B,

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\text{tang.} \frac{1}{2}(A + B)}{\text{tang.} \frac{1}{2}(A - B)};$$

whence

$$\frac{a + b}{a - b} = \frac{\text{tang.} \frac{1}{2}(A + B)}{\text{tang.} \frac{1}{2}(A - B)}; \qquad (136)$$

or

$$a + b : a - b = \text{tang.} \tfrac{1}{2}(A + B) : \text{tang.} \tfrac{1}{2}(A - B).$$

82. *Problem. To solve a triangle, when two of its sides and the included angle are given.* [B., p. 43.]

Solution. Let the two sides a and b (fig. 1) be given, and the included angle C; to solve the triangle.

First. To find the two unknown angles. Subtract the given angle C from 180°, and the remainder is the sum of A and B, for the sum of the three angles of a triangle is 180°; that is,

$$A + B = 180° - C,$$

and $\quad \frac{1}{2}(A + B) = 90° - \frac{1}{2} C = \text{complement of } \frac{1}{2} C.$

The difference of A and B is then found by (136)

$$a + b : a - b = \text{tang.} \tfrac{1}{2}(A + B) : \text{tang.} \tfrac{1}{2}(A - B).$$

But we have

$$\text{tang.} \tfrac{1}{2}(A + B) = \text{cotan.} \tfrac{1}{2} C;$$

whence

$$\text{tang.} \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \text{tang.} \tfrac{1}{2}(A + B) = \frac{a - b}{a + b} \text{cotan.} \tfrac{1}{2} C; \quad (137)$$

in which the *acute* value given in the tables must be taken for $\frac{1}{2}(A - B)$, being made positive when $a > b$, so that tang. $\frac{1}{2}(A - B)$ comes out positive, and, by (112), negative when $a < b$, so that tang. $\frac{1}{2}(A - B)$ comes out negative.

The angle A is then found by adding $\frac{1}{2}(A - B)$ to $\frac{1}{2}(A + B)$, and the angle B by subtracting $\frac{1}{2}(A - B)$ from $\frac{1}{2}(A + B)$.

Secondly. The third side is found by the proportion

$$\sin. A : \sin. C = a : c;$$

whence $\quad c = \dfrac{a \sin. C}{\sin. A}.$

83. Examples.

1. Given two sides of a triangle equal to 99.341 and 1.234, and their included angle equal to 169° 58′; to solve the triangle.

Solution. Making $a = 99.341$, $b = 1.234$, $C = 169° 58'$; we have $\frac{1}{2}(A + B) = 90° - \frac{1}{2} C = 5° 1'$; and, by (137),

$a + b = 100.575$	(ar. co.) 7.99751
$a - b = 98.107$	1.99170
$\frac{1}{2}(A + B) = 5° 1'$	tang. 8.94340
$\frac{1}{2}(A - B) = 4° 53' 39''$	tang. 8.93261
$A = 9° 54' 39''$	
$B = 0° 7' 21''$	
$a = 99.341$	1.99712
$C = 169° 58'$	sin. 9.24110
$A = 9° 54' 39''$	cosec. 10.76519
$c = 100.56$	2.00241

Ans. The third side $= 100.56$

The other angles $= \begin{cases} 9° 54' 39'' \\ 0° 7' 21'' \end{cases}$

2. Given two sides of a triangle equal to 10.121 and 15.421, and the included angle equal to 41° 2′; to solve the triangle.

Ans. The other side $= 10.236$

The other angles $= \begin{cases} 40° 28' 28'' \\ 98° 29' 32'' \end{cases}$

84. *Theorem. Either side of a triangle is to the sum of the other two as their difference is to the difference of the segments of the first side made by a perpendicular from the opposite vertex, if the perpendicular fall within the triangle, or to the sum of the distances from the extremities of the base to the foot of the perpendicular, if it fall without the triangle.* [B., p. 14.]

Proof. Let AC (figs. 12 and 13) be the side of the triangle ABC on which the perpendicular is dropped, and BP the perpendicular.

From B as a centre, with a radius equal to BC, the shorter of the other two sides, describe the circumference $CC'E'E$. Produce AB to E' and AC to C', if necessary.

Then, since AC and AB are secants, they are inversely proportional to their parts without the circle; that is, in both figures,

$$AC : AE' = AE : AC'.$$

But

$$AE' = AB + BE' = AB + BC$$
$$AE = AB - BE = AB - BC,$$

and

$$\text{(fig. 12) } AC' = AP - PC' = AP - PC$$
$$\text{(fig. 13) } AC' = AP + PC' = AP + PC;$$

whence

$$\text{(fig. 12) } AC : AB + BC = AB - BC : AP - PC$$
$$\text{(fig. 13) } AC : AB + BC = AB - BC : AP + PC.$$

85. ***Problem.*** *To solve a triangle, when its three sides are given.* [B., p. 43.]

Solution. Suppose a perpendicular BP (fig. 2 or 3) dropped on the side b from the opposite vertex.

First find the value of the fourth term x of the proportion,

$$b : c + a = c - a : x;$$

and we have, by figs. 2 and 3 and § 84,

$$b = AP \pm PC,$$
$$x = AP \mp PC;$$

in which the upper signs correspond to the case (fig. 2) of the perpendicular falling within the triangle, and the lower signs to the case (fig. 3) of its falling without. If $x < b$, we have the former case; if $x > b$, the latter. In either case, by finding the half sum of b and x, we have AP; and by finding the half difference of b and x, we have PC.

Then, in triangles ABP and BCP, we have

$$\cos. A = \frac{AP}{c} = \frac{b + x}{2c};$$

and

$$\text{(fig. 2) } \cos. C = \frac{PC}{a} = \frac{b - x}{2a},$$

or, by (99), (fig. 3) $\cos. C = -\cos. PCB = -\frac{PC}{a} = \frac{b - x}{2a}.$

The third angle B is found by subtracting the sum of A and C from 180°.

86. *Corollary.* The above proportion gives

$$x = \frac{(c+a)(c-a)}{b} = \frac{c^2 - a^2}{b},$$

which, added to b, gives, by § 85,

$$2\,AP = b + x = b + \frac{c^2 - a^2}{b} = \frac{b^2 + c^2 - a^2}{b}.$$

Hence,

$$AP = \frac{b^2 + c^2 - a^2}{2\,b},$$

$$\cos. A = \frac{AP}{c} = \frac{b^2 + c^2 - a^2}{2\,b\,c}; \tag{138}$$

and, by reduction and transposition,

$$a^2 = b^2 + c^2 - 2\,b\,c\cos. A; \tag{139}$$

that is, *the square of either side of a triangle is equal to the sum of the squares of the other two sides diminished by twice their product multiplied by the cosine of the included angle.*

87. *Corollary.* The above proposition, when applied to the right triangle, becomes the Pythagorean proposition. For, if A is the right angle and a the hypothenuse, $\cos. A = 0$, by (66), and (139) becomes

$$a^2 = b^2 + c^2;$$

but if c is the hypothenuse, $\cos. A = \frac{b}{c}$, by (4), and (139) becomes

$$a^2 = b^2 + c^2 - 2\,b^2 = c^2 - b^2.$$

88. *Corollary.* Add unity to both sides of (138), and we have

$$1 + \cos. A = 1 + \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{b^2 + 2\,bc + c^2 - a^2}{2\,bc}$$

$$= \frac{(b+c)^2 - a^2}{2\,bc}. \tag{140}$$

Since the numerator of (140) is the difference of two squares, it may be separated into two factors, and we have

$$1 + \cos. A = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Now, representing half the sum of the three sides of a triangle by s, we have

$$2s = a + b + c, \tag{141}$$

and

$$2s - 2a = 2(s-a) = a + b + c - 2a = b + c - a. \tag{142}$$

If we substitute these values in the above equation, it becomes

$$1 + \cos. A = \frac{4s(s-a)}{2bc} = \frac{2s(s-a)}{bc}. \tag{143}$$

But, by (55),

$$1 + \cos. A = 2(\cos. \tfrac{1}{2} A)^2.$$

Hence

$$2(\cos. \tfrac{1}{2} A)^2 = \frac{2s(s-a)}{bc}$$

or

$$(\cos. \tfrac{1}{2} A)^2 = \frac{s(s-a)}{bc} \tag{144}$$

$$\cos. \tfrac{1}{2} A = \sqrt{\left(\frac{s(s-a)}{bc}\right)}. \tag{145}$$

Since A may represent either of the angles, provided a represents the opposite side, we have similar equations for the angles B and C; that is,

$$\cos. \tfrac{1}{2} B = \sqrt{\left(\frac{s(s-b)}{ac}\right)} \tag{146}$$

$$\cos. \tfrac{1}{2} C = \sqrt{\left(\frac{s(s-c)}{ab}\right)}; \tag{147}$$

and (145 - 147), which correspond to B., p. 14, prop. LXI, may be used to calculate the angles of a triangle when the three sides are known; each half angle being taken less than 90°, so that the whole angle may be less than 180°.

89. *Corollary.* Subtract both sides of (138) from unity, and we have

$$1 - \cos. A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - b^2 + 2bc - c^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc}. \tag{148}$$

Since the numerator of (148) is the difference of two squares, it may be separated into two factors, as follows,

$$1 - \cos. A = \frac{(a-b+c)\ (a+b-c)}{2\,b\,c}.$$

But, from (141),

$$2\,s - 2\,b = 2\,(s-b) = a+b+c-2\,b = a-b+c \qquad (149)$$

$$2\,s - 2\,c = 2\,(s-c) = a+b+c-2\,c = a+b-c. \qquad (150)$$

If we substitute these values in the above equation, it becomes

$$1 - \cos. A = \frac{4\,(s-b)\ (s-c)}{2\,b\,c} = \frac{2\,(s-b)\ (s-c)}{b\,c}. \qquad (151)$$

But, by (56),

$$1 - \cos. A = 2\,(\sin. \tfrac{1}{2} A)^2.$$

Hence, by reduction,

$$\sin. \tfrac{1}{2} A = \sqrt{\left(\frac{(s-b)\ (s-c)}{b\,c}\right)}. \qquad (152)$$

In the same way, we have

$$\sin. \tfrac{1}{2} B = \sqrt{\left(\frac{(s-a)\ (s-c)}{a\,c}\right)} \qquad (153)$$

$$\sin. \tfrac{1}{2} C = \sqrt{\left(\frac{(s-a)\ (s-b)}{a\,b}\right)}; \qquad (154)$$

and these formulas give a *third method* of solving a triangle, when the three sides are known.

90. *Corollary.* The quotients of (152, 153, and 154) divided by (145, 146, and 147), are by (7)

$$\text{tang. } \tfrac{1}{2} A = \sqrt{\left(\frac{(s-b)\ (s-c)}{s\,(s-a)}\right)} \qquad (155)$$

$$\text{tang. } \tfrac{1}{2} B = \sqrt{\left(\frac{(s-a)\ (s-c)}{s\,(s-b)}\right)} \qquad (156)$$

$$\text{tang. } \tfrac{1}{2} C = \sqrt{\left(\frac{(s-a)\ (s-b)}{s\,(s-c)}\right)}; \qquad (157)$$

which furnish a *fourth method* of solution, when the sides are given.

91. The product of (143) by 151) is

$$1-(\cos. A)^2=\frac{4\,s\,(s-a)\,(s-b)\,(s-c)}{b^2\,c^2}.$$

But, by (9),

$$1-(\cos. A)^2=(\sin. A)^2.$$

Hence

$$(\sin. A)^2=\frac{4\,s\,(s-a)\,(s-b)\,(s-c)}{b^2\,c^2},$$

or

$$\sin. A=\frac{2\sqrt{[s\,(s-a)\,(s-b)\,(s-c)]}}{b\,c}. \qquad (158)$$

Similar formulas may be given for sin. B and sin. C; and we thus have a *fifth method* of solution, when the sides are given.

92. *Scholium.* The problem is impossible, if the given value of either side exceed the sum of the other two.

93. Examples.

1. Given the three sides of a triangle equal to 12.348, 13.561, and 14.091; to solve the triangle.

Solution. First Method.

Make (fig. 2 or 3) $a=12.348$, $b=13.561$, $c=14.091$.

Then, by § 85,

$b=13.561$ (ar. co.)	8.86771		
$c+a=26.439$	1.42224		
$c-a=1.743$	0.24130		
$x=3.3982$	0.53125	Since we find $x<b$, the case is that of fig. 2.	
$\frac{1}{2}(b+x)=AP=8.4796$	0.92838		
$\frac{1}{2}(b-x)=PC=5.0814$			0.70598
$c=14.091$ (ar. co.)	8.85106		
$a=12.348$			(ar. co.) 8.90840
$A=53°\ 0'$	cos. 9.77944		
$C=65°\ 42'$			cos. 9.61438

$$B=180°-(A+C)$$
$$=180°-118°\ 42'=61°\ 18'.$$

Second Method.

By (145, 146, and 147),

$a = 12.348$		(ar. co.) 8.90840	(ar. co.) 8.90840
$b = 13.561$	(ar. co.) 8.86771		(ar. co.) 8.86771
$c = 14.091$	(ar. co.) 8.85106	(ar. co.) 8.85106	
$s = 20.000$	1.30103	1.30103	1.30103
$s-a = 7.652$	0.88377		
$s-b = 6.429$		0.80882	
$s-c = 5.909$			0.77151
	2) 19.90357	2) 19.86931	2) 19.84865
cos.	9.95179	9.93466	9.92433
	$\frac{1}{2}A = 26° 30'$,	$\frac{1}{2}B = 30° 39'$,	$\frac{1}{2}C = 32° 51'$
	$A = 53° 0'$,	$B = 61° 18'$,	$C = 65° 42'$.

$$\textit{Ans.}\quad \text{The angles} = \begin{cases} 53° \ 0' \\ 61° \ 18' \\ 65° \ 42'. \end{cases}$$

The third, fourth, and fifth methods, furnished by (152 – 154), (155 – 157), and (158), might also be applied.

2. Given the three sides of a triangle equal to 17.856, 13.349, and 11.111; to solve the triangle.

$$\textit{Ans.}\quad \text{The angles} = \begin{cases} 93° \ 19' \ 16'' \\ 48° \ 16' \ 24'' \\ 38° \ 24' \ 20''. \end{cases}$$

CHAPTER VII.

LOGARITHMIC AND TRIGONOMETRIC SERIES.

94. *Problem. To develop the expression*

$$(1+i)^{\frac{x}{i}} \tag{159}$$

in which x *is finite, and* i *is any infinitesimal, into a series arranged according to powers of* x.

Solution. Since the binomial theorem is applicable to the development of all powers, it gives at once

$$(1+i)^{\frac{x}{i}} = 1 + \frac{x}{i}i + \frac{x}{i}\left(\frac{x}{i}-1\right)\frac{i^2}{1.2}$$

$$+ \frac{x}{i}\left(\frac{x}{i}-1\right)\left(\frac{x}{i}-2\right)\frac{i^3}{1.2.3} + \&c. \tag{160}$$

But $\frac{x}{i}$ is infinite and gives, therefore,

$$\frac{x}{i}-1 = \frac{x}{i}, \quad \frac{x}{i}-2 = \frac{x}{i}, \ \&c. \tag{161}$$

which, substituted in (160), give

$$(1+i)^{\frac{x}{i}} = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. \tag{162}$$

95. *Corollary.* When $x = 1$, (162) becomes

$$(1+i)^{\frac{1}{i}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. \tag{163}$$

which we may denote by e.

This quantity e *is one of frequent occurrence in analysis, and is celebrated on account of its having been adopted by Napier as the base of his system of logarithms,* which were called by him *hyperbolic logarithms,* but are known as the *Naperian logarithms.*

The value of e is easily computed, from the consideration that it is the *sum of the series* (163) *of which the first term is unity and each succeeding term is obtained by dividing the preceding term by the number of the place of this preceding term.*

Thus

1)1.000000
2)1.000000
3) .500000
4) .166667
5) .041667
6) .008333
7) .001389
8) .000198
9) .000025
.000003

$$(1+i)^{\frac{1}{i}} = e = 2.71828; \qquad (164)$$

which gives the value of e to five places. The sixth place is neglected, in the sum of the decimals, as being uncertain.

96. *Corollary.* The xth power of e is by (164 and (162)

$$e^x = \left((1+i)^{\frac{1}{i}}\right)^x = (1+i)^{\frac{x}{i}} = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c. \qquad (165)$$

97. *Corollary.* The ith power of (164) is

$$e^i = 1 + i. \qquad (166)$$

98. *Corollary.* The logarithm of (166) is

$$\log.\ (1+i) = i \log.\ e, \qquad (167)$$

in which, since i represents any infinitesimal, we may substitute $-i$, and thus we have

$$\log.\ (1-i) = -i \log.\ e. \qquad (168)$$

99. *Problem.* *To develop log.* (1 — x) *into a series of terms arranged according to the powers of* x.

Solution. Let the series be denoted as follows,

$$\log.\ (1-x) = A + A_1 x + A_2 x^2 + \&c. \ldots + A_n x^n + \&c.; \qquad (169)$$

in which the coefficients do not involve x, and the number below the coefficient denotes the power of x to which it belongs.

First. To find the value of A; let

$$x = 0,$$

which, by the principles of logarithms, reduces (169) to

$$\log.\ 1 = A = 0, \qquad (170)$$

and this term may, therefore, be dropped in the second member of (169).

Secondly. To find the value of A_1; let

$$x = i.$$

Then, in the second member of (169), each term is infinitely smaller than the preceding term and may be neglected in comparison with it, because

$$i : 1 = i^2 : i = i^3 : i^2 = \&c.;$$

and the whole second member may be reduced to its first term, $A_1\, i$; so that, by (168),

$$\log.\ (1-i) = A_1\, i = -i \log.\ e \qquad (171)$$

$$A_1 = -\log.\ e., \qquad (172)$$

Thirdly. To find the value of any coefficient, A_n; let r, r', r'', r''', . . . $r^{[n-1]}$ be the n roots of the equation

$$x^n = 1, \text{ or } x^n - 1 = 0, \qquad (173)$$

and by the theory of equations, we have for all values of x

$$x^n - 1 = (x-r)\ (x-r')\ (x-r'')\ \&c. \qquad (174)$$

Moreover the product of the negatives of the roots of an equation is equal to the constant term, which is, in this case, -1; that is,

$$-1 = (-r)\,(-r')\,(-r'')\ \&c. \tag{175}$$

The quotient of (174) by (175) is

$$1 - x^n = \frac{x-r}{-r} \cdot \frac{x-r'}{-r'} \cdot \frac{x-r''}{-r''} \cdot \&c.$$

$$= \left(1 - \frac{x}{r}\right) \left(1 - \frac{x}{r'}\right) \left(1 - \frac{x}{r''}\right) \&c. \tag{176}$$

the logarithm of which is

$$\log.\,(1 - x^n) = \log.\left(1 - \frac{x}{r}\right) + \log.\left(1 - \frac{x}{r'}\right)$$
$$+ \log.\left(1 - \frac{x}{r''}\right) + \&c. \tag{177}$$

But by substituting x^n for x, in (169), and the values of A and A_1 found above, we have

$$\log.\,(1 - x^n) = -\log.\,e\,x^n + A_2\,x^{2n} + \&c. \tag{178}$$

and any term of the second member of (177), as the first, is by (169)

$$\log.\left(1 - \frac{x}{r}\right) = -\log.\,e\,\frac{x}{r} + \&c. \ldots . + A_n \frac{x^n}{r^n}. \tag{179}$$

Since r is a root of the equation (173), that is, since

$$r^n = 1, \tag{180}$$

the term of (179) multiplied by x^n becomes $A_n\,x^n$, which is independent of the particular root r, r', &c., and, therefore, the same for each term of the second member of (177). The sum of all the terms of the second member of (177) which are multiplied by x^n is equal to either of them multiplied by their number, which is n; that is, it is

$$n\,A_n\,x^n, \tag{181}$$

Hence this term must be equal to the term of (178) which is multiplied by x^n; or

$$n\,A_n\,x^n = -\log.\,e\,x^n \tag{182}$$

$$A_n = -\frac{1}{n}\log. e; \tag{183}$$

that is,

$$A_2 = -\tfrac{1}{2}\log. e, \qquad A_3 = -\tfrac{1}{3}\log. e, \text{ \&c.};$$

and the resulting value of (169) is

$$\log. (1-x) = \log. e\,(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \text{\&c.}) \tag{184}$$

100. *Corollary.* By reversing the sign of x in (184), we have

$$\log. (1+x) = \log. e\,(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \text{\&c.}) \tag{185}$$

101. *Corollary.* The remainder of (185) diminished by (184) is

$$\log. \frac{1+x}{1-x} = 2\log. e\,(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \text{\&c.}) \tag{186}$$

102. *Corollary.* Since

$$a + x = a\left(1 + \frac{x}{a}\right)$$

we have, by (185),

$$\log. (a+x) = \log. a + \log. \left(1 + \frac{x}{a}\right)$$

$$= \log. a + \log. e\left(\frac{x}{a} - \tfrac{1}{2}\frac{x^2}{a^2} + \tfrac{1}{3}\frac{x^3}{a^3} - \tfrac{1}{4}\frac{x^4}{a^4} + \text{\&c.}\right) \tag{187}$$

103. *Corollary.* Equations (184), (185), and (187) may be used in calculating logarithmic tables. But, for this purpose, log. e must first be obtained; that is, by the definition of logarithms, we must solve the equation

$$10^x = e = 2.71828,$$

which gives

$$\log. e = x = 0.43429. \tag{188}$$

104. Examples.

1. Find the logarithm of 1.1.

Solution. By making, in (185),

$$x = 0.1$$

we have

$$\tfrac{1}{2}x^2 = 0.005000$$
$$\tfrac{1}{3}x^3 = 0.000333$$
$$\tfrac{1}{4}x^4 = 0.000025$$
$$\tfrac{1}{5}x^5 = 0.000002$$

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 = 0.09536$$

$$\log.\,(1+x) = (0.09536)\,(\log.\,e) = 0.09536 \times 0.43429 = 0.04139$$

2. Find the logarithm of 625, knowing that

$$\log.\,624 = 2.79518.$$

Solution. In this case we have, in (187),

$$a = 624,\ x = 1,\ \frac{x}{a} = \frac{1}{624}.$$

and $\frac{x}{a}$ is so small that its square and higher powers may be neglected in (187), whence

$$\log.\,625 = \log.\,624 + \frac{\log.\,e}{624}$$

$$= 2.79518 + \frac{0.43429}{624} = 2.79518 + 0.00070$$

$$= 2.79588.$$

3. Find the logarithm of .9. *Ans.* -0.04576 or $\bar{1}.95424$.

4. Find the logarithm of 1.01. *Ans.* 0.00432.

5. Find the logarithm of 1.095. *Ans.* 0.03941.

6. Find the logarithm of 1.003. *Ans.* 0.00130.

7. Find the logarithm of 463, knowing that

$$\log.\,462 = 2.66464.$$

Ans. 2.66558.

8. Find the logarithm of 1291, knowing that

$$\log. 1290 = 3.11059.$$

Ans. 3.11093.

9. Find the logarithm of 123.6, knowing that

$$\log. 123 = 2.08991$$

Ans. 2.09202.

105. *Problem. To express sines and cosines by means of exponential functions.*

Solution. The first member of the equation

$$\cos.^2 x + \sin.^2 x = 1 \tag{189}$$

may be written $\cos.^2 x - (-\sin.^2 x)$; that is, the difference of the two squares $\cos.^2 x$ and $(-\sin.^2 x)$, of which the roots are $\cos. x$ and $\sin. x . \sqrt{-1}$. This first member is, therefore, the product of the sum and difference of these two roots, or (189) may be written

$$(\cos. x + \sin. x . \sqrt{-1})(\cos. x - \sin. x . \sqrt{-1}) = 1.$$

The logarithm of this equation is

$$\log. (\cos. x + \sin. x . \sqrt{-1}) + \log. (\cos. x - \sin. x . \sqrt{-1}) = 0$$

or

$$\log. (\cos. x + \sin. x . \sqrt{-1}) = -\log. (\cos. x - \sin. x . \sqrt{-1}). \tag{190}$$

Denote either member of (190) by y, so that

$$\left.\begin{aligned} \log. (\cos. x + \sin. x . \sqrt{-1}) &= y, \\ \log. (\cos. x - \sin. x . \sqrt{-1}) &= -y, \end{aligned}\right\} \tag{191}$$

or

$$\cos. x + \sin. x . \sqrt{-1} = 10^y, \cos. x - \sin. x . \sqrt{-1} = 10^{-y}. \tag{192}$$

The sum of the last two equations is

$$2 \cos. x = 10^y + 10^{-y}. \tag{193}$$

Hence, by (55 and 56),

$$\cos.^2 \tfrac{1}{2} x = \tfrac{1}{2}(1 + \cos. x) = \tfrac{1}{4}(2 + 2\cos. x) = \tfrac{1}{4}(10^y + 2 + 10^{-y})$$

$$-\sin.^2 \tfrac{1}{2} x = \tfrac{1}{2}(\cos. x - 1) = \tfrac{1}{4}(2\cos. x - 2) = \tfrac{1}{4}(10^y - 2 + 10^{-y}),$$

of which the square roots are

$$\cos. \tfrac{1}{2} x = \tfrac{1}{2}(10^{\frac{1}{2}y} + 10^{-\frac{1}{2}y})$$

$$\sin. \tfrac{1}{2} x . \sqrt{-1} = \tfrac{1}{2}(10^{\frac{1}{2}y} - 10^{-\frac{1}{2}y}),$$

and the sum of these two equations is

$$\cos. \tfrac{1}{2} x + \sin. \tfrac{1}{2} x . \sqrt{-1} = 10^{\frac{1}{2}y}. \tag{194}$$

The comparison of (194) with the first equation of (192) shows that x may be changed into $\frac{1}{2} x$, provided that y is changed into $\frac{1}{2} y$. The same changes may, therefore, also be made in (194), or $\frac{1}{2} x$ may be changed into its half, that is, into $\frac{1}{4} x$, provided $\frac{1}{2} y$ is changed into $\frac{1}{4} y$; which gives

$$\cos. \tfrac{1}{4} x + \sin. \tfrac{1}{4} x . \sqrt{-1} = 10^{\frac{1}{4}y}. \tag{195}$$

A repetition of this change gives

$$\cos. \tfrac{1}{8} x + \sin. \tfrac{1}{8} x . \sqrt{-1} = 10^{\frac{1}{8}y}. \tag{196}$$

By continuing this process, x may be divided by any power of 2, however great, provided y is divided by the same power. Let, then,

$$m = 2^n \tag{197}$$

and we have

$$\cos. \frac{x}{m} + \sin. \frac{x}{m} . \sqrt{-1} = 10^{\frac{y}{m}}; \tag{198}$$

the logarithm of which is

$$\log. \left(\cos. \frac{x}{m} + \sin. \frac{x}{m} . \sqrt{-1}\right) = \frac{y}{m}. \tag{199}$$

But if, in (197), n is made infinite, m will also be infinite, and $\frac{x}{m}$

will be an infinitesimal, of which the cosine is unity and the sine is equal to its arc in the circle of which radius is unity; that is, (199) becomes, if the angle is expressed as in (16),

$$\log.\ (1 + \frac{x}{m} \sqrt{-1}) = \frac{y}{m}. \tag{200}$$

But, again, since $\frac{x}{m}$ is an infinitesimal, (200) becomes by means of (167),

$$\log.\ e \,.\, \frac{x}{m} \sqrt{-1} = \frac{y}{m}, \text{ or } y = x\sqrt{-1} \,.\, \log.\ e, \tag{201}$$

which substituted in (191) gives

$$\log.\,(\cos.\,x + \sin.\,x \,.\, \sqrt{-1}) = x\sqrt{-1} \,.\, \log.\ e = \log.\ e^{x\sqrt{-1}}$$
$$\log.(\cos.\,x - \sin.\,x \,.\, \sqrt{-1}) = -x\sqrt{-1} \,.\, \log.\,e = \log.\ e^{-x\sqrt{-1}} \tag{202}$$

or

$$\left.\begin{array}{l}\cos.\ x + \sin.\ x \,.\, \sqrt{-1} = e^{x\sqrt{-1}} \\ \cos.\ x - \sin.\ x \,.\, \sqrt{-1} = e^{-x\sqrt{-1}}\end{array}\right\} \tag{203}$$

106. *Corollary.* Half the sum of (203) is

$$\cos.\ x = \tfrac{1}{2}\,(e^{x\sqrt{-1}} + e^{x\sqrt{-1}}), \tag{204}$$

and half their difference, multiplied by $\sqrt{-1}$, is

$$\sin.\ x = -\tfrac{1}{2}\,(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}})\sqrt{-1}. \tag{205}$$

107. *Problem. To develop cos.* x *and sin.* x *in terms arranged according to powers of* x.

Solution. Since we have

$$(x\sqrt{-1})^2 = -x^2, (x\sqrt{-1})^3 = -x^3\sqrt{-1}, (x\sqrt{-1})^4 = x^4, \&c. \tag{206}$$

the substitution of $x\sqrt{-1}$ for x in (165) gives

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} \,.\, - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3}$$
$$+ \frac{x^4}{1.2.3.4} + \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \&c. \tag{207}$$

which gives, by reversing the sign of x,

$$e^{-x\sqrt{-1}} = 1 - x\sqrt{-1}. - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3}$$

$$+ \frac{x^4}{1.2.3.4} - \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \&c. \qquad (208)$$

Half the sum of (207) and (208) is, by (204),

$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c. \qquad (209)$$

Half their difference, multiplied by $\sqrt{-1}$, is, by (205),

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c. \qquad (210)$$

which are the series required. But it must not be forgotten that, in the second member of these equations, x is expressed *in terms of the radius as unity*, as in (16).

108. *Corollary*. Equations (209) and (210) can be used for calculating tables of sines and cosines.

109. Examples.

1. Find the sine and cosine of 13° 25′.

Solution. In this case, since 13° 25′ = 805′, x, or the arc of 13° 25′ in the circle of which radius is unity, is 805 times the arc of 1′; that is, by (13),

$$x = 805 \sin. 1' = 805 \times 0.000290888 = 0.234165$$

$$\frac{x^2}{1.2} = 0.027416, \qquad \frac{x^3}{1.2.3} = 0.002140$$

$$\frac{x^4}{1.2.3.4} = 0.000125, \qquad \frac{x^5}{1.2.3.4.5} = 0.000006$$

Hence $\cos. x = 0.97271$ $\qquad$ $\sin. x = 0.23203$

2. Find the sine and cosine of 6° 10′.

Ans. sin. 6° 10′ = 0.10742

cos. 6° 10′ = 0.99421

NAVIGATION AND SURVEYING.

NAVIGATION AND SURVEYING.

CHAPTER I.

PLANE SAILING.

1. *The figure of the earth* is considered, in Navigation, to be that of a perfect *sphere*, from which, in fact, it differs but slightly; and very small portions of its surface are regarded as *plane*. [B., p. 46.]

The earth performs a daily revolution around one of its diameters, which is called the *earth's axis*. [B., p. 48.]

The extremities of this axis on the surface of the earth are the *terrestrial poles;* one being the *north pole*, and the other the *south pole*. [B., p. 48.]

2. The section of the earth or the circumference of the section made by a plane which passes through the earth's centre and is perpendicular to its axis is the *terrestrial equator*. [B., p. 48.]

Parallels of latitude are the circumferences of small circles the planes of which are parallel to the equator.

3. *Meridians* are the circumferences of great circles which pass from one pole to the other. [B., p. 48.]

The *first meridian* is one arbitrarily assumed, to which all others are referred. In most countries, that has been taken as the first meridian which passes through the capital of the country.

4. The *latitude* of a place is its angular distance from the equator, the vertex of the angle being at the centre of the

earth; or it is the arc which is comprehended between the place and the equator, on the meridian passing through the place. [B., p. 48.]

If D (fig. 18) is the centre of the earth, C one of the poles, $A'B'$ a portion of the equator, and $B'C$ a portion of a meridian, the latitude of the place B is the angle $B'DB$ or the arc $B'B$.

Latitude is commonly expressed in degrees, &c. and is reckoned north and south of the equator from 0° to 90°; one direction being sometimes regarded as positive, and the other as negative. Thus, the latitude of Melbourne (Australia) is 37° 48′ S., or — 37° 48′ N.

5. The *difference of latitude* of two places is the angular distance between the parallels of latitude in which they are respectively situated, the vertex of the angle being at the centre of the earth; or it is the arc which is comprehended between the parallels of latitude, on any meridian. [B., p. 52.]

The difference of latitude of two places is equal to the difference of their latitudes, if they are on the same side of the equator, and to the sum of their latitudes, if they are on opposite sides of the equator. [B., p. 50.]

Difference of latitude is often regarded as a length and expressed in terms of the *nautical mile*, which is equal to a minute of arc measured on the circumference of a great circle. Thus, the diff. of lat. of San Francisco and Melbourne is 37° 49′ + 37° 48′ = 37° 49′ — (— 37° 48′) = 75° 37′ = 4537 miles.

6. The *longitude* of a place is the angle made by the plane of the meridian which passes through the place with the plane of the first meridian; or it is the arc of the equator comprehended between these two meridians. [B., p. 48.]

If $A'C$ (fig. 18) is a part of the first meridian, the longitude of B is the angle $A'DB'$ or the arc $A'B'$.

Longitude is reckoned east and west of the first meridian from 0° to 180°, or only towards the west from 0° to 360°. Thus, the longitude of Melbourne from Greenwich is 144° 59′ E or — 144° 59′ W or (360° — 144° 59′) W = 215° 1 W.

7. The *difference of longitude* of two places is the angle contained between the planes of their meridians ; or it is the arc of the equator comprehended between their meridians.

The difference of longitude of two places is equal to the difference of their longitudes, if they are on the same side of the first meridian, and to the sum of their longitudes, if they are on opposite sides of the first meridian, unless their sum be greater than 180°, *in which case the sum must be subtracted from* 360° *to give the difference of longitude.* [B., p. 50.]

Thus, the diff. of long. of Melbourne and San Francisco is 360° — (122° 31′ + 144° 59′) = 215° 1′ — 122° 31′ = 92° 30′ = 5550 miles.

8. A *rhumb line*, or *rhumb*, is a line drawn on the surface of the earth so as to cross every meridian at the same angle. Any two places can be connected by a rhumb line, and the length of the rhumb line is called the *nautical distance* between them. [B., p. 52.]

The parallels of latitude and the equator are rhumb lines running at right angles with the meridians, and any meridian is a rhumb line running north and south. In general, however, the rhumb line is not an arc of a circle, but, when indefinitely produced, it winds round and round the earth, somewhat after the manner of the thread of a screw, being always convex to the equator. Any very small part of it is, sensibly, a straight line.

The shortest distance between two places is that which is measured on the arc of a great circle. Ships are, therefore, sometimes navigated on great circles, but more commonly on rhumb lines, because the increase of distance is, in most cases, small, while there are several practical advantages in favor of rhumb sailing. In this treatise, rhumb sailing alone is considered, and the word *distance* is always used to denote the *nautical* distance.

9. The *course* of a ship at any time is the angle which her path at that time makes with the meridian she is crossing. The *bearing* of two places from each other is the angle at which the rhumb line connecting them crosses the meridians. [B., p. 52.]

When a ship sails on a rhumb, her course is everywhere the same and equal to the bearing of the place reached from the place left.

10. The *departure* of two places from each other is the actual amount of easting or westing made by a ship in sailing on a rhumb from one place to the other. If the places are so near each other that their meridians may be considered as parallel, the departure is obviously the distance of either place from the meridian of the other. But if the distance is great, let it be divided into very small portions, and the departure of the places is the sum of the departures corresponding to all these portions; and, since the portions may be made as small as we please, this method of finding the departure can be carried to an unlimited degree of accuracy. [B., pp. 52, 66.]

Thus, to find the departure of the places A and B (fig. 71); draw the rhumb line AB, divide it into small portions at the points a, b, c, &c., draw the meridians PA, PB, Pa, Pb, Pc, &c., and the parallels of latitude AA', BB', am, bn, cp, &c., so that ma is the departure of A and a, nb that of a and b, &c.; then

$$\text{dep. of } A \text{ and } B = ma + nb + pc + \&c.$$

The departure of A and B must be distinguished from the meridional distances AA' and BB', and also from the diff. long., which, when expressed in miles, is the meridional distance LL' measured on the equator.

11. For the purpose of expressing the course, navigators are in the habit of dividing the quadrant into eight equal parts called *points*, and of subdividing the points into halves and quarters. A point, therefore, is equal to one eighth of 90°, or to 11° 15′. Names are given to the directions determined by the different points, as in fig. 14, which represents the face of the card of the *Mariner's Compass*. [B., p. 52.]

The *Mariner's Compass* consists of this card, attached to a magnetic needle, which has the property of constantly pointing toward the north, and thereby shows the ship's course. [B., p. 52.]

Other ways of expressing the course are easily understood. Thus, N. 30° E. means 30° from N. towards E.

B., p. 53 contains a table of the degrees and minutes which correspond to every *quarter-point* of the compass; and B., Table XXV gives the log. sine, &c., for every quarter-point.

12. *Plane Sailing* embraces those problems of Navigation which involve only the Nautical Distance, the Course, or Bearing, the Difference of Latitude, and the Departure. It is a method of calculating any two of these quantities, when the other two are known. [B., p. 52.]

13. *Problem. To find the difference of latitude and the departure when the course and the distance are known.* [B., p. 54.]

Solution. First. Take the case where the distance is so small that the curvature of the earth's surface may be neglected: — Let AB (fig. 15) be the distance. Draw through A the meridian AC, and let fall on it the perpendicular BC. The angle A is the course, AC is the difference of latitude, and CB is the departure. Then, as in Pl. Trig. § 32,

$$\text{diff. lat.} = \text{dist.} \times \text{cos. course}, \qquad (211)$$

$$\text{departure} = \text{dist.} \times \text{sin. course}. \qquad (212)$$

Secondly. When the distance is great, as AB (fig. 71): — Divide it into small portions, as in § 10. Then AB', the difference of latitude of A and B, is evidently equal to the sum of the partial differences of latitude which correspond to the distances $A\,a$, &c. Hence, and by § 10,

$$\text{diff. lat.} = A\,m + a\,n + b\,p + \text{\&c.}$$

$$\text{departure} = m\,a + n\,b + p\,c + \text{\&c.}$$

But, since AB is a rhumb, each of the angles $m\,A\,a$, $n\,a\,b$, $p\,b\,c$, &c. is equal to the given course. Hence the right triangles $A\,m\,a$, $a\,n\,b$, $b\,p\,c$, &c. give

$$A\,m = A\,a \times \text{cos. course}, \qquad m\,a = A\,a \times \text{sin. course};$$

$$a\,n = a\,b \times \text{cos. course}, \qquad n\,b = a\,b \times \text{sin. course};$$

$$b\,p = b\,c \times \text{cos. course}, \qquad p\,c = b\,c \times \text{sin. course}; \text{ \&c. \&c.}$$

Adding each of these sets of equations, we have

$$\text{diff. lat.} = A\,m + a\,n + b\,p + \text{\&c.}$$

$$= (A\,a + a\,b + b\,c + \text{\&c.}) \times \text{cos. course},$$

$$\text{departure} = m\,a + n\,b + p\,c + \text{\&c.}$$

$$= (A\,a + a\,b + b\,c + \text{\&c.}) \times \text{sin. course}.$$

But

$$A\,a + a\,b + b\,c + \text{\&c.} = AB = \text{distance}.$$

Hence

$$\text{diff. lat.} = \text{dist.} \times \text{cos. course},$$

$$\text{departure} = \text{dist.} \times \text{sin. course};$$

precisely the same with (211) and (212); so that the diff. lat., dep., dist., and course have, in all cases, the same relative magnitude as if they formed the right triangle of fig. 15.

Hence all the problems of Plane Sailing may be solved by this right triangle. [B., p. 52.]

Tables of difference of latitude and departure, such as B., Tables I. and II., may be calculated by (211) and (212).

14. *Problem. To find the distance and the difference of latitude, when the course and the departure are known.* [B., p. 55.]

Solution. There are given (fig. 15) the angle A and the side CB. Hence, as in Pl. Trig. § 33,

$$\text{distance} = \text{departure} \times \text{cosec. course}, \qquad (213)$$

$$\text{diff. lat.} = \text{departure} \times \text{cotan. course}. \qquad (214)$$

15. *Problem. To find the distance and the departure when the course and the difference of latitude are known.* [B., p. 55.]

Solution. There are given (fig. 15) the angle A and the side AC. Then, as in Pl. Trig. § 34,

$$\text{distance} = \text{diff. lat.} \times \text{sec. course}, \qquad (215)$$

$$\text{departure} = \text{diff. lat.} \times \text{tang. course}. \qquad (216)$$

16. *Problem. To find the course and the difference of latitude, when the distance and the departure are known.* [B., p. 57.]

Solution. There are given (fig. 15) the hypothenuse AB and the side CB. Then, as in Pl. Trig. § 35,

$$\text{sin. course} = \frac{\text{departure}}{\text{distance}}, \qquad (217)$$

$$\text{diff. lat.} = \sqrt{[(\text{dist.})^2 - (\text{departure})^2]}. \qquad (218)$$

17. *Problem. To find the course and the departure when the distance and the difference of latitude are known.* [B., p. 56.]

Solution. There are given (fig. 15) the hypothenuse AB and the leg AC. Then, as in Pl. Trig. § 35,

$$\text{cos. course} = \frac{\text{diff. lat.}}{\text{distance}}, \qquad (219)$$

$$\text{departure} = \sqrt{[(\text{dist.})^2 - (\text{diff. lat.})^2]}. \qquad (220)$$

18. ***Problem.*** *To find the course and the distance when the departure and the difference of latitude are known.* [B., p. 57.]

Solution. There are given (fig. 15) the legs AC and BC. Then, as in Pl. Trig. § 36,

$$\text{tang. course} = \frac{\text{departure}}{\text{diff. lat.}}, \tag{221}$$

$$\text{dist.} = \text{diff. lat.} \times \text{sec. course.} \tag{222}$$

19. EXAMPLES.

1. A ship sails from latitude 3° 45′ S., upon a course N. by E., a distance of 2345 miles; to find the latitude at which she arrives and the departure which she makes.

Ans. Latitude = 34° 34′ N.
Departure = 457 miles.

2. A ship sails from latitude 62° 19′ N., upon a course W. N. W., till she makes a departure of 1000 miles; to find the latitude at which she arrives and the distance sailed.

Ans. Latitude = 69° 13′ N.
Distance = 1082 miles.

3. The bearing of Paris from Athens is N. 54° 56′ W.; find the distance and departure of these two places from each other.

Ans. Distance = 1135 miles.
Departure = 929 miles.

4. A ship sails, from latitude 72° 3′ S., a distance of 2000 miles, upon a course between the north and the west, and makes a departure of 1000 miles; find the latitude at which she arrives and the course.

Ans. Latitude = 43° 11′ S.
Course = N. 30° W.

5. The distance from New Orleans to Portland is 1256 miles; find the bearing and departure.

Ans. Bearing = N. 49° 18′ E.
Departure = 952 miles.

6. The departure of Boston from Canton is 8786 miles; find the bearing and distance.

Ans. Bearing = N. 82° 31′ E.
Distance = 8862 miles.

CHAPTER II.

TRAVERSE SAILING.

20. *A traverse* is an irregular track made by a ship which sails on several different courses in succession.

The object of *Traverse Sailing* is to reduce a traverse to a single course; that is, to find the single track which is equivalent to the combination of several successive tracks; when the whole distance sailed is so small that the curvature of the earth's surface may be neglected. This is called *working the traverse.* [B., p. 59.]

21. *Problem. To reduce several successive tracks of a ship to one, when the curvature of the earth's surface may be neglected.* [B., p. 59.]

Solution. Suppose the ship to start from the point A (fig. 17), and to sail first from A to B, then from B to C, then from C to E, and lastly from E to F; to find the bearing and distance of F from A. Call the differences of latitude corresponding to the 1st, 2d, 3d, and 4th tracks, the 1st, 2d, 3d, and 4th differences of latitude; and call the corresponding departures the 1st, 2d, 3d, and 4th departures. The whole northing or southing made by the ship on her successive courses is evidently equal to the difference of latitude of the place of arrival and that of starting; and, *if we neglect the earth's curvature and consider the meridians as parallel,* the whole easting or westing made is equal to that which would have been made on a direct course; that is, to the departure of the places; or, in the case of fig. 17,

$$\text{diff. lat. of } A \text{ and } F = \text{1st diff. lat.} - \text{2d diff. lat.} + \text{3d diff. lat.} - \text{4th diff. lat.}$$

$$\text{dep. of } A \text{ and } F = \text{1st dep.} - \text{2d dep.} - \text{3d dep.} + \text{4th dep.}$$

Hence, *the difference of latitude of the place of arrival and the place of starting is found by taking the difference between the sum of the northings made on the northerly courses and the sum of the* *southings made on the southerly courses; and their departure is found by taking the difference between the sum of the eastings made on the easterly courses and the sum of the westings made on the westerly courses. When the difference of latitude and the departure are known, the direct course and the nautical distance can be found by* § 18.

22. The calculations of traverse sailing are usually put into a tabular form, as in the following example. In the *first* column of the table are the numbers of the tracks; in the *second* and *third* columns are the courses and distances; in the *fourth* and *fifth* columns are the differences of latitude, the column headed N. corresponding to the northerly courses; and that headed S. to the southerly courses; in the *sixth* and *seventh* columns are the departures, the column headed E. corresponding to the easterly courses, and that headed W. to the westerly courses. [B., p. 59.]

23. Examples.

1. A ship sails on several successive tracks, in the order and with the courses and distances of the first three columns of the following table; find the bearing and distance of the place the ship is in from that which she left.

No.	Course.	Dist.	N.	S.	E.	W.
1	N. N. E.	30	27.7		11.5	
2	N. W.	80	56.6			56.6
3	West.	60				60.0
4	S. E. by S.	55		45.7	30.6	
5	North	43	43.0			
6	S. by W.	152		149.1		29.7
	Sum of columns,		127.3	194.8	42.1	146.3
				127.3		42.1
			Diff. lat. =	67.5 S.	dep. =	104.2 W.

Dep.	= 104.2		2.01787		
Diff. lat.	= 67.5	(ar. co.)	8.17070		1.82930
Bearing	= 57° 4′	tang.	0.18857	sec.	0.26467
Dist.	= 124.2				2.09397

Ans. Bearing = S. 57° 4′ W.
Distance = 124.2 miles.

2. A ship sails on the following successive tracks, N. E. 12 miles, E. $\frac{1}{4}$ S. 10 miles, S. E. by S. 14 miles, S. 31° W. 7 miles, E. N. E. 25 miles.

Required the bearing and distance of the place reached from the place left.

Ans. Bearing = East
Distance = 45.8 miles.

3 A ship sails on the following successive tracks, South 10 miles, W. S. W. 25 miles, S. W. 30 miles, and West 20 miles; she is bound to a port which is at a distance of 100 miles from the place of starting and bears W. by S.

Required the bearing and distance of the port to which the ship is bound from the place at which she has arrived.

Ans. Bearing = N. 57° 47′ W.
Distance = 40 miles.

CHAPTER III.

PARALLEL SAILING.

24. *Parallel Sailing* considers only the case where the ship sails exactly east or west and therefore remains constantly on the same parallel of latitude. Its object is to find *the change in longitude* corresponding to the ship's track; and, in general, to investigate the relation of the Difference of Longitude of two places on the same parallel to their Departure. [B., p. 63.]

25. *Problem. To find the difference of longitude in parallel sailing.* [B., p. 65.]

Solution. Let AB (fig. 18) be the distance sailed by the ship on the parallel of latitude AB. As the course is exactly east or west, the distance sailed must, by § 10, be itself equal to the departure made.

The latitude of the parallel is $A'DA$, or $A'A$. The angle $AEB = A'DB'$, or the arc $A'B'$, is, by § 7, the difference of longitude. Denote the radius of the earth $DA' = DB' = DA$ by R, and the radius of the parallel $EA = EB$ by r; also the circumference of the earth by C, and that of the parallel by c.

Since AB and $A'B'$ correspond to the equal angles AEB and $A'DB'$, they must be similar arcs and give the proportion,

$$AB : A'B' = c : C,$$

or

$$\text{dep.} : \text{diff. long.} = c : C,$$

But, as circumferences are proportional to their radii,

$$c : C = r : R.$$

Hence, leaving out the common ratio,

$$\text{dep.} : \text{diff. long.} = r : R.$$

Putting the product of the extremes equal to that of the means,

$$r \times \text{diff. long.} = R \times \text{departure.}$$

But, in the triangle ADE, since

$$EAD = A'DA = \text{latitude},$$

we have, from (22),

$$r = R \times \text{cos. lat.},$$

which, substituted in the above equation, gives, if the result is divided by R,

$$\text{diff. long.} \times \text{cos. lat.} = \text{departure (or distance).} \qquad (223)$$

Hence, by (6),

$$\text{diff. long.} = \frac{\text{dep. (or dist.)}}{\text{cos. lat.}} = \text{dep. (or dist.)} \times \text{sec. lat.} \qquad (224)$$

26. *Problem. To find the distance between two places which are upon the same parallel of latitude.*

Solution. This problem is solved at once by (223).

27. The Table, p. 64, of the Navigator, which "shows for every degree of latitude how many miles distant two meridians are whose difference of longitude is one degree," is readily calculated by this formula.

28. *Corollary.* It appears from (223) and (224) that if a right triangle (fig. 18) is constructed of which the hypothenuse is the difference of longitude and one of the acute angles the latitude, the leg adjacent to this angle is the departure. *All the cases of parallel sailing may, then, be reduced to the solution of this triangle.*

29. Examples.

1. A ship sails from Boston 1000 miles exactly east; find the longitude in which she arrives.

Ans. Longitude sought = 48° 20′ W.

2. Find the distance of Barcelona (Spain) from Nantucket (Massachusetts).

Ans. Distance = 3250 miles.

3. Find the distance between two meridians whose difference of longitude is one degree, in the latitude of 45°.

Ans. Distance = 42.43 miles.

4. Find the difference of longitude which corresponds to a departure of one *league*, or three sea miles, in the latitude of 72°.

Ans. Diff. long. = 9′ 42″.

CHAPTER IV.

MIDDLE LATITUDE SAILING.

30. *Middle Latitude Sailing* is an approximate method of solving those problems of rhumb sailing which involve the consideration of the Difference of Longitude. It is properly applicable to cases in which the difference of latitude is small, and consists in calculating the difference of longitude from the departure or the departure from the difference of longitude by the formulas of Parallel Sailing, on the hypothesis that the departure is equal to the distance between the extreme meridians measured at the *Middle Latitude;* that is, at the latitude of the middle point of the rhumb. [B., p. 66.]

If A and B (fig. 71) are situated on the same side of the equator, A being in the higher latitude, their departure is less than the meridional distance BB' and greater than the meridional distance AA', since each of the partial departures, as ma, is less than the corresponding arc of BB' and greater than the corresponding arc of AA'. Hence, the departure of A and B must be equal to the meridional distance measured on some intermediate parallel, DD'; so that the departure and difference of longitude of A and B are the same as those of D and D'. Since the meridional distance regularly increases, as we go from AA' to BB', it is natural to take the middle parallel as an approximation to the position of DD'; and it is evident that if the difference of latitude is small, little error can result from this assumption, especially if the places are near the equator, where the meridians converge but slightly.

It is evident that

$$\text{mid. lat. of two places} = \tfrac{1}{2} \text{ sum of their lats.}$$
$$= \text{either lat.} \pm \tfrac{1}{2} \text{ diff. lat.} \quad (225)$$

31. *Corollary.* By combining the triangle (fig. 15) of Plane Sailing with that (fig. 18) of Parallel Sailing, and making the latitude in the latter equal to the middle latitude, *we obtain a triangle* (fig. 19) *by which all the cases of Middle Latitude Sailing can be solved.*

32. *Problem.* *To find the difference of latitude, the departure, and the difference of longitude, when the course and the distance are known, and the latitude of one extremity of the ship's track.* [B., p. 71.]

Solution. The triangle (fig. 19) gives at once, as in Plane Sailing

$$\text{diff. lat.} = \text{dist.} \times \text{cos. course}$$
$$\text{dep.} = \text{dist.} \times \text{sin. course.}$$

The middle latitude may then be found by (225); and we have, as in (224),

$$\text{diff. long.} = \frac{\text{dep.}}{\text{cos. mid. lat.}} = \text{dep.} \times \text{sec. mid. lat.} \quad (226)$$

or, by substituting the value of the departure,

$$\text{diff. long.} = \text{dist.} \times \text{sin. course} \times \text{sec. mid. lat.} \quad (227)$$

33. *Problem.* *To find the bearing and the distance of two given places from each other.* [B., p. 68.]

Solution. The places being given, their latitudes and longitudes are supposed to be known, so that the diff. lat., mid. lat., and diff. long. are easily found. Then we have (fig. 19), by the principles of the solution of right triangles,

$$\text{departure} = \text{diff. long.} \times \text{cos. mid. lat.} \quad (228)$$

$$\text{tang. bearing} = \frac{\text{departure}}{\text{diff. lat.}} \quad (229)$$

$$\text{dist.} = \text{diff. lat.} \times \text{sec. bearing.} \quad (230)$$

34. *Problem.* *To find the course, the distance, and the difference of longitude, when both latitudes and the departure are given.* [B., p. 70.]

Solution. The difference of longitude is found by (226), the course by (229), and the distance by (230).

35. *Problem.* *To find the departure, the distance, and the difference of longitude, when both latitudes and the course are given.* [B., p. 72.]

Solution. The departure is found by the formula

$$\text{departure} = \text{diff. lat.} \times \text{tang. course;} \quad (231)$$

the distance by (230); and the difference of longitude may be found by (226), or by substituting (231) in (226), as follows,

$$\text{diff. long.} = \text{diff. lat.} \times \text{tang. course} \times \text{sec. mid. lat.} \qquad (232)$$

36. *Problem.* *To find the course, the departure, and the difference of longitude, when both latitudes and the distance are given.* [B., p. 73.]

Solution. The course is found by the formula

$$\text{cos. course} = \frac{\text{diff. lat.}}{\text{dist.}}; \qquad (233)$$

the departure by (212) or by the formula

$$\text{departure} = \sqrt{[(\text{dist.})^2 - (\text{diff. lat.})^2]}; \qquad (234)$$

and the difference of longitude by (226) or (227).

37. *Problem.* *To find the difference of latitude, the distance, and the difference of longitude, when one latitude, the course, and the departure are given.* [B., p. 74.]

Solution. The difference of latitude is found by the formula

$$\text{diff. lat.} = \text{dep.} \times \text{cotan. course}; \qquad (235)$$

the distance by the formula

$$\text{dist.} = \text{dep.} \times \text{cosec. course}; \qquad (236)$$

the mid. lat. by (225); and the difference of longitude by (226).

38. *Problem.* *To find the course, the difference of latitude, and the difference of longitude, when one latitude, the distance, and the departure are given.* [B., p. 75.]

Solution. The course is found by the formula

$$\text{sin. course} = \frac{\text{dep.}}{\text{dist.}}; \qquad (237)$$

the difference of latitude by (211) or by the formula

$$\text{diff. lat.} = \sqrt{[(\text{dist.})^2 - (\text{dep.})^2]}; \qquad (238)$$

and the difference of longitude by (226).

39. *Scholium.* If two places are in opposite latitudes, the middle latitude may evidently differ considerably from the latitude of DD';

though it may still be used with little error. But it is better to divide the rhumb at the equator and use the middle latitude of each part separately, in connexion with the departure and the difference of longitude which correspond to that part; or we may, by an obvious extension of the principle of middle latitude sailing, take for the approximate latitude of DD' a latitude which is intermediate in amount between these two partial middle latitudes and differs less from the middle latitude which corresponds to the greater part of the rhumb, in proportion as that part is the greater. Thus, if the latitudes are l and l', taken without regard to their signs as north and south, the middle latitudes of the two parts of the rhumb are $\frac{1}{2}l$ and $\frac{1}{2}l'$; the lengths of these parts of the rhumb and the corresponding departures are proportional to l and l'; and l_1, the approximate latitude of DD', is found by the proportion

$$(l_1 - \tfrac{1}{2}l') : (\tfrac{1}{2}l - l_1) = l : l',$$

which gives

$$l_1 = \frac{l^2 + l'^2}{2(l + l')}.$$

The approximate latitude of DD' for all cases is expressed by the formula

$$l_1 = \frac{l^2 \mp l'^2}{2(l \mp l')},$$

in which the upper signs or the lower are to be used according as the places are on the same side of the equator or on opposite sides, and l and l' denote the latitudes, taken independently of their signs; for if the places are on the same side of the equator the formula becomes

$$l_1 = \frac{l^2 - l'^2}{2(l - l')} = \tfrac{1}{2}(l + l').$$

40. Examples.

Note. The calculations of Middle Latitude Sailing are rendered accurate by applying to the middle latitude *a correction*, which may be found in the table of B., p. 76 (given in the Useful Tables after p. 329). The method of computing this correction will be explained in the next chapter. The corrected mid. lat. is the true lat. of DD' (fig. 71) and is always a little greater than the actual mid. lat.

1. A ship sailed from Halifax (Nova Scotia) a distance of 2515 miles, upon a course S. 79° 30′ E.; find the place at which she arrived.

Solution. By § 32,

dist. = 2515		3.40054	3.40054
course = 79° 30′		cos. 9.26063	sin. 9.99267
diff. lat. = 458′ =	7° 38′ S.	2.66117	
lat. left	= 44° 40′ N.	m. lat. = 40° 51′	
lat. in	= 37° 2′ N.	cor. = 7′	
		cor. mid. lat. = 40° 58′	sec. 10.12200
diff. long. = 3275′ =	54° 35′ E.		3.51521
long. left	= 63° 35′ W.		
long. in	= 9° 0′ W.		

Ans. The place arrived at is lat. 37° 2′ N., long. 9° 0′ W.; which is one mile south of Cape St. Vincent, in Portugal.

2. Find the bearing and distance of Canton from Washington.

Solution. By § 33,

lat. of Washington	= 38° 53′ N.		long. = 77° 0′ W.		
lat. of Canton	= 23° 8′ N.		long. = 113° 17′ E.		
diff. lat.	= 945′ = 15° 45′,		sum of longs. = 190° 17′		
mid. lat.	= 31° 0′		diff. long. = 169° 43′ = 10183′		
cor.	= 31′				
cor. mid. lat.	= 31° 31′	cos. 9.93069			
diff. long.	= 10183′	4.00788			
diff. lat.	= 945′	ar. co. 7.02457			2.97543
bearing	= S. 83° 47′ W.	tang. 10.96314		sec.	10.96570
dist.	= 8732 miles.				3.94113

3. A ship sails from New York a distance of 650½ miles, upon a course S. E. ¼ S.; find the place at which she arrives.

Ans. 15½ miles to the west of Georgetown, in Bermuda.

4. Find the bearing and distance of Portland (Maine) from New Orleans.

Ans. The bearing. = N. 49° 18′ E.
The distance = 1256 miles.

5. A ship from the Cape of Good Hope sails northwesterly, that is, between north and west, until her latitude is 22° 3′ S., and her departure 3115 miles; find her course, distance sailed, longitude, and distance from Cape St. Thomas (Brazil).

Ans. Course = N. 76° 39′ W.
Distance = 3201 miles.
Longitude = 40° 36′ W.
Distance to the Cape St. Thomas = 22 miles.

6. A ship sails from Boston upon a course E. by N. until she arrives in latitude 45° 20′ N.; find the distance sailed, the longitude reached, and the distance and bearing from Liverpool.

Ans. Distance sailed = 923 miles.
Longitude = 49° 59′ W.
Distance from Liverpool = 1893 miles.
Bearing from Liverpool = S. 75° 9′ W.

7. A ship sails southwesterly from Gibraltar a distance of 1500 miles, when she is in latitude 14° 43′ N.; find her course, the longitude she is in, and her distance from Cape Verde.

Ans. Course = S. 31° 8′ W.
Longitude = 19° 47′ W.
Dist. from Cape Verde = 132 miles.

8. A ship sails from Nantucket upon a course S. 62° 11′ E., until she has made a departure of 2274 miles; find the distance sailed and the place arrived at.

Ans. Distance = 2571 miles.

The place arrived at is 261 miles north of Santa Cruz (Cape Verde Islands).

9. A ship sails southwesterly from Land's End (England) a distance of 3466 miles, when her departure is 3306 miles; find the course and the place arrived at.

Ans. The course = S. 72° 30′ W.
The place arrived at is Charleston (South Carolina).

CHAPTER V.

MERCATOR'S SAILING.

41. *Mercator's Sailing* is an accurate method of solving those problems of rhumb sailing which involve the Difference of Longitude. [B., p. 78.]

42. *Problem. To find the difference of longitude, when both latitudes and the course are known.*

Solution. Let A and B (fig. 71) be the places. Suppose the rhumb AB divided into very small portions $A\,a$, $a\,b$, $b\,c$, &c., which are such that the difference of longitude is the same for each of them. Let

$D =$ the required difference of longitude of B and A,

$d =$ the small difference of longitude which corresponds to either of the small portions of the rhumb,

$L =$ the given latitude of B,

$L' =$ the given latitude of A,

$l =$ the latitude of any one of the points of division, as c,

$l' =$ the latitude of b, the next point towards A,

$C =$ the given course,

$n =$ the number of portions into which BA is divided.

Now, since we suppose the rhumb to be divided into as many parts as we please, we may suppose each of the parts to be so small that the formulas of middle latitude sailing can be applied to it without error; so that we have for any one of them, as $c\,b$, by (232),

$$d = (l' - l) \times \text{tang. } C \times \text{sec. } \tfrac{1}{2}\,(l' + l), \tag{239}$$

or, by dividing by 2 tang. C, we have, by (6),

$$\tfrac{1}{2}\,d \text{ cotan. } C = \frac{\tfrac{1}{2}\,(l' - l)}{\text{cos. } \tfrac{1}{2}\,(l' + l)}. \tag{240}$$

But $\frac{1}{2}\,(l' - l)$ is a very small arc; so that, if it is expressed in minutes, we have, by (14),

$$\tfrac{1}{2}\,(l' - l) \text{ sin. } 1' = \text{sin. } \tfrac{1}{2}\,(l' - l); \tag{241}$$

which, substituted in (240) multiplied by sin. 1′, gives

$$\tfrac{1}{2}\, d \sin. 1' \operatorname{cotan.} C = \frac{\sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l)}. \tag{242}$$

Let now

$$m = \tfrac{1}{2}\, d \sin. 1' \operatorname{cotan.} C = \frac{\sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l)}; \tag{243}$$

and (243) may be written in the usual form of a proportion

$$\sin. \tfrac{1}{2}(l' - l) : \cos. \tfrac{1}{2}(l' + l) = m : 1; \tag{244}$$

whence, by the theory of proportions,

$$\frac{\cos. \frac{1}{2}(l' + l) + \sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l) - \sin. \frac{1}{2}(l' - l)} = \frac{1 + m}{1 - m}. \tag{245}$$

But if in (47), in which A and B may have any values, we take

$$A = 90^\circ - \tfrac{1}{2}(l' + l), \quad B = \tfrac{1}{2}(l' - l), \tag{246}$$

we have

$$A + B = 90^\circ - l, \quad A - B = 90^\circ - l', \tag{247}$$

and (47) becomes

$$\frac{\cos. \frac{1}{2}(l' + l) + \sin. \frac{1}{2}(l' - l)}{\cos. \frac{1}{2}(l' + l) - \sin. \frac{1}{2}(l' - l)} = \frac{\operatorname{cotan.}(45^\circ - \frac{1}{2} l')}{\operatorname{cotan.}(45^\circ - \frac{1}{2} l)}; \tag{248}$$

and, if we put

$$M = \frac{1 + m}{1 - m}, \tag{249}$$

(245) and (248) give

$$\frac{\operatorname{cotan.}(45^\circ - \frac{1}{2} l')}{\operatorname{cotan.}(45^\circ - \frac{1}{2} l)} = M. \tag{250}$$

Now, since the course C is everywhere the same, and since d is assumed to be the same for each portion of the rhumb, m is, by (243), the same for each portion of the rhumb, and, therefore, by (249), M, the ratio of cotan. $(45^\circ - \frac{1}{2} l')$ to cotan. $(45^\circ - \frac{1}{2} l)$, is likewise the same for each portion of the rhumb. Hence the successive values of cotan. $(45^\circ - \frac{1}{2} l)$, for the points $B, \ldots\ldots c, b, a, A$, form *a geometric progression*, of which

cotan. $(45^\circ - \frac{1}{2} L)$ = the first term,
cotan. $(45^\circ - \frac{1}{2} L')$ = the last term,
M = the common ratio,
$n + 1$ = the number of terms.

Therefore, by the theory of geometric progression,

$$\operatorname{cotan.}(45^\circ - \tfrac{1}{2} L') = \operatorname{cotan.}(45^\circ - \tfrac{1}{2} L) \,.\, M^n, \tag{251}$$

and, by logarithms,

$$\log. \operatorname{cotan.}(45^\circ - \tfrac{1}{2} L') - \log. \operatorname{cotan.}(45^\circ - \tfrac{1}{2} L) = \log. M^n. \tag{252}$$

Since the value of d is the same for each portion, we have, by (243),

$$n = \frac{D}{d} = \frac{D \sin. 1'}{2\, m \text{ tang. } C}; \qquad (253)$$

and, if we put

$$e = M^{\frac{1}{2m}}, \qquad (254)$$

we have, by (253),

$$M^n = e^{2mn} = e^{\frac{D \sin. 1'}{\text{tang. } C}} \qquad (255)$$

$$\log. M^n = \frac{D \sin. 1'}{\text{tang. } C} \log. e = D \frac{\log. e}{\text{cosec. } 1' \text{ tang. } C}; \qquad (256)$$

which, substituted in (253), gives by a simple reduction

$$\left[\frac{\text{cosec. } 1'}{\log. e} \log. \text{cotan.} (45^\circ - \tfrac{1}{2} L') - \frac{\text{cosec. } 1'}{\log. e} \log. \text{cotan.} (45^\circ - \tfrac{1}{2} L)\right]$$

$$\times \text{ tang. } C = D. \qquad (257)$$

Now the value of $\frac{\text{cosec. } 1'}{\log. e}$ log. cotan. $(45^\circ - \frac{1}{2} L)$ has been calculated for every minute of latitude and inserted in tables, such as B., Table III. It is called *the Meridional Parts of the Latitude*, and the method of computing it is given in § 44. The algebraic difference between the meridional parts of two latitudes is called *the Meridional Difference of Latitude.*

Hence (257) gives

$$D = \text{diff. long.} = \text{mer. diff. lat.} \times \text{tang. course.} \qquad (258)$$

Since $(45^\circ - \frac{1}{2} L)$ is the complement of $(45^\circ + \frac{1}{2} L)$, we have, by the principles of logarithms,

$$\log. \text{cotan.} (45^\circ - \tfrac{1}{2} L) = \log. \text{tang.} (45^\circ + \tfrac{1}{2} L),$$
$$= - \log. \text{cotan.} (45^\circ + \tfrac{1}{2} L); \qquad (259)$$

and it is evident that $(45^\circ - \frac{1}{2} L)$ and $(45^\circ + \frac{1}{2} L)$ are the halves of the angular distances between the place of which L is the latitude and the two poles of the earth.

In order to apply (257) to the case of two places on opposite sides of the equator, we must consider L as negative, since, in the above solution, the latitude continually increases from L to L'. If $L = -L_1$, we have, by (259),

$$\frac{\text{cosec. } 1'}{\log. e} \log. \cot. (45° - \tfrac{1}{2} L) = - \frac{\text{cosec. } 1'}{\log. e} \log. \cot. (45° - \tfrac{1}{2} L_1)$$

or

$$\text{mer. parts of } L = -\text{mer. parts of } L_1.$$

Hence, if the latitudes are taken without regard to their signs, *the Meridional Difference of Latitude of two places on the same side of the equator is equal to the difference between the meridional parts of their latitudes; and that of two places on opposite sides of the equator is equal to the sum of the meridional parts.*

Since it is supposed, in the above solution, that $(l' - l)$ is expressed in minutes, d is found by (239) in minutes, and, therefore, the value of D given by (257) is expressed in minutes.

43. *Corollary.* It appears from (258) that the difference of longitude is the leg DE (fig. 20) of a right triangle of which AD is equal to the meridional difference of latitude and the angle A to the course. This triangle may be combined with the triangle ABC of Plane Sailing; and *all the cases of Mercator's Sailing are reduced to the solution of these two similar right triangles.*

44. *Problem. To calculate the table of Meridional Parts.*

Solution. I. In finding the value of e, which is involved in the expression for the meridional parts, the portions into which the rhumb is divided are supposed to be *infinitely small.* Hence d is infinitely small, and therefore, by (243), m is also infinitely small.

We have, then, by (249), together with (167) and (168),

$$\log. M = \log. (1 + m) - \log. (1 - m),$$
$$= m \log. e + m \log. e = 2\, m \log. e;$$

which gives

$$e = M^{\frac{1}{2m}},$$

which is identical with (254); so that e in (254) has the same value as in (164); that is

$$e = 2.71828. \tag{260}$$

II. This value of e gives by (13)

$$\frac{\text{cosec. } 1'}{\log. e} = \frac{3437.7}{\log. (2.71828)} = \frac{3437.7}{0.43429} = 7915.7, \tag{261}$$

so that we have by (257)

$$\begin{aligned} \text{mer. parts of } L &= 7915.7 \log. \text{cotan.} (45° - \tfrac{1}{2} L) \\ &= 7915.7 \log. \text{tang.} (45° + \tfrac{1}{2} L), \end{aligned} \tag{262}$$

which agrees with the explanation of Table III. given in the Preface to the Navigator.

45. Examples.

1. Calculate the meridional parts of latitude 45° 48′.

Solution. 2)45° 48′

45° — ½ L = 45° — 22° 54′ = 22° 6′			
22° 6′ log. cotan.	0.39141	log.	9.59263
	7915.7		3.89849
mer. parts of 45° 48′ = 3098			3.49112

2. Calculate the meridional parts of latitude 28° 14′.

Ans. 1767.

3. Calculate the meridional parts of latitude 83° 59′.

Ans. 10127.

46. *Problem. To calculate the correction for middle latitude sailing.*

Solution. If the angle DBC (fig. 19) were exactly what it should be in order that the hypothenuse BD should be the difference of longitude and the leg BC the departure, it would be the corrected middle latitude, or the true latitude of DD' (fig. 71), and we should have

$$\begin{aligned} \text{diff. long.} &= \text{sec. cor. mid. lat.} \times \text{departure} \\ &= \text{sec. cor. mid. lat.} \times \text{diff. lat.} \times \text{tang. course}, \end{aligned} \tag{263}$$

which, compared with (258), gives, by dividing by tang. course,

$$\text{mer. diff. lat.} = \text{sec. cor. mid. lat.} \times \text{diff. lat.} \tag{264}$$

whence
$$\text{sec. cor. mid. lat.} = \frac{\text{mer. diff. lat.}}{\text{diff. lat.}} \tag{265}$$

If from the corrected middle latitude, calculated by this formula, the actual middle latitude is subtracted, the correction of the middle latitude is obtained, and thus a table like that on p. 76 of the Navigator may be computed. The meridional difference of latitude should be obtained for these calculations, not from the tables of meridional parts, but directly from the tables of logarithmic sines, &c., by means of (257) and (262); and when the difference of latitude is less than 14°, tables should be used in which the logarithms are given to seven places of decimals.

47. *Corollary.* A formula adapted to calculation by logarithms of five places can be obtained by the following process.

Let $L_0 =$ the middle latitude $= \frac{1}{2}(L + L')$
$x =$ the correction of mid. lat.
$l_0 =$ the difference of latitude $= L' - L$,

and, by § 42,

$$\text{mer. diff. lat.} = \frac{\text{cosec. } 1'}{\log. e} \log. \frac{\text{cotan. } (45° - \frac{1}{2} L')}{\text{cotan. } (45° - \frac{1}{2} L)}. \tag{266}$$

By changing, in (248), the small letters to large ones, we obtain

$$\log. \frac{\text{cotan. } (45° - \frac{1}{2} L')}{\text{cotan. } (45° - \frac{1}{2} L)} = \log. \frac{\cos. L_0 + \sin. \frac{1}{2} l_0}{\cos. L_0 - \sin. \frac{1}{2} l_0}$$

$$= \log. \frac{1 + \sin. \frac{1}{2} l_0 \sec. L_0}{1 - \sin. \frac{1}{2} l_0 \sec. L_0}. \tag{267}$$

But, by (186),

$$\log. \frac{1 + \sin. \frac{1}{2} l_0 \sec. L_0}{1 - \sin. \frac{1}{2} l_0 \sec. L_0} = 2 \log. e \,[\sin. \tfrac{1}{2} l_0 \sec. L_0$$

$$+ \tfrac{1}{3} (\sin. \tfrac{1}{2} l_0 \sec. L_0)^3 + \tfrac{1}{5} (\sin. \tfrac{1}{2} l_0 \sec. L_0)^5 + \&c.] \tag{268}$$

which gives, by substitution in (267) and (266),

$$\text{mer. diff. lat.} = 2 \cdot \text{cosec. } 1' \left[\text{sin. } \tfrac{1}{2} l_0 \text{ sec. } L_0 \right.$$

$$\left. + \tfrac{1}{3} (\text{sin. } \tfrac{1}{2} l_0 \text{ sec. } L_0)^3 + \&c.\right] \qquad (269)$$

and (265) gives

$$\text{sec. } (L_0 + x) = \frac{\text{cosec. } 1'}{\tfrac{1}{2} l_0} \left[\text{sin. } \tfrac{1}{2} l_0 \text{ sec. } L_0 \right.$$

$$\left. + \tfrac{1}{3} (\text{sin. } \tfrac{1}{2} l_0 \text{ sec. } L_0)^3 + \&c.\right] \qquad (270)$$

48. Examples.

1. Find the correction for middle latitude sailing, when the middle latitude is 35°, and the difference of latitude 14°.

Solution. Greater lat. $= 35° + 7° = 42°$

Less lat. $= 35° - 7° = 28°$

$45° - \tfrac{1}{2}$ gr. lat. $= 24°$	log. cotan. 0.35142
$45° - \tfrac{1}{2}$ less lat. $= 31°$	log. cotan. 0.22123
log. cotan. 24° — log. cotan. 31°	0.13019 log. 9.11458
$\dfrac{\text{cosec. } 1'}{\log. e}$	7915.7 log. 3.89849
diff. lat.	840′ log. ar. co. 7.07572
corrected mid. lat. $= 35°\ 24'$	log. sec. 10.08879

correction $= 35°\ 24' - 35° = 24'$.

2. Find the correction for middle latitude sailing, when the middle latitude is 66°, and the difference of latitude 10°.

Solution. In this case $\frac{1}{2}\, l_0 = 5° = 300'$, $L_0 = 66°$.

	5° sin.	8.94030		
	66° sec.	0.39069		
sin. 5° sec. 66°	$= 0.21428$	9.33099		0.21428
(sin. 5° sec. 66°)^3	$= 0.00984$	7.99297	$\frac{1}{3}(0.00984) =$	0.00328
(sin. 5° sec. 66°)^5	$= 0.00045$	6.65495	$\frac{1}{5}(0.00045) =$	0.00009
(sin. 5° sec. 66°)^7	$= 0.00002$	5.31693	$\frac{1}{7}(0.00002) =$	0.00000
(0.21765)	log.	9.33776		0.21765
300′	ar. co.	7.52288		
1′	cosec.	3.53627		
66° 22′	sec.	0.39691		

cor. of mid. lat. $= 66° \, 22' - 66° = 22'$.

3. Find the correction for middle latitude sailing, when the middle latitude is 30°, and the difference of latitude 4°.

Solution. In this case $\frac{1}{2}\, l_0 = 2° = 120'$, $L_0 = 30°$.

	2° sin.	8.54282	
	30° sec.	0.06247	
sin. 2° sec. 30°	$= 0.040298$	8.60529	0.040298
(sin. 2° sec. 30°)^3	$= 0.000065$	5.81587	0.000022
		8.60552	0.040320
	120′ ar. co.	7.92082	
	1′ cosec.	3.53627	
	30° 2′ sec.	0.06261	

cor. of mid. lat. $= 30° \, 2' - 30° = 2'$.

4. Find the correction for middle latitude sailing, when the middle latitude is 60°, and the difference of latitude 16°.

Ans. 46′.

5. Find the correction for middle latitude sailing, when the middle latitude is 8°, and the difference of latitude 16°.

Ans. 77′.

6. Find the correction of the middle latitude and also that of l_1 (found as in § 39), when the middle latitude is 2°, and the difference of latitude 32°.

Ans. Cor. of mid. lat. = 450′; cor. of $l_1 = 82'$.

7. Find the corrections of the middle latitude and of l_1, when the middle latitude is 0°, and the difference of latitude 32°.

Ans. Cor. of mid. lat. = 557′; cor. of $l_1 = 77'$.

8. Find the correction for middle latitude sailing, when the middle latitude is 21°, and the difference of latitude 3°.

Ans. 1′.

9. Find the correction for middle latitude sailing, when the middle latitude is 24°, and the difference of latitude 6°.

Ans. 5′.

10. Find the correction for middle latitude sailing, when the middle latitude is 15°, and the difference of latitude 12°.

Ans. 26′.

49. *Problem. To find the bearing and the distance from each other of two given places.* [B., p. 79.]

Solution. We have by (fig. 20) for the bearing,

$$\text{tang. bearing} = \frac{\text{diff. long.}}{\text{mer. diff. lat.}}, \tag{271}$$

and the distance is found by (230).

50. *Problem. To find the course, the distance, and the difference of longitude, when both latitudes and the departure are given.* [B., p. 80.]

Solution. The course is found by (229), the difference of longitude by (258), and the distance by (230).

51. *Problem. To find the distance and the difference of longitude, when both latitudes and the course are given.* [B., p. 82.]

Solution. The distance is found by (230), and the difference of longitude by (258).

52. *Problem. To find the course and the difference of longitude, when both latitudes and the distance are given.* [B., p. 83.]

Solution. The course is found by (233), and the difference of longitude by (258).

53. *Problem. To find the distance, the difference of latitude, and the difference of longitude, when one latitude, the course, and the departure are given.* [B., p. 84.]

Solution. The distance is found by (236), the difference of latitude by (235), and the difference of longitude by (258).

54. *Problem. To find the course, the difference of latitude, and the difference of longitude, when one latitude, the distance, and the departure are given.* [B., p. 85.]

Solution. The course is found by (237), the difference of latitude by (238), and the difference of longitude by (258) or by the following proportion deduced from the similar triangles of (fig. 20),

$$\text{diff. lat.} : \text{dep.} = \text{mer. diff. lat.} : \text{diff. long.} \qquad (272)$$

55. Examples.

1. A ship sails from Boston a distance of 6743 miles, upon a course S. 46° 57½′ E.; to find the place at which she arrives.

Solution.

dist.	= 6743		3.82885	
course.	= 46° 57½′	cos.	9.83412	tang. 10.02971
diff. lat.	= 76° 42′ S. = 4602′,		3.66297	m. d. lat. = 5007, 3.69958
lat. left	= 42° 20′ N. mer. p.		2809	diff. long. = 5362′, 3.72929
lat. in	= 34° 22′ S. mer. p.		2198	= 89° 22′ E.
	mer. diff. lat. =		5007	long. left = 70° 53′ W.
				long. in = 18° 29′ E.

Ans. The place reached is the Cape of Good Hope.

2. Find the bearing and distance from Moscow to St. Helena.

Solution.

Moscow,	lat. 55° 45′ N.	mer. parts 4047	long. 37° 34′ E.
St. Helena,	lat. 15° 55′ S.	mer. parts 968	long. 5° 43′ W.
	diff. lat. = 71° 40′	mer. diff. lat. = 5015	d. long. = 43° 17′
	= 4300′		= 2597′

mer. diff. lat.	= 5015	(ar. co.)	6.29973		
diff. long.	= 2597		3.41447		
bearing	= S. 27° 23′ W.	tang.	9.71420	sec.	10.05157
		diff. lat. =	4300		3.63347
		dist. =	4842 miles		3.68504

Ans. The bearing = S. 27° 23′ W.; the distance = 4842 miles.

3. A ship sails from a position 200 miles to the east of Cape Horn a distance of 3636 miles, upon a course N. N. E.; find the position at which she has arrived.

Ans. It has arrived at the equator in the longitude of 33° 12′ W.

4. Required the bearing and distance of Botany Bay from London.

Ans. Bearing = S. 57° 28′ E.
Distance = 9544 miles.

5. A ship sails northwesterly from Lima until she arrives in the latitude 23° 8′ N., and has made a departure of 9967 miles; find the place at which she has arrived.

Ans. Canton.

6. A ship sails from Disappointment Island in the North Pacific Ocean, upon a course S. 61° 16½′ E., until she has arrived in latitude 14° 10′ S.; find the place at which she has arrived.

Ans. The Disappointment Islands in the South Pacific Ocean.

7. A ship sails from Smeerenburg Harbor (Spitzbergen) a distance of 8979 miles southwesterly, when she has arrived in latitude 62° 30′ S.; find the place at which she has arrived.

Ans. Yankee Straits in New South Shetland.

8. A ship sails from the Cape of Good Hope, upon a course S. 82° 12′ E., until she has made a departure of 10951 miles; find the position at which she has arrived.

Ans. Her position is 203 miles south of Cape Horn.

9. A ship sails southeasterly from the South Point of the Great Bank of Newfoundland a distance of 2812 miles, when she has made a departure of 799½ miles; find the position at which she has arrived.

Ans. Her position is 208 miles north of Cape St. Roque.

56. *Mercator's Chart* is a map of the earth's surface or of any part of it, constructed on the principle of *representing departure by difference of longitude and difference of latitude by meridional difference of latitude.* [B., p. 87.]

In this chart, the equator and the parallels of latitude are represented by parallel straight lines, and the meridians by straight lines perpendicular to the equator. The distance between any two meridians is proportional to their difference of longitude (expressed in miles), and the distance of any parallel from the equator is in the same proportion to the meridional parts of its latitude (which are also to be regarded as expressed in miles). The position of any place on the chart is determined by finding the point of intersection of its parallel with its meridian.

Mercator's Chart gives a distorted view of the earth's surface as a whole, since the scale on which the regions of the earth are represented increases continually from the equator towards either pole. But as both dimensions are increased the same ratio, any small territory is given with approximate correctness. The chart is, however, excellently adapted to the purposes of the navigator, in sailing on a rhumb. For the spherical figure $AB'B$ or $AA'B$ (fig. 71) formed by the rhumb with the meridian of one of its extremities and the parallel of the other is converted, on the chart, into the corresponding triangle ADE (fig. 20) of Mercator's Sailing; the rhumb becoming a straight line, since it crosses all the meridians at the same angle; and the course or bearing remaining unchanged, since each of the small triangles, $A\ m\ a$, &c., is converted into a similar triangle. Hence the mariner can easily estimate, by means of the chart, the bearing of one given place from another or the position which has been reached by sailing on a given course. The nautical distance is not correctly given on the chart; but it can be estimated by methods explained in the Navigator. [B., p. 88.]

The ease of laying down rhumbs on Mercator's chart furnish one of the reasons for preferring rhumb sailing to great circle sailing. Another reason is found in the fact that the problems of rhumb sailing can be solved by plane trigonometry; and a third in the use of the mariner's compass in steering. Great circle sailing is used, however, in some long voyages; but it consists practically in sailing on a succession of rhumbs, approximating to the arc of a great circle; for as long as the ship's head is kept on any given point of the compass, it is plain that she is sailing on a rhumb.

Professor Chauvenet has invented a very ingenious chart, founded on the properties of the stereographic projection of the earth, for showing the courses to be taken and the distance to be sailed in great circle sailing.

CHAPTER VI.

SURVEYING.

57. *Surveying* is the art of the mensuration of portions of the earth's surface. An accurate survey of extensive territories or coasts involves the knowledge of the true figure of the earth, ascertained with the utmost possible exactness. Such surveys belong to the department of *Geodesy*. But we are here to consider only the determination of the areas of portions of the earth's surface which are so small that they can be regarded as plane.

The measure commonly used by land-surveyors is *Gunter's chain*. The chain is divided into 100 *links*, and is equal to 4 rods, or 66 feet, or $\frac{1}{80}$ of a statute mile. The *square chain* (that is, the surface equal to the square of which the side is a chain) is consequently equal to 10000 square links, or 16 square rods, or 4356 square feet, or $\frac{1}{6400}$ of a square mile, or $\frac{1}{10}$ of an acre.

58. *Problem. To find the area of a triangular field, when its angles and one of its sides are known.*

Solution. Let ABC (fig. 2 or 3) be the triangle to be measured and c the given side. The area of the triangle is equal to half the product of its base by its altitude, or

$$\text{area of } ABC = \tfrac{1}{2}\, b p. \tag{273}$$

But, by (130),

$$\sin. C : \sin. B : : c : b,$$

whence

$$b = \frac{c \sin. B}{\sin. C};$$

and, by (131),

$$p = c \sin. A.$$

Substituting in (273), we have

$$\text{area of } ABC = \frac{c^2 \sin. A \sin. B}{2 \sin. C}. \tag{274}$$

59. *Problem. To find the area of a triangular field, when two of its sides and the included angle are known.*

Solution. Let ABC (fig. 2 or 3) be the triangle to be measured, b and c the given sides, and A the given angle. Then, by (273),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, p,$$

and, by (131),

$$p = c \sin. A.$$

Hence

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin. A; \qquad (275)$$

that is, *the area of a triangle is equal to half the continued product of two of its sides and the sine of the included angle.*

60. *Problem. To find the area of a triangular field, when its three sides are known.*

Solution. Let ABC (fig. 1) be the given triangle. Then, by (275),

$$\text{area of } ABC = \tfrac{1}{2}\, b\, c \sin. A;$$

but, by (158),

$$\sin. A = \frac{2\sqrt{[s(s-a)(s-b)(s-c)]}}{b\,c},$$

in which s denotes the half sum of the three sides of the triangle. Hence

$$b\,c \sin. A = 2\sqrt{[s(s-a)(s-b)(s-c)]};$$

and

$$\text{area of } ABC = \sqrt{[s(s-a)(s-b)(s-c)]}; \qquad (276)$$

that is, *to find the area of a triangular field, subtract each side separately from the half sum of the sides, and the square root of the continued product of the half sum and the three remainders is the required area.*

61. EXAMPLES.

1. Given the three sides of a triangular field, equal to 45.56 ch., 52.98 ch., and 61.22 ch.; to find its area.

Solution. Let $a = 45.56$ ch., $b = 52.98$ ch., $c = 61.22$ ch.

$2s =$	159.76 ch.	
$s =$	79.88 ch.	1.90244
$s - a =$	34.32 ch.	1.53555
$s - b =$	26.90 ch.	1.42975
$s - c =$	18.66 ch.	1.27091
	2	6.13865
Area of $ABC =$	1173.07 sq. ch.	3.06932.

Ans. The area = 117 A. 1 R. 9 r.

2. Given the three sides of a triangular field equal to 32.56 ch., 57.84 ch., and 44.44 ch.; to find its area.

Ans. The area = 71 A. 3 R. 12 r.

3. Given one side of a triangular field equal to 17.95 ch., and the adjacent angles equal to 100° and 70°; to find its area.

Ans. The area = 85 A. 3 R. 17 r.

4. Given two sides of a triangular field equal to 12.34 ch. and 17.97 ch., and the included angle equal to 44° 56′; to find its area.

Ans. The area = 7 A. 3 R. 13 r.

62. *Problem. To find the area of an irregular field bounded by straight lines.*

First Method of Solution. Divide the field into triangles in any manner best suited to the nature of the ground. Measure all those sides and angles which can be measured conveniently, remembering that three parts of each triangle, one of which is a side, must be known to determine it.

But it is desirable to measure more than three parts of each triangle, when it can be done; because the comparison of them with each other will often serve to correct the errors of observation. Thus, if the three angles were measured, and their sum were found to differ from 180°, there must be an error of measurement equivalent to the difference; and the error, if small, might be divided between the angles; but if it were large, it would show the observations were so inaccurate that they must be taken again.

The area of each triangle is to be calculated by one of the preceding formulas, and the sum of the areas of the triangles is the area of the whole field.

This method of solution is general and may be applied to surfaces of any extent, provided each triangle is so small as not to be affected by the earth's curvature.

Second Method of Solution. Let $ABCEFHA$ (fig. 21) be the field to be measured. Starting from its most easterly or its most westerly point, the point A for instance, measure successively round the field the bearings and lengths of all its sides. Through A draw the meridian NS, on which let fall the perpendiculars BB', CC', EE', FF', and HH'. Also draw $CB''E''$, EF'', and HF''', parallel to NS.

Then the area of the required field is

$$ABCEFHA = AC'CEFF'A - [AC'CBA + AHFF'A].$$

But

$$AC'CEFF'A = C'CEE' + E'EFF';$$

and

$$AC'CBA + AHFF'A = C'CBB' + B'BA + AHH' + H'HFF'.$$

Hence

$$ABCEFHA = [C'CEE' + E'EFF'] - [C'CBB' + B'BA + AHH' + H'HFF'];$$

or doubling and changing a very little the order of the terms,

$$\left.\begin{array}{l} 2\,ABCEFHA = [2\,C'CEE' + 2\,E'EFF'] - \\ [2\,B'BA + 2\,C'CBB' + 2\,H'HFF' + 2\,AHH']. \end{array}\right\} \qquad (277)$$

Again, by the principles of the measurement of triangles and trapezoids,

$$\left.\begin{array}{lll} 2\,B'BA = B'B & \times & AB' \\ 2\,C'CBB' = (B'B + C'C) & \times & B'C' \\ 2\,C'CEE' = (C'C + E'E) & \times & C'E' \\ 2\,E'EFF' = (E'E + F'F) & \times & E'F' \\ 2\,H'HFF' = (F'F + H'H) & \times & F'H' \\ 2\,AHH' = H'H & \times & H'A. \end{array}\right\} \qquad (278)$$

So the determination of the required area is now reduced to the calculation of the several lines in the second members of (278). But the rest of the solution may be more easily comprehended by means of the following table, which is precisely similar in its arrangement to the table actually used by surveyors, when calculating areas by this process.

Sides.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
AB	AB′		B′B		B′B	B′B	2 B′BA	
BC	B′C′			BB″	C′C	B′B + C′C	2 C′CBB′	
CE		C′E′	E″E		E′E	C′C + E′E		2 C′CEE′
EF		E′F′	F″F		F′F	E′E + F′F		2 E′EFF′
FH	F′H′			FF‴	H′H	F′F + H′H	2 H′HFF′	
HA	H′A			HH′	0	H′H	2 AHH′	

In the *first* column of the table are the successive sides of the field.

In the *second* and *third* columns are the differences of latitude of the several sides; the column headed N. corresponding to the sides running in a northerly direction, and that headed S. corresponding to those running in a southerly direction.

These two columns are calculated by the formula

$$\text{Diff. lat.} = \text{dist.} \times \cos. \text{bearing}.$$

In the *fourth* and *fifth* columns are the departures of the several sides; the column headed E. corresponding to the sides running in an easterly direction, and that headed W. to those running in a westerly direction.

These two columns are calculated by the formula

$$\text{Departure} = \text{dist.} \times \sin. \text{bearing}.$$

In the *sixth* column, headed *Departure*, are the departures of the several vertices of the field from the vertex *A*. This column is calculated from the two columns E. and W. in the following manner. *The first number in column Departure is* *the same as the first in the two columns E. and W.; and every succeeding number in column Departure is obtained by adding the corresponding number in columns E. and W., if it is of the same column with the first number in those two columns, to the previous number in column Departure, or by subtracting it from that previous number, if it is not of the same column with the first number in columns E. and W.*

Thus

$$\begin{aligned}
B'B &= B'B \\
C'C &= B'B'' & &= B'B - BB'' \\
E'E &= E'E'' + E''E &&= C'C + E''E \\
F'F &= F'F'' + F''F &&= E'E + F''F \\
H'H &= F'F''' &&= F'F - FF''' \\
0 & &&= H'H - HH'.
\end{aligned}$$

In the *seventh* column, headed *Sum*, are the first factors of the second members of (278). This column is calculated from column Departure in the following manner. *The first number in column Sum is the same as the first in column Departure; and every other number in column Sum is the sum of the corresponding number in column Departure added to the previous number in column Departure*, as is evident from simple inspection.

In the *eighth* and *ninth* columns are the values of the areas which compose the first members of (278). *These columns are calculated by multiplying the numbers in column Sum by the corresponding numbers in columns N. and S.*, which contain the second factors of the second members of (278). *The products are written in the column of North Areas when the second factors are taken from column N., and in that of South Areas when the second factors are taken from column S.*

If we compare the columns of North and South Areas with (277), we find that all those areas which are preceded by the negative sign are the same with those in the column of North Areas; while all those which are preceded by the positive sign belong to the column of South Areas. *To obtain, therefore, the value* of the second member of (277), that is, *of double the required area, we have only to find the difference between the sums of the columns of North and South Areas.* [B., p. 107.]

63. *Corollary.* The columns N., S., E., and W. are those which would be calculated in Traverse Sailing, if a ship was supposed to start from the point A and proceed round the sides of the field till it returned to the point A. The difference of the sums of columns N. and S. is, then, by Traverse Sailing, the difference of latitude of the point from which the ship starts and the point at which she arrives; and the difference of columns E. and W. is the departure of the same two points. But as both the points are here the same, their difference of latitude and their departure must be nothing; or

Sum of column N. = sum of column S.

Sum of column E. = sum of column W.

But when, as is almost always the case, the sums of these columns differ from each other, the difference must arise from errors of observation. If the error is great, new observations must be taken; but if it is small, it may be divided among the sides by the following proportion: —

The sum of the sides : *each side* = *whole error* :

error corresponding to that side. (279)

The errors corresponding to the sides are then to be subtracted from the differences of latitude or departures which are in the larger column, and added to those which are in the smaller column.

64. Examples.

1. Given the bearings and lengths of the sides of a field, as in the three first columns of the following table; to find its area.

Solution. The table is computed by § 62.

No.	Bearing.	Dist.	N.	S.	E.	W.	Cor. N.	Cor. E.	N.	S.	E.	W.	Dep.	Sum.	N. Areas.	S. Areas.
1	N. 45° W.	21 ch.	14.85			14.85	.02	.05	14.83			14.90	14.90	14.90	220.9670	
2	N. 24° E.	32 ch.	29.24		18.01		.03	.08	29.21		12.93		1.97	16.87	492.7727	
3	S. 86° W.	54 ch.		3.76		53.87	.05	.14		3.81		54.01	55.98	57.95		220.7895
4	South.	10 ch.		10.00			.01	.03		10.01		.03	56.01	111.99		1121.0199
5	S. 70° W.	11 ch.		3.76		10.84	.01	.03		3.77		10.87	66.88	122.89		461.4103
6	S. 20° E.	99 ch.		93.03	33.86		.10	.26		93.18	33.60		32.78	99.16		9234.7708
7	East.	6 ch.			6.00		.00	.02			5.98		26.80	59.58		
8	N. 22° E.	72 ch.	66.76		26.98		.08	.18	66.68		26.80		0.00	26.80	1787.0240	
			110.85 110.55	110.55	79.85 79.06	79.06	.30	.79	110.72	110.72	79.81	79.81			2500.7637	11037.9905 2500.7637
			.30	N.	.79	E.										2)8537.2268
																10)4268.6134
																426.8613
																4
																3.4452
																40
																17.8080

Ans. **The Area = 426 A. 3 R. 18 r.**

2. Given the lengths and bearings of the sides of a field, as follows; to find its area.

1st side;	N. 17° E.;	25 ch.
2nd side;	East;	28 ch.
3rd side;	South;	54 ch.
4th side;	S. 4° W.;	22 ch.
5th side;	N. 33° W.;	62 ch.

Ans. The area = 167 A. 3 R. 21 r.

65. *Problem. To find the area of a field bounded by sides irregularly curved.*

Solution. Let $ABCEFHIKL$ (fig. 22) be the field to be measured, the boundary $ABCEFHIKL$ being irregularly curved. Take any points C and F so that when we join AC, CF, and FL, the field $ACFL$, bounded by straight lines, may not differ much from the given field.

Find the area of $ACFL$ by either of the preceding methods, and then measure the parts included between the curved and the straight sides by the following method of *offsets.*

Take the points a, b, c, d, so that the lines Aa, ab, bc, cd, dC may be sensibly straight. Let fall on AC the perpendiculars aa', bb', cc', dd'. Measure these perpendiculars and also the distances Aa', $a'b'$, $b'c'$, $c'd'$, $d'C$.

The triangles Aaa', Cdd', and the trapezoids $aba'b'$, $bcb'c'$, $cdc'd'$ are then easily calculated, and their sum is the area of ABC.

In the same way may the areas of CEF, FHI, and IKL be calculated; and then the required area is found by the equation.

$$ABCEFHIKL = ACFL - ABC + CEF + FHI - IKL.$$

Example.

Given (fig. 22) $Aa' = 5$ ch., $a'b' = 2$ ch., $b'c' = 6$ ch., $c'd' = 1$ ch., $d'C = 4$ ch.; also $aa' = 3$ ch., $bb' = 2$ ch., $cc' = 2.5$ ch., $dd' = 1$ ch.; to find the area of ABC.

Ans. Required area = 2 A. 3 R. 36 r.

CHAPTER VII.

HEIGHTS AND DISTANCES.

66. The plane of *the sensible horizon* at any place is the plane which is tangent to the earth's surface at that place.

Any line or plane which is parallel to the plane of the horizon is said to be *horizontal;* and any line or plane which is perpendicular to the plane of the horizon is said to be *vertical.*

The visible horizon for any observer is the circumference of a small circle of the earth which limits his view of the earth's surface.

The plane of the sensible horizon coincides with the surface of tranquil water, when this surface is so small that its curvature can be neglected; and it is perpendicular to the *plumb line.*

67. *The angle of elevation* of an object is the vertical angle which a line drawn to the object from the place of the observer makes with the horizontal plane at that place, when the object is above this horizontal plane; *the angle of depression* is the same angle, when the object is below the horizontal plane.

The bearing of an object from the place of the observer is the horizontal angle which the vertical plane passing through the place and the object makes with the plane of the meridian of the place.

Various instruments have been devised for estimating the direction of any visible object from the observer, with reference to the plane of the horizon or to that of the meridian or to the directions of other visible objects. The most important of these instruments in land-surveying is the *theodolite,* which consists of a telescope, capable of being rotated on its stand, about a vertical axis, into the same vertical plane with any visible object, and also of being rotated in that plane, about a horizontal axis perpendicular to it. By measuring these rotations, we can measure the horizontal angle made by the vertical plane of the object with the plane of the meridian (which is indicated by the compass) or with any other vertical plane, and also the angle of elevation or depression of the object. Other instruments, such as the *quadrant*, the *sextant*, and the *azimuth compass*, are used on shipboard for measuring angles.

68. *Problem. To determine the height of a vertical tower situated on a horizontal plane.* [B., p. 94.]

Solution. Observation. Let AB (fig. 23) be the tower whose height is to be determined. Measure off the distance BC on the horizontal plane of any convenient length. At the point C observe the angle of elevation BCA.

Calculation. We have, then, given, in the right triangle ACB, the angle C and the base BC, as in § 34 of Pl. Trig., and the leg AB is found by (26).

EXAMPLE.

At the distance of 95 feet from a tower, the angle of elevation of the tower is found to be $48° 19'$. Required the height of the tower.

Ans. 106.69 feet.

69. *Problem. To find the height of a vertical tower situated on an inclined plane.*

Solution. Observation. Let AB (fig. 24) be the tower, situated on the inclined plane BC. Observe the angle B which the tower makes with the plane. Measure off the distance BC of any convenient length. At the point C, observe the angle BCA by which the top of the tower is elevated above the inclined plane.

Calculation. In the oblique triangle ABC, there are given the side BC and the two adjacent angles B and C, and BA may be found as in § 73 of Plane Trigonometry.

EXAMPLE.

Given (fig. 24) $BC = 89$ feet, $B = 113° 12'$, $C = 23° 27'$; to find BA.

Ans. $BA = 51.595$ feet.

70. *Problem. To find the distance of an inaccessible object.* [B., pp. 89 and 95.]

Solution. Observation. Let B (fig. 2) be the point the distance of which is to be determined, and A the place of the observer. Measure off the distance AC of any convenient length, and observe the angles A and C.

Calculation. AB and CB are found by § 73 of Pl. Trig.

71. *Corollary.* The perpendicular distance PB of the point B from the line AC and the distance AP and PC are found, in the triangles ABP and BPC, by § 32 of Pl. Trig.

72. *Corollary.* Instead of directly observing the angles A and C, the bearings of the lines AB, AC, and CB may be observed, when the plane ABC is horizontal; and the angles A and C are then easily determined, since the meridians may be considered as parallel.

73. EXAMPLES.

1. An observer sees a cape which bears N. by E.; after sailing 30 miles N. W., he sees the same cape bearing east; find the distance of the cape from the two points of observation.

Ans. The first distance = 21.63 miles.
The second dist. = 25.43 miles.

2. Two observers, stationed on directly opposite sides of a cloud, observe the angles of elevation to be 44° 56′ and 36° 4′, their distance apart being 700 feet; find the distance of the cloud from each observer and its perpendicular altitude.

Ans. Distances from observers = 417.2 feet, and = 500.6 ft.
Height = 294.7 feet.

3. The angle of elevation of the top of a tower at one station is observed to be 68° 19′, and at another station, 546 feet farther from the tower, the angle of elevation is 32° 34′; find the height and distance of the tower, the two points of observation being supposed to be in the same horizontal plane with the foot of the tower.

Ans. The height . . . . = 467.44 ft.
The distance from the nearest point of observ. = 185.86 ft.

74. *Problem. To find the distance of an object from the foot of a tower of known height, the observer being at the top of the tower.*

Solution. Observation. Let the tower be AB (fig. 23) and the object C. Measure the angle of depression HAC.

Calculation. Since

$$ACB = HAC,$$

we know in the triangle ACB the leg AB and the opposite angle C, so that we can find BC as in § 33 of Pl. Trig.

EXAMPLE.

Given the height of the tower $= 150$ feet, and the angle of depression $= 17° 25'$; to find the distance from the foot of the tower.

Ans. 478.16 feet.

75. *Problem. To find the height of an inaccessible object above a horizontal plane, by means of observations taken at any two points in that plane.* [B., p. 96.]

Solution. Observation. Let A (fig. 25) be the object, and let D be the foot of the perpendicular dropped from A on the horizontal plane. At two different stations in the horizontal plane, B and C, whose distance apart and bearing from each other are known, observe the bearings of the object, which are the same as the angles made by BD and CD with the meridians of B and C. Also observe the angle of elevation of A at one of the stations, as B.

Calculation. In the triangle BCD, the side BC and its adjacent angles are known, so that BD is found by § 73 of Pl. Trig. In the right triangle ABD, the height DA is, then, computed by § 34 of Pl. Trig.

EXAMPLE.

At one station, the bearing of a cloud is N. N. W., and its angle of elevation 50° 35′. At a second station, whose bearing from the first station is N. by E. and distance 5000 feet, the bearing of the cloud is W. by N. Find the height of the cloud.

Ans. 7316.5.

76. *Problem. To find the distance of two objects whose relative position is known.* [B., p. 90.]

Solution. Observation. Let B and C (fig. 1) be the two known objects, and A the position of the observer. Observe the bearings of B and C from A.

Calculation. In the triangle ABC, the side BC and the three angles are known. The sides AB and AC are found by § 73 of Pl. Trig.

EXAMPLE.

The bearings of the two objects are, of the first N. E. by E., and of the second E. by S.; the known distance of the first object from the second is 23.25 miles, and the bearing N. W.; find their distance from the observer.

Ans. The distance of the first object is = 18.27 miles.
That of the second object = 32.25 miles.

77. *Problem. To find the distance apart of two objects separated by an impassable barrier, and their bearing from each other.* [B., p. 91.]

Solution. Observation. Let A and B (fig. 1) be the objects the distance and bearing of which from each other is sought. Measure the distances and bearings from any point C to both A and B.

Calculation. In the triangle ABC, the two sides AC and BC and the included angle C are known. The side AB and the angles A and B may be found by § 82 of Pl. Trig.

EXAMPLE.

Two ships sail from the same port, the one N. 10° E. a distance of 200 miles, the second N. 70° E. a distance of 150 miles; find their bearing and distance from each other.

Ans. The distance . . . . . . = 180.3 miles.
The bearing of the first ship from the second = N. 36° 6′ W.

78. *Problem. To find the distance apart of two inaccessible objects situated in the same plane with the observer, and their bearing from each other.* [B., p. 92.]

Solution. Observation. Let A and B (fig. 26) be the two inaccessible objects. At two stations, C and E, observe the bearings of A and B; and observe the bearing and distance of C from E.

Calculation. In the triangle AEC, we have the side CE and the angles ECA and AEC, so that CA is found by § 73 of Pl. Trig.

In the same way CB is calculated from the triangle BCE.

Lastly, in the triangle ABC, we know the two sides CA and CB and the included angle BCA.

Hence AB and the angles BAC and CBA are found by Pl. Tr. § 82.

EXAMPLE.

An observer from a ship saw two headlands; the first bore E. N. E., and the second N. W. by N. After he had sailed N. by W. 16.25 miles, the first headland bore E. and the second N. W. by W.; find the bearing and distance of the first headland from the second.

Ans. Distance = 55.89 miles.
Bearing = S. 80° 42′ E.

79. *Problem. To find the distance of an object of known height, which is just seen in the visible horizon.*

Solution. I. *If light moved in a straight line*, and if A (fig. 27) were the eye of the observer, and B the object, the straight line APB would be that of the visual ray. The point P, at which the ray touches the curved surface CPD of the earth, is the point of the visible horizon at which the object is seen. The distances PA and PB may be calculated separately, when the heights CA and DB are known. For this purpose, let O be the earth's centre, let BD be produced to E, and let

$$h = CA, \quad H = DB,$$
$$l = PA, \quad L = PB,$$
$$R = \text{the earth's radius.}$$

Since BP is a tangent and BOE a secant to the earth, we have

$$EB : PB = PB : DB;$$

and DB is so small in comparison with the radius that we may take

$$EB = ED = 2\,R,$$

and the above proportion becomes

$$2\,R : L = L : H;$$

whence

$$L^2 = 2\,RH, \quad L = \sqrt{(2\,RH)}, \tag{280}$$

$$H = \frac{L^2}{2\,R}; \tag{281}$$

and in the same way

$$l^2 = 2\,R\,h, \quad l = \sqrt{(2\,R\,h)}, \tag{282}$$

$$h = \frac{l^2}{2\,R}. \tag{283}$$

II. In consequence, however, of its refraction by the earth's atmosphere, light does not move in a straight line near the earth's surface, but *in a line curved towards the earth's centre, which line is nearly an arc of a circle whose radius is seven times the earth's radius;* so that for the point of contact P and the distances l and L, the positions of the eye and of the object are A' and B'. Now if we put

$$B'B = H', \quad DB' = H_1 = H - H'$$
$$CA' = h_1,$$

we can find the value of H' with sufficient accuracy by changing in (281) R into $7\,R$, which gives

$$H' = \frac{L^2}{14\,R} = \tfrac{1}{7}\,H$$

$$H_1 = H - H' = \tfrac{6}{7}\,H = \frac{3\,L^2}{7\,R}, \tag{284}$$

whence

$$L = \sqrt{(\tfrac{7}{3}\,RH_1)}. \tag{285}$$

III. In calculating the value of L by (285), it is usually desired in statute miles, while the height H_1 is given in feet. Now the radius of the earth is, as given in the Preface to the Navigator, page v,

$$R = 20911790 \text{ feet}, \tag{286}$$

whence

$$\tfrac{7}{3}\,R = 48794177 \text{ feet},$$

$$\log. \sqrt{(\tfrac{7}{3}\,R)} = \tfrac{1}{2} \log. \tfrac{7}{3}\,R = 3.84418,$$

and

$$\log. (L \text{ in feet}) = 3.84418 + \tfrac{1}{2} \log. (H_1 \text{ in feet}).$$

But

$$L \text{ in miles} = \frac{L \text{ in feet}}{5280},$$

so that $\log. L \text{ in miles} = \log. L \text{ in feet} - 3.72263$

$$= 0.12155 + \tfrac{1}{2} \log. H_1 \text{ in feet}; \quad (287)$$

which agrees with the formula given in the Preface to the Navigator for calculating Table X.

IV. Table X may be used for finding L and l, when H_1 and h_1 are given, and then the required distance is the sum of L and l.

80. *Corollary.* Table X gives the correction for the error which is committed in § 68 by neglecting the earth's curvature, for it is evident that to the height PB (fig. 28) of the object above the visible level must be added the height CP of the level above the curved surface of the earth, as in B., p. 95.

81. Examples.

1. Calculate the distance in Table X at which an object can be seen from the surface of the earth, when its height is 5000 feet.

Solution.

$$\tfrac{1}{2} \log. 5000 = \tfrac{1}{2} (3.69897) = 1.84948$$
$$\text{constant log.} = 0.12155$$
$$\text{dist.} = 93.5 \text{ m. (as in Table X)} \quad 1.97103$$

2. Being on a hill 200 feet above the sea, I see just appearing in the horizon the top of a mast, which I know to be 150 feet above water; how far distant is it?

Solution. By Table X,

200 feet corresponds to 18.71 miles.

150 feet corresponds to 16.20 miles.

The distance is 34.91 miles.

3. At the distance of $7\tfrac{1}{2}$ statute miles from a hill the angle of elevation of its top is 2° 13′; find its height in feet, the observer being 20 feet above the sea.

10*

Solution.

2° 13′	tang. 8.58779
7½ miles = 39600	4.59770
1533 feet	3.18549

observer's dist. from hill = 7.50
height 20 gives observer's dist. from horizon = 5.92

dist. of hill beyond horizon = 1.58,
which gives 1 foot correction.

Ans. 1534 feet.

4. Calculate the distance in Table X, when the height is 450 feet.

Ans. 28.06 miles.

5. Upon a height of 5000 feet, the top of a hill, one mile high, is just visible in the horizon; how far distant is the hill?

Ans. 189.6 miles.

6. At the distance of 25 statute miles from a mountain the angle of elevation of its top is 3°; find its height, the observer being 60 feet above the intervening sea.

Ans. 7042 feet.

SPHERICAL TRIGONOMETRY.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS.

1. *Spherical Trigonometry* treats of the solution of *spherical triangles.*

A Spherical Triangle is a portion of the surface of a sphere included between three arcs of great circles.

2. *The sides* of a spherical triangle are the measures of the angles formed, at the centre of the sphere, by the lines of intersection of their planes; and they are said to be *acute or obtuse,* according as they are less or greater than 90°.

The angles of a spherical triangle are the same as the angles formed by the planes of the sides; for any two sides are perpendicular, at their point of intersection, to the line of intersection of their planes.

The solution of spherical triangles in which any of the sides or angles are greater than 180° can always be reduced to the solution of spherical triangles in which all the parts are less than 180°; and, in this treatise, the discussion is limited to the latter class of triangles, and when values greater than 180° are found, in the solution of a triangle, for its unknown parts, they are rejected.

The student is directed to Chauvenet's Trigonometry for a chapter on the "Solution of the General Spherical Triangle."

3. Besides the usual method of denoting sides and angles

by degrees, minutes, &c., another method of denoting them is so often used in Spherical Astronomy that it will be found convenient to explain it here.

The circumference is supposed to be divided into 24 equal arcs called *hours*, each hour is divided into 60 *minutes of time*, each minute into 60 *seconds of time*, and so on.

Hours, minutes, seconds, &c. of time are denoted by *h*, *m*, *s*, &c.

4. *Problem. To convert degrees, minutes, &c. of arc into hours, minutes, &c. of time.*

Solution. Since

$$360° = 24^h$$

we have

$$15° = 1^h, \quad 1° = \tfrac{1}{15}{}^h = 4^m,$$

$$15' = 1^m, \quad 1' = 4^s,$$

$$15'' = 1^s, \quad 1'' = 4^t.$$

Hence

$$a° = 4\,a^m, \quad a' = 4\,a^s, \quad a'' = 4\,a^t;$$

so that *to convert degrees, minutes, &c. of arc into time, multiply by* 4, *and change the marks* ° ′ ″ *respectively into* $^{m\,s\,t}$.

5. *Corollary. To convert time into degrees, minutes, &c. of arc, multiply the hours by* 15 *for degrees, and divide the minutes, seconds, &c. of time by* 4, *changing the marks* $^{m\,s\,t}$ *into* ° ′ ″.

The turning of degrees, minutes, &c. of arc into time and the reverse may be at once performed by table XXI of the Navigator.

6. EXAMPLES.

1. Convert 225° 47′ 38″ into time.

Solution. By § 4.

$$225° = 900^m = 15^h$$
$$47' = 188^s = 3^m\ 8^s$$
$$38'' = 152^t = 2^s\ 32^t$$
$$\overline{225°\ 47'\ 38'' = 15^h\ 3^m\ 10^s\ 32^t}$$

By Table XXI.

$$15^h$$
$$3^m\ 8^s$$
$$2^s\ 32^t$$
$$\overline{15^h\ 3^m\ 10^s\ 32^t}$$

2. Convert $17^h\ 19^m\ 13^s$ into degrees, minutes, &c. of arc.

Solution. By § 5.

$$17^h = 255°$$
$$19^m\ 13^s = 4°\ 48'\ 15''$$
$$17^h\ 19^m\ 13^s = 259°\ 48'\ 15''$$

By Table XXI.

$$17^h\ 16^m = 259°$$
$$3^m\ 12^s = 48'$$
$$1^s = 15''$$
$$17^h\ 19^m\ 13^s = 259°\ 48'\ 15''$$

3. Convert 12° 34′ 56″ into time. *Ans.* $50^m\ 19^s\ 44^t$.

4. Find the difference of longitude, in time, of Portland and San Francisco.

Ans. $3^h\ 29^m\ 16^s$.

5. Convert $3^h\ 2^m\ 12^s$ into degrees, minutes, &c. of arc.

Ans. 45° 33′.

6. Convert $11^h\ 59^m\ 59^s$ into degrees, minutes, &c. of arc.

Ans. 179° 59′ 45″.

7. When an arc is given in time, its log. sine, &c. can be found directly from Table XXVII, by means of the column headed *Hour P. M.*, in which twice the time is given, so that the double of the angle must be found in this column.

The use of the table of proportional parts for these columns is explained upon page 35 of the Navigator. When the time exceeds 6^h, the difference between it and 12^h or 24^h must be used.

EXAMPLES.

1. Find the log. cosine of $19^h\ 33^m\ 11^s$.

Solution.

$$24^h - 19^h\ 33^m\ 11^s = 4^h\ 26^m\ 49^s$$
$$2 \times (4^h\ 26^m\ 49^s) = 8^h\ 53^m\ 38^s$$

$8^h\ 53^m\ 36^s$ P. M.	cos.	9.59720
prop. parts of 2^s		7
$8^h\ 53^m\ 38^s$ P. M.	cos.	9.59713

2. Find the angle in time of which the log. tang. is 10.12049.

$7^h\ 2^m\ 40^s$ P. M tang.	10.12026
7^s prop. parts	23
2) $7^h\ 2^m\ 47^s$ P. M	10.12049

Ans. $3^h\ 31^m\ 23\frac{1}{2}^s$

3. Find the log. sine of $3^h\ 12^m\ 2^s$. *Ans.* 9.87113.

4. Find the log. cosine of $11^h\ 3^m\ 13^s$. *Ans.* 9.98653_n.

5. Find the log. tang. of $15^h\ 0^m\ 9^s$. *Ans.* 10.00057.

6. Find the log. cotan. of $23^h\ 59^m\ 59^s$. *Ans.* 10.57183_n.

7. Find the angle in time whose log. secant is 10.23456.
Ans. $3^h\ 37^m\ 26^s$.

8. Find the angle in time whose log. cosecant is 10.12346.
Ans. $3^h\ 15^m\ 15^s$.

8. An *isosceles* spherical triangle is one which has two of its sides equal.

An *equilateral* spherical triangle is one which has all its sides equal.

9. A spherical *right* triangle is one which has a right angle; all other spherical triangles are called *oblique*.

We shall in spherical trigonometry, as we did in plane trigonometry, attend first to the solution of right triangles.

CHAPTER II.

SPHERICAL RIGHT TRIANGLES.

10. *Problem. To investigate some relations between the sides and angles of a spherical right triangle.*

Solution. The importance of this problem is obvious; for, unless some relations were known between the sides and the angles, they could not be determined from each other, and there could be no such thing as the solution of a spherical triangle.

Let, then, ABC (fig. 29) be a spherical right triangle, right-angled at C. Call the hypothenuse AB, h; and call the legs CB and AC, opposite the angles A and B respectively, a and b.

Let O be the centre of the sphere. Join OA, OB, OC.

The angle of the planes BOA and COA is, by § 2, equal to the angle A. The angle of the planes BOC and BOA is equal to the angle B. The angle of the planes BOC and AOC is equal to the angle C, that is, to a right angle; these two planes are, therefore, perpendicular to each other.

Moreover, the angle AOB, measured by AB, is equal to AB, or h; COB is equal to its measure CB, or a; and AOC is equal to its measure AC, or b.

Through any point A' of the line OA, suppose a plane $B'A'C'$ to pass perpendicular to OA. Its intersections $A'C'$ and $A'B'$ with the planes COA and BOA must be perpendicular to OA', because they are drawn through its foot in the plane $B'A'C'$.

As the plane $B'A'C'$ is perpendicular to OA, it must be perpendicular to the plane AOC, which contains OA; and its intersection $B'C'$ with the plane BOC, which is also perpendicular to AOC, must likewise be perpendicular to AOC. Hence $B'C'$ must be perpendicular to $A'C'$ and OC', which pass through its foot in the plane AOC.

The triangles $OA'B'$, $OA'C'$, $OC'B'$, and $A'C'B'$ are then right-angled, the first two at A', and the last two at C'; and the comparison of them leads to the desired equations, as follows: —

First. We have, from triangle $OA'B'$, by (4),

$$\cos.\ A'OB' = \cos.\ h = \frac{OA'}{OB'};$$

and, from triangles $OA'C'$ and $OC'B'$,

$$\cos.\ A'OC' = \cos.\ b = \frac{OA'}{OC'},$$

$$\cos.\ C'OB' = \cos.\ a = \frac{OC'}{OB'}.$$

The product of the last two equations is

$$\cos.\ a \cos.\ b = \frac{OA'}{OC'} \times \frac{OC'}{OB'} = \frac{OA'}{OB'};$$

hence, from the equality of the second members of these equations,

$$\cos.\ h = \cos.\ a \cos.\ b. \qquad (288)$$

Secondly. From triangle $A'C'B'$ we have, by (4), and the fact that the angle $C'A'B'$ is equal to the inclination of the two planes AOC and BOA,

$$\cos.\ C'A'B' = \cos.\ A = \frac{A'C'}{A'B'};$$

and, from triangles $OA'C'$ and $OA'B'$, by (4),

$$\text{tang.}\ AOC' = \text{tang.}\ b = \frac{A'C'}{OA'},$$

$$\text{cotan.}\ A'OB' = \text{cotan.}\ h = \frac{OA'}{A'B'}.$$

The product of these equations is

$$\text{tang.}\ b\ \text{cotan.}\ h = \frac{A'C'}{OA'} \times \frac{OA'}{A'B'} = \frac{A'C'}{A'B'};$$

hence

$$\cos.\ A = \text{tang.}\ b\ \text{cotan.}\ h. \qquad (289)$$

Thirdly. Corresponding to the preceding equation between the hypothenuse h, the angle A, and the adjacent side b, there must be a precisely similar equation between the hypothenuse h, the angle B, and the adjacent side a; which is

$$\cos. B = \text{tang.}\, a \text{ cotan.}\, h. \tag{290}$$

Fourthly. From triangles $OC'B'$, $OA'B'$, and $A'C'B'$, by (4),

$$\sin. C'OB' = \sin. a = \frac{C'B'}{OB'},$$

$$\sin. A'OB' = \sin. h = \frac{A'B'}{OB'},$$

$$\sin. C'A'B' = \sin. A = \frac{C'B'}{A'B'}.$$

The product of these last two equations is

$$\sin. h \sin. A = \frac{A'B'}{OB'} \times \frac{C'B'}{A'B'} = \frac{C'B'}{OB'};$$

hence

$$\sin. a = \sin. h \sin. A. \tag{291}$$

Fifthly. The preceding equation, between h, the angle A, and the opposite side a, leads to the following corresponding one, between h, the angle B, and the opposite side b;

$$\sin. b = \sin. h \sin. B. \tag{292}$$

Sixthly. From triangles $OA'C'$, $A'C'B'$, and $OC'B'$, by (4),

$$\sin. A'OC' = \sin. b = \frac{A'C'}{OC'},$$

$$\text{cotan.}\, C'A'B' = \text{cotan.}\, A = \frac{A'C'}{C'B'},$$

$$\text{tang.}\, C'OB' = \text{tang.}\, a = \frac{C'B'}{OC'}.$$

The product of these last two equations is

$$\text{cotan.}\, A \text{ tang.}\, a = \frac{A'C'}{C'B'} \times \frac{C'B'}{OC'} = \frac{A'C'}{OC'};$$

hence

$$\sin. b = \text{cotan.}\, A \text{ tang.}\, a. \tag{293}$$

Seventhly. The preceding equation, between the angle A, the opposite side a, and the adjacent side b, leads to the following corresponding one, between the angle B, the opposite side b, and the adjacent side a;

$$\sin. a = \text{cotan.}\, B \,\text{tang.}\, b. \tag{294}$$

Eighthly. From (7),

$$\text{tang.}\, a = \frac{\sin. a}{\cos. a},$$

$$\text{tang.}\, b = \frac{\sin. b}{\cos. b};$$

which, substituted in (294) and (293), give

$$\sin. a = \frac{\text{cotan.}\, B \sin. b}{\cos. b},$$

$$\sin. b = \frac{\text{cotan.}\, A \sin. a}{\cos. a}.$$

Multiplying the first of these equations by cos. b, and the second by cos. a, we have

$$\sin. a \cos. b = \text{cotan.}\, B \sin. b,$$

$$\sin. b \cos. a = \text{cotan.}\, A \sin. a.$$

The product of these equations is

$$\sin. a \sin. b \cos. a \cos. b = \text{cotan.}\, A \,\text{cotan.}\, B \sin. a \sin. b;$$

which, divided by sin. a sin. b, becomes

$$\cos. a \cos. b = \text{cotan.}\, A \,\text{cotan.}\, B.$$

But, by (288),

$$\cos. h = \cos. a \cos. b;$$

hence

$$\cos. h = \text{cotan.}\, A \,\text{cotan.}\, B. \tag{295}$$

Ninthly. We have, by (288) and (292),

$$\cos. a = \frac{\cos. h}{\cos. b},$$

$$\sin. B = \frac{\sin. b}{\sin. h},$$

the product of which is, by (7) and (8),

$$\cos. a \sin. B = \frac{\sin. b \cos. h}{\cos. b \sin. h} = \frac{\sin. b}{\cos. b} \cdot \frac{\cos. h}{\sin. h}$$

$$= \text{tang. } b \text{ cotan. } h.$$

But, by (289),

$$\cos. A = \text{tang. } b \text{ cotan. } h;$$

hence $$\cos. A = \cos. a \sin. B. \qquad (296)$$

Tenthly. The preceding equation, between the side a, the opposite angle A, and the adjacent angle B, leads to the following similar one, between the side b, the opposite angle B, and the adjacent angle A;

$$\cos. B = \cos. b \sin. A. \qquad (297)$$

11. *Corollary.* The ten equations (288 – 297) have, by a most happy artifice, been reduced to two very simple theorems, called, from their celebrated inventor, *Napier's Rules.*

In these rules, the complements of the hypothenuse and of the angles are used instead of the hypothenuse and the angles themselves, and the right angle is neglected.

There are, then, five parts: the legs, the complement of the hypothenuse, and the complements of the angles. Either part may be called the *middle part.* The two parts including the middle part on each side, are called the *adjacent parts;* and the other two parts are called the *opposite parts.* The two rules are as follows:—

I. *The sine of the middle part is equal to the product of the tangents of the two adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the two opposite parts.* [B., p. 438.]

Proof. To demonstrate the preceding rules, it is only necessary to compare all the equations which can be deduced from them with those previously obtained (288 – 297).

Let there be the spherical right triangle ABC (fig. 30), right-angled at C.

First. If co. h were made the middle part, then, by the above rules, co. A and co. B would be adjacent parts, and a and b opposite parts ; and we should have

$$\sin. (\text{co. } h) = \text{tang. } (\text{co. } A) \text{ tang. } (\text{co. } B),$$
$$\sin. (\text{co. } h) = \cos. a \cos. b ;$$

or

$$\cos. h = \text{cotan. } A \text{ cotan. } B,$$
$$\cos. h = \cos. a \cos. b ;$$

which are the same as (295) and (288).

Secondly. If co. A were made the middle part, then co. h and b would be adjacent parts, and co. B and a opposite parts ; and we should have

$$\sin. (\text{co. } A) = \text{tang. } (\text{co. } h) \text{ tang. } b,$$
$$\sin. (\text{co. } A) = \cos. (\text{co. } B) \cos. a ;$$

or

$$\cos. A = \text{cotan. } h \text{ tang. } b,$$
$$\cos. A = \sin. B \cos. a ;$$

which are the same as (289) and (296).

In like manner, if co. B were made the middle part, we should have

$$\cos. B = \text{cotan. } h \text{ tang. } a,$$
$$\cos. B = \sin. A \cos. b ;$$

which are the same as (290) and (297).

Thirdly. If a were made the middle part, then co. B and b would be the adjacent parts, and co. A and co. h the opposite parts ; and we should have

$$\sin. a = \text{tang. } (\text{co. } B) \text{ tang. } b,$$
$$\sin. a = \cos. (\text{co. } A) \cos. (\text{co. } h) ;$$

or

$$\sin. a = \text{cotan. } B \text{ tang. } b,$$
$$\sin. a = \sin. A \sin. h ;$$

which are the same as (294) and (291).

In like manner, if b were made the middle part, we should have

$$\sin. b = \text{cotan. } A \text{ tang. } a,$$

$$\sin. b = \sin. B \sin. h;$$

which are the same as (293) and (292).

Having thus made each part successively the middle part, the ten equations which we have obtained must be all the equations included in Napier's Rules; and we perceive that they are identical with the ten equations (288 – 297).

12. *Theorem. The three sides of a spherical right triangle are either all less than* 90°, *or else, one is less while the other two are greater than* 90°; unless one of them is equal to 90°, as in § 16.

Proof. If the two legs a and b are both acute or both obtuse, the factors of the second member of (288) are, by Pl. Trig. § 62, both positive or both negative. In either case, the first member of (288) must be positive; and consequently h must be less than 90°, or acute.

If one of the legs a and b is acute, and the other obtuse, the two factors of the second member of (288) have opposite signs; so that the first member is negative, and h is greater than 90°, or obtuse.

13. *Theorem. The hypothenuse of a spherical right triangle differs less from* 90° *than does either of the legs;* the case of either side equal to 90° being excepted.

Proof. The factors cos. a and cos. b of the second member of the equation (288) are, by Pl. Trig. §§ 9 and 61, or § 71, each less, in absolute value, than unity. Their product, neglecting the signs, must then be less than either of the factors, as cos. a for instance; or

$$\cos. h < \cos. a;$$

and therefore, by Pl. Trig. §§ 69 and 70, or § 71, h must differ less from 90° than a does.

14. *Theorem. An angle and its opposite leg in a spherical right triangle must be both acute or both obtuse or both equal to* 90°.

Proof. In the second member of (296), the factor sin. B is, by Pl. Trig. § 62, always positive, since all the sides and angles are supposed to be less than 180°. Therefore, the first member, cos. A, and the other factor, cos. a, of the second member must be both positive or both negative, unless they are both equal to zero; that is, A and a must be both acute or both obtuse or both equal to 90°.

15. *Theorem. Either angle of a spherical right triangle differs less from* 90° *than its opposite leg;* unless both are equal to 90°.

Proof. Since the second member of (296) is the product of the two fractions cos. a and sin. B, each of which is, absolutely, less than unity, the first member must, in absolute value, be less than either of them. Thus, neglecting the signs,

$$\cos. A < \cos. a;$$

hence A differs less from 90° than a does.

16. *Theorem. When in a spherical right triangle either side is equal to* 90°, *one of the other two sides is also equal to* 90°; *and each side of the triangle is equal to its opposite angle.*

Proof. I. If either of the legs is equal to 90°, the corresponding factor of the second member of (288) is, by (66), equal to zero; which, substituted in (288), gives

$$\cos. h = 0,$$

or, by (66),

$$h = 90°.$$

Again, if we have

$$h = 90°,$$

it follows, from (66) and (288), that

$$0 = \cos. a \cos. b,$$

and therefore either cos. a or cos. b must be zero; that is, either a or b must be equal to 90°.

II. When either of the three sides is equal to 90°, it follows from the preceding proof that

$$h = 90°;$$

which substituted in (291) produces, by (67),

$$\sin. a = \sin. A;$$

so that a is equal either to A or, by Pl. Trig. § 61, to the supplement of A. But, by Sph. Trig. § 14, a cannot be equal to the supplement of A; and therefore $a = A$.

17. *Corollary.* When both the legs of a spherical right triangle are equal to 90°, all the sides and angles are equal to 90°. In this case, the triangle is called *quadrantal.*

18. *Theorem. When two of the angles of a spherical triangle are equal to* 90°, *each side of the triangle is equal to its opposite angle.*

Proof. For in this case, one of the factors of the second member of the equation (295) must, by (68), be equal to zero, since either A or B is 90°; hence (295) gives

$$\cos. h = 0;$$

or, by (66),

$$h = 90°;$$

and the remainder of the proposition follows from § 16.

19. *Corollary.* When all the angles of a spherical triangle are equal to 90°, all the sides are also equal to 90°; and the triangle is quadrantal.

20. *Theorem. The sum of the angles of a spherical right triangle is greater than* 180° *and less than* 360°; *and each angle is less than the sum of the other two.*

Proof. I. It is proved in Geometry that the sum of the angles of any spherical triangle is greater than 180°.

II. It is proved in Geometry that each angle of any spherical triangle is greater than the difference between two right angles and the sum of the other two angles. Hence, if the sum of the two angles A and B is greater than 180°, we have

$$C, \text{ or } 90° > A + B - 180°,$$

or
$$A + B < 270°,$$

or
$$A + B + 90° < 360°;$$

that is, the sum of the three angles is less than 360°; and in case the sum of the angles A and B is less than 180°, the sum of the three angles is obviously less than 360°.

III. Since the sum of the three angles is greater than 180°, we have

$$90° + A + B > 180°,$$

or
$$A + B > 90°;$$

that is, if the right angle is the greatest of the three angles, the greatest angle is less than the sum of the other two.

But if one of the other angles A is the greatest of the three angles, we have, since each angle is greater than the difference between 180° and the sum of the other two,

$$B > 90° + A - 180°,$$

or
$$B > A - 90°,$$

or
$$A < B + 90°;$$

so that, in every case, each angle is less than the sum of the other two.

21. To solve a spherical right triangle, two parts must be known in addition to the right angle. From the two known parts the other three parts are to be determined, separately, by equations derived from Napier's Rules; each equation containing the two given parts and one of those which are to be determined. *The three parts which are to enter into any single equation are either all adjacent to each other, in which case the middle one is taken as the* MIDDLE PART, *and the other two are, by* § 11, ADJACENT PARTS; *or one is separated from the other two, and then the part which stands by itself is the* MIDDLE PART, *and the other two are, by* § 11, OPPOSITE PARTS. *The desired equation can then be formed by* § 11, *and the unknown part determined.*

22. *Problem. To solve a spherical right triangle, when the hypothenuse and one of the angles are given.*

Solution. Let ABC (fig. 30) be the right triangle, right-angled at C; and let the sides be denoted as in § 10. Let h and A be given; to solve the triangle.

First. To find the other angle B. The three parts which are to enter into the same equation are co. h, co. A, and co. B; and, by § 21, as they are all adjacent to each other, co. h is the middle part, and co. A and co. B are adjacent parts. Hence, by Napier's Rules,

$$\text{sin. (co. } h) = \text{tang. (co. } A) \text{ tang. (co. } B),$$

or

$$\text{cos. } h = \text{cotan. } A \text{ cotan. } B;$$

and, by (6),

$$\text{cotan. } B = \frac{\text{cos. } h}{\text{cotan. } A} = \text{cos. } h \text{ tang. } A.$$

Secondly. To find the opposite leg a. The three parts are co. A, co. h, and a; of which, by § 21, a is to be taken as the middle part, and co. h and co. A are the opposite parts. Hence, by Napier's Rules,

$$\text{sin. } a = \text{cos. (co. } h) \text{ cos. (co. } A),$$

or

$$\text{sin. } a = \text{sin. } h \text{ sin. } A.$$

Thirdly. To find the adjacent leg b. The three parts are co. A, co. h, and b; of which co. A is the middle part, and co. h and b are the adjacent parts. Hence, by Napier's Rules,

$$\text{sin. (co. } A) = \text{tang. (co. } h) \text{ tang. } b,$$

or

$$\text{cos. } A = \text{cotan. } h \text{ tang. } b;$$

and, by (6),

$$\text{tang. } b = \frac{\text{cos. } A}{\text{cotan. } h} = \text{tang. } h \text{ cos. } A.$$

23. *Scholium.* In the use of the above formulas and of all the formulas of Spherical Trigonometry, the precepts of Pl. Trig. § 62, with regard to the signs of the trigonometric functions, must be carefully observed, in order that we may determine whether the unknown parts are to be taken acute or obtuse. These precepts are sufficient for determining B and b, in the above solution; but a, being found by means of its sine, is left doubtful. But, by § 14, a must be taken acute or obtuse according as the given value of A is acute or obtuse.

We can easily test the results of the solution of a spherical right triangle, by comparing them with §§ 12 – 20 or with the ten equations (288 – 297).

24. *Scholium.* When h and A are both given equal to 90°, a is, by (67), equal to 90°, in the above solution, and the values of cotan. B and tang. b are *indeterminate*, since cos. h, cotan. A, cos. A, and cotan. h are then, by (66) and (68), all equal to zero. Now, it is geometrically evident that, in this case, the problem has *an infinite number of solutions;* that is, that an infinite number of triangles can be formed in which A, C, a, and h are each equal to 90°; but b and B are, by § 16, subject to the limitation that they are equal to each other in each solution.

If the given value of h is equal to 90°, while that of A differs from 90°, B and b are each equal to 90°, and a is equal to A, as in § 16.

The problem is *impossible*, by § 18, if the given value of h differs from 90°, while that of A is equal to 90°; and, in this case, the formulas give B and b equal to 0° or 180°, and a equal to h or 180° — h; that is, the triangle degenerates into either *a single arc* or *a lunary surface.*

25. Examples.

1. Given in the spherical right triangle (fig. 30) $h = 145°$ and $A = 23° 28'$; to solve the triangle.

Solution.

h, cos. 9.91336$_n$,*	sin. 9.75859,	tang. 9.84523$_n$
A, tang. 9.63761,	sin. 9.60012,	cos. 9.96251
B, cotan. 9.55097$_n$;	a sin. 9.35871;	b tang. 9.80774$_n$

Ans. $B = 109° 34' 33''$, $a = 13° 12' 12''$, $b = 147° 17' 15''$.

2. Given in the spherical right triangle (fig. 30) $h = 32° 34'$ and $A = 44° 44'$; to solve the triangle.

Ans. $B = 50° 8' 21''$,
$a = 22° 15' 43''$,
$b = 24° 24' 19''$.

* The letter n placed after a logarithm indicates that it is the logarithm of a negative quantity; and it is plain that, when the number of such logarithms to be added together is even, the sum is the logarithm of a positive quantity; but when the number is odd, the sum is the logarithm of a negative quantity.

26. *Problem. To solve a spherical right triangle, when its hypothenuse and one of its legs are given.*

Solution. Let ABC (fig. 30) be the triangle, h the given hypothenuse, and a the given leg.

First. To find the opposite angle A; a is the middle part, and co. A and co. h are the opposite parts. Hence, by Napier's Rules,

$$\sin. a = \sin. h \sin. A;$$

and, by (6),

$$\sin. A = \frac{\sin. a}{\sin. h} = \sin. a \text{ cosec. } h.$$

Secondly. To find the adjacent angle B; co. B is the middle part, and co. h and a are the adjacent parts. Hence, by Napier's Rules,

$$\cos. B = \text{tang. } a \text{ cotan. } h.$$

Thirdly. To find the other leg b; co. h is the middle part, and a and b are the opposite parts. Hence, by Napier's Rules,

$$\cos. h = \cos. a \cos. b;$$

and, by (6),

$$\cos. b = \frac{\cos. h}{\cos. a} = \sec. a \cos. h.$$

27. *Solution.* When h and a are both equal to 90°, it may be shown, as in § 24, that the values of B and b are *indeterminate;* and the problem has *an infinite number of solutions.*

28. *Scholium.* The problem is *impossible*, by § 13, when the given value of h differs more from 90° than that of a; and such values would give sin. A, cos. B, and cos. b each greater than unity, in the above solution.

29. EXAMPLE.

Given in the spherical right triangle (fig. 30) $a = 141° \; 11'$ and $h = 127° \; 12'$; to solve the triangle.

Ans. $A = 128° \; 5' \; 54''$,
$B = 52° \; 21' \; 45''$,
$b = 39° \; 6' \; 23''$.

30. *Problem. To solve a spherical right triangle, when one of its legs and the opposite angle are given.*

Solution. Let ABC (fig. 30) be the triangle, a the given leg, and A the given angle.

First. To find the hypothenuse h; a is the middle part, and co. h and co. A are the opposite parts. Hence

$$\sin. a = \sin. h \sin. A;$$

and, by (6),

$$\sin. h = \frac{\sin. a}{\sin. A} = \sin. a \operatorname{cosec.} A.$$

Secondly. To find the other angle B; co. A is the middle part, and a and co. B are the opposite parts. Hence

$$\cos. A = \cos. a \sin. B;$$

and, by (6),

$$\sin. B = \frac{\cos. A}{\cos. a} = \sec. a \cos. A.$$

Thirdly. To find the other leg b; b is the middle part, and a and co. A are the adjacent parts. Hence

$$\sin. b = \operatorname{tang.} a \operatorname{cotan.} A.$$

31. *Scholium.* Since h, B, and b are found by means of their sines, they may be taken either acute or obtuse, consistently with the formulas. In fact, there are *two triangles* ABC (fig. 31) and $A'BC$, formed by producing the sides AB and AC till they meet at A', both of which satisfy the conditions of the problem. For the side BC, or a, is common to both these triangles, the angle A' is, by § 2, equal to A, and the angle BCA', being the supplement of ACB, is, like ACB a right angle.

Now ABA' and ACA' are semicircumferences. Hence BA', or h', is the supplement of AB, or h; $A'C$, or b', is the supplement of CA, or b; and $A'BC$ is the supplement of CBA. One set of values, then, of the unknown parts corresponds to the triangle ABC, and the other set to $A'BC$; and the values must be distributed between the two triangles conformably to §§ 12 and 14.

32. *Corollary.* When the given values of a and A are equal, the above formulas give

$$\sin. h = 1, \ \sin. B = 1, \ \sin. b = 1;$$

or, by (67),

$$h = 90°, \quad B = 90°, \quad b = 90°;$$

and the case is that of § 16.

33. *Corollary.* When a and A are both equal to 90°, the values of b and B are *indeterminate*, as in § 24, and h is also equal to 90°.

34. *Scholium.* The problem is *impossible*, by § 14, when the given values of a and A are one acute and the other obtuse; or, by §§ 15 and 16, when A differs more from 90° than does a; or, by § 17, when A is equal to 90°, while a differs from 90°. In the first of these cases, the above formulas give sin. B and sin. b negative, so that B and b can be neither acute nor obtuse; in the second case, we have sin. $A <$ sin. a, cos. $A >$ cos. a, and tang. $A <$ tang. a, so that sin. h, sin. B, and sin. b are greater than unity; and, in the third case, B and b are equal to 0° or 180°, and h is equal to a or $180° - a$.

35. Example.

Given in the spherical right triangle (fig. 30) $a = 35°\ 44'$ and $A = 37°\ 28'$; to solve the triangle.

$$Ans. \quad \left.\begin{aligned} h &= 73°\ 45'\ 15'' \\ B &= 77°\ 54' \\ b &= 69°\ 50'\ 24'' \end{aligned}\right\} \text{ or } \left\{\begin{aligned} h &= 106°\ 14'\ 45'' \\ B &= 102°\ \ 6' \\ b &= 110°\ \ 9'\ 36''. \end{aligned}\right.$$

36. *Problem. To solve a spherical right triangle, when one of its legs and the adjacent angle are given.*

Solution. Let ABC (fig. 30) be the triangle, a the given leg, and B the given angle.

First. To find the hypothenuse h; co. B is the middle part, and co. h and a are adjacent parts. Hence

$$\cos. B = \text{tang. } a \text{ cotan. } h;$$

and, by (6),

$$\text{cotan. } h = \frac{\cos. B}{\text{tang. } a} = \text{cotan. } a \cos. B.$$

Secondly. To find the other angle A; co. A is the middle part, and co. B and a are opposite parts. Hence

$$\cos. A = \cos. a \sin. B.$$

Thirdly. To find the other leg b; a is the middle part, and co. B and b are adjacent parts. Hence

$$\sin.\ a = \text{tang.}\ b\ \text{cotan.}\ B;$$

and, by (6),

$$\text{tang.}\ b = \frac{\sin.\ a}{\text{cotan.}\ B} = \sin.\ a\ \text{tang.}\ B.$$

37. Example.

Given in the spherical right triangle (fig. 30) $a = 118°\ 54'$ and $B = 12°\ 19'$; to solve the triangle.

Ans. $h = 118°\ 20'\ 20''$,
$A = 95°\ 55'\ 2''$,
$b = 10°\ 49'\ 17''$.

38. *Problem. To solve a spherical right triangle, when its two legs are given.*

Solution. Let ABC (fig. 30) be the triangle, a and b the given legs.

First. To find the hypothenuse h; co. h is the middle part, a and b are opposite parts. Hence

$$\cos.\ h = \cos.\ a\ \cos.\ b.$$

Secondly. To find either of the angles, as A; b is the middle part, and co. A and a are adjacent parts. Hence

$$\sin.\ b = \text{tang.}\ a\ \text{cotan.}\ A;$$

and, by (6),

$$\text{cotan.}\ A = \frac{\sin.\ b}{\text{tang.}\ a} = \text{cotan.}\ a\ \sin.\ b.$$

In the same way,

$$\text{cotan.}\ B = \text{cotan.}\ b\ \sin.\ a.$$

39. EXAMPLE.

Given in the spherical right triangle (fig. 30) $a = 1°$ and $b = 100°$; to solve the triangle.

Ans. $h = 99° \; 59' \; 54''$,
$A = 1° \; 0' \; 56''$,
$B = 90° \; 10' \; 35''$.

40. *Problem. To solve a spherical right triangle, when the two oblique angles are given.*

Solution. Let ABC (fig. 30) be the triangle, A and B the given angles.

First. To find the hypothenuse h; co. h is the middle part, and co. A and co. B are adjacent parts. Hence

$$\cos. h = \text{cotan.} A \text{ cotan.} B.$$

Secondly. To find either of the legs, as a; co. A is the middle part, and co. B and a are the opposite parts. Hence

$$\cos. A = \cos. a \sin. B;$$

and, by (6),

$$\cos. a = \frac{\cos. A}{\sin. B} = \cos. A \text{ cosec. } B.$$

In the same way,

$$\cos. b = \text{cosec. } A \cos. B.$$

41. *Scholium.* The problem is, by § 20, ***impossible***, when the sum of the given values of A and B is less than 90° or greater than 270°, or when their difference is greater than 90°. In either of these cases, the above formulas give cos. h, cos. a, and cos. b greater than unity, in absolute value.

42. EXAMPLE.

Given in the spherical right triangle (fig. 30) $A = 135°$ and $B = 60°$; to solve the triangle.

Ans. $h = 125° \; 15' \; 53''$,
$a = 144° \; 44' \; 13''$,
$b = 45° \; 0' \; 5''$.

CHAPTER III.

SPHERICAL OBLIQUE TRIANGLES.

43. *Theorem. The sines of the sides, in any spherical triangle, are proportional to the sines of the opposite angles.* [B., p. 439.]

Proof. Let ABC (fig. 32 or 33) be the given triangle. Denote by a, b, c, the sides respectively opposite to the angles A, B, C. From either of the vertices, as B, let fall the perpendicular BP upon the opposite side AC. Then, in the right triangle ABP, if we make BP the middle part, co. c and co. BAP are the opposite parts. Hence by Napier's Rules,

$$\sin. BP = \sin. c \sin. BAP = \sin. c \sin. A.$$

For BAP is either the same as A (fig. 32), or it is its supplement (fig. 33), and in either case it has the same sine, by (98).

Again, in the right triangle BPC, if we make BP the middle part, co. a and co. C are the opposite parts. Hence, by Napier's Rules,

$$\sin. BP = \sin. a \sin. C;$$

and, from the two preceding equations,

$$\sin. c \sin. A = \sin. a \sin. C,$$

which may be written as a proportion, as follows: —

$$\sin. a : \sin. A = \sin. c : \sin. C.$$

In the same way, it must be true that

$$\sin. a : \sin. A = \sin. b : \sin. B.$$

44. *Theorem. Bowditch's Rules for Oblique Triangles.* If, in a spherical triangle, two right triangles are formed by a perpendicular let fall from either of its vertices upon the opposite side; and if, in the two right triangles, the middle parts are so taken that the perpendicular is one of the adjacent parts in each of them; then

The sines of the middle parts in the two triangles are proportional to the tangents of the other adjacent parts.

But, if, in the two right triangles, the middle parts are so taken that the perpendicular is one of the opposite parts in each of them ; then

The sines of the middle parts are proportional to the cosines of the other opposite parts. [B., p. 439.]

Proof. Let M denote the middle part in one of the right triangles, and m the middle part in the other right triangle, and let p denote the perpendicular.

First. If the middle parts M and m are so taken that the perpendicular is an adjacent part in both triangles, we have, by Napier's Rules, if A and a denote the other two adjacent parts in the two triangles,

$$\sin. M = \text{tang. } A \text{ tang. } p,$$
$$\sin. m = \text{tang. } a \text{ tang. } p;$$

whence

$$\frac{\sin. M}{\sin. m} = \frac{\text{tang. } A \text{ tang. } p}{\text{tang. } a \text{ tang. } p} = \frac{\text{tang. } A}{\text{tang. } a}.$$

or $$\sin. M : \sin. m = \text{tang. } A : \text{tang. } a.$$

Secondly. If M and m are so taken that the perpendicular is an opposite part in both the triangles, we have, by Napier's Rules, letting O and o denote the other two opposite parts in the two triangles,

$$\sin. M = \cos. O \cos. p,$$
$$\sin. m = \cos. o \cos. p;$$

whence

$$\frac{\sin. M}{\sin. m} = \frac{\cos. O \cos. p}{\cos. o \cos. p} = \frac{\cos. O}{\cos. o},$$

or $$\sin. M : \sin. m = \cos. O : \cos. o.$$

45. It is sometimes found useful, in the solution of a spherical triangle, to refer to the following propositions, which are proved in Geometry.

I. *The sum of the sides of a spherical triangle is less than* 360°.

II. *Each side is less than the sum of the other two.*

III. *The sum of the angles is greater than* 180°.

IV. *Each angle is greater than the difference between* 180° *and the sum of the other two.*

V. *Of any two sides, that is the greater which is opposite the greater angle.*

VI. *Of any two sides, that which differs the most from* 90° *is opposite to the angle which differs the most from* 90°; *and this side and its opposite angle are either both acute or both obtuse.*

46. *Problem. To solve a spherical triangle, when two sides and the included angle are given.* [B., p. 440.]

Solution. Let ABC (fig. 32 or 33) be the triangle, a and b the given sides, and C the given angle. Let BP be a perpendicular dropped from B on AC, and falling either within or without the triangle.

First. To find the segment CP. In the right triangle BPC, we know the hypothenuse a and the angle C. Hence, we have, by Napier's Rules, taking co. C as the middle part and CP and co. a as adjacent parts,

$$\text{tang. } CP = \text{cos. } C \text{ tang. } a; \tag{298}$$

and, according as CP is less or greater than the given side b, we shall know that the perpendicular BP falls within or without the triangle.

Secondly. The segment PA is the difference between CA and CP; that is, according as the perpendicular falls within or without the triangle,

$$\text{(fig. 32) } PA = b - CP, \quad \text{or (fig. 33) } PA = CP - b. \tag{299}$$

Thirdly. To find the side c. If, in the triangle BPC, co. a is the middle part, CP and PB are opposite parts; and if, in the triangle APB, co. c is the middle part, BP and PA are the opposite parts. Hence, by Bowditch's Rules, since the perpendicular is an opposite part in both triangles,

$$\text{cos. } CP : \text{cos. } PA = \text{sin. (co. } a) : \text{sin. (co. } c),$$

or

$$\text{cos. } CP : \text{cos. } PA = \text{cos. } a : \text{cos. } c. \tag{300}$$

Fourthly. To find the angle A. If, in the triangle BPC, CP is the middle part, co. C and BP are adjacent parts; and if, in the triangle APB, PA is the middle part, co. BAP and BP are adjacent parts. Hence, by Bowditch's Rules, since the perpendicular is an adjacent part in both triangles,

$$\sin. CP : \sin. PA = \text{cotan.}\ C : \text{cotan.}\ BAP; \qquad (301)$$

and the angle A is the same as BAP (fig. 32), when the perpendicular falls within the triangle; or it is the supplement of BAP (fig. 33), when the perpendicular falls without the triangle.

Fifthly. B is found by means of § 43, which gives

$$\sin. c : \sin. C = \sin. b : \sin. B. \qquad (302)$$

47. *Scholium.* In using (298), (300), and (301), we must carefully attend to the signs of the several functions, in order to determine, by Pl. Trig. § 62, whether CP, c, and BAP are acute or obtuse.

B, being found by means of its sine, cannot be thus determined; but all ambiguity is avoided by taking that one of the two given sides which differs the most from 90° as the side b; since, in that case, by § 45, proposition VI, B will be acute or obtuse according as b is acute or obtuse.

48. *Corollary.* Since, by (111) and (36),

$$\cos. (b \mathbin{\overline{\smile}} CP) = \cos. (CP - b) = \cos. b \cos. CP + \sin. b \sin. CP,$$

(300) gives, by (299),

$$\cos. CP : \cos. b \cos. CP + \sin. b \sin. CP = \cos. a : \cos. c;$$

$$\cos. c = \cos. a \cos. b + \sin. b \cos. a \ \text{tang.}\ CP. \qquad (303)$$

But, by (298),

$$\cos. a \ \text{tang.}\ CP = \cos. C \ \text{tang.}\ a \cos. a = \cos. C \sin. a; \qquad (304)$$

which, substituted in (303), gives

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C. \qquad (305)$$

Since c and C may denote any side and its opposite angle, we have also

$$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A, \qquad (306)$$

$$\cos. b = \cos. c \cos. a + \sin. c \sin. a \cos. B; \qquad (307)$$

and these three equations are *fundamental equations of Spherical Trigonometry.*

49. *Corollary.* In like manner, we can deduce from (301) the equation

$$\text{cotan. } A \text{ sin. } C = \text{cotan. } a \text{ sin. } b - \text{cos. } b \text{ cos. } C; \qquad (307\,a)$$

which represents a group of six other *fundamental equations of Spherical Trigonometry.*

50. *Corollary.* We have, by (55),

$$\text{cos. } C = -1 + 2\,(\text{cos. } \tfrac{1}{2}\,C)^2,$$

which, substituted in (305), gives, by (35),

$$\text{cos. } c = \text{cos. } (a + b) + 2 \text{ sin. } a \text{ sin. } b\,(\text{cos. } \tfrac{1}{2}\,C)^2, \qquad (308)$$

from which the value of the side c can readily be found by using the table of Natural Sines; and which, like (305), may be applied to either of the three sides.

We have, by (56),

$$\text{cos. } C = 1 - 2\,(\text{sin. } \tfrac{1}{2}\,C)^2.$$

which, substituted in (305), gives, by (36),

$$\text{cos. } c = \text{cos. } (a - b) - 2 \text{ sin. } a \text{ sin. } b\,(\text{sin. } \tfrac{1}{2}\,C)^2, \qquad (309)$$

which can be used like formula (308) to find the side c; and which may be applied to either of the three sides.

51. *Corollary.* The use of formula (309) is much facilitated by means of the column of Rising in Table XXIII of the Navigator. This column contains the values of

$$\begin{aligned}\text{log. } 2\,(\text{sin. } \tfrac{1}{2}\,C)^2 &= 2 \text{ log. sin. } \tfrac{1}{2}\,C + \text{log. } 2\\ &= 2 \text{ log. sin. } \tfrac{1}{2}\,C + 0.30103. \qquad (310)\end{aligned}$$

But the decimal point is supposed to be changed so as to correspond to the table of Natural Sines, that is, 5 is added to the logarithm; and 20 is to be subtracted from the value of 2 log. sin. $\frac{1}{2}\,C$, as obtained from Table XXVII, as is evident from Pl. Trig. § 30. So that the column Rising of Table XXIII is constructed by the formula

$$\begin{aligned}\text{log. Ris. } C &= 2 \text{ log. sin. } \tfrac{1}{2}\,C + 5.30103 - 20\\ &= 2 \text{ log. sin. } \tfrac{1}{2}\,C - 14.69897, \qquad (311)\end{aligned}$$

which agrees with the explanation in the Preface to the Navigator.

We have then the following rule for finding the third side by using Table XXIII, when two sides and the included angle are given.

Add together the log. Rising of the given angle and the log. sines of the two given sides. The sum is the logarithm of a number which is to be subtracted from the natural cosine of the difference of the two given sides (regard being had to the sign of this cosine). The difference is the natural cosine of the required side.

52. Examples.

1. Calculate the value of log. Ris. of $67° = 4^h\ 28^m$.

Solution.

$\frac{1}{2}(4^h\ 28^m) = 2^h\ 14^m$	sin.	9.74189
		2
		19.48378
		$-$14.69897
log. Ris. $4^h\ 28^m$		= 4.78481

2. Given in a spherical triangle two sides equal to 138° 32′ and 45° 54′, and the included angle equal to 98° 44′; to solve the triangle.

Solution. I. Of the two given sides, the former differs most from 90°. Therefore, let

$$a = 45°\ 54',\quad b = 138°\ 32',\quad C = 98°\ 44'.$$

Then, by (298),

$C = 98°\ 44'$	cos.	9.18137_n
$a = 45°\ 54'$	tang.	10.01365
$CP = 171°\ 5'\ 43''$	tang.	9.19502_n

Since $CP > b$, the case is that of fig. 33, and by (299),

$$PA = 171°\ 5'\ 43'' - 138°\ 32' = 32°\ 33'\ 43''.$$

By (300),

$CP = 171°\ 5'\ 43''$	sec.	10.00527_n
$PA = 32°\ 33'\ 43''$	cos.	9.92573
$a = 45°\ 54'$	cos.	9.84255
$c = 126°\ 25'\ 7''$	cos.	9.77355_n

By (301),

$CP = 171° \ 5' \ 43''$	cosec.	10.81024
$PA = \ 32° \ 33' \ 43''$	sin.	9.73095
$C = \ 98° \ 44'$	cotan.	9.18644_n
$BAP = 118° \ 6' \ 26''$	cotan.	9.72763_n

$A = 180° - 118° \ 6' \ 26'' = 61° \ 53' \ 34''$.

By (302),

$c = 126° \ 25' \ 7''$	cosec.	10.09436
$C = \ 98° \ 44'$	sin.	9.99494
$b = 138° \ 32'$	sin.	9.82098
$B = 125° \ 34' \ 30''$	sin.	9.91028

Ans. $c = 126° \ 25' \ 7''$
$A = \ 61° \ 53' \ 34''$
$B = 125° \ 34' \ 30''$.

II. The third side is thus calculated by means of (308).

2	log.	0.30103
$a = \ 45° \ 54'$	log. sin.	9.85620
$b = 138° \ 32'$	log. sin.	9.82098
$\frac{1}{2} C = \ 49° \ 22'$	2 log. cos.	19.62744

$0.40332 = 2 \sin. a \sin. b (\cos. \frac{1}{2} C)^2$ log. 9.60565

$-0.99701 = \text{Nat. cos.} (a + b) = \text{N. cos.} 184° \ 26' = -\text{N. cos.} 4° \ 26'$

$-0.59369 = \text{Nat. cos.} 126° \ 25' \ 10''$.

Ans. $c = 126° \ 25' \ 10''$.

III. The third side is thus calculated by (309) and Table XXIII.

$C = \ 98° \ 44' = 6^h \ 34^m \ 56^s$		log. Ris.	5.06139
$a = \ 45° \ 54'$		log. sin.	9.85620
$b = 138° \ 32'$		log. sin.	9.82098
$-2 \sin. a \sin. b (\sin. \frac{1}{2} C)^2$	-54774	log.	4.73857
$a - b = \ 92° \ 38'$ N. cos.	$-\ 4594$		
$c = 126° \ 25' \ 8''$ N. cos.	-59368		

3. Calculate the log. Ris. of $11^h\ 12^m\ 20^s$. *Ans.* 5.29632.

4. Given in a spherical triangle two sides equal to 125° and 100°, and the included angle equal to 45°; to solve the triangle.

Ans. The third side = 47° 55′ 52″.

The other two angles = 128° 42′ 48″, and = 69° 43′ 48″.

53. *Problem. To solve a spherical triangle, when one of its sides and the two adjacent angles are given.* [B., p. 440.]

Solution. Let ABC (fig. 32 or 33) be the triangle, a the given side, and B and C the given angles. From B let fall on AC the perpendicular BP.

First. To find CBP. We know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by Napier's Rules, if we make co. a the middle part,

$$\text{cotan. } CBP = \text{cos. } a \text{ tang. } C; \tag{312}$$

and if CBP is less than the given angle B, the perpendicular BP falls within the triangle; if CBP is greater than B, the perpendicular falls without.

Secondly. PBA is the difference between CBA and CBP, that is,

$$\left.\begin{array}{ll} & (\text{fig. } 32)\ PBA = B - CBP, \\ \text{or} & (\text{fig. } 33)\ PBA = CBP - B. \end{array}\right\} \tag{313}$$

Thirdly. To find the angle A. If, in the triangle PBC, co. C is the middle part, PB and co. CBP are the opposite parts; and if, in the triangle ABP, co. BAP is the middle part, PB and co. PBA are the opposite parts. Hence, by Bowditch's Rules,

$$\text{cos. (co. } CBP) : \text{cos. (co. } PBA) = \text{sin. (co. } C) : \text{sin. (co. } BAP),$$

$$\text{or} \quad \text{sin. } CBP : \text{sin. } PBA = \text{cos. } C : \text{cos. } BAP; \tag{314}$$

and the angle A is either BAP or its supplement, according as the perpendicular falls within or without the triangle.

Fourthly. To find the side c. If, in the triangle PBC, co. CBP is the middle part, PB and co. a are the adjacent parts; and if, in the triangle ABP, co. PBA is the middle part, PB and co. c are the adjacent parts. Hence, by Bowditch's Rules,

$$\text{cos. } CBP : \text{cos. } PBA = \text{cotan. } a : \text{cotan. } c. \tag{315}$$

Fifthly. The side b is found by the proportion of § 43,

$$\sin. A : \sin. a = \sin. B : \sin. b. \qquad (316)$$

54. *Scholium.* We must carefully attend to the signs of the several functions, in the use of (312), (314), and (315), in order to determine, by Pl. Trig. § 62, whether CBP, BAP, and c are acute or obtuse.

The side b, being found by means of its sine, cannot be thus determined; but if we take that one of the two given angles which differs the most from 90° as the angle B, the side b must, by § 45, proposition VI, be acute or obtuse according as B is acute or obtuse.

55. *Corollary.* If we denote by $A'B'C'$ the triangle which is *polar* to ABC, A' being the pole of the side a, B' of b, and C' of c, we have, by Geometry, for the sides and angles of the polar triangle,

$$a' = 180° - A, \quad b' = 180° - B, \quad c' = 180° - C,$$
$$A' = 180° - a, \quad B' = 180° - b, \quad C' = 180° - c.$$

We have given, then, in the problem of § 53, the sides b' and c' and the included angle A' of the triangle $A'B'C'$; so that we might find the unknown parts of the triangle ABC of that problem by solving its polar triangle $A'B'C'$ by § 46. In fact, we are thus led to the same formulas as those of § 53.

It is evident that the great circle of which $B'P'$ is an arc passes through the point B, since it is perpendicular to c', of which B is the pole, and that it is perpendicular to c, since it passes through B', the pole of c. The relations of the segments $A'P'$ and $P'C'$ to the angles CBP and PBA are, then, easily determined.

56. *Corollary.* If formula (306) is applied to the polar triangle $A'B'C'$, it gives

$$\cos. a' = \cos. b' \cos. c' + \sin. b' \sin. c' \cos. A',$$

or
$$\cos. (180° - A) = \cos. (180° - B) \cos. (180° - C) + \sin. (180° - B) \sin. (180° - C) \cos. (180° - a),$$

or, by (98) and (99),

$$-\cos. A = (-\cos. B)(-\cos. C) + \sin. B \sin. C (-\cos. a),$$

or
$$\cos. A = -\cos. B \cos. C + \sin. B \sin. C \cos. a; \qquad (317)$$

and this equation, which may be applied to either of the three angles of a spherical triangle, is another *fundamental equation of Spherical Trigonometry.*

57. *Corollary.* In like manner, we can obtain, from the six equations represented by (307 a), six equations of the form

$$\cot. c \sin. a = \cot. C \sin. B + \cos. B \cos. a;$$

but, as they can also be obtained from the equations (307 a) by simple transposition, they do not constitute a distinct group of equations.

58. *Corollary.* In the same way (308) gives for $A'B'C'$, if we change c to a', a to b', b to c', and C to A',

$$\cos. a' = \cos. (b' + c') + 2 \sin. b' \sin. c' (\cos. \tfrac{1}{2} A')^2,$$

or $$\cos. (180° - A) = \cos. (360° - [B + C])$$
$$+ 2 \sin. (180° - B) \sin. (180° - C) (\cos. [90° - \tfrac{1}{2} a])^2,$$

or, by (98), (99), and (123), and since $\cos. (90° - \frac{1}{2} a) = \sin. \frac{1}{2} a$,

$$\cos. A = - \cos. (B + C) - 2 \sin. B \sin. C (\sin. \tfrac{1}{2} a)^2; \quad (318)$$

which, like (317), may be applied to either of the three angles, and which may be used, like (309), in connexion with Table XXIII, to find the value of the unknown angle in the problem of § 53.

In like manner, (309) gives, for $A'B'C'$,

$$\cos. a' = \cos. (b' - c') - 2 \sin. b' \sin. c' (\sin. \tfrac{1}{2} A')^2,$$

or, by (98), (99), and (111), or (36),

$$\cos. A = - \cos. (C - B) + 2 \sin. B \sin. C (\cos. \tfrac{1}{2} a)^2$$
$$= - \cos. (B - C) + 2 \sin. B \sin. C (\cos. \tfrac{1}{2} a)^2; \quad (319)$$

which may also be applied to either of the three angles, and which may be used to find the value of the unknown angle in the problem of § 53.

59. Examples.

1. Given in a spherical triangle one side equal to 175° 27′, and the two adjacent angles equal to 126° 12′ and 109° 16′; to solve the triangle.

Solution. I. Observing that B should denote the angle which differs most from 90°, let

$$a = 175° \; 27', \quad B = 126° \; 12', \quad C = 109° \; 16'.$$

Then, by (312),

$a = 175° \; 27'$	cos.	9.99863_n
$C = 109° \; 16'$	tang.	0.45650_n
$CBP = \; 19° \; 19' \; 24''$	cotan.	0.45513

Since $CBP < B$, the perpendicular falls within the triangle, as in fig. 32. Hence, by (313),

$$PBA = 126° \; 12' - 19° \; 19' \; 24'' = 106° \; 52° \; 36''.$$

By (314),

$CBP = \; 19° \; 19' \; 24''$	cosec.	10.48031
$PBA = 106° \; 52' \; 36''$	sin.	9.98088
$C = 109° \; 16'$	cos.	9.51847_n
$BAP = 162° \; 36'$	cos.	9.97966_n

$$A = BAP = 162° \; 36'.$$

By (315),

$CBP = \; 19° \; 19' \; 24''$	sec.	10.02518
$PBA = 106° \; 52' \; 36''$	cos.	9.46287_n
$a = 175° \; 27'$	cotan.	1.09920
$c = \; 14° \; 30' \; 10''$	cotan.	0.58725_n

By (316),

$A = 162° \; 36'$	cosec.	10.52427
$a = 175° \; 27'$	sin.	8.89943
$B = 126° \; 12'$	sin.	9.90685
$b = 167° \; 38' \; 21''$	sin.	9.33055

Ans. $A = 162° \; 36'$

$b = 167° \; 38' \; 21''$

$c = \; 14° \; 30' \; 10''.$

II. The third angle is thus calculated by (318) and Table XXIII.

$a = 175° 27' = 11^h 41^m 48^s$			log. Ris. 5.30035
$B = 126° 12'$			log. sin. 9.90685
$C = 109° 16'$			log. sin. 9.97497
-2 sin. B sin. C (sin. $\frac{1}{2}a)^2$		$-$ 152114	5.18217
$B + C = 235° 28'$	$-$N. cos.	$+$ 56689	
$A = 162° 36' 7''$	N. cos.	$-$ 95425	

III. The third angle is thus calculated by means of (319).

2			log. 0.30103
$\frac{1}{2}a = \frac{1}{2}(175° 27') = 87° 43' 30''$			2 log. cos. 17.19748
$B = 126° 12'$			log. sin. 9.90685
$C = 109° 16'$			log. sin. 9.97497
2 sin. B sin. C (cos. $\frac{1}{2}a)^2$		$+$ 0.00240	7.38033
$B - C = 16° 56'$	$-$N. cos.	$-$ 0.95664	
$A = 162° 36'$	N. cos.	$-$ 0.95424	

2. Given in a spherical triangle one side $= 45° 54'$, and the two adjacent angles $= 125° 37'$ and $= 98° 44'$; to solve the triangle.

Ans. The third angle $= 61° 55' 2''$.

The other two sides $= 138° 34' 17''$, and $= 126° 26' 11''$.

60. *Problem. To solve a spherical triangle, when two sides and an angle opposite one of them are given.* [B., p. 439.]

Solution. Let ABC (fig. 32 or 33) be the triangle, a and c the given sides, and C the given angle. From B let fall on AC the perpendicular BP.

First. To find CP. We know, in the right triangle PBC, the side a and the angle C. Hence, by Napier's Rules, as in § 46,

$$\text{tang. } CP = \text{cos. } C \text{ tang. } a. \qquad (320)$$

Secondly. To find PA. If, in the triangle PBC, co. a is the middle part, CP and BP are the opposite parts; and if, in the triangle ABP, co. c is the middle part, PA and PB are the opposite parts. Hence, by Bowditch's Rules,

$$\text{cos. } a : \text{cos. } c = \text{cos. } CP : \text{cos. } PA. \qquad (321)$$

Thirdly. To find b. It is evident that this problem, like the corresponding one of Plane Trigonometry § 75, may have *two solutions* for the same values of the given parts, in one of which the perpendicular falls within the triangle, as in fig. 32, and

$$b = CP + PA, \tag{322}$$

while in the other the perpendicular falls without the triangle, as in fig. 33, and

$$b = CP - PA. \tag{323}$$

But, if PA is greater than CP, the second solution is impossible, and, if $CP + PA$ is greater than 180°, that is, if PA is greater than the supplement of CP, the first solution is impossible.

Fourthly. A and B may be found by the proportions

$$\sin. c : \sin. C = \sin. a : \sin. A \tag{324}$$

$$\sin. c : \sin. C = \sin. b : \sin. B. \tag{325}$$

61. *Scholium.* In determining CP and PA by (320) and (321), the precepts of Pl. Trig. § 62 concerning the signs of the trigonometric functions must be carefully attended to.

Since A is found by means of its sine, two supplementary values of A are given by (324). *These two values correspond to the two solutions of the problem,* since BAP, which is equal to A in fig. 32 and its supplement in fig. 33, must have the same value in both solutions. Also, as PB is the leg opposite C in the right triangle BPC, and the leg opposite BAP in the right triangle BPA, therefore, by § 14, C, PB, and BAP are, in both solutions, either all acute or all obtuse. Hence, *in the first solution* (fig. 32) *C and A are both acute or both obtuse, and in the second solution* (fig. 33) *one of them is acute and the other is obtuse.*

The two values of b are to be substituted separately in (325); and, for each value of b, (325) gives two values of B, of which the correct one is to be selected by the rules of § 45. But, instead of using (325), we can find CBP by (312), PBA by (315), and B by the equation

$$B = CBP \pm PBA.$$

62. *Scholium.* If the given value of c differs less from 90° than that of a; that is, if $\sin. c > \sin. a$; *only one solution of the problem* is possible, since, by § 45, prop. VI, a and A must be both acute or both obtuse. In this case, (321) gives, since $\cos. c < \cos. a$, if absolute values only are considered,

$$\cos. PA < \cos. CP.$$

If, then, a and C are *alike;* that is, both acute or both obtuse; CP is, by (320), acute; so that $PA > CP$, and $PA < 180° - CP$. But, if a and C are *unlike;* that is, one acute and the other obtuse; CP is, by (320), obtuse; so that $PA < CP$, and $PA > 180° - CP$. Therefore, in the former case, the first solution is the possible one, and, in the latter case, the second solution is the possible one.

If c differs more from 90° than a; that is, if $\sin. c < \sin. a$; (321) gives, in absolute value,

$$\cos. PA > \cos. CP.$$

If, then, c and C are unlike, the first ratio of (321) is opposite in sign to the second member of (320) and, therefore, to $\cos. CP$; so that (321) gives $\cos. PA$ negative; that is, PA is obtuse; and we have $PA > CP$, $PA > 180° - CP$, and *neither solution of the problem is possible.* This also appears from § 45, prop. VI. But if c and C are alike, PA is acute; and we have $PA < CP$, $PA < 180° - CP$, and *both solutions are possible.*

If, however, $\sin. c < \sin. a \sin. C$; that is, if $\sin. c < \sin. BP$; (324) gives $\sin. A > 1$; so that *the problem is impossible.* In this case, (321) will also give $\cos. PA > 1$.

63. *Scholium.* The various cases of this problem may be compared with those of Pl. Trig. § 75; a, c, and C, in this problem, corresponding respectively with b, a, and A, in that.

64. Examples.

1. Given in a spherical triangle one side = 35°, a second side = 142°, and the angle opposite the second side = 176°; to solve the triangle.

Solution. Let $a = 35°$, $c = 142°$, $C = 176°$.

Since c differs less from 90° than a, while a and C are unlike, the second solution alone is possible. Then, by (320) and (321),

$C = 176°$	cos.	9.99894_n		
$a = 35°$	tang.	9.84523	sec.	10.08664
$CP = 145°\ 3'\ 56''$	tang.	9.84417_n	cos.	9.91371_n
$c = 142°$			cos.	9.89653_n
$PA = 37°\ 56'\ 30''$			cos.	9.89688

By (323),

$$b = 145°\ 3'\ 56'' - 37°\ 56'\ 30'' = 107°\ 7'\ 26''.$$

By (324) and (325),

$c = 142°$	cosec.	10.21066	cosec.	10.21066
$C = 176°$	sin.	8.84358	sin.	8.84358
$a = 35°$	sin.	9.75859		
$b = 107° 7' 26''$			sin.	9.98031
	sin.	8.81283	sin.	9.03455
(§ 45, props. VI, IV.)	$A = 3° 43' 34''$;		$B = 6° 12' 58''$.	

Ans. $b = 107° 7' 26''$
$A = 3° 43' 34''$
$B = 6° 12' 58''$.

2. Given in a spherical triangle one side $= 120°$, a second side $= 135°$, and the angle opposite the second side $= 155°$; to solve the triangle.

Ans. The third side $= 94° 38' 18''$ or $= 17° 58' 54''$.
The first angle $= 142° 14' 22''$ or $= 37° 45' 38''$.
The third angle $= 135° 11' 14''$ or $= 12° 36' 31''$.

3. Given in a spherical triangle one side $= 54°$, a second side $= 22°$, and the angle opposite the second side $= 30°$; to solve the triangle.

Ans. The question is impossible.

65. *Problem. To solve a spherical triangle, when two angles and a side opposite one of them are given.* [B., p. 440.]

Solution. Let ABC (fig. 32 or 33) be the triangle, A and C the given angles, and a the given side.

From B let fall on AC the perpendicular BP. This perpendicular must fall within the triangle, if A and C are either both obtuse or both acute; but it falls without, if one is obtuse and the other acute. This is evident from § 61.

First. CBP may be found, as in § 53, by (312).

Secondly. PBA may be found by (314); that is,

$$\cos. C : \cos. BAP = \sin. CBP : \sin. PBA. \quad (326)$$

Thirdly. To find B. We have

$$\text{(fig. 32) } B = CBP + PBA. \quad (327)$$

$$\text{or (fig. 33) } B = CBP - PBA; \quad (328)$$

according as A and C are alike or unlike.

Fourthly. The sides c and b may be found by the proportions

$$\sin. A : \sin. a = \sin. C : \sin. c, \tag{329}$$

$$\sin. A : \sin. a = \sin. B : \sin. b. \tag{330}$$

66. *Scholium.* Since PBA is found by means of its sine, two values of PBA, supplements of each other, are given by (326). These two values correspond to *two solutions of the problem.* For, if the triangle CBA' (fig. 72) is a solution of the problem, and if the angles PBA' and PBA'' are supplements of each other, CBA'' is likewise a solution. For, if $A'B$ and $A'C$ are produced till they meet at A''', they form a lunary surface, in which the angle $A''' =$ the angle A'; and, since $A'''BP =$ the supplement of $PBA' = PBA''$, the right triangles $A'''PB$ and $A''PB$ are symmetrical, and $BA''P = BA'P =$ the given angle A; so that CBA'' contains the given values of A, C, and a. But both solutions are alike in respect to the position of the perpendicular within or without the triangle.

If, when the perpenaicular falls within the triangle, either value of PBA is greater than the supplement of CBP, or if, when the perpendicular falls without, either value is greater than CBP, that value must be rejected, and the corresponding solution is impossible.

Two supplementary values of c are given by (329), which evidently correspond to the two solutions of the problem; and since, by § 14, C and PB are alike, and PBA and PA are alike, therefore, in each solution, by § 12, C, PBA, and c must be either all acute or else one acute and the other two obtuse.

Each value of B gives two values of b, by (330), of which the true value is to be selected by the rules of § 45. But, instead of using (330), we may find CP by (298), PA by (301), and b by (322) or (323).

67. *Scholium.* If A differs less from 90° than C; that is, if $\sin. A > \sin. C$; we have $\cos. BAP < \cos. C$, and (326) gives

$$\sin. PBA < \sin. CBP.$$

Hence the acute value of PBA is less, and the obtuse value of PBA is greater, than CBP and than the supplement of CBP; so that, whether BP falls within or without, *the problem has but one solution;* namely, that in which PBA is acute. This also follows from § 45, prop. VI.

If A differs more from 90° than C; that is, if $\sin. A < \sin. C$; (326) gives

$$\sin. PBA > \sin. CBP.$$

Hence, if CBP is acute; that is, by (312), if a and C are alike; both values of PBA are greater than CBP and less than its supplement but, if CBP is obtuse; that is, if a and C are unlike; both values of PBA are less than CBP and greater than its supplement. But B is found by (327), when A and C are alike, and by (328), when A and C are unlike. Therefore, if a and A are alike, ***both solutions are possible;*** and, if a and A are unlike, ***neither solution is possible.***

If, however, sin. $A <$ sin. C sin. a; that is, if sin. $A <$ sin. BP; ***the problem is impossible,*** since (329) gives sin. $c > 1$. In this case, (326) also gives sin. $PBA > 1$.

68. *Scholium.* The formulas of § 65 might also be obtained by applying those of § 60 to the polar triangle $A'B'C'$, supposing a', c', and A' to be known; and the various cases of the two problems correspond to each other. The ambiguity of solution arises, in the former problem, from doubt whether the perpendicular BP falls within or without the triangle, and, in this, from doubt whether PBA is acute or obtuse. Nevertheless, the two solutions of the former problem correspond to the two solutions of this.

69. Examples.

1. Given in a spherical triangle one angle $= 95°$, a second angle $= 104°$, and the side opposite the first angle $= 138°$; to solve the triangle.

Solution. Let $A = 95°$, $C = 104°$, $a = 138°$.

Since A and C are both obtuse, the perpendicular falls within the triangle; and, since A differs less from 90° than C, PBA is acute, and there is only one solution. Then, by (312) and (326),

$a = 138°$	cos.	9.87107_n		
$C = 104°$	tang.	10.60323_n	sec.	10.61632_n
$CBP = 18° 32' 49''$	cotan.	10.47430	sin.	9.50254
$BAP = 95°$			cos.	8.94030_n
$PBA = 6° 34' 49''$			sin.	9.05916

By (327),

$$B = 18° 32' 49'' + 6° 34' 49'' = 25° 7' 38''.$$

By (329) and (330),

$A = 95°$	cosec.	10.00166	cosec.	10.00166
$a = 138°$	sin.	9.82551	sin.	9.82551
$C = 104°$	sin.	9.98690		
$B = 25° \; 7' \; 38''$			sin.	9.62801
	sin.	9.81407	sin.	9.45518
(§ 45, prop. VI.)		$c = 139° \; 19' \; 40''$;		$b = 16° \; 34' \; 19''$.

Ans. $b = 16° \; 34' \; 19''$
$c = 139° \; 19' \; 40''$
$B = 25° \; 7' \; 38''$.

2. Given in a spherical triangle one angle $= 135°$, a second angle $= 60°$, and the side opposite the first angle $= 155°$; to solve the triangle.

Ans. The third angle $= 94° \; 38' \; 18''$ or $= 17° \; 58' \; 54''$.
The second side $= 37° \; 45' \; 38''$ or $= 142° \; 14' \; 22''$.
The third side $= 135° \; 11' \; 14''$ or $= 12° \; 36' \; 31''$.

70. *Problem. To solve a spherical triangle, when its three sides are given.* [B., p. 440.]

Solution. Equation (305) gives, by transposition and division,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}, \qquad (331)$$

whence the value of the angle C may be calculated; and in the same way either of the other angles.

71. *Corollary.* An equation more easy for calculation by logarithms may be obtained from (308), which gives, by transposition and division,

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{\cos. c - \cos. (a + b)}{\sin. a \sin. b}. \qquad (332)$$

Now, letting s denote half the sum of the sides, or

$$s = \tfrac{1}{2} (a + b + c); \qquad (333)$$

if we make, in the general formula (42),

$$M = \tfrac{1}{2} (a + b + c) = s,$$
$$N = \tfrac{1}{2} (a + b - c) = s - c;$$

we have
$$M + N = a + b,$$
$$M - N = c;$$

and (42) becomes

$$\cos. c - \cos. (a + b) = 2 \sin. s \sin. (s - c);$$

which, substituted in (332), gives

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{2 \sin. s \sin. (s - c)}{\sin. a \sin. b}, \tag{334}$$

$$\cos. \tfrac{1}{2} C = \sqrt{\left(\frac{\sin. s \sin. (s - c)}{\sin. a \sin. b}\right)}. \tag{335}$$

The angles A and B may, in like manner, be found by the two following equations, which are easily deduced from (335),

$$\cos. \tfrac{1}{2} A = \sqrt{\left(\frac{\sin. s \sin. (s - a)}{\sin. b \sin. c}\right)}. \tag{336}$$

$$\cos. \tfrac{1}{2} B = \sqrt{\left(\frac{\sin. s \sin. (s - b)}{\sin. c \sin. a}\right)}. \tag{337}$$

72. *Corollary.* Another equation, equally simple in calculation, can be obtained from (309), which gives, by transposition and division,

$$2 (\sin. \tfrac{1}{2} C)^2 = \frac{\cos. (a - b) - \cos. c}{\sin. a \sin. b}, \tag{338}$$

whence C can be found by the aid of Table XXIII.

73. *Corollary.* If, in (42), we make

$$M = \tfrac{1}{2} (a - b + c) = s - b,$$
$$N = \tfrac{1}{2} (-a + b + c) = s - a;$$

we have
$$M + N = c,$$
$$M - N = a - b;$$

and (42) becomes

$$\cos. (a - b) - \cos. c = 2 \sin. (s - a) \sin. (s - b);$$

which, substituted in (338), gives

$$2 (\sin. \tfrac{1}{2} C)^2 = \frac{2 \sin. (s - a) \sin. (s - b)}{\sin. a \sin. b}, \tag{339}$$

$$\sin. \tfrac{1}{2} C = \sqrt{\left(\frac{\sin. (s - a) \sin. (s - b)}{\sin. a \sin. b}\right)}. \tag{340}$$

In the same way we might deduce the following equations :

$$\sin.\ \tfrac{1}{2}\,A = \sqrt{\left(\frac{\sin.\ (s-b)\ \sin.\ (s-c)}{\sin.\ b\ \sin.\ c}\right)}. \qquad (341)$$

$$\sin.\ \tfrac{1}{2}\,B = \sqrt{\left(\frac{\sin.\ (s-c)\ \sin.\ (s-a)}{\sin.\ c\ \sin.\ a}\right)}. \qquad (342)$$

74. *Corollary.* The quotient of (341), divided by (336), is, by (7),

$$\text{tang.}\ \tfrac{1}{2}A = \frac{\sin.\ \tfrac{1}{2}A}{\cos.\ \tfrac{1}{2}A} = \sqrt{\left(\frac{\sin.\ (s-b)\ \sin.\ (s-c)}{\sin.\ s\ \sin.\ (s-a)}\right)}. \qquad (343)$$

In the same way,

$$\text{tang.}\ \tfrac{1}{2}\,B = \sqrt{\left(\frac{\sin.\ (s-c)\ \sin.\ (s-a)}{\sin.\ s\ \sin.\ (s-b)}\right)}. \qquad (344)$$

$$\text{tang.}\ \tfrac{1}{2}\,C = \sqrt{\left(\frac{\sin.\ (s-a)\ \sin.\ (s-b)}{\sin.\ s\ \sin.\ (s-c)}\right)}. \qquad (345)$$

75. Examples.

1. Given in the spherical triangle ABC the three sides equal to 46°, 72°, and 68°; to solve the triangle.

Solution.	I. By (336),	by (337),	by (335),
$a = 46°$ cosec.		10.14307	10.14307
$b = 72°$ cosec.	10.02179		10.02179
$c = 68°$ cosec.	10.03283	10.03283	
$s = 93°$ sin.	9.99940	9.99940	9.99940
$s - a = 47°$ sin.	9·86413		
$s - b = 21°$ sin.		9.55433	
$s - c = 25°$ sin.			9.62595
	2) 19.91815	2) 19.72963	2) 19.79021
cos.	9.95908	9.86482	9.89510

$\tfrac{1}{2}\,A = 24°\ 29'\ 5''$, $\tfrac{1}{2}\,B = 42°\ 54'\ 8''$, $\tfrac{1}{2}\,C = 38°\ 14'\ 21''$

$A = 48°\ 58'\ 10''$, $B = 85°\ 48'\ 16''$, $C = 76°\ 28'\ 42''$.

II. By Table XXIII and equation (338).

$a - b = -26°$	N. cos.	89879	a log. cosec.	0.14307
$c = 68°$	N. cos.	37461	b log. cosec.	0.02179
		52418	log.	4.71948
$C = 5^h\ 5^m\ 55^s = 76°\ 28'\ 45''$.			log. Ris.	4.88434

In like manner, we find $A = 48° 58'$, $B = 85° 48' 15''$.

2. Given in a spherical triangle the three sides equal to 3°, 4°, and 5°; to solve the triangle.

Ans. The three angles are 36° 54′ 15″, 53° 10′ 9″, and 90° 2′ 5″.

76. Napier obtained two theorems for the solution of a spherical triangle, when a side and the two adjacent angles are given, by which the two sides can be calculated without the necessity of calculating the third angle. [B., p. 441.] These theorems, which are given in §§ 79 and 80, and are called *Napier's first and second Analogies*, can be obtained from equations (343 - 345) by the assistance of the following lemmas.

77. *Lemma. If we have an equation of the form*

$$\frac{\text{tang. } M}{\text{tang. } N} = \frac{x}{y}, \tag{346}$$

we can deduce from it the equation

$$\frac{\sin. (M+N)}{\sin. (M-N)} = \frac{x+y}{x-y}. \tag{347}$$

Proof. We have from (7)

$$\text{tang. } M = \frac{\sin. M}{\cos. M}, \text{ and tang. } N = \frac{\sin. N}{\cos. N};$$

which, substituted in (346), give

$$\frac{\sin. M \cos. N}{\cos. M \sin. N} = \frac{x}{y}.$$

This equation is the same as the proportion

$$\sin. M \cos. N : \cos. M \sin. N = x : y;$$

whence, by the theory of proportions,

$$\sin. M \cos. N + \cos. M \sin. N : \sin. M \cos. N - \cos. M \sin. N$$
$$= x + y : x - y,$$

or, by (33) and (34),

$$\sin. (M+N) : \sin. (M-N) = x + y : x - y;$$

which may be written in the form of an equation, as in (347).

78. *Lemma. If we have an equation of the form*

$$\text{tang. } M \text{ tang. } N = \frac{x}{y}, \tag{348}$$

we can deduce from it the equation

$$\frac{\cos.\ (M+N)}{\cos.\ (M-N)}=\frac{y-x}{y+x}. \qquad (349)$$

Proof. We have, by (348) and (7),

$$\frac{\sin.\ M \sin.\ N}{\cos.\ M \cos.\ N}=\frac{x}{y}.$$

This equation is the same as the proportion

$$\cos.\ M \cos.\ N : \sin.\ M \sin.\ N = y : x;$$

whence, by the theory of proportions,

$$\cos.\ M \cos.\ N - \sin.\ M \sin.\ N : \cos.\ M \cos.\ N + \sin.\ M \sin.\ N = y - x : y + x,$$

or, by (35) and (36),

$$\cos.\ (M+N) : \cos.\ (M-N) = y - x : y + x;$$

which may be written as in (349).

79. *Theorem. The sine of half the sum of two angles of a spherical triangle is to the sine of half their difference as the tangent of half the interjacent side is to the tangent of half the difference of the opposite sides;* that is, in the spherical triangle ABC (fig. 32 or 33),

$$\sin.\ \tfrac{1}{2}(A+C) : \sin.\ \tfrac{1}{2}(A-C) = \text{tang.}\ \tfrac{1}{2} b : \text{tang.}\ \tfrac{1}{2}(a-c). \qquad (350)$$

Proof. The quotient of (343) divided by (345) is, by an easy reduction,

$$\frac{\text{tang.}\ \frac{1}{2} A}{\text{tang.}\ \frac{1}{2} C}=\frac{\sin.\ (s-c)}{\sin.\ (s-a)}. \qquad (351)$$

Hence, by § 77,

$$\frac{\sin.\ \frac{1}{2}(A+C)}{\sin.\ \frac{1}{2}(A-C)}=\frac{\sin.\ (s-c)+\sin.\ (s-a)}{\sin.\ (s-c)-\sin.\ (s-a)}. \qquad (352)$$

If, in equation (47), we put

$$A = s - c = \tfrac{1}{2}(a+b-c),$$
$$B = s - a = \tfrac{1}{2}(-a+b+c);$$

we have

$$A + B = b,$$
$$A - B = a - c;$$

and (47) becomes

$$\frac{\sin.\ (s-c)+\sin.\ (s-a)}{\sin.\ (s-c)-\sin.\ (s-a)}=\frac{\text{tang.}\ \frac{1}{2} b}{\text{tang.}\ \frac{1}{2}(a-c)}.$$

This equation, substituted in the second member of (352), gives

$$\frac{\sin.\ \frac{1}{2}(A+C)}{\sin.\ \frac{1}{2}(A-C)}=\frac{\text{tang.}\ \frac{1}{2}b}{\text{tang.}\ \frac{1}{2}(a-c)}; \qquad (353)$$

which is the same as (350).

80. *Theorem. The cosine of half the sum of two angles of a spherical triangle is to the cosine of half their difference as the tangent of half the interjacent side is to the tangent of half the sum of the opposite sides;* that is, in the spherical triangle ABC (fig. 32 or 33),

$$\cos.\ \tfrac{1}{2}(A+C):\cos.\ \tfrac{1}{2}(A-C)=\text{tang.}\ \tfrac{1}{2}b:\text{tang.}\ \tfrac{1}{2}(a+c). \qquad (354)$$

Proof. The product of (343) and (345) is, by a simple reduction,

$$\text{tang.}\ \tfrac{1}{2}A\ \text{tang.}\ \tfrac{1}{2}C=\frac{\sin.\ (s-b)}{\sin.\ s}.$$

Hence, by § 78,

$$\frac{\cos.\ \frac{1}{2}(A+C)}{\cos.\ \frac{1}{2}(A-C)}=\frac{\sin.\ s-\sin.\ (s-b)}{\sin.\ s+\sin.\ (s-b)}. \qquad (355)$$

If, in equation (47) inverted, we put

$$A=s\quad=\tfrac{1}{2}(a+b+c),$$
$$B=s-b=\tfrac{1}{2}(a-b+c);$$

we have

$$A+B=a+c,$$
$$A-B=b;$$

and (47) becomes

$$\frac{\sin.\ s-\sin.\ (s-b)}{\sin.\ s+\sin.\ (s-b)}=\frac{\text{tang.}\ \frac{1}{2}b}{\text{tang.}\ \frac{1}{2}(a+c)}.$$

This equation, substituted in (355), gives

$$\frac{\cos.\ \frac{1}{2}(A+C)}{\cos.\ \frac{1}{2}(A-C)}=\frac{\text{tang.}\ \frac{1}{2}b}{\text{tang.}\ \frac{1}{2}(a+c)}; \qquad (356)$$

which is the same as (354).

81. *Scholium.* In using (350) and (354), the signs of the terms must be attended to, by reference to Pl. Trig. §§ 62 and 64; and it will be seen that, by (354), $a+c$ is greater or less than 180°, according as $A+C$ is greater or less than 180°.

82. Examples.

1. Given in a spherical triangle two angles $= 158°$ and $= 98°$, and the interjacent side $= 144°$; to find the other two sides.

Solution. By (350) and (354),

$\frac{1}{2}(A + C) = 128°$	cosec. 10.10347	sec. 10.21066_n
$\frac{1}{2}(A - C) = 30°$	sin. 9.69897	cos. 9.93753
$\frac{1}{2}b = 72°$	tang. 10.48822	tang. 10.48822
$\frac{1}{2}(a - c) = 62° \ 53' \ 2''$	tang. 10.29066	
$\frac{1}{2}(a + c) = 103° \ 0' \ 24''$		tang. 10.63641_n

Ans. $a = 165° \ 53' \ 26''$,
$c = 40° \ 7' \ 22''$.

2. Given in a spherical triangle two angles $= 126° \ 12'$ and $= 109° \ 16'$, and the interjacent side $= 175° \ 27'$; to find the other two sides.

Ans. $167° \ 38' \ 19''$ and $14° \ 30' \ 11''$.

83. *Problem. To solve a spherical triangle, when its three angles are given.* [B., p. 441.]

Solution. When the angles of the triangle ABC are given, the sides of its polar triangle $A'B'C'$ are readily found. The desired solution may, then, be obtained by applying to $A'B'C'$ any of the methods of §§ 70 – 74.

84. *Corollary.* Applying (331) to $A'B'C'$, we have, by (98) and (99),

$$\cos. c = \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B}; \tag{357}$$

which may also be derived from (317), and which may be used to find either side, when the angles are given.

85. *Corollary.* If we put

$$S = \tfrac{1}{2}(A + B + C), \tag{358}$$

we have, in the polar triangle $A'B'C'$,

$$s' = 270° - S,$$

$s' - a' = \text{co.}(S - A), s' - b' = \text{co.}(S - B), s' - c' = \text{co.}(S - C)$;

so that equations (335 - 337), when applied to $A'B'C'$, give by (34), (78), (79), and (98),

$$\sin.\tfrac{1}{2}a = \sqrt{\left(\frac{-\cos.S\cos.(S-A)}{\sin.B\sin.C}\right)}, \qquad (359)$$

$$\sin.\tfrac{1}{2}b = \sqrt{\left(\frac{-\cos.S\cos.(S-B)}{\sin.C\sin.A}\right)}, \qquad (360)$$

$$\sin.\tfrac{1}{2}c = \sqrt{\left(\frac{-\cos.S\cos.(S-C)}{\sin.A\sin.B}\right)}. \qquad (361)$$

86. *Corollary.* Equations (340 - 342), applied to $A'B'C'$, give

$$\cos.\tfrac{1}{2}a = \sqrt{\left(\frac{\cos.(S-B)\cos.(S-C)}{\sin.B\sin.C}\right)}, \qquad (362)$$

$$\cos.\tfrac{1}{2}b = \sqrt{\left(\frac{\cos.(S-C)\cos.(S-A)}{\sin.C\sin.A}\right)}, \qquad (363)$$

$$\cos.\tfrac{1}{2}c = \sqrt{\left(\frac{\cos.(S-A)\cos.(S-B)}{\sin.A\sin.B}\right)}, \qquad (364)$$

87. *Corollary.* Equations (343 - 345), applied to $A'B'C'$, give

$$\text{tang.}\tfrac{1}{2}a = \sqrt{\left(\frac{-\cos.S\cos.(S-A)}{\cos.(S-B)\cos.(S-C)}\right)}, \qquad (365)$$

$$\text{tang.}\tfrac{1}{2}b = \sqrt{\left(\frac{-\cos.S\cos.(S-B)}{\cos.(S-C)\cos.(S-A)}\right)}, \qquad (366)$$

$$\text{tang.}\tfrac{1}{2}c = \sqrt{\left(\frac{-\cos.S\cos.(S-C)}{\cos(S-A)\cos.(S-B)}\right)}; \qquad (367)$$

and it should be observed that, if A, B, and C are each less than 180° and conform to § 45, prop. III, the quantities under the radicals in (359 - 367) are all positive.

88. *Corollary.* Equation (332), applied to the polar triangle, gives

$$2(\sin.\tfrac{1}{2}c)^2 = \frac{-\cos.C - \cos.(A+B)}{\sin.A\sin.B}, \qquad (368)$$

which may be used with Table XXIII, like equation (338).

89. Example.

Given in a spherical triangle the three angles equal to 177°, 176°, and 175°; to solve the triangle.

Ans. The three sides are 143° 5′ 45″, 126° 49′ 51″, and 89° 57′ 55″.

90. *Theorem. The sine of half the sum of two sides of a spherical triangle is to the sine of half their difference as the cotangent of half the included angle is to the tangent of half the difference of the other two angles;* that is, in ABC (fig. 32 or 33),

$$\sin. \tfrac{1}{2}(a+c) : \sin. \tfrac{1}{2}(a-c) = \text{cotan.} \tfrac{1}{2} B : \text{tang.} \tfrac{1}{2}(A-C). \quad (369)$$

Proof. This theorem is at once obtained by applying (350) to the polar triangle.

91. *Theorem. The cosine of half the sum of two sides of a triangle is to the cosine of half their difference as the cotangent of half the included angle is to the tangent of half the sum of the other two angles;* or, in ABC (fig. 32 or 33),

$$\cos. \tfrac{1}{2}(a+c) : \cos. \tfrac{1}{2}(a-c) = \text{cotan.} \tfrac{1}{2} B : \text{tang.} \tfrac{1}{2}(A+C). \quad (370)$$

Proof. This theorem is at once obtained by applying (354) to the polar triangle.

92. *Corollary.* These two theorems, similar to §§ 79 and 80, were given by Napier for the solution of the case in which two sides and the included angle are given; and they are known as *Napier's third and fourth Analogies.* By means of them the other two angles can be found without the necessity of calculating the third side. [B., p. 441.] In using them, regard must be had to the signs of the terms, by means of Pl. Trig. §§ 62 and 64.

93. Example.

1. Given in a spherical triangle two sides = 138° 32′ and = 45° 54′, and the included angle = 98° 44′; to find the other angles.

Solution. By (369) and (370),

$\frac{1}{2}(a+c)$	$= 92° 13'$	cosec.	10.00033	sec.	11.41253_n
$\frac{1}{2}(a-c)$	$= 46° 19'$	sin.	9.85924	cos.	9.83927
$\frac{1}{2}B$	$= 49° 22'$	cotan.	9.93354	cotan.	9.93354
$\frac{1}{2}(A-C)$	$= 31° 50' 29''$	tang.	9.79311		
$\frac{1}{2}(A+C)$	$= 93° 44' 2''$			tang.	11.18534_n

Ans. $A = 125° 34' 31''$,
$C = 61° 53' 33''$.

2. Given in a spherical triangle two sides $= 100°$ and $= 125°$, and the included angle $= 45°$; to find the other two angles.

Ans. $69° 43' 49''$ and $128° 42' 51''$.

94. The problems of *Great-Circle Sailing* are easily reduced to problems in the solution of a spherical triangle on the surface of the earth; the three vertices of this triangle being one of the terrestrial poles and the two extremities of the great-circle track. [B., p. 452.]

Chauvenet's elegant chart, the "*Great-Circle Protractor,*" has been already mentioned, on page 96.

95. EXAMPLE.

A ship sails from one mile south of Cape St. Vincent (Portugal) on a course N. 61° 38′ W. and continues on a great circle for 2472 miles; to find the place arrived at, the course at the end of the voyage, and the situation of the northernmost point of the track.

Ans. The place reached is Halifax; the course at the end of the voyage is S. 81° W.; and the northernmost point of the track is in lat. 45° 23′ N., long. 50° 53′ W.

SPHERICAL ASTRONOMY.

SPHERICAL ASTRONOMY.

CHAPTER I.

THE CELESTIAL SPHERE AND ITS CIRCLES.

1. *Astronomy* is the science which treats of the heavenly bodies.

2. *Mathematical Astronomy* is the science which treats of the positions and motions of the heavenly bodies.

The elements of position of a heavenly body are (Geo. § 8) distance and direction.

3. *Spherical Astronomy* regards only one of the elements of position, namely, direction, and usually refers all directions to the centre of the earth.

4. In Spherical Astronomy, all the stars may, then, be regarded as at the same distance from the earth's centre, upon the surface of a sphere, which is called the *celestial sphere*.

Upon this imaginary sphere are supposed to be drawn various circles, which are divided into the well known classes of *great* and *small* circles. [B., p. 47.]

" All angular distances on the surface of the sphere, to an eye at the centre, are measured by arcs of *great* circles." [B., p. 48.]

5. "*Secondaries* to a great circle are great circles which pass through its poles, and are consequently perpendicular to it." [B., p. 48.]

6. "If the plane of the *terrestrial equator* be produced to the celestial sphere, it marks out a circle called the *celestial equator;* and if the axis of the earth be produced in like manner, it becomes the *axis* of the celestial sphere; and the points of the heavens to which it is produced are called the *poles*, being the poles of the celestial equator."

"The star nearest to the north pole is called the *north pole star*. [B., p. 48.]

7. "Secondaries to the celestial equator are called *circles of declination;* of these, 24, which divide the equator into equal parts of 15° each, are called *hour circles*."

"Small circles, parallel to the celestial equator, are called *parallels* of declination." [B., p. 48.]

The parallels of declination correspond, therefore, to the terrestrial parallels of latitude, and the circles of declination to the terrestrial meridians. A certain point of the celestial equator has been fixed by astronomers, and is called the *vernal equinox*. The circle of declination which passes through the vernal equinox bears to the other circles of declination the same relation which the first meridian does to other terrestrial meridians.

8. "The *declination* of a star is its angular distance from the celestial equator," measured upon its circle of declination. [B., p. 49.]

9. The *right ascension* of a star is the arc of the equator intercepted between its circle of declination and the vernal equinox. [B., p. 49.]

Right ascension is either estimated in degrees, minutes, &c., from 0° to 360°; or in hours, minutes, &c. of time, 15 degrees being allowed for each hour, as in Sph. Trig. § 3.

The positions of the stars are completely determined upon the celestial sphere, when their right ascensions and declinations are known. Catalogues of the stars have accordingly been given, containing their right ascensions and declinations. [B., Table VIII., p. 80.]

10. "The *sensible horizon* is that circle in the heavens, whose plane touches the earth at the spectator."

"The *rational horizon* is a great circle of the celestial sphere parallel to the sensible horizon." [B., p. 48.]

11. The radius, which is drawn to the observer, is called the *vertical* line.

The point, where the vertical line meets the celestial sphere *above* the observer, is called the *zenith;* the opposite point, where this line meets the sphere *below* the observer, is called the *nadir.*

Hence the vertical line is a radius of the celestial sphere perpendicular to the horizon; and the zenith and nadir are the poles of the horizon. [B., p. 48.]

12. Circles whose planes pass through the vertical line are called *vertical* circles. [B., p. 48.]

The vertical circles are secondaries to the horizon.

13. The vertical circle at any place, which is also a circle of declination, is called the *celestial meridian* of that place. [B., p. 48.]

The plane of the celestial meridian of a place is the same with that of the terrestrial meridian.

14. The points, where the celestial meridian cuts the horizon, are called the *north* and *south* points. [B., p. 48.]

The north point corresponds to the north pole, and the south point to the south pole.

15. The vertical circle, which is perpendicular to the meridian, is called the *prime vertical.* [B., p. 48.]

16. The points, where the prime vertical cuts the horizon, are called the *east* and *west* points. [B., p. 48.]

"To an observer, whose face is directed towards the south, the east point is to his left hand, and the west to his right hand. Hence the east and west points are 90° distant from the north and south. These four are called the *cardinal* points."

"The meridian of any place divides the heavens into two hemispheres, lying to the east and west; that lying to the east is called the *eastern* hemisphere, and the other the *western* hemisphere."

17. The *altitude* of a star is its angular distance from the horizon, measured upon the vertical circle passing through the star. [B., p. 48.]

18. The *azimuth* of a star is the arc of the horizon intercepted between its vertical circle and the north or south point. [B., p. 48.]

A star may be found without difficulty, when its altitude and azimuth are known. But these elements of position are constantly varying.

CHAPTER II.

THE DIURNAL MOTION.

19. "STARS are distinguished into two kinds, *fixed* and *wandering*." [B., p. 45.]

Most of the stars are fixed, that is, retain constantly almost the same relative position; so that the same celestial globes and maps continue to be accurate representations of the firmament for many years. This is a fact of fundamental importance, and furnishes the fixed points for arriving at a complete knowledge of the celestial motions. Small changes of position, have, indeed, been detected even in the fixed stars, as will be shown in the course of this treatise; but these changes are too small to disturb the general fact; they are, indeed, too small ever to have been detected, if the positions of the stars had been subject to great variations.

20. Of the wandering stars there are eleven, which are called *planets*. They are *Mercury* (☿), *Venus* (♀), *the Earth* (⊕), *Mars* (♂), *Vesta* (⚶), *Juno* (⚵), *Pallas* (⚴), *Ceres* (⚳), *Jupiter* (♃), *Saturn* (♄), and *Uranus* (♅). [B., p. 45.]

21. For the sake of remembering the stars with greater ease, they have been divided into groups called *constellations:* and to give distinctness to the constellations, they have been supposed to be circumscribed by the outlines of some figure which they were imagined to resemble. [B., p. 45.]

The stars have also been distinguished according to their brilliancy, as of the *first*, *second*, &c. magnitude.

Proper names have been given to the constellations and to the most remarkable stars.

The catalogues and the maps of the stars are now so accurate, that no new star could appear without being detected; and any change in the place of any of the larger stars would be immediately discovered.

22. All the stars appear to have a common motion, by which they are carried round the earth from east to west in 24 hours. This rotation of the heavens, or of the celestial sphere, is called the *diurnal motion.*

By its diurnal motion, the celestial sphere rotates, with the most perfect uniformity, about its axis. The pole star would, therefore, if it were exactly at the pole, remain stationary; but since it is not exactly at the pole, it revolves in a very small parallel of declination about the stationary pole.

Any star in the equator revolves in the plane of the equator, and all other stars revolve in the planes of the parallels of declination in which they are situated.

If O (fig. 34) is the place of the observer, $NESW$ his horizon, Z his zenith, P and P' the poles, the star which is at the distance from P,

$$PM = PM'$$

will appear to describe the circumference $MH'M'H$. It will rise in the east at H and set at H', if the distance PM' from the pole is greater than the altitude PN of the pole. But if its distance from the pole

$$PL = PL'$$

is less than PN, the star will not set, but will describe a circle above the horizon; and if its distance from the pole

$$PG = PG'$$

is greater than the greatest distance PS from the pole to the horizon, the star will never rise so as to be seen by the observer at O, but will describe a circle below the horizon.

23. The time which it takes a star to pass from any position round again to the same position, is called a *sideral* day, that is, literally, a star-day. This day is divided into 24 hours,

and clocks regulated to this time are said to denote *sideral time.* [B., p. 147.]

24. Each point of the celestial equator passes the meridian once in a sideral day; and the arc contained between two hour circles passes it in a sideral hour. The sideral time, therefore, which has elapsed since the vernal equinox was upon the equator is equal to the right ascension of the meridian expressed in time. [B., p. 208.]

The meridian changes its right ascension at each instant, precisely *as if* the celestial sphere were stationary, and the observer, with his meridian and zenith, were carried uniformly round the earth's axis from west to east once in a sideral day.

25. The angle ZPB (fig. 35) which the circle of declination of the star makes with the meridian is called its *hour angle.*

While the star moves from the point C in the meridian to the point B with an uniform motion, the arc PC is carried to the position PB, and the angle CPB is described with an uniform motion. This angle converted into time is, then, the sideral time since the passage of the star over the meridian.

26. *Corollary.* The difference of the right ascensions of the star and of the meridian is the hour angle of the star.

27. The distance of a star from the east or west point of the horizon, at the time of its rising or setting, is the *amplitude* of the star. [B., p. 48.]

28. *Problem. To find the altitude and azimuth of a star, when its declination and hour angle are known, and also the latitude of the place.*

Solution. If P (fig. 35) is the pole, Z the zenith, and B the star, we have

$$PZ = \text{polar dist. of zenith} = \text{co. latitude} = 90° - L,$$

$PN = 90° - PZ = L,$

$PB =$ polar dist. of star $= p,$

$=$ co. declination of star, when it is on the same side of the equator with the pole.

$= 90° +$ declination of star, when it is on the different side of the equator from the pole.

$= 90° \mp D,$

$ZB =$ zenith dist. of star $= z,$

$=$ co. altitude of star, when it is above the horizon.

$= 90° +$ depression of star, when it is below the horizon.

$ZPB = *$'s hour angle $= h,$

$PZB =$ azimuth of star counted from the direction of the elevated pole,

$= a =$ azimuth, when less than 90°.

$= 180° -$ azimuth, when greater than 90°.

There are, then, given in the spherical triangle PZB, the two sides PZ and PB, and the included angle ZPB; so that the side BZ and the angle PZB can be calculated by Sph. Trig. § 46.

If we let fall the perpendicular BC upon PZ,

$$\text{tang. } PC = \text{cos. } h \text{ tang. } (90° \mp D) = \pm \text{ cos. } h \text{ cotan. } D \quad (371)$$

$$CZ = PZ - PC = 90° - (L + PC),$$

or

$$= PC - PZ = (L + PC) - 90°. \quad (372)$$

Hence, by (300),

$$\text{cos. } PC : \text{sin. } (L + PC) = \pm \text{ sin. } D : \text{cos. } z; \quad (373)$$

in which formulas the upper sign is used when the star is upon the same side of the equator with the elevated pole, that is, when D and L are of the same name; and, by (301),

$$\text{sin. } PC : \pm \text{ cos. } (L + PC) = \text{cotan. } h : \text{cotan. } a. \quad (374)$$

29. *Corollary.* When the altitude and azimuth are both to be found, the calculation by the above method is as short as by any

other; but when, as is usually the case, the altitude only is required, the following method is preferable.

We have

$$PZ + PB = 180° - L \mp D = 180° - (L \pm D)$$

$$PB - PZ = \mp D + L = (L \mp D);$$

whence, by (308) and (309),

$$\cos. z = -\cos. (L \pm D) + 2 \cos. D \cos. L (\cos. \tfrac{1}{2} h)^2 \quad (375)$$

$$\cos. z = \cos. (L \mp D) - 2 \cos. D \cos. L (\sin. \tfrac{1}{2} h)^2, \quad (376)$$

which may be used at once, and (376) may be calculated by the aid of the column of Rising in Table XXIII. The rule obtained from (376) is the same with that on p. 250 of the Navigator, remembering that when the star is above the horizon

$$\cos. z = \sin. \ast\text{'s alt.} \quad (377)$$

But when the star is below the horizon

$$\cos. z = -\sin. \ast\text{'s depression.} \quad (378)$$

30. *Corollary.* If the given hour angle is $6^h = 90°$, the problem is at once reduced to the solution of a right triangle. We in this case have, by Napier's Rules,

$$\cos. z = \sin. L \cos. p,$$

or

$$\sin. \ast\text{'s alt.} = \pm \sin. L \sin. D \quad (379)$$

$$\text{cotan. } a = \cos. L \text{ cotan. } p$$

$$\text{cotan. } \ast\text{'s azimuth} = \pm \cos. L \text{ tang. } D. \quad (380)$$

The upper sign is to be used in formulas (379) and (380), when the declination is of the same name with the latitude; otherwise the lower sign. In the former case, therefore, the star is above the horizon when its hour angle is six hours, and on the same side of the prime vertical with the elevated pole; but, in the latter case, it is below the horizon, and on the same side of the prime vertical with the depressed pole.

31. *Corollary.* If the star is in the celestial equator, as in (fig. 36), we have in the right triangle BZQ,

$$BQ = BPQ = h$$

$$ZQ = L$$

$$QZB = 180° - a,$$

whence $\cos. z = \cos. L \cos. h,$

or $\sin. \ast\text{'s alt.} = \cos. L \cos. h \qquad (381)$

$$\text{cotan.} (180° - a) = \sin. L \text{ cotan. } h,$$

or $\text{cotan. } a = - \sin. L \text{ cotan. } h. \qquad (382)$

Hence, if the hour angle is less than six hours, the star which moves in the celestial equator is above the horizon and on the same side of the prime vertical with the depressed pole; but if the hour angle is greater than six hours, this star is below the horizon and on the same side of the prime vertical with the elevated pole.

32. *Corollary.* If the place is at the equator, as in (fig. 37), the celestial equator of ZE is the prime vertical, so that if the hour circle PB is produced to C, we have in the right triangle ZBC,

$$ZC = ZPB = h$$

$$BZC = 90° - a$$

$$BC = D,$$

whence $\cos. z = \cos. D \cos. h,$

or $\sin. \ast\text{'s alt.} = \cos. D \cos. h \qquad (383)$

$$\text{cotan.} (90° - a) = \sin. h \text{ cotan. } D,$$

or $\text{tang. } a = \sin. h \text{ cotan. } D; \qquad (384)$

so that the star is above the horizon when the hour angle is less than six hours, and below the horizon when the hour angle is greater than six hours.

33. Examples.

1. Find the altitude and azimuth of Aldebaran to an observer at Boston, in the year 1830, when the hour angle of this star is 3^h 25^m 12^s.

Solution. We find by Tables VIII and LIV

$D = 16° 10'$ N. $L = 42° 21'$ N.

Hence

$h = 3^h\ 25^m\ 12^s$	log. col. Ris.			4.57375
$L = 42° 21'$	cos.			9.86867
$D = 16° 10'$	cos.			9.98248
		26601		4.42490
$L - D = 26° 10'$	nat. cos.	89752		
alt. $= 39° 10'$	nat. sin.	63151	sec.	10.11052
	$h = 51° 18'$		sin.	9.89233
			D cos.	9.98248
azimuth from South $= 75° 11'$			sin.	9.98533

2. Find the altitude and azimuth of Aldebaran at Boston, in the year 1830, six hours after it has passed the meridian.

Solution. By formulas (379) and (380),

$L = 42° 21'$	sin.	9.82844	cos.	9.86867
$D = 16° 10'$	sin.	9.44472	tang.	9.46224
alt. $= 10° 49'$	sin.	9.27316		
azimuth from north $= 77° 54'$			cotan.	9.33091

3. Find the altitude and azimuth of a star in the celestial equator to an observer at Boston, when the hour angle of the star is $3^h\ 25^m\ 12^s$.

Solution. By formulas (381) and (382),

$L = 42° 21'$	cos.	9.86867	sin.	9.82844
$h = 51° 18'$	cos.	9.79605	cotan.	9.90371
alt. $= 27° 31'$	sin.	9.66472		
azimuth from South $= 61° 39'$			cotan.	9.73215

4. Find the altitude and azimuth of Aldebaran to an observer at the equator, in the year 1830, when the hour angle of the star is $3^h\ 25^m\ 12^s$.

Solution. By formulas (383) and (384),

$D = 16° 10'$	cos.	9.98248	cotan.	10.53776
$h = 51° 18'$	cos.	9.79605	sin.	9.89233
alt. $= 36° 54'$	sin.	9.77853		
azimuth from North $= 69° 37'$			tang.	10.43009

5. Find the altitude and azimuth of Fomalhaut to an observer at Boston, in the year 1840, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . . $= 11° 59'$.
Its azimuth from the South $= 26° 51'$.

6. Find the altitude and azimuth of Dubhe to an observer at Boston, in the year 1840, when its hour angle is $9^h\ 30^m$.

Ans. Its altitude . . . $= 19° 11'$.
Its azimuth from the North $= 17° 15'$.

7. Find the altitude and azimuth of Fomalhaut to an observer at Boston, in the year 1840, when its hour angle is 6^h.

Ans. Its depression below the horizon $= 19° 58'$.
Its azimuth from the South $= 66° 30'$.

8. Find the altitude and azimuth of Dubhe to an observer at Boston, in the year 1840, when its hour angle is 6^h.

Ans. Its altitude . . . $= 36° 44'$.
Its azimuth from the North $= 35° 2'$.

9. Find the altitude and azimuth of a star in the celestial equator to an observer at Stockholm, when the hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . . $= 25° 58'$.
Its azimuth from the South $= 34° 45'$.

10. Find the altitude and azimuth of a star in the celestial equator to an observer at Stockholm, when the hour angle is $9^h\ 30^m$.

Ans. Its depression below the horizon $= 23° 51'$.
Its azimuth from the North $= 41° 45'$.

11. Find the altitude and azimuth of Fomalhaut to an observer at the equator, in the year 1840, when its hour angle is $2^h\ 3^m\ 20^s$.

Ans. Its altitude . . . $= 47°\ 45'$.

Its azimuth from the South $= 41°\ \ 4'$.

12. Find the altitude and azimuth of Dubhe to an observer at the equator, in the year 1840, when its hour angle is $9^h\ 30^m$.

Ans. Its depression below the horizon $= 21°\ 24'$.

Its azimuth from the North $= 17°\ 30'$.

34. In the triangle ZPB (fig. 35) other parts might be given instead of the two sides ZP, PB, and the included angle P, and the triangle might be resolved. Of the problems thus derived, we shall only, for the present, consider two cases.

35. *Problem. To find a given star's hour angle and altitude, when it is upon the prime vertical.*

Solution. The angle PZB is, in this case, a right angle, and if we use the preceding notation, we have

$$\cos. h = \text{cotan.}\ L\ \text{cotan.}\ p = \pm\ \text{cotan.}\ L\ \text{tang.}\ D \qquad (385)$$

$$\cos. z = \cos. p\ \text{cosec.}\ L,$$

or

$$\sin. \ast\text{'s alt.} = \pm \sin. D\ \text{cosec.}\ L\,; \qquad (386)$$

so that when the declination and latitude are of the same name, the hour angle is less than 6 hours, and the star is above the horizon; but when the declination and latitude are of different names, the hour angle is greater than 6 hours, and the star is below the horizon.

36. *Scholium.* The problem is, by Sph. Trig. § 27, impossible, when the declination is greater than the latitude; so that, in this case, the star is never exactly east or west of the observer.

37. *Scholium.* The problem is, by Sph. Trig. § 28, indeterminate, when the latitude and declination are both equal to zero; so that, in this case, the star is always upon the prime vertical.

38. Examples.

1. Find the hour angle and altitude of Aldebaran, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. The hour angle $= 4^h\, 45^m\, 44^s$.
The altitude $= 24°\, 26'$.

2. Find the hour angle and altitude of Fomalhaut, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. The hour angle . . $= 8^h\, 40^m\, 51^s$.
The depression below the horizon $= 48°\, 49'$.

3. Find the hour angle and altitude of Dubhe, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. Dubhe is never upon the prime vertical of Boston.

4. Find the hour angle and altitude of Canopus, when it is exactly east or west of an observer at Boston, in the year 1840.

Ans. Canopus is never upon the prime vertical of Boston.

39. *Problem. To find the hour angle and amplitude of a star, when it is in the horizon.*

Solution. In this case the side ZB (fig. 35) of the triangle ZPB is 90°. The corresponding angle of the polar triangle is, therefore, a right angle, and the polar triangle is a right triangle, of which the other two angles are

$$180° - PZ = 180° - (90° - L) = 90° + L,$$

and

$$180° - PB = 180° - (90° \mp D) = 90° \pm D.$$

The hypothenuse of the polar triangle is $180° - h$, and the leg, opposite the angle, $90° \pm D$, is $180° - a$.

Hence, by Sph. Trig. § 40, and Pl. Trig. § 60 and 62,

$$-\cos. h = \pm \text{tang. } L \text{ tang. } D,$$

or

$$\cos. h = \mp \text{tang. } L \text{ tang. } D \qquad (387)$$

$$-\cos. a = \mp \sin. D \sec. L,$$

or

$$\cos. a = \pm \sin. D \sec. L; \qquad (388)$$

in which the upper sign is used when the latitude and declination have the same name, and the lower sign when they have different names; so that in the former case the hour angle is greater than 6 hours, and the azimuth is counted from the direction of the elevated pole; but in the latter case, the hour angle is less than 6 hours, and the azimuth is counted from the direction of the depressed pole. The amplitude is the difference between the azimuth a and 90°. Hence

$$\cos. \ast\text{'s azim.} = \sin. \ast\text{'s amp.} = \sin D \sec. L. \qquad (389)$$

40. *Scholium.* The problem is, by Sph. Trig. § 41, impossible, when the sum of the declination and latitude is greater than 90°; so that, in this case, the star does not rise or set.

41. Examples.

1. Find the hour angle and amplitude of Aldebaran, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. The hour angle $= 7^h\ 1^m\ 21^s$.
The amplitude $= 22°\ 9'$ N.

2. Find the hour angle and amplitude of Fomalhaut, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. The hour angle $= 3^h\ 50^m\ 18^s$.
The amplitude $= 43°\ 19'$ S.

3. Find the hour angle and amplitude of Dubhe, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. Dubhe neither rises nor sets at Boston.

4. Find the hour angle and amplitude of Canopus, when it rises or sets, to an observer at Boston, in the year 1840.

Ans. Canopus neither rises nor sets at Boston.

CHAPTER III.

THE MERIDIAN.

42. THE intersection of the plane of the meridian with that of the horizon, is called the *meridian line.*

43. *Problem. To determine the meridian line.*

Solution. First Method. Stars obviously rise to their greatest altitude in the plane of the meridian; so that if their progress could be traced with perfect accuracy, and the instant of their rising to their greatest height be observed, the direction of the meridian line could be exactly determined. But stars, when they are at their greatest height, change their altitude so slowly, that this method is of but little practical value.

Second Method. A star is evidently at equal altitudes, when it is at equal distances from the meridian on opposite sides of it. If, therefore, the direction and altitude of a star are observed before it comes to the meridian; and if its direction is also observed, when it has descended again to the same altitude, after passing the meridian; the horizontal line, which bisects the angle of the two horizontal lines drawn in the direction thus determined, is the meridian line.

Third Method. [B., p. 147.] The time which elapses between the superior and inferior passage of a star over the meridian is just half of a sideral day. If, then, a telescope were placed so as to revolve on a horizontal axis in the plane of the meridian, the two intervals of time between three successive passages of a star over the central wire, must be exactly equal. But if the vertical plane of the telescope is not that of the meridian, these two intervals will not be equal, and the position of the telescope must be changed until they become equal.

Thus, if $Z\ M\ m\ N$ (fig. 88) is the plane of the meridian, $Z\ S\ s\ T$ that of the vertical circle described by the telescope, $M\ S\ W\ s\ m\ E$ the circle of declination described by the star about the pole P; this star will be observed at the points S and s instead of at the points M and m. Now the star describes the circle of declination with an uniform motion, and therefore the arc SP moves uniformly with the star around the pole, so that the angle SPM is proportional to the time of its description; that is, the angle SPM, reduced to time, denotes the sideral time of its description.

Let then

$T =$ the sideral time of describing the arc SM,

$t =$ the sideral time of describing the arc $s\ m$,

$I =$ interval from the observation at S to that at s,

$i =$ interval from the observation at s to that at S,

$\delta i =$ the difference of these two intervals;

we have then, in sideral time,

$$I = 12^h - T - t = 12^h - (T + t)$$

$$i = 12^h + T + t = 12^h + (T + t)$$

$$\delta i = i - I = 2\ (T + t); \qquad (390)$$

so that if T and t were equal to each other, and they are nearly so in the case of the pole-star, we should have

$$\delta i = 4\ T = 4\ t$$

$$T = t = \tfrac{1}{4}\,\delta i;$$

that is, *the time of describing the arc* MS *or* $m\ s$ *is nearly one quarter part of the difference between the intervals.*

But the error of this result can be calculated without much difficulty. For this purpose, let

$L =$ the latitude of the place $= 90° - PZ$,

$p =$ the polar distance of the star $= PS = P\ s$,

$a =$ the azimuth of $ZST = TN = TZN$.

The arcs MS and $m\ s$ are so small, that they do not differ sensibly from the arcs of great circles drawn from S and s perpendicular to ZPN.

If, then, in the two right triangles PSM and ZSM, PM and ZM are the middle parts, SM, co. SZM, and co. SPM are the adjacent parts, so that

$$\begin{aligned}\sin. PM : \sin. ZM &= \text{cotan. } SPM : \text{cotan. } SZM \\ &= \frac{1}{\text{tang. } SPM} : \frac{1}{\text{tang. } SZM} \\ &= \text{tang. } SZM : \text{tang. } SPM.\end{aligned}$$

But
$$ZM = ZP - PM = 90° - L - p,$$

and the angle SZM and SPM are so small, that they are sensibly proportional to their tangents, whence

$$\sin. p : \cos. (p + L) = a : SPM, \tag{391}$$

or
$$\begin{aligned}a : SPM &= \sin. p : \cos. p \cos. L - \sin. p \sin. L \\ &= 1 : \text{cotan. } p \cos. L - \sin. L,\end{aligned}$$

and if T is expressed in sideral hours

$$T \,.\, 15° = SPM = a \text{ cotan. } p \cos. L - a \sin. L.$$

In like manner, we find

$$t \,.\, 15° = s\, P\, m = a \text{ cotan. } p \cos. L + a \sin. L.$$

Hence, by (390),

$$\begin{aligned}(T + t)\, 15° &= \tfrac{1}{2}\, \delta\, i \,.\, 15° = 2\, a \text{ cotan. } p \cos. L \\ a \text{ cotan. } p \cos. L &= \tfrac{1}{4}\, \delta\, i \,.\, 15° \\ T \,.\, 15° &= \tfrac{1}{4}\, \delta\, i \,.\, 15° - a \sin. L \\ t \,.\, 15° &= \tfrac{1}{4}\, \delta\, i \,.\, 15° + a \sin. L \\ a &= \tfrac{1}{4}\, \delta\, i \,.\, 15° \text{ tang. } p \sec. L \qquad (392) \\ T &= \tfrac{1}{4}\, \delta\, i - \tfrac{1}{4}\, \delta\, i \text{ tang. } p \text{ tang. } L \\ t &= \tfrac{1}{4}\, \delta\, i + \tfrac{1}{4}\, \delta\, i \text{ tang. } p \text{ tang. } L,\end{aligned}$$

so that the correction is

$$\tfrac{1}{4}\, \delta\, i \text{ tang. } p \text{ tang. } L, \tag{393}$$

which is to be added to the quarter interval at the lower transit; and to be subtracted from the quarter interval at the upper transit.

This correction is proportional to the quarter interval, so that if it is computed for any supposed value of this interval, it may be computed for any other interval by a simple proportion. Now Table A, page 151, of the Navigator, is the value of this correction, when the interval is 1000^s. It may be observed, that it is not necessary that this time should be sideral time, because all the terms of the values of T and t are expressed in the same time, which may be that of the clock.

The azimuth a is given in Table B [B., p. 151], and may be computed from the formula (392). But the interval in the formula is supposed to be sideral time, whereas the time of the table is that called *solar time*, to which clocks are usually regulated, and which is soon to be described; all that need be known for the present is, that an interval of sideral time is reduced to solar time by Table LII of the Navigator, or by the formula

$$\frac{\text{an interval of solar time}}{\text{an interval of sideral time}} = 0.9972695. \qquad (394)$$

Fourth Method. [B., p. 149.] This method of determining the meridian is by means of two known circumpolar stars, which differ nearly 12 hours in right ascension. The upper passage of one of these stars is to be observed, and the lower passage of the other. Then any deviation in the plane of the instrument from the meridian, will evidently produce contrary effects upon the observed times of transit, exactly as in the upper and lower transits of the same star. The time, which elapses between the two observations, will differ from the time which should elapse by the sum of the effects of the deviation upon the two stars. In the use of this method, therefore, the time of the clock must be known, so that it can readily be reduced to sideral time.

The deviations in the time of passage of a star, corresponding to any azimuth, can be calculated by means of equation (391). For this formula gives for the time of describing the arc SM.

$$T \,.\, 15^\circ = a \cos. (p + L) \operatorname{cosec.} p,$$

or

$$T = \tfrac{1}{15}\; a \;\cos.\; (p + L) \operatorname{cosec.} p\,; \qquad (395)$$

which may be used if T is expressed in sideral seconds, and the arc a in seconds of space. But if T is expressed in solar time, we have, by (394),

$$T = 0.0664846\; a \cos. (p + L) \operatorname{cosec.} p. \qquad (396)$$

In the same way the value of t for an inferior passage is found to be

$$t = 0.0664846\, a \cos. (p - L) \text{ cosec. } p. \qquad (397)$$

Now, since these values of T and t are proportional to the azimuth a, their values may be computed for a given value of the azimuth, as 1000″, and arranged in a table like Table C, p. 152 of the Navigator, and their values for any other azimuth can be obtained by a simple proportion.

Fifth Method. [B., p. 149] This method consists in observing the transits of two stars, which differ but little in right ascension. The error in the position of the telescope is, in this case, equal to the difference in the errors of the observed transits, instead of the sum as in the preceding method.

44. In making calculations where angles are introduced as factors, some labor, in reducing them to the same denomination, is often saved by means of a table of Proportional Logarithms, such as Table XXII of the Navigator.

This table was particularly designed for reducing lunar distances, given in the Nautical Almanac, for every 3 houses to any intermediate time. It contains, on this account, the logarithm of the ratio of 3 hours to each angle expressed in time; that is, if A is the angle.

$$\text{Prop. log. } A = \log. \frac{3^h}{A} = \log. 3^h - \log. A = \log. 180^m - \log. A$$
$$= \log. 10800^s - \log. A, \qquad (398)$$

so that if A in the second member is reduced to seconds,

$$\text{Prop. log. } A = 4.03342 - \log. A \text{ in seconds}; \qquad (399)$$

neglecting the right hand figure, so as to retain only four decimal places. This agrees with the explanation of the table in the Introduction to the Navigator; and it is evident that it is immaterial whether the angles, whose ratios are sought, are given in time or in degrees, &c.

Suppose, now, that the logarithm of the ratio of two angles is sought, A and a; we have, evidently,

$$\log. \frac{A}{a} = \log. A - \log. a = \text{Prop. log. } a - \text{Prop. log. } A; \quad (400)$$

so that if this ratio, which we will denote by M, were known, and if a were known, A might be calculated by the formula

$$\text{Prop. log. } A = \text{Prop. log. } a - \text{log. } M$$
$$= \text{Prop. log. } a + \text{(ar. co.) log. } M; \qquad (401)$$

which is, therefore, the formula for calculating the value of A, given by the equation.

$$A = a\,M. \qquad (402)$$

Finally, the use of formula (401) is facilitated by remembering that *the arithmetical complements of the logarithms of the sine, cosine, tangent, cotangent, secant, and cosecant of an angle, are respectively the logarithms of its cosecant, secant, cotangent, tangent, cosine, and sine.*

45. Examples.

1. Calculate the proportional logarithm of 0° 5′ 45″.

Solution. By (399),

		4.03342
0° 5′ 45′ = 345″.		2.53782
Prop. log. 5′ 45″	=	14.596

as in Table XXII.

2. Calculate the corrections of Tables A and B [B., p. 151], when the latitude is 42°, and the polar distance of the star is 30°.

Solution. By means of proportional logarithms, and equations (392) and (393),

$\frac{1}{4}$. 1000ˢ = 4ᵐ 10ˢ	Prop. log.	1.6355		1.6355
L = 42°	cotan.	10.0456	cos.	9.8711
30°	cotan.	10.2386		10.2385
corr. A = 130ˢ = 2ᵐ 10ˢ	Prop. log.	1.9197		
0.0664846				8.8227
corr. B = 48′ 41″			Prop. log.	0.5679

3. Calculate the corrections of Table C [B., p. 152] for the pole-

star and the latitude of 30°, when the polar distance of this star is 1° 32′ 37″.

Solution. By (396) and (397),

0.0664846	8.82273	8.82273
$a = 1000''$	3.00000	3.00000
$p = 1° 32' 37''$	cosec. 11.5694	11.56964
$p + L = 31° 32' 37''$	cos. 9.93056	
$p - L = -28° 27' 23''$		9.94407
corr. C upper trans. $= 2103^s$	3.32293	
corr. C lower trans. $= 2170^s$		3.33644

4. An observer in Boston in the year 1840, wishing to determine his meridian line, observed three successive transits of β Cephei over the central vertical wire of his transit instrument, by means of a clock regulated to solar time, and found them to occur as follows; the first upper transit at $7^h\ 45^m\ 28^s$ P. M., the next inferior transit the next day at $7^h\ 41^m$ A. M., the third transit at $7^h\ 41^m\ 32^s$ P. M. What were the times of the star's passing the meridian the second day? and what was the azimuth error in the position of the instrument?

Solution.

The first interval $= 19^h\ 41^m - 7^h\ 45^m\ 28^s = 11^h\ 55^m\ 32^s$

The second interval $= 19^h\ 41^m\ 32^s - 7^h\ 41^m = 12^h\ 0^m\ 32^s$.

Hence $\delta i = 5^m = 300^s$.

Now $L = 42°\ 21'$, $D = 69°\ 52'$, $p = 20°\ 8'$.

Hence, by Tables A and B

$$\text{corr. } A = 83^s \times 0.3 = 25^s,$$

$$\text{corr. } B = 31'\ 6'' \times 0.3 = 9'\ 19'';$$

so that the error in the time of the upper transit is

$$\tfrac{1}{4} \cdot 300^s - 25 = 75^s - 25^s = 50^s,$$

and the error in the lower transit is

$$\tfrac{1}{4} \cdot 300 +^s 25^s = 75^s + 25^s = 100^s = 1^m\ 40^s.$$

The times of the star's passing the meridian the second day were, then,

$$7^h\,41^m + 1^m\,40^s = 7^h\,42^m\,40^s \text{ A. M.}$$

and
$$7^h\,41^m\,32^s - 50^s = 7^h\,40^m\,42^s \text{ P. M.}$$

the error in the azimuth of the instrument was 9′ 19″ to the west of north.

5. An observer at Boston, wishing to determine his meridian line, on the morning of January 1, 1840, observed, by means of a clock regulated to solar time, the superior transit of γ Ursæ Majoris at $5^h\,6^m\,54^s$ A. M., and the inferior transit of Polaris at $6^h\,12^m\,23^s$ A. M. What was the azimuth error in the position of the transit instrument?

Solution. The interval between these two transits is

$$6^h\,12^m\,23^s - 5^h\,6^m\,54^s = 1^h\,5^m\,29^s.$$

But, by the Nautical Almanac,

12^h + R. A. of Polaris	$= 13^h\ 1^m\ 59^s$
R. A. of γ Ursæ Majoris	$= 11^h\ 45^m\ 25^s$
Sideral Interval	$= 1^h\ 16^m\ 34^s$
Solar Interval	$= 1^h\ 16^m\ 22$
Observed Interval	$= 1^h\ 5^m\ 29^s$
Error of Interval	$= 10^m\ 53^s = 653^s.$

Now for 1000″ of azimuth error, and the latitude of Boston, Table C gives, since

Dec. of γ Ursæ Majoris	= 54° 35′
Error of lower trans. of Polaris . .	$= 1866^s$
Error of upper trans. of γ Ursæ Majoris .	$= 25^s$
Sum of errors	$= 1891^s$

Then the proportion

$$1891^s : 653^s = 1000'' : \text{azimuth error},$$

gives

$$\text{azimuth error} = 345'' = 5'\,45'' \text{ W.}$$

6. An observer at Boston wishing to determine his meridian line, in the evening of December 17, 1839, observed by means of a clock regulated to solar time, the superior transit of α Cassiopeæ at $6^h\ 48^m\ 35^s$ P. M., and that of Poloris at $6^h\ 53^m\ 15^s$ P. M. What was the azimuth error in the position of the transit instrument?

Solution. By the Nautical Almanac,

R. A. of Polaris	$= 1^h\ 2^m\ 26^s$
R. A. of α Cassiopeæ	$= 0^h\ 31^m\ 28^s$
Sideral Interval	$= 0^h\ 30^m\ 58^s$
Solar Interval	$= 0^h\ 30^m\ 53^s$
Observed Interval	$= 0^h\ 4^m\ 40^s$
Error of Interval	$= 0^h\ 26^m\ 13^s = 1573^s$.

Now Table C gives, for 1000″ of azimuth error and the latitude of Boston, since

Dec. of α Cassiopeæ	$= 55°\ 40'$
Error of trans. of Polaris	$= 1777^s$
Error of trans. of α Cassiopeæ	$= 26^s$
Diff. of errors	$= 1751^s$.

Then, the proportion

$$1751^s : 1573^s = 1000'' : \text{azimuth error}$$

gives

$$\text{azimuth error} = 900'' : = 15'\ 0''\ \text{E}.$$

7. Calculate the proportional logarithm of 0° 2′ 33″.

Ans. 1.8487.

8. Calculate the proportional logarithm of 2° 59′ 12″.

Ans. 0.0019.

9. Calculate the corrections of Tables A and B, when the latitude is 54°, and the star's polar distance 20°.

Ans. Corr. A. $= 125^s$
Corr. B. $= 38'\ 48''$.

10. Calculate the corrections of Table C, when the latitude is 20°, and the polar distance 5°.

Ans. For the upper transit, corr. C $= 691^s$.

For the lower transit, corr. C $= 737^s$.

11. An observer at Boston, in the year 1840, wishing to determine his meridian line, observed three successive transits of Polaris, by means of a clock regulated to solar time. The first lower transit was observed at 6^h A. M., the next transit at $6^h\ 2^m\ 11^s$ P. M., and the second lower transit at $5^h\ 56^m\ 4^s$ A. M. What was the time of the star's passing the meridian the second morning? and what was the azimuth error in the position of the instrument?

Ans. The time of the third merid. trans. was $5^h\ 58^m\ 49^s$ A. M.

The azimuth error $= 15'\ 27''$ W.

12. An observer at Boston, wishing to determine his meridian line by means of a clock regulated to solar time, observed the inferior transit of Polaris on April 4, 1839, at 0^h A. M., and the superior transit of η Ursæ Majoris at $0^h\ 53^m\ 59^s$ A. M. What was the azimuth error in the position of his transit instrument?

The R. A. of Polaris is $1^h\ 0^m\ 50^s$, that of η Ursæ Majoris is $13^h\ 41^m\ 14^s$, and the declination of η Ursæ Majoris is 50° 7′ N.

Ans. The azimuth error $= 7'\ 18''$ W.

13. An observer at Boston, wishing to determine his meridian line, in the evening of May 1, 1839, observed, by means of a clock regulated to solar time, the lower transit of Polaris at $9^h\ 49^m\ 22^s$ P. M., and that of α Cassiopeæ at $9^h\ 52^m$ P. M. What was the azimuth error of the instrument?

The R. A. of Polaris	$= 1^h\ 0^m\ 56^s$.
The R. A. of α Cassiopeæ	$= 0^h\ 31^m\ 22^s$.
The Dec. of α Cassiopeæ	$= 55°\ 39'$ N.

Ans. The azimuth error $= 18'\ 34''$ W.

CHAPTER IV.

LATITUDE.

46. *Problem. To find the latitude of a place.*

Solution. The latitude of the place is evidently, from (fig. 34), equal to the altitude of the pole; so that this problem is the same as to find the altitude of the pole, which would be done without difficulty if the pole were a visible point of the celestial sphere.

First Method. By Meridian Altitudes. [B., p. 166 – 175.]

Observe the altitude of a star at its transit over the meridian, and let

$A =$ the altitude of the star,

$A' =$ ✱'s dist. from point of horizon below the pole;

then, if the notation of § 28 is used, it is evident, from (fig. 34), that

$$L = A' \mp p; \qquad (403)$$

the upper sign being used when the transit is a superior one, and the lower sign when it is an inferior one.

I. Suppose the observed transit to be a superior one; then, if it passes upon the side of the zenith opposite to the pole, we have

$$A' = 180° - A, p = 90° \mp D,$$

and (403) becomes

$$L = 90° - (A \mp D) = (90° - A) \pm D = z \pm D; \qquad (404)$$

the upper sign being used when the declination and latitude are of the same name, and the lower sign when they are of different names.

But if the star passes upon the same side of the zenith with the pole, we have

$$A' = A, p = 90° - D,$$

and (403) becomes

$$L = (A + D) - 90° = D - (90° - A) = D - z. \qquad (405)$$

II. If the transit is an inferior one, we have

$$A' = A, p = 90° - D,$$

and (403) becomes

$$L = (A - D) + 90° = A + (90° - D). \qquad (406)$$

Equations (404) and (405) agree with the rule of Case I, [B., p. 166], and (406) with Case II, [B., p. 167.]

III. If both transits are observed, and if A' and A are referred to the upper transits, and

$$A_1 = \text{the altitude at the lower transit,}$$

we have, by (403),

$$L = A' - p$$
$$L = A_1 + p,$$

the sum of which is

$$L = \tfrac{1}{2}(A' + A_1); \qquad (407)$$

so that the latitude is determined in this case without knowing the star's declination.

Second Method. By a Single Altitude.

Observe the altitude and the time of the observation.

I. If the star is considerably distant from the meridian, we have given in the triangle PBZ (fig. 35), PB, BZ, and BPZ to find PZ, which may be solved by Sph. Trig. § 60, and gives by the notation of § 28,

$$\text{tang. } PC = \cos. h \text{ tang. } p = \pm \cos. h \text{ cotan. } D \qquad (408)$$
$$\cos. ZC = \cos. PC \,.\, \cos. z \sec. p$$
$$= \pm \cos. PC \,.\, \cos. z \text{ cosec } D, \qquad (409)$$

in which the upper sign is used if the declination and latitude are of the same name, otherwise the lower sign.

$$90° - L = PZ = PC \pm ZC$$
$$L = 90° - (PC \pm ZC); \qquad (410)$$

in which both signs may be used if they give values of L contained between 0° and 90°, and in this case other data must be resorted to, in order to determine which is the true value of L.

Scholium. The problem is, by Sph. Trig. § 61, impossible, if the altitude is greater than the declination, when the hour angle is more than six hours.

II. If the latitude is known within a few miles, it may be exactly calculated by means of (376), or

$$\cos. z = \cos. [90° - (L + p)] - 2 \cos. L \cos. D (\sin. \tfrac{1}{2} h)^2. \quad (411)$$

But if A is the star's observed altitude, and A_1 its meridian altitude at its upper transit, (403) gives

$$A_1 = L + p, \text{ or} = 180° - (L + p),$$

and (411) becomes, by transposition,

$$\sin. A_1 = \sin. A + 2 \cos. L \cos. D (\sin. \tfrac{1}{2} h)^2; \quad (412)$$

from which the meridian altitude may be calculated by means of Table XXIII, as in the Rule. [B., p. 200.]

III. A formula can also be obtained from (340), which is particularly valuable when the star is, as it always should be in these observations, near the meridian.

In this case we have in (340) applied to PBZ

$$2s = 90° - L + p + x = 180° - L + p - A \quad (413)$$

$$\begin{aligned} 2s - 2PZ &= L + p - A \\ &= A_1 - A \text{ or} = 180° - (A_1 + A) \quad (414) \end{aligned}$$

$$\begin{aligned} 2s - 2PB &= 180° - L - p - A \\ &= 180° - (A_1 + A) \text{ or} = A_1 - A; \quad (415) \end{aligned}$$

and if these values are substituted in (340), after it is squared and freed from fractions, they give

$$(\sin. \tfrac{1}{2} h)^2 \cos. L \cos. D = \sin. \tfrac{1}{2} (A_1 - A) \cos. \tfrac{1}{2} (A_1 + A), \quad (416)$$

or

$$\sin. \tfrac{1}{2} (A_1 - A) = (\sin. \tfrac{1}{2} h)^2 \cos. L \cos. D \sec. \tfrac{1}{2} (A_1 + A); \quad (417)$$

and if, in the second member of this equation, the value of A_1 is used, which is obtained from the approximate value of the latitude, the difference between the observed and the meridian altitudes may be found at once; and this difference is to be added to the observed altitude to obtain the meridian altitude.

IV. If the star is very near the meridian, $\frac{1}{2}(A_1 - A)$ and $\frac{1}{2} h$ will be so small, that we may put

$$\sin. \tfrac{1}{2}(A_1 - A) = \tfrac{1}{2} A' - A) \sin. 1''$$

$$\sin. \tfrac{1}{2} h = \tfrac{1}{2} h \sin. 1^s = \tfrac{15}{2} h \sin. 1'';$$

which, substituted in (417) give, by supposing A_1 equal to A in the second member, which is very nearly the case,

$$A_1 - 1 = \tfrac{15}{2} h^2 \sin. 1^s \cos. L \cos. D \sec. A_1. \qquad (418)$$

This value of $A_1 - A$ is proportional to h^2, so that if it were calculated for

$$h = 1^s,$$

any other value might be calculated by multiplying by h^2. Now Table XXXII, of the Navigator, contains the values of $A_1 - A$ for all latitudes and for all declinations less than 24°, excepting a few latitudes in which the meridian transit of the observed body is too near the zenith for this observation to be accurate; and Table XXXIII contains all the values of h^2, where h is less than 13^m.

V. If the observed star is very near the pole, we have in (408)

$$\text{tang. } PC = \cos. h \text{ tang. } p; \qquad (419)$$

so that as p is very small, PC must be likewise small, and we have

$$\cos. h = \frac{\text{tang. } PC}{\text{tang. } p} = \frac{PC}{p}$$

$$PC = p \cos. h; \qquad (420)$$

and, by Pl. Trig. § 22,

$$\cos. PC = 1, \sin. D = \cos. p = 1,$$

whence, by (409), and (410),

$$\cos. ZC = \cos. z, \; ZC = z,$$

$$L = 90° - PC - ZC = 90° - z - PC$$

$$= A - p \cos. h; \qquad (421)$$

so that p cos. h may be regarded as a correction to be subtracted from A when it is positive, that is, when the hour angle is less than 6 hours, or greater than 18 hours; and it is to be added when the hour angle is greater than 6 hours and less than 18 hours.

The table [B., p. 206] for the pole star was calculated for the year 1840, when

$$\text{its R. A.} = 1^h\, 2^m; \text{ its dec.} = 88^\circ\, 27' \text{ nearly.}$$

VI. The method of determining the latitude by means of the pole star is so accurate in practice, that tables are given in the Nautical Almanac for correcting the observed altitude for differences of latitude, and for changes in the right ascension and declination of the star. Of these corrections the *first* is the same as that of the Navigator, and is computed from (421) by using the pole star's mean right ascension and declination for the year; and the *third* is the correction for the change in the star's right ascension and declination. Both of these corrections may, however, be full as readily obtained by direct computation from (421), if the actual right ascension and declination of the star are at once substituted in the formula. The *second* correction of the Nautical Almanac arises from the error in supposing ZC to be equal to z, and is so small that the mean right ascension and declination of the pole star may be used in its computation.

We have, then, in the right triangle BPC, since p and BC are small,

$$\frac{BC}{p} = \frac{\sin. BC}{\sin. p} = \sin. h,$$

or

$$BC = p \sin. h;$$

and the right triangle BCZ gives, since

$$BZ = 90^\circ - A$$

$$CZ = 90^\circ - PC - L = 90^\circ - p \cos. h - L$$

$$\cos. BZ = \cos. CZ \cos. BC$$

$$\sin. A = \sin. (p \cos. h + L) \cos. BC$$

$$\sin. (p \cos. h + L) - \sin. A = \sin. (p \cos. h + L)\,(1 - \cos. BC)$$

$$2 \cos. \tfrac{1}{2}\,(p \cos. h + L + A) \sin. \tfrac{1}{2}\,(p \cos. h + L - A)$$

$$= 2 \sin. (p \cos. h + L) \sin.^2 \tfrac{1}{2}\, BC;$$

or, since A differs but little from L, and p and BC are small,

$$\cos. L . (p \cos. h + L - A) \sin. 1'' = \tfrac{1}{2} \sin. L . (BC)^2 \sin.^2 . 1''$$

$$p \cos. h + L - A = \tfrac{1}{2} p^2 \tan. L . \sin.^2 . h \sin. 1'', \qquad (422)$$

which gives the required second correction, and this method of computing the latitude is most accurate when h is nearly 6 or 18 hours.

VII. The formula (417) may, however, be used directly for observations of the pole star more readily than the tables of the Nautical Almanac, and gives at once

$$L = A p + p (\sin. \tfrac{1}{2} h)^2 \cos. L \sec. \tfrac{1}{2} (A + L + p, \qquad (423)$$

and is most accurate when h is small.

VIII. By applying (417) to the lower transit of the pole star, that is, substituting its supplement for h, and making

$$A_1 = L - p,$$

it becomes

$$L = A + p - p (\cos. \tfrac{1}{2} h)^2 \cos. L \sec. \tfrac{1}{2} (A + M - p), \qquad (424)$$

which is most accurate when h is nearly 12 hours.

Third Method. By Circummeridian Altitudes.

I. If several altitudes are observed near the meridian, each observation may be reduced separately by (417) and (418), and the mean of the resulting latitudes is the correct latitude.

II. But if (418) is used, the mean of the values of $A_1 - A$ is evidently obtained by multiplying the mean of the values of h^2 by the constant factor; and if to the mean of the values of $A_1 - A$, the mean of the values of A is added, the sum is the mean of the value of A_1, whence precisely the same mean of resulting latitude is obtained as by the former method, but with much less calculation.

III. If the star is changing its declination in the course of the observations, this change may, in all cases which can occur if the hour angle is small, be neglected in the value of cos. D. But the value of A_1 will not, in this case, be at each observation equal to the meridian altitude, but will differ from it by the difference of the star's declination. Let the change of the star's declination in one minute be denoted by δD, which is positive when the star is approaching the elevated pole; and if h is the star's hour angle at the time of observation, which is negative before the star arrives at the meridian and afterwards positive, the whole change of declination is $h \delta D$, so that the correct meridian altitude is

$$A_1 - h \delta D.$$

The mean of the values of the corrected meridian altitude is, therefore, equal to the mean of the values of A_1 diminished by the mean of the values of $h \delta D$; and, if H denotes the mean of the hour angles h (regard being had to their signs), the correct meridian altitude is the mean of the values of A_1 diminished by $H \delta D$.

Fourth Method By Double Altitudes.

I. Let two altitudes of a star, which does not change its declination, be observed, and the intervening time. Then (fig. 39) let Z be the zenith, P the pole, S and S' the positions of the star; join ZS, ZS', PS, PS', and $SS'M$; draw PT to the middle T of SS', join ZT, and draw ZV perpendicular to PT. Let

$$p = PS = PS' = 90° - D, \; SPS' = \text{elapsed time} = h$$
$$ST = A = S'T, \; PT = 90° - B$$
$$A_1 = 90° - ZS, \; A_1' = 90° - ZS'$$
$$ZTP = T, \; ZT = F, \; ZV = C$$
$$TV = Z, \; PV = 90° - E;$$

in which D and B are positive, when the latitude and declination are of the same name, but negative, if they are of contrary names; Z is positive, if the zenith is nearer the elevated pole than the point M.

Now the triangle TPS gives

$$\sin. A = \sin, PS \sin. SPT = \cos. D \sin. \tfrac{1}{2} h$$
$$\cos. PS = \cos. PT \cos. A, \text{ or } \sin. D = \sin. B \cos. A, \qquad (425)$$
or
$$\text{cosec. } A = \sec. D \text{ cosec. } \tfrac{1}{2} h \qquad (426)$$
$$\text{cosec. } B = \cos. A \text{ cosec. } D. \qquad (427)$$

The triangles ZTS and ZTS' give

$$\sin. A_1 = \cos. F \cos. A - \sin. F . \sin. A \sin. T, \qquad (428)$$
$$\sin. A_1' = \cos. F \cos. A + \sin. F . \sin. A \sin. T. \qquad (429)$$

The sum and difference of which is, by (43) and (44),

$$\sin.\tfrac{1}{2}(A_1+A_1')\cos.\tfrac{1}{2}(A_1'-A_1)=\cos.F\cos.A, \qquad (430)$$

$$\sin.\tfrac{1}{2}(A_1'-A_1)\cos.\tfrac{1}{2}(A_1+A_1')=\sin.F\sin.A\sin.T. \qquad (431)$$

But triangle ZTV gives

$$\sin.C=\sin.F\sin.T, \qquad (432)$$

$$\cos.F=\cos.C\cos.Z; \qquad (433)$$

which, substituted in (430) and (431), give

$$\sin.C=\sin.\tfrac{1}{2}(A_1'-A_1)\cos.\tfrac{1}{2}(A_1+A_1')\operatorname{cosec.}A, \qquad (434)$$

$$\sec.Z=\cos.A\cos.C\sec.\tfrac{1}{2}(A_1-A_1')\operatorname{cosec.}\tfrac{1}{2}(A_1'+A_1). \qquad (435)$$

But $$PV=PT-TV,$$

or $$90^\circ-E=90^\circ-B-Z$$

$$E=B+Z. \qquad (436)$$

Lastly, triangle ZPV gives

$$\cos.PZ=\cos.ZV\cos.PV$$

$$\sin.L=\cos.C\sin.E. \qquad (437)$$

Equations (426, 427, 434 – 437) correspond to the rule and formula given in the Navigator. [B., p. 180.]

II. Another method of calculating the values of B, C, and Z, has been given, which dispenses with A and one opening of the tables, and may therefore be preferred by some computers, although it requires one more logarithm. Triangle TPS gives

$$\operatorname{tang.}PT=\cos.\tfrac{1}{2}h\operatorname{tang.}PS,$$

or $$\operatorname{cotan.}B=\cos.\tfrac{1}{2}h\operatorname{cotan.}D. \qquad (438)$$

The substitution of (426) in (434) gives

$$\sin.C=\cos.\tfrac{1}{2}(A_1+A_1')\tfrac{1}{2}\sin.(A_1'-A_1)\sec.D\operatorname{cosec.}\tfrac{1}{2}h. \qquad (439)$$

Triangle PTS gives

$$\cos.A=\sin.D\operatorname{cosec.}B; \qquad (440)$$

which, substituted in (435), gives (441)

$$\sec.Z=\cos.C.\sin.D\operatorname{cosec.}B\operatorname{cosec.}\tfrac{1}{2}(A_1+A_1')\sec.\tfrac{1}{2}(A_1'-A_1).$$

Corollary. The hour angle ZPT is the mean between the hour angles ZPS and ZPS, and if we put

$$ZPT = H,$$

the triangle ZPV gives

$$\text{tang. } H = \text{tang. } C \text{ sec. } E, \tag{442}$$

as in [B., p. 181.]

III. When the latitude is known within a few miles. In this case let

$$L' = \text{the assumed latitude,}$$

and the triangle ZPV gives

$$\text{sin. } C = \text{cos. } L' \text{ sin. } H; \tag{443}$$

whence, by (439), (444)

$$\text{sin. } H = \text{cos. } \tfrac{1}{2}(A_1 - A_1') \text{ sin. } \tfrac{1}{2}(A_1' - A_1) \text{ sec. } L' \text{ sec. } D \text{ cosec. } \tfrac{1}{2} h.$$

$$ZPS' = H - \tfrac{1}{2} h, \tag{445}$$

whence the hour angle ZPS', corresponding to the observation at S', is known, and the latitude may be found by the method of a single altitude.

IV. *Douwes's Method.* Formula (444) is, by (44),

$$2 \text{ sin. } H = (\text{sin. } A_1' - \text{sin. } A_1) \text{ sec. } L' \text{ sec. } D \text{ cosec. } \tfrac{1}{2} h. \tag{446}$$

The combination of the formulas (446, 445,) and the method of computing the latitude by a single altitude, corresponds exactly to the rule given in the Navigator. [B., p. 185.]

The log. cosec. $\frac{1}{2} h$ is not only given in Table XXVII, but also in Table XXIII, where it is called the log. $\frac{1}{2}$ elapsed time of $\frac{1}{2} h$.

The value of

$$\begin{aligned} \text{log. } 2 \text{ sin. } H - 5 &= \text{log. sin. } H + \text{log. } 2 - 5 \\ &= \text{log. sin. } H - \text{ar. co. log. } 2 + 5 \\ &= \text{log. sin. } H - 4.69897 \\ &= 5.30103 - \text{log. elapsed time of } H \end{aligned} \tag{447}$$

is inserted in Table XXIII, and is called the log. middle time of H. The 5 is subtracted from log. 2 sin H, on account of the different values of the radius in Tables XXIV and XXVII.

Scholium. When the calculated latitudes differ much from the assumed latitude, the calculation must be gone over again, with the calculated latitude instead of the assumed latitude. This labor may be avoided by noticing, in the course of the original calculation, the difference which would arise from a change of 10′ in the value of the assumed latitude, and calculating the correction of the latitude by the rule of double position. The error of the hypothesis is, in each case, the excess of the calculated above the assumed latitude, and the proportion is

$$\text{diff. of errors : diff. of hyp.} = \text{least error : corr. of hyp.} \quad (448)$$

V. If the star has increased its declination a little during the interval between the observations, the second altitude will also be increased, and will require a reduction, before applying either of these methods, in which the declination is supposed to be unchanged; or else the first declination and the first altitude must be increased.

Thus if $S a$ is the increase of declination, and if $a b$ is drawn perpendicular to ZS, $S b$ will be the increase of altitude. By putting

$$S a = \delta D, \; S b = \delta A,$$

we have

$$\delta A = \cos. S \,.\, \delta D, \quad (449)$$

or, from the triangle ZSP,

$$\delta A = \frac{\sin. L - \sin. A_1 \sin. D}{\cos. A_1 \cos. D} \,.\, \delta D, \quad (450)$$

and, by (41) and (42),

$$\delta A_1 = \frac{2 \sin. L - \cos. (A_1 - D) + \cos. (A_1 + D)}{\cos. (A_1 - D) + \cos. (A_1 + D)} \, \delta D, \quad (451)$$

in which D is to be negative, when the latitude and declination are of contrary names. Hence the value of δA_1 can be computed by this formula, and it is to be added to the first altitude when the declination is increasing, and subtracted when the declination is decreasing. Since the value of δA_1 is proportional to δD, it may be computed for some assumed valued of δD, and arranged in a table like Table XLVI of the Navigator, and the value of δA_1 can be computed from this table by a simple proportion. The rest of the calculation can be conducted according to the preceding methods, as in [B., p. 189.]

VI. If two stars are observed, whose declinations are quite different. Then, if P (fig. 40) is the pole, Z the zenith, S and S' the places of the stars.

$$A_1 = 90° - ZS = \text{the less altitude,}$$

$$A_1' = 90° - ZS = \text{the greater altitude,}$$

$$D = 90° - PS = \text{the declination of star at } S,$$

$$D' = 90° - PS' = \text{the declination of star at } S',$$

$$H = SPS' = \text{hour angle} = \text{interv. of sideral time.}$$

Then, in the triangle PSS', PS, PS', and H, are given to find

$$SS' = C, \text{ and } S'SP = 90° - F.$$

Next, in the triangle ZSS', the three sides are known, to find the angle

$$ZSS' = Z.$$

Hence $$ZSP = 90° - G = 90° - F - Z$$

$$G = F + Z.$$

Lastly, in the triangles ZSP, CS, SP, and the included angle ZSP are given to find

$$ZP = 90° - L.$$

This solution is precisely similar to the Rule in [B., p. 193]; and it is easy to prove the rules for the signs which are there given.

VII. If the distance SS' were observed, the angles ZSS' and $S'SP$ might be found from the triangles ZSS' and $S'SP$, in which the sides are all known, and the rest of the calculation would be as in the last method, and this method corresponds exactly to the Rule in [B., p. 197.]

47. Examples.

1. The correct meridian altitude of Aldebaran was found by

observation, in the year 1838, to be 55° 45′, when its bearing was south; what was the latitude?

Solution.

The zenith distance	=	34° 15′ N.
The declination	=	16° 11′ N.
The latitude	=	50° 26′ N.

2. The correct meridian altitude of Canopus was found by observation, in the year 1839, to be 16° 25′, when its bearing was south; what was the latitude?

Solution.

The zenith distance	=	73° 35′ N.
The declination	=	52° 36′ S.
The latitude	=	20° 59′ N.

3. The correct meridian altitude of Dubhe was found by observation, in the year 1830, to be 50° 45′, when its bearing was north; what was the latitude?

Solution.

The zenith distance	=	39° 15′ S.
The declination	=	62° 40′ N.
The latitude	=	23° 25′ N.

4. If the correct meridian altitude of Dubhe, at its greatest elevation, were found by observation, in the year 1830, to be 50° 45′, when its bearing was south; what would be the latitude?

Solution.

The zenith distance	=	39° 15′ N.
The declination	=	62° 40′ N.
The latitude	=	107° 55′ N.

The problem is impossible.

5. The correct meridian altitude of Dubhe, at its least elevation, was found by observation, in the year 1830, to be 50° 45′; what was the latitude?

Solution.

The polar distance	=	27° 20′
The altitude	=	50° 45′
The latitude	=	78° 5′ N.

6. The correct meridian altitudes of Dubhe, at its greatest and least elevations, which were on opposite sides of the zenith, were found by observation to be 72° 4′ and 53° 16′; what was the latitude?

Solution.

The greatest altitude	=	72° 4′
The least altitude	=	53° 16′
Diff. of altitudes	=	18° 48′
180° — Diff. of altitudes	=	161° 12′
Latitude	=	80° 36′

7. The correct meridian altitudes of Dubhe, at its greatest and least altitudes, which were on the same side of the zenith, were found by observation to be 15° 1′ and 69° 41′; what was the latitude?

Solution.

Greatest alt.	=	69° 41′
Least alt.	=	15° 1′
Sum of alts.	=	84 42′
Latitude	=	42° 21′ N.

8. In a northern latitude, the altitude of Aldebaran was found by observation, in the year 1839, to be 25° 38′, when its hour angle was $4^h\ 12^m\ 20^s$; what was the latitude?

Solution. By (408, 409, 410),

$h = 4^h\ 12^m\ 20^s$	cos. 9.65580	
$D = 16°\ 11'$	cotan. 10.53729	cosec. 10.55484
$90° - PO = 32°\ 40'$	cotan. 10.19309	sin. 9.73215
	$A = 25°\ 38'$	sin. 9.63610
$ZC = 33°\ 6'$		cos. 9.92309
$L = 65°\ 46'$ N.		

9. In lat. 65° 40′ N. nearly, the latitude of Aldebaran was found by observation, in the year 1839, to be 25° 38′, when its hour angle was $4^h\ 12^m\ 20^s$; what was the true latitude?

Solution. I. By (412),

	$65° 40'$	cos. 9.61494
	$16° 11'$	cos. 9.98244
	$4^h 12^m 20^s$	log. Ris. 4.73823
	Nat. num. 21657	4.33561
$25° 38'$	Nat. sin. 43261	
$49° 31'$ N.	Nat. cos. 64918	
$16° 11'$ N.		
$65° 42'$ N. = the latitude.		

Had the assumed latitude been taken $10'$ more, the calculated latitude would have been $65° 48\frac{1}{2}'$ N.; hence, by (448),

$$3\tfrac{1}{2} : 1\tfrac{1}{2} = 10' : 4' = \text{corr. of second hypothesis,}$$

or the latitude $= 65° 46'$ N., as in the preceding example.

II. By (417),

	$\frac{1}{2} h = 2^h 6^m 10^s$	2 log. sin. 9.43720
	$L = 65° 40'$	cos. 9.61494
	$D = 16° 11'$	cos. 9.98244
	$A_1 = 40° 31'$	
$A = 25° 38'$	$A = 25° 38'$	
$A_1 A — = 14° 51'$	$\frac{1}{2}(A_1 + A) = 33° 4\frac{1}{2}'$	sec. 10.07678
$A_1 = 40° 29'$	$\frac{1}{2}(A_1 - A) = 7° 23\frac{1}{2}'$	sin. 9.11136
corr. $A_1 =$	$2' =$ corr. lat.	

Lat. $= 65° 40' + 2' = 65° 42'$ as before.

10. Calculate the variation of a star's altitude in one minute from the meridian, when the declination is $12°$ N., and the latitude $5°$ N.

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Solution. If $A_1 - A$ is required in seconds, (418) gives

$$A_1 - A = 450 \sin. 1^m \cos. L \cos. D \sec. A_1$$

log. 450 sin. 1^m = log. 450 + log. sin. 1^m		
= 2.65321 + 7.63982 =		0.29303
$D = 12°$	cos.	9.99040
$L = 5°$	cos.	9.99834
$A_1 = 83°$	sec.	0.91411
$A_1 - A = 15''.7$, as in Table XXXII.		1.19588

11. Calculate the tabular number for $11^m\ 48^s$ in Table XXXIII.

Solution.	$11^m\ 48^s = 708^s$	log.	2.85003
	60^s	log.	1.77815
			1.07188
			2
	139.2, as in Table XXXIII.		2.14376

12. In lat. 45° 28′ N. nearly, the correct altitude of Aldebaran was found by observation, in the year 1839, to be 60° 40′ 20″, when its hour angle was $7^m\ 17^s$. What was the true latitude, if the declination of Aldebaran was 16° 11′ 9″.2 N.?

Solution.	From Table XXXII	2″.7
	From Table XXXIII	53
	2′ 23″.1 =	143″.1
	60° 40′ 20″	
	Third alt. = 60° 42′ 43″.1	
	Dec. = 16° 11′ 9″.2	
	Lat. = 45° 28′ 26″.1 N.	

13. In lat. 40° N. nearly, the sum of ten correct central altitudes of the sum, when its declination was 20° S. were 300° 6′ 40″. The hour angles of these observations were $4^m\ 15^s$, 3^m, $2^m\ 6^s$, $1^m\ 8^s$, 30^s, 50^s, $1^m\ 12^s$, $2^m\ 15^s$, $3^m\ 10^s$, $4^m\ 25^s$. What is the true latitude, if the change of declination is neglected?

Solution. The numbers of Table XXXIII are

4^m	15^s	gives	18.1
3	0		9.0
2	6		4.5
1	8		1.3
0	30		0.2
0	50		0.7
1	12		1.4
2	15		5.1
3	10		10.0
4	25		19.5

Sum = 69.7
Mean = 6.97
Table XXXII gives 1″.6

11″

Mean of observations = 30° 0′ 40″

Merid. alt. = 30° 0′ 51″
Dec. = 20° S.

Lat. = 39° 59′ 9″ N.

14. At Göttingen, in lat. 51° 32′ N. nearly, the correct central altitudes of the sun on the 11th of March, 1794, were by observation

34° 54′ 46″ when the hour angle was	— 9^m 41^s
34 55 26	— 8 19
34 56 8	— 6 39
34 56 31	— 5 16
34 56 53	— 3 49
34 57 6	— 2 47
34 57 18	0 19
34 57 11	2 5
34 57 3	3 9
34 56 48	4 36
34 56 26	6 8

The sun's meridian declination was 3° 30′ 38″ S., and it was decreasing at the rate of 0″.98 in a minute. What is the true latitude?

Solution. The mean of the altitude is	34° 56′ 30″.5;
that of the numbers of Table XXXIII is 30″.0; which, multiplied by 1″.5 from Table XXXII, gives	45″.0
The mean of the hour angles is, regarding their signs, — 1ᵐ 50ˢ, which, multiplied by 0″.98, gives by (418), for the correction of the meridian altitude . . .	1″.8
The meridian altitude	= 34° 57′ 17″.3
The declination	= 3° 30′ 38″ S.
The latitude	= 51° 32′ 4″.7 N.

which agrees exactly with the calculations of Littrow in his Astronomy.

15. Calculate the correction for the altitude of the pole star [B., p. 206], when the right ascension of the zenith is $2^h\ 7^m$.

Solution. By (421),

$h = 2^h\ 7^m - 1^h\ 2^m = 1^h\ 5^m$	sec. 0.0177
$p = 1°\ 33'$	Prop. log. 0.2868
Corr. alt. = 1° 29′, as in the table,	Prop. log. 0.3045

16. When the right ascension of the zenith was $7^h\ 9\frac{1}{2}^m$, the altitude of the pole star was observed at Newburyport to be 42° 44′. What is the latitude of Newburyport?

Solution.

The correction of table	= 0° 3′
Altitude . . .	= 42° 44′
Latitude . . .	= 42° 47′

17. Calculate the log. elapsed time and log. middle time of Table XXIII for $3^h\ 7^m\ 10^s$.

Solution. By Table XXVII and (447);

$3^h\ 7^m\ 10^s$ cosec. 0.13635 = log. elapsed time

5.30103

5.16368 = log. mid. time.

18. Calculate the variation of the altitude of a star arising from the change of 100 seconds in the declination, when the latitude is 40°, the declination 10, and the altitude 30°.

Solution. By (451),

$L = 40°$, $2 \times$ Nat. sin.	1.2856	1.2856	
$A_1 - D = 20°$ Nat. cos.	0.9397	$-$ 0.9397	0.9397
$A_1 + D = 40°$ Nat. cos.	0.7660	0.7660	$-$ 0.7660
	1.7057	1.1119	1.4593

1.7057 (ar. co.)	9.7681	9.7681
$100'' \times 1.1119$	2.0461	
$100'' \times 1.4593$		2.1641
$65''$ = var. when D is $+$,	1.8142	
$86''$ = var. when D is $-$,		1.9322

19. The moon's correct central altitude was found, by observation, to be 53° 43′, when her declination was 14° 16′ N. After an interval, in which the hour angle was $1^h\ 44^m\ 15^s$, her correct central altitude was 42° 29′, and her declination 13° 52′ N. The latitude was 48° 50 nearly; what was it exactly?

Solution. Table XLVI gives, for the second alt. . 83″

Whole change of declination . . . 24′

Correction of second altitude . . . 20′

Corrected second alt. = 42° 49′, dec. = 14° 16′ N.

I. By Bowditch's first method.

$1^h\,44^m\,15^s$	cosec. 0.64689					
14° 16′	sec. 0.01360		cosec. 0.60830			
A	cosec. 0.66049		cos. 9.98937		cos. 9.98937	
B = 14° 38′ N.			cosec. 0.59767			
	cos. 9.82326	½ sum alts. =	48° 16′		cosec. 0.12712	
	sin. 8.97762	½ diff. alts. =	5° 27′		sec. 9.00197	
C	sin. 9.46137		cos. 9.98102		cos. 0.98102	
Z = 37° 19° N.					sec. 0.09948	
E = 51° 57′ N.			sin. 9.89624			
Latitude = 48° 55′ N.			sin. 9.87726			

II. By the method (438 – 441).

$1^h\,44^m\,15^s$	cos. 9.98867	cosec. 0.64689	
14° 16′	cotan. 0.59469	sec. 0.01360	sin. 9.39170
B = 14° 38′ N.	cotan. 0.58336		cosec. 0.59767
½ sum alts. = 48° 16′		cos. 9.82326	cosec. 0.12712
½ diff. alts. = 5° 27′		sin. 8.97762	sec. 0.00197
C	cos. 9.98102	sin. 9.46137	cos. 9.98102
Z = 37° 17′ N.			sec. 0.09948
E = 51° 57′ N.	sin. 9.89624		
Lat. = 48° 55′ N.	sin. 9.87726		

III. By Douwes's method.

			48° 50′ sec. 0.18161
53° 43′ N. sin.	80610		14° 16′ sec. 0.01360
42° 49′ N. sin.	67965		log. ratio 0.19521
	12645		log. 4.10192
$\frac{1}{2}$ (1^h 44^m 15^s) =	52^m $7\frac{1}{2}^s$		log. el. time 0.64674
	1^h 44^m $15\frac{1}{2}^s$		log. mid. time 4.94387
	52^m 8^s		log. ris. 3.41097
			log. ratio 0.19521
		1643	log. 3.21576
		80610	
34° 39½′ N.	N. cos.	82253	
14° 16′ N.			

Lat. = 48° 55½′ N.

IV. By Bowditch's fourth method.

	1^h 44^m 15^s sec. 0.04657		tan. 9.68938
	14° 16′ N. tan. 9.40531	sin. 9.39170	
A =	15° 44½′ S. tan. 9.45188	cosec. 0.56485	cos. 9.98326
	13° 52′ N.		
B =	1° 56½′ S.	cos. 9.99975	cosec. 1.47003
C =	25° 16½′ cosec. 0.36961	cos. 9.95630	
		F = 4° 6½′ N.	cotan. 1.14367
	53° 43′	Z = 51° 38′ N.	
		G = 55° 44½′ N.	sin. 9.91724
	42° 29′ sec. 0.13225	sin. 9.82955	cotan. 0.03820
½ sum =	60° 44′ cos. 9.68920	I sec. 0.12938	tan. 9.95544
Rem. =	7° 1′ sin. 9.08692	K sin. 9.91823	I = 42° 4′ N.
	2) 19.27798	lat. sin. 9.87716	13° 52′ N.
½ Z =	25° 49′ N. sin. 9.63899	lat. = 48° 54½′ N.	K = 55° 56′ N.

20. The correct meridian altitude of Aldebaran was, by observation, 56° 25′ 40″ bearing south, and its declination at the time of the observation was 16° 8′ 44″ N.; what was the latitude?

Ans. 49° 43′ 4″ N.

21. The correct meridian altitude of Sirius was 70° 59′ 33″ bearing north, and its declination 16° 28′ 9″ S.; what was the latitude?

Ans. 35° 28′ 36″ S.

22. The meridian altitude of the sun's centre was 25° 38′ 30″ bearing south, and its declination 22° 18′ 14″ S.; what was the latitude?

Ans. 42° 3′ 16″ N.

23. The meridian altitude of the planet Jupiter was 50° 20′ 8″ bearing south, and its declination 18° 47′ 37″ N.; what was the latitude?

Ans. 58° 27′ 29″ N.

24. The altitude of the pole star was 30° 1′ 30″ below the pole, and its polar distance 1° 38′ 2″; what was the latitude?

Ans. 31° 39′ 32″ N.

25. The altitude of Capella on the meridian below the pole was 9° 52′ 42″, and its polar distance 44° 11′ 33″; what was the latitude?

Ans. 54° 4′ 15″ N.

26. The meridian altitude of the sun's centre was 7° 9′ 11″ below the pole, and its declination 23° 8′ 17″ N.; what was the latitude?

Ans. 74° 0′ 54″ N.

37. The two meridian altitudes of a northern circumpolar star were 61° 49′ 13″ and 47° 24′ 27″; what was the latitude?

Ans. 54° 36′ 50″ N.

28. In a northern latitude, the altitude of the sun's centre was 54° 9′, when its hour angle was 32^m 40^s, and its declination 11° 17′ N.; what was the latitude?

Ans. 46° 27′ N.

29. In latitude 49° 15′ N. nearly, the altitude of the sun's centre was 14° 15′, when its hour angle was 1^h 40^m, and its declination 23° 28′ S.; what was the true latitude?

Ans. 48° 55′ N.

30. Calculate the variation of a star's altitude in one minute from the meridian, when the declination is 3° and the latitude 7°.

Ans. It is 27″.9 when the dec. and lat. are of the same name, and 11″.2 when they are of contrary names.

31. Calculate the tabular number for 13^m 59^s in Table XXXIII.

Ans. 168.6.

32. In lat. 50° 30′ N. nearly, the altitude of Sirius was 22° 59′ 36″, when its hour angle was 4^m 15^s, and its declination 16° 29′ 11″ S.; what was the true latitude?

Ans. 50° 30′ 49″ N.

33. In lat. 20° 27′ N. nearly, the sum of seven altitudes of Sirius was 371° 21′; the hour angles of the observations were 7^m, 5^m 3^s, 2^m 12^s, 9^s, 3^m, 4^m 6^s, 8^m 13^s; what was the true latitude, if the declination of Sirius was 16° 29′ 30″ S.?

Ans. 20° 26′ 18″ N.

34. In lat. 60° N. nearly, the sum of twelve central altitudes of the moon was 590°; the hour angles of the observations were — 9^m 3^s, — 7^m 40^s, — 6^m 12^s, — 5^m 30^s, — 3^m 2^s, — 1^m, — 12^s, — 50^s, 1^m 59^s, 4^m, 7^m 30^s, 10^m; the moon's meridian declination was 19° 0′ 58″.4 N., and her change of declination for one minute 13″.875; what was the true latitude?

Ans. 59° 50′ 2″.6 N.

35. Calculate the correction for the altitude of the pole star [B., p. 206], when the right ascension of the zenith is 9^h 7^m.

Ans. 48′.

36. The altitude of the pole star was 25° 9′, when the right ascension of the zenith was 21^h 47^m; what was the latitude?

Ans. 24° 8′ N.

37. Calculate the log. elapsed time and log. middle time of Table XXIII for $5^h\ 58^m\ 10^s$.

Ans. Log. elapsed time = 0.00001
Log. middle time = 5.30102.

38. Calculate the variation of the altitude of a star arising from the change of 100 seconds in declination, when the latitude is 60°, the declination 20°, the altitude 30°, and the declination and latitude of the same name.

Ans. 85.″

39. Calculate the variation of the altitude of a star arising from the change of 100 seconds in declination, when the latitude is 50°, the declination 24°, and the altitude 20°.

Ans. It is 73″ when the lat. and dec. are of the same name, and 105″ when they are of contrary names.

40. The sun's correct central altitudes were found by observation to be 30° 13′ and 50° 4′; his declination was 20° 7′ N., and the interval of solar time between the observations was $2^h\ 55^m\ 32^s$; the assumed latitude was 56° 29′ N.; what was the true latitude?

Ans. 56° 47′ N.

41. The sun's correct central altitude was 41° 33′ 12″, his declination 14° N.; after an interval of $1^h\ 30^m$, his correct central altitude was 50° 1′ 12″, and declination 13° 58′ 38″ N.; the assumed latitude was 52° 5′ N.; what was the true latitude?

Ans. 52° 5′ N.

42. The moon's correct central altitude was 55° 38′, her declination 0° 20′ S.; after an interval in which the hour angle was $5^h\ 30^m\ 49^s$, her correct central altitude was 29° 57′, and her declination 1° 10′ N.; the assumed latitude was 23° 25′ S.; what was the true latitude?

Ans. 23° 24′ S.

43. The sun's correct central altitude was 16° 6′, his declination 8° 18′ N.; after an interval in which the hour angle was 3^h, his correct central altitude was 42° 14′ 9″, and his declination 8° 15′ N.; the assumed latitude was 49° N.; what was the true latitude?

Ans. 48° 50′ N.

44. The moon's correct central altitude was 35° 21′, and her declination 5° 31′ 6″ S.; after an interval in which the hour angle was $2^h\ 20^m$, her correct central altitude was 70° 1′, and her declination 5° 28′ 54″ S.; the assumed latitude was 1° 30′ S.; what was the true latitude?

Ans. 1° 33′ S.

45. The altitude of Capella was 60° 45′ 36″, and her declination 45° 48′ 21″ N.; at the same instant, the altitude of Sirius was 17° 54′ 12″, and his declination 16° 28′ 40″ S.; the hour angle between the stars was $1^h\ 33^m\ 37^s$, and the latitude was about 53° 15′ N.; what was the true latitude?

Ans. 53° 19′ N.

46. The altitude of α Bootis was 50° 3′ 39″, and its declination 20° 10′ 56″ N.; the altitude of α Aquilæ was 41° 27′, and its declination 8° 22′ 35″ N.; the difference of the hour angles of the observations was $5^h\ 35^m\ 5\frac{1}{2}^s$, and the assumed latitude 38° 27′ N.; what was the true latitude?

Ans. 38° 28′ N.

47. The distance of the centres of the sun and moon was found, by observation, to be 75°; the sun's central altitude was 37° 40′; the moon's central altitude was 55° 20′; the sun's declination was 0° 17′ S.; the moon's declination was 0° 36′ N.; what was the latitude, supposing it to be north?

Ans. 23° 24′ N.

48. The observation has been supposed stationary, in the preceding observations; but if he is in motion, his second altitude will differ from the altitude for this time at the first station, by the number of minutes by which the observer has approached the star or receded from it; so that the correction arising from this change of place is obviously computed by the method in [B., p. 183.]

49. In observing the meridian altitude of a star, the position of the meridian has been supposed to be known; but if it were not known the meridian altitude can be distinguished from any other altitude from the fact that it is the greatest or the least altitude; so that it is only necessary to observe the greatest or the least altitude of the star.

50. But if the star changes its declination, the greatest altitude ceases to be the meridian altitude. Let h denote the hour angle of the star at the time of observation. Then if the star did not change its declination, and if B were the number of seconds given by Table XXXII for the diminution of altitude in one minute from the meridian passage, $h^2 B$ would be the diminution of altitude in h minutes. But, since h is small, the altitude, at this time, is increased by the change of declination; so that if A is the number of minutes by which the star changes its declination in one hour, that is, the number of seconds by which it changes its declination in one minute, $h\ A$ will be the increase of altitude in the time of h, so that the altitude at the time h exceeds the meridian altitude by

$$h\ A - h^2\ B. \tag{452}$$

If, then, h denotes the time of the greatest altitude, and $h + \delta\ h$ a time which differs very slightly from the greatest altitude; the greatest altitude exceeds the altitude at the time $h + \delta\ h$ by the quantity

$$\begin{aligned}(h\ A - h^2\ B) &- [(h + \delta\ h)\ A - (h + \delta\ h)^2\ B] \\ &= \delta\ h\ [(-A + 2\ B\ h) + B\ \delta\ h],\end{aligned} \tag{453}$$

and $\delta\ h$ can be supposed so small that $B\ \delta\ h$ may be insensible, and (453) becomes

$$\delta\ h\ (-A + 2\ B\ h). \tag{454}$$

Now $-A + 2\ B\ h$ cannot be negative, because h is supposed to correspond to the greatest altitude, and cannot be less than the altitude at the time $h + \delta\ h$. Neither can $-A + 2\ B\ h$ be positive, for the altitude at the time h exceeds that at the time $h - \delta\ h$ by the quantity

$$-\delta\ h\ (-A + 2\ B\ h),$$

which, in this case, would be negative, and the altitude at the time $h - \delta\ h$ would exceed the greatest altitude. Since, then, $-A + 2\ B\ h$ can neither be greater nor less than zero, we must have

$$-A + 2\ B\ h = 0$$

or

$$h = \frac{A}{2\ B}, \tag{455}$$

and this value of h, substituted in (452), gives

$$\frac{A^2}{2B} - \frac{A^2}{4B} = \frac{A^2}{4B} \tag{456}$$

for the excess of the greatest altitude above the meridian altitude.

51. If the observer were not at rest, his change of latitude will affect his observed greatest altitude in the same way in which it would be affected by an equal change in the declination of the star; so that the calculation of the correction on this account may be made by means of (455) and (456) precisely as in [B., p. 169.]

52. Examples.

1. An observer sailing N. N. W. 9 miles per hour, found by observation, the greatest central altitude of the moon bearing south, to be 54° 18′; what was the latitude, if the moon's declination was 6° 30′ S., and her increase of declination per hour 16′.52?

Solution.

	☽'s zenith dist.	=	35° 42′ N.
	☽'s dec.	=	6° 30′ S.
	Approx. lat.	=	29° 12′ N.
☽'s increase of dec. per hour		=	16′.52 S.
Ship's change of lat.		=	8′.3
	A	=	24.82, $A^2 = 616.0$
By Table XXXII	B	=	2.9, $4B = 11.6$

Corr. of gr. alt. = corr. of lat. = 52″ = 1′ nearly.

Lat. = 29° 12′ + 1′ = 29° 13′ N.

2. An observer sailing south $12\frac{1}{2}$ miles per hour, found, by observation, the greatest central altitude of the moon bearing south, to be 25° 15′; what was the latitude, if the moon's declination was 1° 12′ N., and her increase of declination per hour 18′.5?

Ans. 66° 1′ N.

CHAPTER V.

THE ECLIPTIC.

53. THE careful observation of the sun's motion shows this body to move nearly in the circumference of a great circle. This circle is called *the ecliptic.* [B., p. 48.]

54. The angle which the ecliptic makes with the equator is called the *obliquity of the ecliptic.*

55. The points, where the ecliptic intersects the equator, are called the *equinoctial points;* or the *equinoxes.* The point through which the sun *ascends* from the southern to the northern side of the equator, is called the *vernal equinox;* and the other equinox is called the *autumnal equinox.*

The points 90° distant from the ecliptic are called the *solstitial points*, or the *solstices.* [B., p. 49.]

56. The circumference of the ecliptic is divided into twelve equal parts, called *signs*, beginning with the vernal equinox, and proceeding with the sun from west to east.

The names of these signs are *Aries* (♈), *Taurus* (♉), *Gemini* (♊), *Cancer* (♋), *Leo* (♌), *Virgo* (♍), *Libra* (♎), *Scorpio* (♏), *Sagittarius* (♐), *Capricornus* (♑), *Aquarius* (♒), *Pisces* (♓). The vernal equinox is therefore *the first point*, or beginning of Aries, and the autumnal equinox is the first point of Libra; the first six signs are north of the equator, and the last six south of the equator. The northern solstice is the first point of Cancer, and the southern solstice the first point of Capricorn. [B., p. 49.]

57. Secondary circles, drawn perpendicular to the ecliptic, are called *circles of latitude.*

The circle of latitude drawn through the equinoxes is called *the equinoctial colure.*

The circle of latitude drawn through the solstices is called *the solstitial colure.* [B., p. 49.]

Corollary. The solstitial colure is also a secondary to the equator, so that it passes through the poles of both the equator and the ecliptic.

58. Small circles, drawn parallel to the equator through the solstitial points, are called *tropics.*

The northern tropic is called the *tropic of Cancer;* the southern tropic the *tropic of Capricorn.*

Small circles, drawn at the same distance from the poles which the tropics are from the equator, are called *polar circles.*

The northern polar circle is called the *arctic circle*, the southern the *antarctic.*

59. The *latitude of a star* is its distance from the ecliptic measured upon the circle of latitude, which passes through the star. If the observer is supposed to be at the earth, the latitude is called *geocentric latitude*; but if he is at the sun, it is *heliocentric latitude.* [B., p. 49.]

60. The *longitude of a star* is the arc of the ecliptic contained between the circle of latitude drawn through the star and the vernal equinox. [B., p. 50.]

Corollary. The longitude and right ascension of the first point of Cancer are each equal to 6^h, and those of the first point of Capricorn are each equal to 18^h.

61. The *nonagesimal point* of the ecliptic is the highest point at any time.

Corollary. The distance of the nonagesimal from the zenith is therefore equal to the distance of the zenith from the ecliptic, that is, to the *celestial latitude of the zenith;* and the longitude of the nonagesimal is the *celestial longitude of the zenith.*

62. *Problem. To find the latitude and longitude of a star, when its right ascension and declination are known.*

Solution. Let P (fig. 35) be the north pole of the equator, Z the north pole of the ecliptic, and B the star. Then EQW will be the equator, $NESW$ the ecliptic, and $NPZS$ the solstitial colure, so that the point S is the southern solstice, and N the northern solstice. Now if the arc PB be produced to cut the equator at M, and ZB to cut the ecliptic at L; the angle ZPB is measured by the arc QM, that is, by the difference of the right ascensions of Q and M, or by the difference of the ✱'s right ascension and 18^h; that is,

$$ZPB = 18^h - \text{R. A.} = 24^h - (6^h + \text{R. A.})$$

or

$$= \text{R. A.} - 18^h = (\text{R. A.} + 6^h) - 24^h$$

or

$$= 24^h + \text{R. A.} - 18^h = \text{R. A.} + 6^h.$$

In the same way

$$PZB = NL = \text{Long.} - 90^\circ$$

or

$$= 360^\circ - (\text{Long.} - 90^\circ)$$

$$= -(\text{Long.} - 90^\circ),$$

in which the first values of ZPB and PZB correspond to the star's being east of the solstitial colure; the second and third values to the star's being west of the colure. We also have

$$PB = 90^\circ - \text{Dec.}$$

$$BZ = 90^\circ - \text{Lat.}$$

$$PZ = 90^\circ - ZQ = QS$$

$$= \text{obliquity of ecliptic} = \pm E, \tag{457}$$

in which the declination and latitude are positive when north, and negative when south, and E has the same sign with R. A. $- 12^h$.

The present problem does not, then, differ from that of § 28, and if we put

$$\pm A = PC - 90^\circ,$$

in which the upper sign is used, when R. A. — 12^h is positive, and otherwise the lower sign, we have, by (298, 299, and 300),

$$\text{tang. } PC = \mp \text{cotan. } A = \text{cos. (R. A.} + 6^h) \text{ cotan Dec.}$$

$$= - \text{sin. R. A. cotan. Dec.} \qquad (458)$$

in which the signs are used as in the preceding equation; so that A and Dec. are always positive or negative at the same time. Instead of (458), its reciprocal may be used, which is

$$\mp \text{tang. } A = - \text{cosec. R. A. tang. Dec.} \qquad (459)$$

If, then,

$$B = E + A, \qquad (460)$$

we have

$$AP = \mp E - 90° \mp A = \mp B - 90° \qquad (461)$$

or

$$= 90° \pm A \pm E = 90° \pm B,$$

in which the upper or lower signs are used, as in (457). Hence

$$\text{cos. } PC : \text{cos. } AP = \mp \text{sin. } A : \mp \text{sin. } B = \text{sin. } A : \text{sin. } B$$

$$= \text{sin. Dec. : sin. Lat.} \qquad (462)$$

so that, since Dec. and A are both positive or both negative, B and Lat. must also be both positive or both negative. Again,

$$\text{sin. } PC : \text{sin. } PA = \text{cos. } A : \pm \text{cos. } B \qquad (463)$$

$$= \pm \text{cotan. (R. A} + 6^h) : \pm \text{cotan. (Long.} - 90°)$$

$$= \pm \text{tang. R. A.} : \pm \text{tang. Long.}$$

in which the signs may be neglected, and Long. is to be found in the same quadrant with R. A., unless the foot P of the perpendicular falls within the triangle; in which case the first value of AP (461) is used, so that B is obtuse. In this case, the longitude is in the adjacent quadrant on the same side of the solstitial colure with the right ascension. These results agree with the Rule in [B., p. 145].

63. *Corollary.* The latitude and longitude of the zenith, that is, the zenith distance and longitude of the nonagesimal, might be found by the same method. But another rule can be used, which is of peculiar advantage, where these quantities are often to be calculated

for the same place. We have by (369) and (370), calling B the zenith, and putting

$$T = 24^h - ZPB \text{ or} = ZPB$$

$$F = \tfrac{1}{2}(PZB - ZBP) \text{ or} = 180° - \tfrac{1}{2}(PZB + ZBP) \quad (464)$$

$$G = \tfrac{1}{2}(PZB + ZBP) \text{ or} = 180° - \tfrac{1}{2}(PZB + ZPB) \quad (465)$$

$$\text{tang. } F = -\text{cosec. } \tfrac{1}{2}(PB + PZ) \sin. \tfrac{1}{2}(PB - PZ) \cot. \tfrac{1}{2} T$$

$$= \text{tang. } (24^h - F) \quad (466)$$

$$\text{tang } G = -\text{sec. } \tfrac{1}{2}(PB + PZ) \cos. \tfrac{1}{2}(PB - PZ) \cot. \tfrac{1}{2} T \quad (467)$$

$$90° + F + G = PZB + 90° \text{ or} = 360° - PZB + 90° \quad (468)$$

$$= \text{Long. or} = 360° + \text{Long.} \quad (469)$$

in which the first member of (466) is used when PB is greater than PZ, and the third when PB is less than PZ, that is, within the north polar circle; and the second members of (464, 465, 468) correspond to the position of the zenith at the east of the solstitial colure, but the third members to the west of the colure.

Again, by (354),

$$\text{tang. } \tfrac{1}{2}(90° - \text{lat.}) = \text{tang. } \tfrac{1}{2} \text{ alt. nonagesimal}$$

$$= \cos. G. \sec. F \text{ tang. } \tfrac{1}{2}(PB + PZ), \quad (470)$$

and the preceding formulas correspond to the rule in [B., p. 402].

64. *Scholium.* The rule with regard to the values of G appears to be a little different, but the difference is only apparent, for it follows from (467), that G and $12^h - \frac{1}{2} T$ are, at the same time, both acute or both obtuse, unless

$$\tfrac{1}{2}(PB + PZ) > 90°,$$

or

$$PB > 180° - PZ, \quad (471)$$

which corresponds to the south polar circle.

65. The abridged method of calculating the altitude and longitude of the nonagesimal [B., p. 403], only consists in the previous computation of the values.

$$A = \text{log. } [\text{cos. } \tfrac{1}{2} (PB - PZ) \text{ sec. } \tfrac{1}{2} (PB + PZ)] \qquad (472)$$

$$C = \text{log. tang. } \tfrac{1}{2} (PB + PZ) \qquad (473)$$

$$B = \text{log. tang. } \tfrac{1}{2} (PB - PZ) - C \qquad (474)$$

$$= \text{log. } [\text{tang. } \tfrac{1}{2} (PB - PZ) \text{ cotan. } \tfrac{1}{2} (PB + PZ)]$$

$$= \text{log. } [\text{cosec. } \tfrac{1}{2} (PB + PZ) \text{ sin. } \tfrac{1}{2} (PB - PZ)] - A,$$

whence

$$\text{log. } [\text{cosec. } \tfrac{1}{2} (PB + PZ) \text{ sin. } \tfrac{1}{2} (PB - PZ)] = B + A \qquad (475)$$

and

$$\text{log. tang. } G = A + \text{log. } (- \text{cotan. } \tfrac{1}{2} T) \qquad (476)$$

$$\text{log. tang. } F = A + B + \text{log. } (- \text{cotan. } \tfrac{1}{2} T) \qquad (477)$$

$$= \text{log. tang. } G + B$$

$$\text{log. tang. } \tfrac{1}{2} \text{ alt. non.} = \text{log. cos. } G + \text{log. sec. } F + C. \qquad (478)$$

66. The rule in [B., p. 436] for finding right ascension and declination, when the longitude and latitude are given, may be obtained by a process precisely similar to that for the rule before it.

67. Examples.

1. Calculate the latitude and longitude of the moon, when its right ascension is $4^h\ 42^m\ 56^s$, and its declination 27° 21′ 58″ N., and the obliquity of the ecliptic 23° 27′ 45″.

Solution.	27° 21′ 58″ N.	tang. 9.71400
$4^h\ 42^m\ 56^s$	tang. 0.45650	cosec. 0.02503
$A =$ 28°44 ″ 12″ N.	sec. 0.05708	tang. 9.73903
$E =$ 23° 27′ 45″ S.		
$B =$ 5° 16′ 27″ N.	cos. 9.99816	tang. 8.96524
long. = 72° 53′ 31″	tang. 0.51174	sin. 9.98034
lat. = 5° 2′ 33″ N.		tang. 8.94558

2. Calculate the values of A, B, and C, for the obliquity 23° 77′ 40″, and the reduced latitude of 42° 12′ 2″ N.

Solution.	Polar dist. $= 47^\circ\ 47'\ 58''$	
$47^\circ\ 47'\ 58''$		
$23^\circ\ 27'\ 40''$		
$\frac{1}{2}$ sum $= 35^\circ\ 37'\ 49''$	sec. 0.09002	tang. 9.85535 $= C$
$\frac{1}{2}$ diff. $= 12^\circ\ 10'\ 9''$	cos. 9.99013	tang. 9.33374
	$A = 0.08015,$	$B = 9.47839$

3. Calculate the altitude and longitude of the nonagesimal, when the right ascension of the meridian is $19^h\ 50^m$, the latitude $42^\circ\ 12'\ 2''$ N., and the obliquity $23^\circ\ 27'\ 40''$.

Solution. $T = 19^h\ 50^m + 6^h - 24^h = 1^h\ 50^m$

$\frac{1}{2}(1^h\ 50^m)$	cotan. 0.61137	
	$A = 0.08015$	
$G = 101^\circ\ 30'\ 2''$	tang. 0.69152	cos. 9.29968
90°	$B = 9.47839$	$C = 9.85535$
$F = 124^\circ\ 4'\ 3''$	tang. 0.16991	sec. 0.25168
long. $= 315^\circ\ 34'\ 5''$	$14^\circ\ 18'\ 40''$	tang. 9.40671
	alt. $= 28^\circ\ 37'\ 20''$.	

4. Calculate the latitude and longitude of the moon, when its right ascension is $18^h\ 27^m\ 12^s$, and its declination $27^\circ\ 49'\ 38''$ S., and the obliquity of the ecliptic $23^\circ\ 27'\ 45''$.

Ans. The ☽'s long. $= 276^\circ\ 1'\ 44''$

Its lat. $= 4^\circ\ 30'\ 27''$ S.

5. Calculate the values of A, B, and C, for Albany in reduced latitude $42^\circ\ 27'\ 39''$, and for the obliquity $23^\circ\ 27'\ 40''$.

Ans. $A = 0.07965$

$B = 9.47565$

$C = 9.85327$

6. Calculate the longitude and altitude of the nonagesimal, when the obliquity of the ecliptic is 23° 27′ 40″, the latitude 42° 12′ 2″ N., and the R. A. of the meridian $10^h\ 10^m$.

Ans. The long. = 138° 30′ 25″

alt. = 61° 18′ 46″.

7. Calculate the moon's right ascension and declination, when its latitude is 5° 0′ 7″ N., its longitude 64° 54′ 1″, and the obliquity of the ecliptic 23° 27′ 45″.

Ans. Its R. A. = $4^h\ 7_m\ 46^s$.

Its Dec. = 26° 3′ 0″ N.

68. *Problem. To find the declination of a star.*

Solution. I. Observe its meridian altitude, and its declination is at once found by one of the equations (404 – 406).

II. If the star does not set, and both its transits are observed, we have

$$p = 90^\circ - \text{Dec.} = \tfrac{1}{2}(A_1 - A'). \tag{478}$$

69. *Problem. To find the position of the equinoctial points.*

Solution. Since the right ascension of all stars is counted from the vernal equinox, and since the two equinoxes are 12^h apart, the present problem is the same as to find the right ascension of some one of the stars, which may afterwards serve as a fixed point for determining the right ascension of the other stars.

Observe the declination of the sun for several successive noons near the equinox, until two noons are found between which its declination has changed its sign; and observe also the instant of the sun's transit across the meridian on these days, by a clock whose rate of going is known. Then, by supposing the sun's motions in declination and right ascension to be uniform at this time, which they nearly are, the time of the equinox, that is, of the sun's being in the equator, is found by the proportion

the whole change of declination : either declination = the sideral interval between the transits — 24^h : the sideral interval between the transits of the equinox and that of the sun at this declination; . . . . (479)

and this interval is the difference between the right ascensions of the sun at this declination and the equinox. If the passage of a star had been observed in the same day, the right ascension of the star would have been the interval of sideral time of its passage after that of the vernal equinox.

70. Examples.

1. If the sun's declination is found at one transit to be $7' \, 9''.5$ S., and at the next transit to be $16' \, 31''.1$ N.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 38^s.21$?

Solution.

$$7' \, 9''.5 + 16' \, 31''.1 = 23' \, 40''.6 = 1420''.6 = \text{ar. co. } 6.84753$$

$$16' \, 31''.1 = 991''.1 \qquad 2.99612$$

$$3^m \, 38^s.21 = 218^s.21 \qquad 2.33887$$

$$\odot\text{'s R. A.} = 0^h \, 2^m \, 32^s.2 = 152^s.2 \qquad 2.18252$$

2. If the sun's declination is found at one transit to be $18' \, 38''.8$ S., and at the next transit to be $5' \, 3''.2$ N.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 38^s.4$?

Ans $0^h \, 0^m \, 46^s.6$.

3. If the sun's declination is found at one transit to be $5' \, 57''.9$ N., and at the next transit to be $17' \, 26''.3$ S.; what is the sun's right ascension at the second transit, if the sideral interval of the transits is $24^h \, 3^m \, 35^s.71$?

Ans. $12^h \, 2^m \, 40^s.7$.

71. *Problem. To find the obliquity of the ecliptic.*

Solution. Observe the right ascension and declination of the sun, when he is nearly at his greatest declination; that is, when his ascension is nearly 6^h or 18^h. If he were observed at exactly his greatest declination, the observed declination would obviously be the required obliquity. But for any other time, the sun's declination and right ascension are the legs of a right triangle, of which the obliquity of the ecliptic is the angle opposite the declination. Hence

$$\text{tang. ☉'s Dec.} = \text{sin. ☉'s R. A. tang. obliq.} \qquad (480)$$

Now if we put

$$h = \text{the diff. of ☉'s R. A. and R. A. of solstice,}$$

we have

$$\cos. h = \frac{\text{tang. ☉'s dec.}}{\text{tang. obliq.}} \qquad (481)$$

and by (346) and 347),

$$\frac{\text{sin. (obliq.} - \text{☉'s dec.)}}{\text{sin. (obliq.} + \text{☉'s dec.)}} = \frac{1 - \cos. h}{1 - \cos. h} = \frac{2 \sin.^2 \frac{1}{2} h}{2 \cos.^2 \frac{1}{2} h}$$

$$= \text{tang.}^2 \tfrac{1}{2} h \qquad (482)$$

$$\text{sin. (obliq.} - \text{☉'s dec.)} = \text{(obliq.} - \text{☉'s dec.) sin. } 1''$$

$$= \text{tang.}^2 \tfrac{1}{2} h \text{ sin. (obl.} + \text{☉'s dec.)} \qquad (483)$$

$$\text{obl.} - \text{☉'s dec.} = \text{cosec. } 1'' \tan.^2 \tfrac{1}{2} h \text{ sin. (obl.} + \text{☉'s dec.)} \qquad (484)$$

$$= \tfrac{1}{4} h^2 \text{ cosec. } 1'' \tan.^2 1^s \text{ sin. (obl.} + \text{☉'s dec.)}$$

and the second member of 484 may be regarded as a correction in seconds to be added to the ☉'s dec. to obtain the obliquity, and the obliquity in the second member need only be known approximately.

72. Examples.

1. The right ascensions and declinations of the sun on several successive days were as follows :

June 19,	R. A. $= 5^h\ 50^m\ 53^s$	Dec. $= 23°\ 26'\ 45''.2$ N.
20	5 55 3	23 27 27 .3
21	5 59 12	23 27 44 .7
22	6 3 21	23 27 37 .3
23	6 7 31	23 27 4 .6

To find the obliquity of the ecliptic.

Solution. Assnme for the obliquity the greatest observed declination, or 23° 27′ 45″, and the corrections of all the observations may be computed in the same way as that of the first, which is thus found,

$\frac{1}{4}$ cosec. 1″ tang.^{2} 1^s = $\frac{225}{4}$ tang. 1″	6.43570
$h = 9^m\ 7^s = 547^s$ 2 log.	5.47598
23° 26′ 45″ + 23° 27′ 45″ = 46° 54′ 30″ sin.	9.86348
cor. dec. = 59″.59	1.77516
23° 26′ 45″.2	
obliquity = 23° 27′ 44″.8	= 23° 27′ 44″.8
In the same way the 2d observation gives	23 27 44 .9
the 3d observation gives	23 27 45 .2
the 4th observation gives	23 27 45 .3
the 5th observation gives	23 27 45 .3
sum =	117° 18′ 45″.5
The mean =	23° 27′ 45″.1

2. The right ascensions and declinations of the sun on several successive days, were as follows:

Dec. 20	☉'s R. A. = $17^h\ 51^m\ 17^s$	23° 26′ 48″.4 S.
21	17 55 40	23 27 30 .9
22	18 0 7	23 27 44 .0
23	18 4 33	23 27 29 .5
24	18 9 0	23 26 45 .5

what was the obliquity?

Ans. 23° 47′ 44″.7.

CHAPTER VI.

PRECESSION AND NUTATION.

73. The ecliptic is not a fixed, but a moving plane; and its observed position in the year 1750 is here adopted as a *fixed plane*, to which its situation at any other time is referred.

The motion of the ecliptic is shown by the changes in the latitudes of the stars.

74. Celestial motions are generally separated into two portions, *secular* and *periodical*.

Secular motions are those portions of the celestial motions which either remain nearly unchanged, or else are subject to a nearly uniform increase or diminution, which lasts for so many ages that their limits and times of duration have not yet been determined with any accuracy.

Periodical motions are those whose limits are small, and periods so short that they have been determined with considerable accuracy.

75. The *true position* of a heavenly body or of a celestial plane is that which it actually has; its *mean position* is that which it would have if it were freed from the effects of its periodical motions.

The mean position is, consequently, subject to all the secular changes.

76. The *mean ecliptic* has, from the time of the earliest observations, been approaching the plane of the equator at a

little less than the half of a second each year, thus causing a diminution of the obliquity of the ecliptic.

Let NAA' (fig. 41) be the fixed plane of 1750, and NA_1 the mean ecliptic for the number of years t after 1750. Let A be the vernal equinox of 1750, and AQ the equator. Let

$$\Pi = NA \text{ and } \pi = \text{the angle } ANA_1;$$

then, upon the authority of Bessel, the point of intersection N of the ecliptic, which is called the *node* of the ecliptic, with the fixed plane, has a retrograde motion, by which it approaches A at the annual rate of 5''.18, and if this point could have existed in 1750, its longitude would have been 171° 36′ 10″, so that

$$\Pi = 171^\circ\ 36'\ 10'' - 5''.18\ t. \qquad (485)$$

Moreover, the angle which the mean ecliptic makes with the fixed plane increases at the annual rate of 0''.48892, but this rate of increase is itself decreasing at such a rate, that at the time t this angle is

$$\pi = 0''.48892\ t - 0''.0000030719\ t^2. \qquad (486)$$

77. *Problem. To find the change of the mean latitude of a star, which arises from the motion of the ecliptic.*

Solution. Let

$$L = \text{the } \star\text{'s lat. in } 1750$$
$$\delta L = \text{its change of lat.}$$
$$\Lambda = \text{its long. in } 1750 - 171^\circ\ 36'\ 10'' + 5.''18\ t \qquad (487)$$
$$= \text{its long. referred to the node of the ecliptic}$$
$$\delta \Lambda = \text{its change of long. from the node;}$$

then, if Z (fig. 42) is the pole of the fixed plane, P that of the ecliptic, and B the star; we have

$$PZ = \pi,\ ZB = 90^\circ - L,\ PB = 90^\circ - L - \delta L$$
$$PZB = 90^\circ + \Lambda,\ P = 90^\circ - \Lambda - \delta \Lambda.$$

Draw ZC perpendicular to PB, and we have, since PZ, PC, and CZ are very small,

$$PC = PZ \cos. P = \pi \sin. (\Lambda + \delta \Lambda),$$

or $$PC = \pi \sin. A$$

$$\cos. PZ : \cos. PC = \cos. BZ : \cos. BC,$$

or $$BZ = BC$$

$$PC = PB - BZ = -\delta L = \pi \sin. A.$$

$$\delta L = -\pi \sin. A$$

$$= -(0''.48892\, t - 0''.0000030719\, t^2) \sin. A. \quad (488)$$

Again, the triangle ZBP gives, by (354),

$$\cos. \tfrac{1}{2}(PZB + P) : \cos. \tfrac{1}{2}(PZB - P) = \tan. \tfrac{1}{2}\pi : \tan. \tfrac{1}{2}(PB + BZ)$$

But

$$\tfrac{1}{2}(PZB + P) = 90° - \tfrac{1}{2}\delta A, \quad \tfrac{1}{2}(PZB - P) = A + \tfrac{1}{2}\delta A,$$

whence $$\delta A = \pi \cos. A \text{ tang. } L$$

$$= (0''.48892\, t - 0''.0000030719\, t^2) \cos. A \text{ tang. } L. \quad (489)$$

78. The *mean celestial equator* has a motion by which its node upon the fixed plane moves from the node of the ecliptic at the annual rate of about 50″, while its inclination to the fixed plane has a very small increase proportioned to the square of the time from 1750.

Thus if AQ (fig. 41) is the equator of 1750, and $A'Q'$ that for the time t, so that A is the vernal equinox of 1750, and A_1 that for the time t.

Let $$\psi = AA', \quad \omega = NA'Q',$$

then A' moves from A at the annual rate of 50″.340499, and this rate is diminishing, so that at the time t

$$\psi = 50''.340499\, t - 0''.0001217945\, t^2, \quad (490)$$

and the value of ω in the year 1750 was

$$\omega' = 23° \; 28' \; 18'',$$

and is increasing at a rate proportioned to the square of the time, so that

$$\omega = \omega' + 0''.00000984233\, t^2. \quad (491)$$

79. *Problem. To find the change of the mean obliquity of the ecliptic and that of longitude.*

Solution. Let (fig. 41)

$$NA_1Q' = \omega_1, \quad NA_1 = \psi_1 + \Pi;$$

then, by (369) and (370),

$$\frac{\sin.\,[\Pi + \frac{1}{2}(\psi + \psi_1)]}{\sin.\,\frac{1}{2}(\psi - \psi_1)} = \frac{\text{tang.}\,\frac{1}{2}(\omega + \omega_1)}{\text{tang.}\,\frac{1}{2}\pi} \qquad (492)$$

$$\frac{\cos.\,[\Pi + \frac{1}{2}(\psi + \psi_1)]}{\cos.\,\frac{1}{2}(\psi - \psi_1)} = \frac{\text{tang.}\,\frac{1}{2}(\omega_1 - \omega)}{\text{tang.}\,\frac{1}{2}\pi}. \qquad (493)$$

Now in calculating the parts of $\psi_1 - \psi$ and $\omega_1 - \omega$ which are proportional to the time, we may, since ψ and ψ_1 differ but little, as well as ω and ω_1, and since π is small, put

$$\Pi + \tfrac{1}{2}(\psi + \psi_1) = \Pi, \quad \sin.\,\tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1)\sin.\,1''$$

$$\text{tang.}\,\tfrac{1}{2}\pi = \tfrac{1}{2}\pi\,\text{tang.}\,1'' = \tfrac{1}{2}\pi\sin.\,1'' = \tfrac{1}{2}(0''.48892)\,t\sin.\,1''$$

$$\tfrac{1}{2}(\omega + \omega_1) = \omega', \text{tang.}\,\tfrac{1}{2}(\omega_1 - \omega) = \tfrac{1}{2}(\omega_1 - \omega)\sin.\,1''$$

$$\cos.\,\tfrac{1}{2}(\psi - \psi_1) = 1;$$

which, substituted in (492) and (493), give

$$\psi - \psi_1 = 0''.48892\,t\sin.\,\Pi\,\text{cotan.}\,\omega' \qquad (494)$$

$$\omega_1 - \omega = 0''.48892\,t\cos.\,\Pi; \qquad (495)$$

which are thus computed,

0''.48892	9.68924	9.68924
171° 36' 10''	cos. 9.99532$_n$	sin. 9.16446
— 0''.48368	9.68456$_n$	
	23° 28' 18''	cotan. 0.36229
— 0''.164431		9.21599

that is, $\omega_1 - \omega = -\,0''.48368\,t$ (496)

$$\psi - \psi_1 = \quad 0''.164431\,t \qquad (497)$$

or $\omega_1 = 23°\,28'\,18'' - 0''.48368\,t$ (498)

$$\psi_1 = 50''.340499\,t - 0''.164431\,t = 50''.176068\,t. \qquad (499)$$

But, in computing the parts of $\omega_1 - \omega$ and $\psi - \psi_1$ which depend upon t^2, we need only retain the part depending upon t^2 in the value of tang. $\frac{1}{2}\pi$, and neglect these parts in the other terms of (492) and (493); and we thus have

$$\sin. [\Pi + \tfrac{1}{2}(\psi + \psi_1)] = \sin. (\Pi + 45''.08\, t) \qquad (500)$$
$$= \sin. \Pi + 45''.08\, t \sin. 1'' \cos. \Pi$$
$$\cos. [\Pi + \tfrac{1}{2}(\psi + \psi_1)] = \cos. \Pi - 45''.08\, t \sin. 1'' \sin. \Pi \quad (501)$$
$$\tan. \tfrac{1}{2}\pi = \tfrac{1}{2}\pi \sin. 1'' = \tfrac{1}{2} \sin. 1''(0''.48892\, t - 0''.0000030719\, t^2) \quad (502)$$
$$\text{cotan.} \tfrac{1}{2}(\omega + \omega_1) = \text{cotan.} (\omega' - 0''.24184\, t) \qquad (503)$$
$$= \frac{1 + 0''.24184\, t \sin. 1'' \text{tang.} \omega'}{\text{tang.} \omega' - 0''.24184\, t \sin. 1''}$$
$$= \text{cotan.} \omega' + 0''.24184\, t \sin. 1'' (1 + \text{cotan.}^2 \omega')$$
$$= \text{cotan.} \omega' + 0''.24184\, t \sin. 1'' \text{cosec.}^2 \omega'$$
$$\cos. \tfrac{1}{2}(\psi - \psi_1) = 1, \quad \sin. \tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1) \sin. 1''$$
$$\sin. \tfrac{1}{2}(\omega_1 - \omega) = \tfrac{1}{2}(\omega_1 - \omega) \sin. 1'',$$

which, substituted in (492) and (493), give

$$\psi - \psi_1 = 0''.164431\, t + 0''.48892\, t^2 \sin. 1''\ 45''.08 \cos. \Pi \text{ cotan.} \omega'$$
$$+ 0''.48892\, t^2 \sin. 1'' \times 0''.24184 \sin. \Pi \text{ cosec.}^2 \omega'$$
$$- 0''.0000030719\, t^2 \sin. \Pi \text{ cotan.} \omega'$$
$$\omega_1 - \omega = - 0''.48368\, t - 0''.48892\, t^2 \sin. 1''\, 45''.08 \sin. \Pi$$
$$- 0''.0000030719\, t^2 \cos. \Pi; \qquad (504)$$

which are thus computed,

0''.48892	9.68924		
1''	sin. 4.68557		
45''.08	1.65398		
171° 36' 10''	sin. 9.16446		cos. 9.99532n
— 0''.000015605	5.19325		
		0''.0000030719	4.48741
+ 0''.000003039			4.48273n
— 0''.000012566			

$0''.0000030719$	4.48741		
$171^\circ\ 36'\ 10''$	sin. 9.16446	cos. 9.99532_n	sin. 9.16446
$23^\circ\ 28'\ 18''$	cotan. 0.36229	0.36229	cosec.^{2} 0.79958
$-\ 0''.000001033$	4.01416	sin. $1''$ 4.68557	4.68557
	$45''.08$	1.65398	
	$0''.48892$	9.68924	9.68924
$-\ 0''.000243445$		6.38640_n	
	$0''.24184$		9.38353
$0''.000000528$			3.72238
$-\ 0''.000243950$			

so that

$$\psi - \psi_1 = 0''.164431\,t - 0''.000243950\,t^2$$

$$\omega_1 - \omega = -\ 0''.48368\,t - 0''.000012566\,t^2$$

$$\psi_1 = 50''.176068\,t - 0''.0001217945\,t^2 + 0''.000243950\,t^2$$

$$= 50''.176068\,t + 0''.000122156\,t^2 \qquad (505)$$

$$\omega_1 = 23^\circ\ 28'\ 18'' - 0''.48368\,t - 0''.000002724\,t^2, \qquad (506)$$

or, more accurately, from Bessel's *Fundamenta Astronomiae*,

$$\psi_1 = 50''.176068\,t + 0''.0001221483\,t^2 \qquad (507)$$

$$\omega_1 = 23^\circ\ 28'\ 18'' - 0''.48368\,t - 0''.00000272295\,t^2. \qquad (508)$$

These values were afterwards changed by Bessel in his *Tabulae Regiomontanae* to

$$\psi = 50''.37572\,t - 0''.0001217945\,t^2 \qquad (509)$$

$$\psi_1 = 50''.21129\,t + 0''.0001221483\,t^2 \qquad (510)$$

$$\omega_1 = 23^\circ\ 28'\ 18'' - 0''.48368\,t - 0''.00000272295\,t^2. \qquad (511)$$

But these formulas were obtained from the physical theory, and are, as Bessel says, subject to errors, on account of the uncertainty with regard to some of the data; so that we shall adopt Poisson's formulas, because they agree in the variation of the obliquity almost exactly with Bessel's observations, and shall change the value of ω' to that determined by Bessel from observations; our formulas are, then,

$$\omega' = 23^\circ\ 28'\ 17''.65 \qquad (512)$$

$$\psi = 50''.37572\ t - 0''.00010905\ t^2 \qquad (513)$$

$$\psi_1 = 50''.22300\ t + 0''.00011637\ t^2 \qquad (514)$$

$$\omega = 23^\circ\ 28'\ 17''.65 + 0''.00008001\ t^2 \qquad (515)$$

$$\omega_1 = 23^\circ\ 28'\ 17''.65 - 0''.45692\ t - 0''.000002242\ t^2. \qquad (516)$$

The formulas at present adopted in the American Ephemeris and Nautical Almanac are those of Peters, given in his *Numerus Constans Nutationis.*

If, now, the value of ψ_1 is added to that of $\delta\ \mathcal{A}$ (489), the resulting value is the total change of a star's mean longitude.

80. The backward motion ψ_1 of the equinoxes is called the *precession of the equinoxes.*

81. *Problem. To find the intersection of the mean equator with the equator of* 1750 *and its inclination to it.*

Solution. Produce AQ and $A'Q'$ (fig. 41) till they meet at T, and let

$$AT = \Phi, \quad A'T = \Phi'$$

and the triangle ATA' gives, by (350, 354, and 369),

$$\cos. \tfrac{1}{2}(\omega' - \omega) : \cos. \tfrac{1}{2}(\omega' + \omega) = \text{tang.} \tfrac{1}{2}\psi : \text{tang.} \tfrac{1}{2}(\Phi' - \Phi) \quad (517)$$

$$\sin. \tfrac{1}{2}(\omega' - \omega) : \sin. \tfrac{1}{2}(\omega' + \omega) = \text{tang.} \tfrac{1}{2}\psi : \text{tang.} \tfrac{1}{2}(\Phi' + \Phi) \quad (518)$$

$$\sin. \tfrac{1}{2}(\Phi' + \Phi) : \sin. \tfrac{1}{2}(\Phi' - \Phi) = \text{cotan.} \tfrac{1}{2} T : \cot. \tfrac{1}{2}(\omega' + \omega) \quad (519)$$

so that t^2 may be neglected in all the terms but ψ, and we have

$$1 : \cos. \omega' = \tfrac{1}{2}\psi \sin. 1'' : \tfrac{1}{2}(\Phi' - \Phi) \sin. 1'' \qquad (520)$$

$$0 : \sin. \omega' = \tfrac{1}{2}\psi \sin. 1'' : \text{tang.} \tfrac{1}{2}(\Phi' + \Phi) \qquad (521)$$

$$1 : \tfrac{1}{2}(\Phi' - \Phi) \sin. 1'' = \text{tang.} \omega' : \tfrac{1}{2} T \sin. 1''. \qquad (522)$$

Hence

$$\tfrac{1}{2}(\Phi' + \Phi) = 90^\circ \qquad (523)$$

$$\tfrac{1}{2}(\Phi' - \Phi) = \tfrac{1}{2}\psi \cos. \omega' \qquad (524)$$

$$T = (\Phi' - \Phi) \text{ tang. } \omega', \qquad (525)$$

which are thus computed,

ω'	cos. 9.96249	cos. 9.96249
25″.18786	1.40120	
23″.103	1.36369	
0″.000054525		5.73660
0″.000050013		5.69909
ω'	tang. 9.63771	9.63771
10″.032	1.00140	
0″.000021717		5.33680

so that

$$\Phi = 90^\circ - 23''.103\, t + 0''.000050013\, t^2 \tag{526}$$

$$T = 20''.0640\, t - 0''.000043434\, t^2. \tag{527}$$

82. *Problem. To find the variation of a star's mean right ascension and declination.*

I. The variation which arises from the change of the equator's inclination may be found precisely in the same way in which the variations of latitude and longitude were found in § 77, for a similar change in the position of the ecliptic; so that formulas (488) and (489) give, by substituting for A, L and π,

$$A = *\text{'s R. A.} - 90^\circ + 23''.103\, t = R - 90^\circ$$

$$L = *\text{'s Dec.} = D, \pi = T$$

$$\delta D = - T \cos. R \tag{528}$$

$$\delta R = T \sin. R \text{ tang. } D; \tag{529}$$

or instead of counting the value of T and t from 1750, they may be reduced to the beginning of each year, and the square of t may then be neglected.

II. The variation in right ascension is to be increased by the change in the position of the equinox arising from its precession, which is thus found. Had the ecliptic remained stationary, the equinox would have removed from A to A', so that if AP is perpendicular to the equator, we should have for the increase of right ascension by (515) and (524),

$$A'P = AA' \cos. AA'P = \psi \cos. \omega \qquad (530)$$
$$= (\Phi' - \Phi)$$
$$= 46''.206\ t - 0''.000100026\ t^2.$$

But the equinox advances upon the equator from the motion of the ecliptic by the arc $A'A_1$, which is thus found. We have, by (350),

$$\cos. \tfrac{1}{2}(\omega_1 - \omega) : \cos. \tfrac{1}{2}(\omega_1 + \omega) = \text{tang.} \tfrac{1}{2} A'A_1 : \text{tang.} \tfrac{1}{2}(\psi - \psi_1).$$

But $\cos. \tfrac{1}{2}(\omega_1 - \omega) = 1$

$$\cos. \tfrac{1}{2}(\omega_1 + \omega) = \cos. (\omega' - 0''.22846\ t)$$
$$= \cos. \omega' + 0''.22846\ t \sin. 1'' \sin. \omega'$$
$$\sec. \tfrac{1}{2}(\omega_1 + \omega) = \sec. \omega' - 0''.22846\ t \sin. 1'' \sin. \omega' \sec.^2 \omega'$$
$$\text{tang.} \tfrac{1}{2} A'A_1 = \tfrac{1}{2} A'A_1 \sin. 1''$$
$$\text{tang.} \tfrac{1}{2}(\psi - \psi_1) = \tfrac{1}{2}(\psi - \psi_1) \sin. 1''$$
$$= \tfrac{1}{2} \sin. 1'' (0''.15272\ t - 0''.00022542\ t^2),$$

whence $A'A_1 = 0''.15272\ t \sec. \omega' - 0''.00022542\ t^2 \sec. \omega'$,

which is thus computed,

0″.15272		9.18390	
ω′	sec.	0.03751	0.03751
0″.1665		9.22141	
		0″.00022542	6.35299
0″.00024575			6.39050

so that

$$A'A_1 = 0''.1665\ t - 0''.00024575\ t^2, \qquad (531)$$

and, by (489) and (490),

$$\delta R = 46''.0395\ t + 0''.00016593\ t^2 + T \sin. R \text{ tang. } D. \qquad (532)$$

83. By the motions of precession and of diminution of the obliquity, the mean pole of the equator is carried round the pole of the ecliptic, gradually approaching it; but the true pole of the equator has a motion round the mean pole, which is called *nutation.* This motion is in an oval, at the centre of which is the mean pole, and is such that the position of the mean equinox differs from that of the true equinox by the longitude

$$\delta \text{ long.} = i \sin. \Omega + i_1 \sin. 2\,\Omega + i_2 \sin 2\,☽ + i_3 \sin. 2\,\odot, \quad (533)$$

where

- ☊ = the mean longitude of that point of intersection of the moon's orbit with the ecliptic, through which the moon ascends from the south to the north side of the ecliptic, and which is called the moon's ascending node,
- ☽ = the moon's true longitude,
- ⊙ = the sun's true longitude.

The values of i, i_1, i_2, i_3, are given differently by different astronomers; and those which are used in the following examples are those of Bailly, adopted in the Nautical Almanacs of 1839, 1840, &c.; viz.,

$$\left. \begin{array}{ll} i = -17''.2985, & i_1 = 0''.2082 \\ i_2 = -0''.2074, & i_3 = -1''.2550. \end{array} \right\} \quad (534)$$

This nutation of the pole causes also the true obliquity of the ecliptic to change from the mean obliquity by the quantity

$$\delta\, \omega_1 = k \cos. \Omega + k_1 \cos. 2\,\Omega + k_2 \cos. 2\,☽ + k_3 \cos. 2\,\odot, \quad (535)$$

in which the values of k, &c., adopted in the following examples are

$$\left. \begin{array}{ll} k = 9''.2500, & k_1 = -0''.0903 \\ k_2 = 0''.0900, & k_3 = 0''.5447. \end{array} \right\} \quad (536)$$

The values of the above coefficients now adopted in the British and American Nautical Almanacs are those given by Peters in his *Numerus Constans Nutationis.*

84. *Corollary.* The effect of nutation upon the right ascensions and declinations of the stars may be computed by § 82, and the formulas which are obtained agree with those given in

the Nautical Almanac, and depend upon the terms called C and D in the Formulas for Reduction in the Almanac; these terms contain also the changes arising from the mean motion of the equinoxes, and the formulas are so reduced that t is counted from the beginning of each year.

85. Example.

1. Find the mean obliquity of the ecliptic for the year 1840, and reduce the formulas for finding the variations of right ascension and declination to the beginning of that year.

Solution. In (516) let $t = 1840 - 1750 = 90$,

and it gives

$$\omega_1 = 23° \, 28' \, 17''.65 - 41''.12 - 0''.02 = 23° \, 27' \, 36''.51.$$

In (527, 528, and 532), let $t = 90 + t'$, and neglect the terms depending upon t'^2, so that

$$T = 30' \, 5''.76 - 0''.35 + 20''.0640 \, t' - 0''.0078 \, t'$$
$$= 30' \, 5''.41 + 20''.0562 \, t',$$

and the mean variations, counted from the beginning of the year, are

$$\delta' D = 20''.0562 \, t' \cos. R$$
$$\delta' R = 46''.0693 \, t' + 20''.0562 \, t' \sin. R \tan g. D.$$

Finally, the variations arising from nutation are thus found. The change in the obliquity of the ecliptic gives at once, from (488) and (489), by referring the positions to the mean ecliptic instead of to that of 1750,

$$\delta' D = \quad \delta \, \omega_1 \sin. R$$
$$\delta' R = - \delta \, \omega_1 \cos. R \tan g. D,$$

and the change in the position of the equinox gives by (525, 528, 529, and 530).

$$T = - \delta \, \Delta \sin. \omega_1$$
$$\delta' D = \delta \, \Delta \sin. \omega_1 \cos. R$$
$$\delta' R = \delta \, \Delta \cos. \omega_1 + \delta \, \Delta \sin. \omega_1 \sin. R \tan g. D.$$

Hence, if we take

$$46''.0693\ C = 46''.0693\ t' + \delta\ \mathcal{A} \cos.\ \omega_1$$
$$c = 46''.0693 + 20''.0562 \sin.\ R \tang.\ D$$
$$c' = 20''.0562 \cos.\ R$$
$$d = \cos.\ R \tang.\ D$$
$$d' = - \sin.\ R,$$

we have

$$C = t' + \frac{\cos.\ \omega_1}{46''.0693}\,\delta\,\mathcal{A} = t' + \frac{\sin.\ \omega_1}{20''.0562}\,\delta\,\mathcal{A}$$
$$= t' - 0.3448 \sin.\ \Omega + 0.00415 \sin.\ 2\ \Omega - 0.00413 \sin.\ 2\ \text{☽} - 0.02502 \sin.\ 2\ \odot,$$

and the entire changes of declination and right ascension are

$$\delta'\ D = C c' - \delta\ \omega_1\ .\ d'$$
$$\delta'\ R = C c - \delta\ \omega_1\ .\ d;$$

which agree with the Formulas for Reduction in the British and American Nautical Almanacs; but the Nautical Almanac for 1840 gives 46''.0206 and 20''.0426 instead of 46''.0693 and 20''.0562.

If, again, we take

$$f = 46''.0693\ C,$$
$$g \cos.\ G = 20''.0562\ C, \quad g \sin.\ G = - \delta\ \omega_1,$$

the above formulas become

$$\delta'\ D = g \cos.\ G \cos.\ R - g \sin.\ G \sin.\ R = g \cos.\ (G + R)$$
$$\delta'\ R = f + g \sin.\ R \cos.\ G \tang.\ D + g \sin.\ G \cos.\ R \tang.\ D$$
$$= f + g \sin.\ (G + R) \tang.\ D,$$

as in the Nautical Almanac.

2. Find the annual variations in the right ascension and declination of α Hydræ for the year 1840, and its true place for mean midnight at Greenwich, January 1, 1840; its mean right ascension for January 1, 1839, being $9^h\ 19^m\ 40^s.620$, and its declination $-7°\ 57'\ 49''.50$; and using the numbers of the Nautical Almanac for 1840.

Solution.

$20''.0426$	1.30195	1.30195
$R = 9^h\ 19^m\ 40^s.620$	cos. 9.88374_n	sin. 9.80872
$\delta' D = -15''.335$	1.18569_n	
$D = -7°\ 57'\ 49''.50$		tang. 9.14584_n
$\delta R = 46''.0206 - 1''.8051$		0.25651_n

$$= 44''.2155 = 2^s.948.$$

Hence its mean place for Jan. 1, 1840, is

$$R = 9^h\ 19^m\ 43^s.568$$

$$D = -7°\ 58'\ 4''.83.$$

To calculate the effects of nutation, we have

$$☊ = 339°\ 40',\quad ☽ = 242°\ 30',\quad ☉ = 281°\ 15'$$

-0.3448 sin. $☊ =$	$0.1205,$	$9''.25$ cos. $☊ =$	$8''.673$
0.00415 sin. $2☊ =$	$-0.0027,$	$-0''.0903$ cos. $2☊ =$	$-0''.068$
-0.00413 sin. $2☽ =$	$-0.0034,$	$0''.0900$ cos. $2☽ =$	$-0''.032$
-0.02502 sin. $2☉ =$	$0.0096,$	$0''.5447$ cos. $2☉ =$	$-0''.504$

$$C = t' + 0.1240, \qquad \delta\omega_1 = 8''.049$$

$$C c' = c' t' + 20''.0426 \times 0.1240 \cos. R$$

$$= c' t' - 15''.335 \times 0.1240 = c' t' - 1''.901$$

$$-\delta\omega \,.\, d' = 8''.049 \sin. R = 5''.181$$

$$C c = c t' + 0.1240 \times 2^s.948 = c t' + 0^s.365$$

$$-\delta\omega\, d = -8''.049 \cos. R \text{ tang. } D = -0''.861 = -0^s.058,$$

whence the variations arising from nutation are

$$\delta' D = 3''.28,\quad \delta' R = 0^s.30,$$

and the true places are

$$D = -7°\ 58'\ 1''.55,\quad R = 9^h\ 19^m\ 43^s.87.$$

3. Find the mean obliquity of the ecliptic for the year 1950, and

reduce the formulas for finding the variations of mean right ascension and declination to the beginning of that year.

Ans. $\omega_1 = 23° 26' 36''.18.$

$$\delta' D = 19''.8903\, t' \cos. R$$

$$\delta' R = 46''.1059\, t' + 19''.8903\, t' \sin. R \text{ tang. } D.$$

4. Find the annual variations in the right ascension and declination of β Ursæ Minoris for the year 1839, and its true place for mean midnight at Greenwich, Aug. 9, 1839; its mean right ascension for Jan. 1, 1839, being $14^h\, 51^m\, 14^s.943$, its declination $74°\, 48'\, 48''.89$ N., the longitude of the moon's ascending node for Aug. 9, 1839, being 347° 17′, that of the moon 144° 2′, and that of the sun 136° 30′, and using the constants of the Nautical Almanac, which give for Aug. 9, 1839,

$$f = 32''.33,\ g = 16''.70,\ G = 327°\ 30'.$$

Ans. Var. in R. A. $= -0^s.277$; var. in Dec. $= 14''.71$;

and for Aug. 9, 1839,

$$R = 14^h\ 51^m\ 16^s.36$$

$$D = 74°\ 48'\ 32''.46.$$

5. Calculate the values of f, g, and G for April 1, 1839, mean midnight at Greenwich, when ☊ $= 354° 10'$, ☉ $= 11° 34'$, and ☽ is neglected.

Ans. $f = 12''.53,\ g = 11''.05,\ G = 299°\ 34'.$

In Table XL of the Navigator, the decimal is neglected, and 20 used instead of 20.0562. Table XLIII is calculated from the formulas of Bessel, which differ a little from those of Bailly used in the Nautical Almanac. The construction of these two tables is sufficiently simple from the calculations already given.

CHAPTER VII.

TIME.

86. THE intervals between the successive returns of the mean place of a star to the meridian are precisely equal, and the mean daily motion of the star is perfectly uniform; so that sideral time is adapted to all the wants of astronomy. The instant, which has been adopted as the commencement of the sideral day, is *the upper transit of the vernal equinox.*

The length of the sideral day, which is thus adopted, differs therefore from the true sideral or *star* day by the daily change in the right ascension of the vernal equinox. But this change is annually about 50″ or 3ˢ.3, so that the daily change is less than 0ˢ.01, and is altogether insensible.

87. *Corollary.* The difference between the sideral time of different places is exactly equal to the difference of the longitude of the places.

88. The interval between two successive upper transits of the sun over the meridian, is called a *solar day;* and the hour angle of the sun is called *solar time.* This is the measure of time best fitted to the common purposes of life.

The intervals between the successive returns of the sun to the meridian, are not exactly equal, but depend upon the variable motion of the sun in right ascension, and can only be determined by an accurate knowledge of this motion.

89. The want of uniformity in the sun's motion in right ascension arises from two different causes.

I. The sun does not move in the equator, but in the ecliptic.

II. The sun's motion in the ecliptic is not uniform.

The variable motion of the sun along the ecliptic, and its deviations from the plane of the mean ecliptic, cannot be distinctly represented, without reference to the variations of its distance from the earth, and to the nature of the curve which it describes. This portion of the subject, therefore, which involves the determination of the sun's exact daily position, that is, the calculation of its *ephemeris*, must be reserved for the *Physical Astronomy*. It is sufficient, for our present purpose, to know that the sun moves with the greatest velocity when it is nearest the earth, that is, in its *perigee;* and that it moves most slowly when it is farthest from the earth, that is, in its *apogee.*

90. The sun arrives at its perigee about 8 days after the winter solstice, and at its apogee about 8 days after the summer solstice. The mean longitude of the perigee at the beginning of the year 1800, was 279° 30′ 5″, and it is advancing towards the eastward at the annual rate of about 11″.8; so that, by adding the precession of the equinoxes, the annual increase of its longitude is about 62″.

91. To avoid the irregularity of time arising from the want of uniformity of the sun's motion, a fictitious sun, called a *mean sun*, is supposed to move uniformly in the ecliptic, at such a rate as to return to the perigee at the same time with the true sun. A *second mean sun* is also supposed to move in the equator at the same rate with the first mean sun, and to return to each equinox at the same time with the first mean sun.

We shall denote the first mean sun by $\odot_1$, and the second mean sun by $\odot_2$.

92. *Corollary*. The right ascension of the second mean sun is equal to the longitude of the first mean sun.

93. The time which is denoted by the second mean sun is

perfectly uniform in its increase, and is called *mean time;* while that which is denoted by the true sun is called *true* or *apparent time*; the difference between mean and true time is called the *equation of time.*

94. The time which it takes the sun to complete a revolution about the earth is called a *year.*

The time which it takes the mean sun to return to the same longitude is the *common* or *tropical year.*

The time which it takes it to return to the same star is the *sideral year;* and the time which it takes it to return to the perigee is the *anomalistic year.*

The length of the mean tropical year is

$$Y = 365^d\ 5^h\ 48^m\ 47^s.808, \qquad (537)$$

so that the daily mean motion of the sun is found by the proportion

$$Y : 1^d = 360^\circ : \text{daily motion} = 59'\ 8''.3302. \qquad (538)$$

95. The fraction of a day is necessarily neglected in the length of the year in common life, and the common year is taken equal to 365^d. By this approximation, the error in four years amounts to

$$23^h\ 15^m\ 11^s.232 = 1^d - 44^m\ 48^s.768, \qquad (539)$$

or nearly a day, aud an additional day is consequently added to the fourth year, which is called the *leap year.* At the end of a century the remaining error amounts to nearly $-\,0^d.75$, which is noticed by the neglect of three leap years in four centuries. For practical convenience, those years are taken as leap years which are exactly divisible by 4, but only those *centurial years* are retained as leap years which are divisible by 400.

96. When the mean sun has returned to the same mean longitude, it has not returned to the same star, because the equinox from which the longitude is counted has retrograded by $50''.223$, so that the mean

sun has 50″.223 farther to go, and the time of describing this arc is the fourth term of the proportion

$$59' \, 8''.3302 : 1^d = 50''.223 : 20^m \, 22^s.786, \qquad (540)$$

so that the length of the sideral year is

$$Y_1 = Y + 20^m \, 22^s.786 = 365^d \, 6^h \, 9^m \, 10^s.594. \qquad (541)$$

97. The length of the mean solar day is also different from that of the sideral day, because when the $\odot_2$, in its diurnal motion, returns to the meridian, it is 59′ 8″.3302 advanced in right ascension; so that 360° 59′ 8″.3302 pass the meridian in a solar day, instead of 360°, which pass in a sideral day. Hence the excess of the solar day above the sideral day, expressed in solar time, is the fourth term of the proportion

$$360^\circ \, 59' \, 8''.3302 : 59' \, 8''.3302 = 1^d : 0^d.0027305$$

$$\text{or } 3^m \, 55^s.9094; \qquad (542)$$

that is, $\quad 1 \text{ sid. day} = 0.9972695 \text{ sol. day},$

or $\quad 24^h \text{ sid. time} = 23^h \, 56^m \, 4^s.0906 \text{ of sotar time}; \qquad (543)$

which agrees with (394) and the table for changing sideral to solar time in the Nautical Almanac and with Table LII of the Navigator.

In the same way this excess expressed in sideral time is the fourth term of the proportion

$$360^\circ : 59' \, 8''.3302 = 1^d : 0^d.002738 \text{ or } 3^m \, 56^s.5554;$$

that is, $\quad 1 \text{ sol. day} = 1.002738 \text{ sid. day}, \qquad (544)$

or $\quad 24^h \text{ sol. time} = 24^h \, 3^m \, 56^s.5554 \text{ sid. time}; \qquad (545)$

which agrees with the table for changing solar to sideral time in the Nautical Almanac and with Table LI of the Navigator. The remainder of Tables LI and LII, as well as the corresponding ones given in the Nautical Almanac, are calculated by simple proportions from the numbers which are given for 24^h.

The sideral day begins with the transit of the true vernal equinox. At the time of the transit of $\odot_2$, then; that is, at *mean noon;* we have the sid. time = R. A. of $\odot_2$ from the equinox

= R. A. of $\odot_2$ from mean equinox

+ Nutation of equinox in R. A.

= sun's mean long. + Nutation in R. A. (546)

98. The sun's mean long. for Jan. 1, 1800, at Paris, was found by Bessel to be 279° 54′ 11″.36. Its longitude for Jan. 1, of any other year t, may thus be found. Let f be the remainder after the division of t by 4; the number of days, then, by which Jan. 1 of the year t is removed from Jan. 1, 1800, is

$$\begin{aligned} 365\tfrac{1}{4}(t-f)+365f &= t \, . \, 365\tfrac{1}{4}-\tfrac{1}{4}f \\ &= Y \, . \, t + t \, . \, 11^m \, 12^s.192-\tfrac{1}{4}f \qquad (547) \\ &= Y \, . \, t+t \, . \, 0^d.00778-\tfrac{1}{4}f. \end{aligned}$$

But in $Y\,t$ days the sun's longitude increases exactly t . 360°, which is to be neglected; and its increase in longitude is

$$59' \, 8''.3302 \; (t \; 0.00778-\tfrac{1}{4}f)=t \, . \, 27''.61-f \, . \, 14' \, 47''.083, \qquad (548)$$

or more accurately from Bessel, the mean longitude E, for the first of January of the year $1800+t$ at Paris, is

$$\begin{aligned} E = 279° \, 54' \, 11''.36 &+ t \, 27''.605844 + t^2 \, . \, 0''.0001221805 \\ &- f \, . \, 14' \, 47''.083. \qquad (549) \end{aligned}$$

The mean longitude is found for the first of January for any other meridian by the following proportion, derived from the interval of time between the $\odot_2$'s passage over this meridian and that of Paris,

$$24^h : \text{long. from Par.} = 59' \, 8''.3302 : \text{change in value of } E. \qquad (550)$$

The sun's mean longitude for any mean noon n of the year after that of the first of Jan. is

$$E+n \, . \, 59' \, 8''.3302. \qquad (551)$$

Hence the sideral time of the mean noon n is

$$S=\frac{E}{15}+n \, . \, 3^m \, 56^s.555348+\text{Nutation in R. A.} \qquad (552)$$

so that the solar time of the transit of the equinox from the preceding noon is

$$24^h - S \text{ (converted into solar time).} \qquad (553)$$

99. Examples.

1. Find the sideral interval which corresponds to 10^h of solar time.

Ans. $10^h \, 1^m \, 38^s.5647$.

2. Find the solar interval which corresponds to 10^h of sideral time.

Ans. $9^h\ 58^m\ 21^s.7044$.

3. Find the sideral interval which corresponds to 10^m of solar time.

Ans. $10^m\ 1^s.6428$.

4. Find the solar interval which corresponds to 10^m of sideral time.

Ans. $9^m\ 58^s.3617$.

5. Find the sideral interval which corresponds to 10^s of solar time.

Ans. $10^s.0274$.

6. Find the solar interval which corresponds to 10^s of sideral time.

Ans. $9^s.9727$.

7. Find the sideral interval which corresponds to $0^s.85$ of solar time.

Ans. $0^s.85233$.

8. Find the solar interval which corresponds to $0^s.85$ of sideral time.

Ans. $0^s.84768$.

9. Find the sun's mean longitude at Greenwich for the mean noon of April 4, 1839, the sideral time at this noon, and the solar time of the transit of the vernal equinox from the preceding noon; the meridian of Greenwich is $9^m\ 21^s.5$ west of that of Paris.

Ans. The sun's mean longitude $= 12°\ 7'\ 3''.02$.
The sideral time of mean noon $= 48^m\ 31^s.27$.
Time of tran. ver. equi. $=$ April 3d, $23^h\ 11^m\ 39^s.68$.

100. *Problem. To find the time by observation.*

Solution. First Method. By equal altitudes.

I. If the star does not change its declination. Observe the times when the star is at equal altitudes before and after passing the meridian; the arithmetical mean between these two times is the time of the star's passing the meridian, which, compared with the known time of this passage, gives the error of the clock at this time, and the correction of this error gives the time of each observation.

II. When the declination of the star is changing, the time of the star's arriving at the observed altitude A is affected; thus if

$L =$ the latitude,

$D =$ the declination at the meridian,

$\delta D =$ the increase of declination from the meridian,

$h =$ the hour angle, supposing no change in the declination,

$\delta h =$ the increase of the hour angle in time,

we have, by (429),

$$\begin{aligned} \sin. A &= \sin. L \sin. D + \cos. L \cos. D \cos. h \qquad (554) \\ &= \sin. L \sin. (D + \delta D) + \cos. L \cos. (D + \delta D) \cos. (h + \delta h) \\ &= \sin. L \sin. D + \delta D \sin. 1'' \sin. L \cos. D + \cos. L \cos. D \cos. h \\ &\quad - \delta D \sin. 1'' \cos. L \sin. D \cos. h - 15 \delta h \sin. 1'' \cos. L \cos. D \sin. h, \end{aligned}$$

whence

$$\begin{aligned} 0 &= \delta D \sin. L \cos. D - \delta D \cos. L \sin. D \cos. h \\ &\quad - 15 \delta h \cos. L \cos. D \sin. h \\ \delta h &= \tfrac{1}{15} \delta D \text{ tang. } L \text{ cosec. } h - \tfrac{1}{15} \delta D \text{ tang. } D \text{ cotan. } h \\ &= \frac{\delta D}{15 \text{ cotan. } L \sin. h} - \frac{\delta D}{15 \text{ cotan. } D \text{ tang. } h}, \qquad (555) \end{aligned}$$

and since the two observations are at nearly the same distance from the meridian, the value of δh is the same for both of them; so that their mean is augmented by δh, and δh is consequently to be subtracted from the mean of the observed times, in order to obtain the true time of the star's passing the meridian.

In calculating the value of δh, its two terms may be calculated separately. Now if $\delta' D$ is the daily variation of the star's declination, we have

$$\delta D = \frac{h \delta' D}{24^h} = \frac{2 h \delta' D}{2 \times 24^h}, \qquad (556)$$

and in using proportional logarithms, the proportional logarithm of the hours and minutes of $2\,h$, which is the elapsed time, may be taken as if they were minutes and seconds, provided the same is done with the 24^h in the denominator. Finally, the value of $\delta\,h$ is reduced from minutes and seconds to seconds and thirds by multiplying by 60, so that if M is taken for the denominator, of either of the parts of (555), this part P is caculated by the formula

$$\text{Prop. log. } P = -\text{Prop. log. } \frac{2 \times 24^m \times 15}{60} + \text{log. } M + \text{Prop. log. } 2\,h$$

$$+ \text{Prop. log. } \delta' D, \qquad (557)$$

which agrees with [B., p. 219], for

$$-\text{Prop. log. } \frac{2 \times 24 \quad \times 15}{60} = -\text{Prop. log. } 12^m = -1.1761$$

$$= 8.8239. \qquad (558)$$

III. If the altitude at the two observations had differed slightly, the mean time would require to be corrected; for this purpose, let

$\varrho\,A =$ the excess of the second altitude above the first,

$\delta\,h -$ the increase of the hour angle,

and we easily deduce from (554)

$$\cos. A\,\delta\,A = -15 \cos. L \cos. D \sin. h\,\delta\,h, \qquad (559)$$

so that

$$\delta\,h = -\frac{\cos. A\,\delta\,A}{15 \cos. L \cos. D \sin. h}\,. \qquad (560)$$

The time of the second observation being thus increased by $\delta\,h$, that of the mean is increased by $\frac{1}{2}\,\delta\,h$, which is, therefore the correction to be subtracted from this mean.

The corrections (555) and (560) must be both of them applied when the star is changing its declination, and at the same time the observed altitudes are slightly different.

Second Method, By a single altitude. [B., p. 208 – 218.]

When a single altitude is observed, there are known in the triangle

PZB (fig. 35), the three sides, to find the hour angle ZPB, which is thus found by (336),

$$s = \tfrac{1}{2}(z + 90° - L + P) \tag{561}$$

$$\cos.\tfrac{1}{2}h = \sqrt{\left(\frac{\sin.s\sin.(s-z)}{\sin.(90°-L)\sin.p}\right)}, \tag{562}$$

which corresponds to [B., p. 210].

The hour angle may also be found by (341); thus, if we put

$$s' = \tfrac{1}{2}(A + L + p), \tag{563}$$

we have

$$s = \tfrac{1}{2}(180° - A - L + p) = 90° - s' + p = 90° - A - L + s'$$

$$s - p = 90° - s', \quad s - (90° - L) = s' - A,$$

whence

$$\sin\tfrac{1}{2}h = \sqrt{\left(\frac{\cos.s'\sin.(s'-A)}{\cos.L\sin.p}\right)}, \tag{564}$$

which corresponds to [B., p. 209].

Third Method. By the distance from a fixed terrestrial object.

If the position of the terrestrial object has been before determined, its hour angle and polar distance may be considered as known.

Hence, if T (fig. 40) is the position of the terrestrial object projected upon the celestial sphere, P the pole, and S the star. Let the distance TS be observed, and let

$$PT = P, \quad PS = p, \quad TS = d,$$

$$TPZ = H, \quad TPS = h', \quad SPZ = h,$$

$$s = \tfrac{1}{2}(P + p + d), \tag{565}$$

we have

$$\sin.\tfrac{1}{2}h' = \sqrt{\left(\frac{\sin.(s-p)\sin.(s-p)}{\sin.P\sin.p}\right)} \tag{566}$$

or

$$\cos.\tfrac{1}{2}h' = \sqrt{\left(\frac{\sin.s\sin(s-d)}{\sin.P\sin.p}\right)}, \tag{567}$$

$$h = H + h'.$$

If the polar distance and hour angle of the terrestrial object is not known, but only its altitude and azimuth, the polar distance and hour angle can be easily found by solving the triangle PZT.

Fourth Method. By a meridian transit. [B., p. 221.]

If the passage of a star is observed over the different wires of a transit instrument, the mean of the observed times is the time of the meridian transit, which should agree with the known time of this transit. This method surpasses all others in accuracy and brevity.

Fifth Method. By a disappearance behind a terrestrial object.

If the instant of a star's disappearance behind a vertical tower has been observed repeatedly with great care, the observed time of this disappearance may afterwards be used for correcting the chronometer. For this purpose, the position of the observer must always be precisely the same. Any change in the right ascension of the star does not affect the star's hour angle, that is, the elapsed time from the meridian transit; this change, consequently, affects the observed time exactly as if the observation were that of a meridian transit.

A small change in the declination of the star affects the hour angle, and therefore the time of observation. Thus, if P (fig. 44) is the pole, Z the zenith, ZSS' the vertical plane of the terrestrial object; then if the polar distance PS is diminished by

$$RS = \delta D,$$

the hour angle ZPS is diminished by the angle

$$SPS' = \delta h,$$

But $S'R$ is nearly perpendicular to SP, and the sides of $SS'R$ are so small, that their curvature may be neglected, whence

$$RS' = \delta D \text{ tang. } S = 15 \cos. D \,.\, \delta h,$$

so that

$$\delta h = \tfrac{1}{15}\, \delta D \text{ tang. } S \text{ sec. } D. \qquad (568)$$

101. Examples.

1. On May 20, 1823, in latitude 54° 20′ N., the sun was at equal altitudes, the observed interval was $6^h\ 1^m\ 36^s$; find the correction for

the mean of the observed times. The sun's declination is 19° 48′ N., and his daily increase of declination 12′ 44″.

Solution.	8.8239		8.8239
54° 20′	cotan. 9.8559	19° 48′	cotan. 04437
$6^h\ 1^m\ 36^s$	sin. 9.8510		tang. 0.0030
$6^m\ 2^s$	P. L. 1.4747		1.4747
12′ 44″	P. L. 1.1503		1.1503
— $12^s.57$	1.1558	$2^s.29$	1.8956
$2^s.29$			

— $10^s.3$ — the required correction.

2. On September 1, 1824, in latitude 46° 50′ N., the interval between the observations, when the sun was at equal altitudes, was $7^h\ 46^m\ 35^s$; the sun's declination was 8° 14′ N., and his daily increase of declination — 21′ 49″; what is the correction for the mean of the observations?

Ans. $16^s.4$.

3. On March 5, 1825, in latitude 38° 34′ N., the interval between the observations, when the sun was at equal altitudes, was $8^h\ 29^m\ 28^s$; the sun's declination was 6° 2′ S., and his daily increase of declination was 23′ 9″; what is the correction for the mean of the observations?

Ans. $15^s.4$.

4. On March 27, 1794, in latitude 51° 32′ N., the interval between the observations, when the sun was at equal altitudes, was $7^h\ 29^m\ 55^s$; the sun's declination was 2° 47′ N., and his daily increase of declination 23′ 26″; what is the correction for the mean of the observations?

Ans. — $21^s.7$.

5. In latitude 20° 26′ N., the altitude of Aldebaran, before arriving at the meridian, was found to be 45° 20′, and, after passing the meridian, to be 45° 10′; the interval between the observations was $7^h\ 16^m\ 35^s$, and the declination of Aldebaran was 16° 10′ N.; what is the correction for the mean of the observations?

Ans. 19^s.

6. In latitude 36° 39′ S., the sun's correct central altitude was found to be 10° 40′, when his declination was 9° 27′ N.; what was the hour angle?

Ans. $4^h\ 36^m\ 9^s$.

7. In latitude 13° 17′ N., the sun's correct central altitude was found to be 36° 37′, when his declination was 22° 10′ S.; what was the hour angle?

Ans. $2^h\ 42^m\ 52_s$.

8. In latitude 50° 56′ 17″ N., the zenith distance of a terrestrial object was found to be 90° 24′ 28″, and its azimuth 35° 47′ 4″ from the south; what were its polar distance and hour angle?

Ans. Its polar distance = 121° 6′ 43″
Its hour angle $= 2^h\ 52^m\ 18^s$.

9. From the preceding terrestrial object, three distances of the sun were found to be 78° 9′ 26″, 77° 39′ 26″, and 77° 29′ 26″, when his declination was 14° 7′ 13″ S.; what were the sun's hour angles, if he was on the opposite side of the meridian from the terrestrial object?

Ans. $2^h\ 45^m\ 49^s$, $2^h\ 43^m\ 26^s$, and $2^h\ 42^m\ 40^s$.

CHAPTER VIII.

LONGITUDE.

102. *Problem. To find the longitude of a place.*

First Method. By terrestrial measurement.

If the longitude of a place is known, that of another place, which is near it, can be found by measuring the bearing and distance; whence the difference of longitude may be calculated by the rules already given in Navigation.

Second Method. By signals.

The stars, by their diurnal motion, pass round the earth once in 24 sideral hours; hence they arrive at each meridian by a difference of sideral time equal to the difference of longitude. In the same way, the sun passes round the earth once in 24 solar hours; so that it arrives at each meridian by a difference of solar time equal to the difference of longitude. The difference of longitude of two places is, consequently, equal to their difference of time. Now if any signal, as the bursting of a rocket, is observed at two places; the instant of this event, as noticed by the clocks of the two places, gives their difference of time.

Third Method. By a chronometer.

The difference of time of two places can, obviously, be determined by carrying a chronometer, whose rate is well ascertained, from one place to the other; and if the chronometer did not change its rate during the passage, this method would be perfectly accurate.

Fourth Method. By an eclipse of one of Jupiter's satellites.
[B., p. 252.]

The signal of the second method cannot be used, when the places are more than 20 or 30 miles apart; and, when the distance is very great, a celestial signal must be used, such as the immersion or emersion of one of Jupiter's satellites. For this purpose, the instant, when any such event would happen to an observer at Greenwich, is inserted in the Nautical Almanac; and the observer at any other place has only to compare the time of his observation with that of the Almanac to obtain his longitude from Greenwich.

Fifth Method. By an eclipse of the moon. [B., p. 253.]

The beginning or ending of an eclipse of the moon may also be substituted for the signal of the second method to determine the difference of time.

Sixth Method. By a meridian transit of the moon. [B., p. 431.]

The motion of the moon is so rapid, that the instant of its arrival at a given place in the heavens may be used for the signal. Of the elements of its position its right ascension is changing most rapidly, and this element is easily determined at the instant of its passage over the meridian by the difference of time between its passage and that of a known star. The instant of Greenwich time, when the moon's right ascension is equal to the observed right ascension, might be determined from the right ascension, which is given in the Nautical Almanac for every hour. But this compution involves the observation of the solar time, whereas the observed interval gives at once the sideral time of the observation.

The calculation is then more simple, by means of the Table of Moon-Culminating stars given in the Nautical Almanac, in which the right ascensions of the suitable stars and of the moon's bright limb are given at the instant of their upper transits over the meridian of Greenwich, and also the right ascension of the moon's bright limb at the instant of its lower transit. Hence the difference between the right ascensions of the moon's limb, at two successive transits, is the change of its right ascension in passing from the meridian of Greenwich to that which is 12^h from Greenwich; so that if the motion in right ascension were perfectly uniform, the right ascension, which corresponded to a given meridian, or the meridian, which corresponded to a given right ascension, might be found by the following simple proportion,

12^h : long. of place = diff. of right ascensions for 12^h : diff. of right ascensions for long. of place, (569)

in which the longitude of the place may be counted from the meridian 12^h from that of Greenwich, provided the change of right ascension for an upper transit is computed from the preceding right ascension, which is that of a lower transit at Greenwich, that is, if the place is in east longitude.

Let then $T =$ long., if west,

or $= 12^h -$ long. (if the long. is east);

and let $A =$ diff of right ascension for the Greenwich transits, which immediately precede and follow the required or observed transit,

and let $\delta A =$ change of right ascension from the preceding Greenwich transit to the observed transit,

and we have, by (569),

$$12^h : T = A : \delta A, \tag{570}$$

whence

$$\delta A = \frac{AT}{12^h}, \text{ and } T = \frac{12^h \, \delta A}{A}, \tag{571}$$

and if T is reduced to seconds, we have

$$\delta A = \frac{AT}{43200} \tag{572}$$

$$\log. \delta A = \log. A + \log. T + (\text{ar. co.}) \log. 43200$$

$$= \log. A + \log. T + 5.36452 \tag{573}$$

and

$$T = \frac{43200 \, \delta A}{A} \tag{574}$$

$$\log. T = 4.63548 + (\text{ar. co.}) \log. A + \log. \delta A, \tag{575}$$

and formulas (573) and (575) agree with the parts of the rules in the Navigator, which depend upon A, and are independent of the want of uniformity in the moon's motion.

The corrections which arise from the change of the moon's motion may be calculated, on the supposition that this motion is uniformly increasing or decreasing so that the mean motion for any interval is equal to the motion which it has at the middle instant of that interval. If we put, then,

$$B = \text{the increase of motion in } 12^h, \tag{576}$$

A is not the mean daily motion for the interval of longitude T and the instant $\frac{1}{2}$ T after the meridian transit at Greenwich, but for the interval 12^h and the instant 6^h after this transit. The mean daily motion for the instant $\frac{1}{2}$ T is therefore,

$$A - \frac{(6^h - \frac{1}{2}\, T)\, B}{12^h}, \tag{577}$$

so that the correction for A is

$$- \frac{(6^h - \frac{1}{2}\, T)\, B}{12^h} = - \frac{(21600^s - \frac{1}{2}\, T)\, B}{43200}, \tag{578}$$

and the correction of $\delta\, A$ in (572) is

$$\delta\, B = - \frac{T\,(21600_s - \frac{1}{2}\, T)}{(43200^s)^2} B = - \frac{T\,(43200 - T)}{2\,(43200)^2}, \tag{579}$$

and the value of $\delta\, B$ is easily calculated and put into tables, like Table XLV of the Navigator.

In correcting the value of T (574), the correction of $\delta\, A$ is to be computed from Table XLV by means of the approximate value of T, and the correction of T is then found by the formula to be

$$\delta\, T = \frac{43200\; \delta\, B}{A}. \tag{580}$$

It only remains, to show how to find the value of B from the Nautical Almanac. Now if A' denotes the motion in right ascension for the 12^h interval of longitude, which precedes that to which A corresponds; and if A'' denotes the motion in right ascension for the 12^h interval of longitude which follows that of A; we have

$$2\, B = A'' - A'$$

$$B = \tfrac{1}{2}\,(A'' - A'), \tag{581}$$

and the calculation agrees entirely with that given in the Navigator.

When the longitude is small, or nearly 12^h, the correction for the variation of motion may be neglected, provided, instead of A, the motion is used which corresponds to the time of the nearest Greenwich transit. Now, in the Nautical Almanac, this motion is given for an hour's interval, of which the middle instant is that of the transit, so that if $H =$ this hourly motion, the motion for the time T may be found by the formula

$$1^h : T = H : \delta A,$$

whence

$$T = \frac{\delta A \times 1^h}{H} = \frac{2600^s \times \delta A}{H} \tag{582}$$

$$\log. T = 3.55630 + \log. \delta A + (\text{ar. co.}) \log. H, \tag{583}$$

which agrees with [B., p. 432].

The formula (583) may be rendered more correct, if the value of H is taken for the instant $\frac{1}{2}$ T of longitude; and the value can be computed precisely in the same way in which the right ascension was computed for the time T, by noticing the want of uniformity in its increase; and the formula thus corrected is accurate for small differnces of longitude.

Seventh Method. By a lunar distance.

The distance of the moon from the sun or a star may be used as the signal; but the true places of these bodies differ from their apparent places, as will be shown in succeeding chapters, so that the observed distance requires to be corrected; and the correction cannot be found without knowing the altitudes of the bodies. It is sufficient, far the present purpose, to know that the difference between the true and apparent places is only a difference of altitude, and not one of azimuth, and that the apparent place of the sun or a star is higher than its true place, while that of the moon is lower. The true distance may, then, be calculated from the observed distance by one of the following methods.

I. Let Z (fig. 45) be the zenith, S the apparent place of the sun or star, and S' the true place, M the apparent place of the moon, M' the true place; let

$$a = \text{the star's apparent alt.} = 90° - ZS$$

$$a' = \text{its true alt.} = 90° - ZS'$$

$$b = \text{the moon's app. alt.} = 90° - ZM$$

$$b' = \text{its true alt.} = 90° - ZM'$$

$$E = \text{the app. dist.} = SM$$
$$E' = \text{the true dist.} = S'M'$$
$$Z = \text{the angle } Z$$
$$\delta a = SS' = a - a'$$
$$\delta b = MM' = b' - b$$
$$\delta b = E' - E.$$

Then the triangles ZSM and $ZS'M'$ give, by (332),

$$2\,(\cos.\tfrac{1}{2}\,Z)^2 = \frac{\cos. E + \cos.(a+b)}{\cos. a \cos. b} = \frac{\cos. E' + \cos.(a'+b')}{\cos. a' \cos. b'}. \quad (584)$$

Let
$$\cos. m = \frac{\cos. a' \cos. b'}{2 \cos. a \cos. b}, \quad (585)$$

and we have, by (584),

$$\cos. E' + \cos.(a'+b') = 2 \cos. m \cos. E + 2 \cos. m \cos.(a+b)$$
$$= \cos.(E+m) + \cos.(E-m) + \cos.(a+b+m) + \cos.(a+b-m)$$
$$\cos. E' = -\cos.(a'+b') + \cos.(E+m) + \cos.(E-m)$$
$$+ \cos.(a+b+m) + \cos.(a+b-m), \quad (586)$$

whence E' can be found by a table of natural sines and cosines, when m has been found from (585).

II. In the same way by (338), we find

$$2\,(\sin.\tfrac{1}{2}\,Z)^2 = \frac{\cos.(a-b) - \cos. E}{\cos. a \cos. b} = \frac{\cos.(a'-b') - \cos. E'}{\cos. a \cos. b} \quad (587)$$

$$\cos.(a'-b') - \cos. E' = 2 \cos. m \cos.(a-b) - 2 \cos. m \cos. E$$
$$= \cos.(a-b+m) + \cos.(a-b-m) - \cos.(E+m) - \cos.(E-m)$$
$$\cos. E' = \cos.(a'-b') - \cos.(a-b+m) - \cos.(a-b-m)$$
$$+ \cos.(E+m) + \cos.(E-m). \quad (588)$$

III. The correction may be separated into two parts, one of which depends only upon the sun or star, and the other upon the moon; and let

$\delta' E =$ the part of δE which depends upon the sun or star,

$\delta'' E =$ the part which depends upon the moon.

Now if the correction were only to be made for the moon, SM would be decreased to SM', whence

$$SM' = E + \delta'' E,$$

and if we put

$$S = ZSM,\ M = ZMS,$$

$$s = \tfrac{1}{2}(a + b + E), \tag{589}$$

the triangles SMM', and SZM give

$$(\sin.\ \tfrac{1}{2} M)^2 = \frac{\cos.\ s \sin.\ (s - a)}{\sin.\ E \cos.\ b}$$

$$= \frac{\sin.\ [E + \tfrac{1}{2}(\delta'' E - \delta b)] \sin.\ \tfrac{1}{2}(\delta'' E + \delta b)}{\sin \delta b \sin.\ E}$$

$$= \frac{\delta'' E + \delta b}{2 \delta b}[1 + \tfrac{1}{2} \text{cotan.}\ E \sin.\ 1''. (\delta'' E - \delta b)] \tag{590}$$

$$60' + \delta'' E = (59' 42'' - \delta b) + \frac{\cos.\ s \sin.\ (s - a)}{\sin.\ E} \cdot \frac{2 \delta b}{\cos.\ b}$$

$$+ 18'' - \tfrac{1}{2} \text{cotan.}\ E \sin.\ 1'' [(\delta'' E)^2 - (\delta b)^2]. \tag{591}$$

The triangles $SS'M$ and SZM give, by (336) and (340),

$$(\cos.\ \tfrac{1}{2} S)^2 = \frac{\cos.\ (s - E) \sin.\ (s - a)}{\sin.\ E \cos.\ a}$$

$$= \frac{\sin.\ [E + \tfrac{1}{2}(\delta' E - \delta a)] \sin.\ \tfrac{1}{2}(\delta' E + \delta a)}{\sin.\ \delta a \sin.\ E}$$

$$= \frac{\delta a + \delta' E}{2 \delta a} \tag{592}$$

$$60' + \delta' E = (60' - \delta a) + \frac{\cos.\ (s - E) \sin.\ (s - a)}{\sin.\ E} \cdot \frac{2 \delta a}{\cos.\ a}. \tag{593}$$

If now $M'K$ and $S'L$ are drawn perpendicular to MS, and $S'L$ to $M'S$, we have nearly

$$S'M' = E + \delta E = SM' + SL' = E + \delta'' E + SL'$$

$$\delta E = \delta'' E + SL' = \delta'' E + \delta' E + (SL' - \delta' E) \tag{594}$$

$$\delta' E = SL = \delta a \cos.\ S \tag{595}$$

$$SL' = \delta a \cos. (S'SL') = \delta a \cos. (S - MSM')$$

$$= \delta a \cos. S + \delta a \sin. S \sin. MSM'$$

$$= \delta' E + S'L \sin. MSM' \quad (596)$$

$$SL' - \delta' E = S'L \sin. MSM'. \quad (597)$$

But from $M'SK$,

$$\sin. MSM' = \frac{\sin. M'K}{\sin. E} = \frac{MK \sin. 1''}{\sin. E} \quad (598)$$

whence

$$SL' - \delta' E = \frac{S'L \times M'K . \sin. 1''}{\sin. E} \quad (599)$$

and

$$\delta E = \delta' E + \delta'' E + \frac{S'L \times M'K . \sin. 1''}{\sin. E} \quad (600)$$

$$2^\circ + \delta E = (60' + \delta' E) + (60' + \delta'' E) + \frac{S'L \times M'K \sin. 1''}{\sin. E}, \quad (601)$$

in which 1° is added to $\delta' E$ and $\delta'' E$, in order to render them positive. Now, of $60' + \delta' E$ (593), the part $60' - \delta a$ is given in Table XVII or Table XVIII; and the remaining term is computed by proportional logarithms, and is the first correction of the First Method of the Navigator. [B., p. 231.] The proportional logarithm of the factor $2 \delta a \sec. a$, is the logarithm of the Table from which $60' - \delta a$ is taken.

In the same way, the two first terms of $60' + \delta'' E$ are taken from Table XIX and (591). The remainder of (591) combined with the third term of (601), is computed and inserted in Table XX of the Navigator.

In calculating Table XX, the value of $\delta'' E$ is used, which is obtained from the two first terms of (591); and $S'L$ and $M'K$ are found from $S'SL$ and MKM' in which the sides are so small that their curvature may be neglected, and we have, nearly,

$$S'L = \sqrt{(\delta a^2 - \delta' E^2)} \quad (602)$$

$$M'K = \sqrt{(\delta b^2 - \delta'' E^2)}. \quad (603)$$

IV. The calculation of the values of δa and δb will be fully explained in subsequent chapters; but we need only remark, in this place, that the value of $\delta\, a$, for a star, is given in Table XII; for the sun, it is the number of Table XII diminished by that of Table XIV; and for a planet, it is that of Table XII diminished by that of Table X, A. The value of $\delta\, b$ is obtained by the formula

$$\delta\, b = P \cos. b = \delta'\, b, \tag{604}$$

in which $\delta'\, b$ is the number of Table XII, and P is the number taken from the Nautical Almanac, and which is called the horizontal parallax. In computing Table XX, the value of P is taken at its mean of 57′ 30″.

In the formulas for the corrections, the zenith distances may be introduced instead of the altitudes, and if we put

$$90° - a = Z,\ 90° - b = z,$$
$$s_1 = \tfrac{1}{2}(z + Z + E), \tag{605}$$

we have, by neglecting the term depending upon the correction of Table XX, as well as the other small quantities,

$$\cos.^2 \tfrac{1}{2} M = \frac{\sin. s_1 \sin. (s_1 - Z)}{\sin. E \sin. z}$$

$$= \frac{\sin. [E + \frac{1}{2}(\delta''\, E + \delta\, b)] \sin. \frac{1}{2}(\delta\, b - \delta''\, E)}{\sin. E \sin. \delta\, b}$$

$$= \frac{\delta\, b - \delta''\, E}{2\, \delta\, b} \tag{606}$$

$$\delta''\, E = \delta\, b - \frac{2 \sin. s_1 \sin. (s_1 - Z)}{\sin. E \sin. z}\, \delta\, b \tag{607}$$

$$\cos.^2 \tfrac{1}{2} S = \frac{\sin. s_1 \sin. (s_1 - z)}{\sin. E \sin. Z} = \frac{\delta'\, E + \delta\, a}{2\, \delta\, a}$$

$$\delta'\, E = -\delta\, a + \frac{2 \sin. s_1 \sin. (s_1 - z)}{\sin. E \sin. Z}\, \delta\, a. \tag{608}$$

Then the second term of the value of $\delta'\, E$ is the first correction of the Third Method of the Navigator [B., p. 242], and the second term of the value of $\delta''\, E$ is the second correction of this method; and the computation from (604, 607, 608) agrees entirely with this method. The third correction is taken from Table XX, as in the first method.

V. Draw ZN perpendicular to MS, so as to make SN acute. In the right triangle ZSN and ZSM let

$$B = 90° - SN, \; B' = 90° + MN, \; A = \tfrac{1}{2}(B' + B), \tag{609}$$

and we have

$$E = MN + SN = B' - B, \tag{610}$$

and, by Bowditch's Rules for oblique triangles,

$$\cos. ZS : \cos. ZM = \cos. NS : \cos. MN,$$

or

$$\sin. a : \sin. b = \sin. B : \sin. B'; \tag{611}$$

and, by the theory of proportions,

$$\frac{\sin. a + \sin. b}{\sin. b - \sin. a} = \frac{\sin. B + \sin. B'}{\sin. B' - \sin. B}$$

that is,

$$\frac{\text{tang.} \tfrac{1}{2}(a + b)}{\text{tang.} \tfrac{1}{2}(b - a)} = \frac{\text{tang.} A}{\text{tang.} \tfrac{1}{2} E} \tag{612}$$

$$\text{tang.} A = \text{tang.} \tfrac{1}{2}(a + b) \, \text{cotan.} \tfrac{1}{2}(b - a) \, \text{tang.} \tfrac{1}{2} E \tag{613}$$

$$B' = A + \tfrac{1}{2} E, \; B = A - \tfrac{1}{2} E, \tag{614}$$

and the right triangles ZSN, MZN, SLS', MKM', give

$$\cos. S = \frac{\delta' E}{\delta a} = \text{cotan.} ZS \, \text{tang.} a \, \text{cotan.} B$$

$$-\cos. M = \frac{\delta'' E}{\delta b} = -\cot. ZM \, \text{tang.} MN = \text{tang.} b \, \text{cotan.} B'$$

$$\delta' E = \delta a \, \text{tang.} a \, \text{cotan.} B \tag{615}$$

$$\delta'' E = \delta b \, \text{tang.} b \, \text{cotan.} B', \tag{616}$$

and the formulas (613-616) correspond to the Fourth Method of the Navigator. [B., p. 243.]

It may be observed, that since cotan. $\tfrac{1}{2}(b - a)$ is the only term of (613) which can change its sign, A is acute when b is greater than a, and obtuse when b is less than a.

VI. The most important of corrections of the distance arise from that term of δb (604), which depends upon the parallax. If we consider this, therefore, as the only correction of the moon's altitude, we may calculate the corrections of the distance arising from it by putting

$$\delta\, b = MM' = P \cos.\, b. \qquad (617)$$

The triangles ZSM and $M'MK$, give then

$$\cos.\, M = -\frac{\delta''\, E}{P \cos.\, b} = \frac{\sin.\, a - \cos.\, E \sin.\, b}{\sin.\, E \cos.\, b} \qquad (618)$$

$$\delta''\, E = -P \sin.\, a \operatorname{cosec.} E + P \operatorname{cotan.} E \sin.\, b, \qquad (619)$$

and if we put

$$\delta_1\, E = P \sin.\, a \operatorname{cosec.} E \qquad (620)$$

$$\delta_2\, E = \pm P \operatorname{cotan.} E \sin.\, b, \qquad (621)$$

in which the signs are taken so that $\delta_2\, E$ is always positive, we have

$$\delta''\, E = -\delta_1\, E \pm \delta_2\, E$$

$$10^\circ + \delta''\, E = (5^\circ - \delta_1\, E) + (5^\circ \pm \delta_2\, E). \qquad (623)$$

Now Table XLVII is a common table of proportional logarithms, like Table XXII; but the angle which is placed at the top of the table is

$$5^\circ - \text{the angle of Table XXII}, \qquad (624)$$

and the angle at the bottom of the table is

$$5^\circ + \text{the angle of Table XXII}; \qquad (625)$$

so that the terms of (623) may be directly obtained from these tables; and this method of computing the corrections which depend upon the moon's parallax agrees with the second method of the Navigator. [B., p. 239.]

The remaining corrections may be computed from the formulas (607 and 608), and the corrections of Table XX may be neglected, provided the value of E is corrected for the parallax. These combined corrections may be inserted in a table like Table XLVIII, which serves for the star, and, by means of the part P, for the sun; or like Tables XLIX and L, which serve for the planets. In calculating those tables, the moon's horizontal parallax is taken at its mean value of 57′ 30″; and the planet's or sun's parallax in altitude is obtained from the formula

$$\delta'\, a = -P \cos.\, a,$$

in which P is the horizontal parallax. The value of P, used in the construction of the part P of Table XLVIII, is 8″.6; that used for Table XLIX is 35″; and since these corrections are proportional to the parallax, they are easily reduced to any other parallax. This reduction is actually made in Table L.

VII. The value of $\delta'' E$ (618), might be found by the formula

$$\delta'' E = -\frac{2 \sin. a - \sin. (b + E) - \sin. (b - E)}{2 \sin. E} P, \qquad (626)$$

which is easily calculated by means of the table of natural sines and cosines.

VIII. The true distance may be obtained from observation by either of the preceding methods, and the time of the observation must be compared with the time when the distance is the same to an observer at Greenwich. Now this latter time can be obtained from the Nautical Almanac by precisely the same process of interpolation which has been applied to the changes of right ascension. The distances are given in the Nautical Almanac for every three hours, and the proportional logarithm of the difference of these distances. If, then, the distance increases uniformly at the rate of increase, F, for every three hours; the interval T, at which it has increased by the quantity F', is found by the proportion

$$F : F' = 3^h : T \qquad (627)$$

$$\text{Prop. log. } T = \text{Prop. log. } F' - \text{Prop. log. } F + \text{Prop. log. } 3^h. \quad (628)$$

But

$$\text{Prop. log. } 3^h = 0; \qquad (629)$$

and if we put

$$\text{Prop. log. } F = Q, \qquad (630)$$

(628) becomes

$$\text{Prop. log. } T = \text{Prop. log. } F' - Q. \qquad (631)$$

If the distance increased uniformly, the value of Q would be invariable; but Q is variable, and must be regarded as belonging to the middle instant of the interval to which it belongs; and it increases while the distance decreases, and the reverse. Let then

$\delta Q =$ the decrease of Q in three hours,

$\delta T =$ the correction of T, arising from the change of Q, and the value of Q for the interval T is

$$Q + \frac{90^m - \frac{1}{2}T}{180^m}\,\delta\,Q = Q + \delta'\,Q, \qquad (632)$$

so that by (631) and (399)

$$\text{Prop. log. } (T + \delta\,T) = \text{Prop. log. } T - \delta'\,Q \qquad (633)$$

$$\log.\ (T + \delta\,T) = \log.\ T + \delta'\,Q \qquad (634)$$

$$\log.\ (T + \delta\,T) - \log.\ T = \log. \left(1 + \frac{\delta\,T}{T}\right) = \delta'\,Q. \qquad (635)$$

But if in (167) we substitute

$$\frac{\delta\,T}{T} = i \qquad (636)$$

we have, by (635),

$$\log.\ e = \left(1 + \frac{\delta\,T}{T}\right) = \frac{T\delta'\,Q}{\delta\,T}, \qquad (637)$$

so that by (632) and (164)

$$\delta\,T = \frac{T\,\delta'\,Q}{\log.\,e} = \frac{(180^m - T)\ T\,\delta\,Q}{2 \times 180^m. \times 0.434} \qquad (638)$$

$$= \frac{(180^m - T)\ T\,\delta\,Q}{156^m}; \qquad (639)$$

and the table [B., p. 245] for correcting by second differences may be calculated by this formula; and, in order to obtain the value of $\delta\,T$ expressed in seconds, the factor T should be expressed in seconds, while $(180^m - T)$ is expressed in minutes; and it must not be forgotten, that the proportional logarithms are decimals.

IX. When the distance is observed for a star whose distance is not given in the Nautical Almanac, the Greenwich time of the observation can be found approximately by adding the assumed longitude, if west, to the observed time, or subtracting it, if east; or the time can be taken from the chronometer, if it is regulated to Greenwich time.

Find, in the Nautical Almanac, the right ascension and declination of the star, and the declination of the moon, for this time. Then, if T and S (fig. 40) are supposed to be the moon and star, and P the pole of the equator, D and D' their declinations, disregarding their names, so that their polar distances are $90° \pm D$ and $90° \pm D'$, and if R' is their difference of right ascension, we have, when their declinations are of the same name, by putting

$$S = \tfrac{1}{2}(D + D' + E), \tag{640}$$

$$\cos.\tfrac{1}{2}R' = \cos.\tfrac{1}{2}SPT = \sqrt{\left(\frac{\cos. S\cos.(S - E)}{\cos. D\cos. D'}\right)}. \tag{641}$$

But if the declinations are of the same name,

$$\sin.\tfrac{1}{2}R'\sin.\tfrac{1}{2}SPT = \sqrt{\left(\frac{\sin. S\sin.(S - E)}{\cos. D\cos. D'}\right)}, \tag{642}$$

and the right ascension of the moon being thus found, the Greenwich time, when it has this right ascension, is easily found from the moon's hourly ephemeris in the Nautical Almanac, and this method is the same with that in [B., p. 428].

X. The latitudes and longitudes may be used instead of the right ascensions and declinations, and the calculation will be as in [B., p. 427]. The variation of daily motion is, in this case, to be regarded precisely as explained in (606 - 611).

XI. The distances of the Nautical Almanac can be calculated from the right ascensions and declinations of the sun, moon, and stars, or their latitudes and longitudes, by resolving the triangles TPS (fig. 40) by either of the methods which have been given, when two sides and the included angle are known, as in [B., p. 434].

In calculating the distance of the sun and moon, the latitude of the sun may be usually neglected; so that if SR (fig. 46) is an arc of the ecliptic, S the sun's place, M the moon's, and MR perpendicular to SR,

$$MR = L = \text{the moon's latitude,}$$

$$SR = L_1 = \text{the diff. of long. of ☉ and ☽,}$$

and

$$\cos. E = \cos. SM = \cos. L\cos. L_1, \tag{643}$$

as in [B., p. 433].

It would, however, be rather more accurate to take

$$L = \text{the diff. of lat. of ☉ and ☽.}$$

XII. The determination of the longitude by solar eclipses and occultations, will be reserved for another chapter.

103. Examples.

1. Calculate the correction of Table XLV, when

$$T = 1^h\ 50^m, \text{ and } B = 9^m = 540^s.$$

Solution.	$1^h\ 50^m$ P. L. $1^m\ 50^s$	ar. co.	8.0080
$12^h - 1^h\ 50^m = 10^h\ 10^m$	P. L. $10^m\ 10^s$	ar. co.	8.7519
	2 P. *L.* 12^m		2.3522
$\frac{1}{2}B = 270^s$			2.4314
corr. $= 34^s.9$			1.5435

2. Calculate the correction of Table XLV, when

$$T = 3^h\ 10^m, \text{ and } B = 11^m.$$

Ans. $64^s.1$.

3. Find the right ascension of the moon's bright limb, Sept. 25, 1830, at the time of the transit over the meridian of New York. The right ascension of the moon for the two preceding and the two following transits at Greenwich are

Sept. 25.	Moon II. L. T.	2^h	0^m	$36^s.69$
	Moon II. U. T.	2	30	38 .08
Sept. 26.	Moon II. L. T.	3	1	33 .18
	Moon II. U. T.	3	33	19 .89
The Longitude of New York is		4	56	4 .5

Ans. $2^h\ 43^m\ 14^s.4$.

4. At a place in west longitude, Oct. 25, 1839, the moon's bright limb passed the meridian $10^m\ 6^s.83$ sidereal time before the star C. Tauri; find the longitude of the place of observation.

The right ascension of the star C. Tauri was $5^h\ 43^m\ 16^s.84$, and those of the moon

Oct. 25.	Moon II. L. T.	$4^h\ 43^m\ 53^s.55$
	Moon II. U. T.	5 18 28 .40
Oct. 26.	Moon II. L. T.	5 52 51 .91
	Moon II. U. T.	6 26 40 .00

Ans. 76° 53′ 33″ W.

5. Find the moon's parallax in altitude, and the correction and logarithm of Table XIX, when the altitude is 40° 40′, and the horizontal parallax is 58′.

Solution.	58′	P. L. 0.4918
	40° 40′	sec. 0.1200
Parallax in alt. =	44′	P. L. 06118
By Table XII. Refrac. =	1′ 6″	9.6990
Corr. = 16′ 48″ * = 59′ 42″ — 42′ 54″		P. L. 0.6228
	Log. of Table XIX =	0.2018

6. Find the correction and logarithm of Table XVII for a star, when the altitude is 13° 15′.

Ans. Corr. = 56′ 2″, Log. = 1.3433.

7. Find the correction and logarithm of Table XVII for Venus or Mars, when the parallax is 20″, and the altitude 24° 30′.

Ans. Corr. = 58′ 14″, Log. = 1.6647.

8. Find the correction and logarithm of Table XVIII, when the altitude is 56°.

Ans. Corr. = 59′ 26″, Log. = 1.9544.

9. Find the correction and logarithm of Table XIX, when the altitude is 70°, and the horizontal parallax 54′.

Ans. Corr. = 41′ 34″, Log. = 0.2299.

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* The numbers of Table XIX are so disposed in the Navigator, that the corrections of proportional parts of parallax are all additive. This is effected by placing each number opposite that parallax, which is 10′ less than the one to which it belongs. There is, therefore, a correction for 0″ of parallax.

10. Compute the value of the auxiliary angle m, in the first and second methods of correcting the lunar distance, when the moon's apparent altitude is 40° 40′, its horizontal parallax 58′, and the sun's apparent altitude 70°.

Solution. The values of m might be computed directly from (585), but it is more convenient to obtain it by some process of approximation. For this purpose let

$$m = 60° + \delta m.$$

and we have

$$2 \cos.\ (60° + \delta m) = \frac{\cos.\ (b + \delta b)}{\cos.\ b} \frac{\cos.\ (a - \delta a)}{\cos.\ a}$$

$$= 2 \cos.\ 60° \cos.\ \delta m - 2 \sin.\ 60° \sin.\ \delta m \qquad (644)$$

$$= (\cos.\ \delta b - \text{tang.}\ b \sin.\ \delta b)(\cos.\ \delta a + \text{tang.}\ a \sin.\ \delta a),$$

in which we may put

$$2 \cos.\ 60° = 1,\ \cos.\ \delta b = 1 - 2 \sin.^2 \tfrac{1}{2} \delta b = 1 - \tfrac{1}{2} \delta b^2 \sin.^2 1''$$

$$\cos.\ \delta m = 1 - \tfrac{1}{2} \delta m^2 \sin.^2 1'',$$

and (644) becomes

$$2 \delta m \sin.\ 60° = \delta b\ \text{tang.}\ b\ . - \delta a\ \text{tang.}\ a \qquad (645)$$

$$+ \tfrac{1}{2} (\delta b^2 - \delta m^2) \sin.\ 1''.$$

But if we take

$$e = 2 \delta b \sec.\ b, \text{ and } e = 2 \delta a \sec.\ a,$$

Prop. log. e is the logarithm of Table XIX, and Prop. log. e' is the corresponding logarithm for the sun, star, or planet; and by (645),

$$\delta m = \tfrac{1}{4} e \sin.\ b\ \text{cosec.}\ 60° - \tfrac{1}{4} e' \sin.\ a\ \text{cosec.}\ 60° \qquad (646)$$

$$+ \tfrac{1}{2} (\delta b^2 - \delta m^2) \sin.\ 1''\ \text{cotan.}\ 60°,$$

whence in the present case

e	P. L. 0.2018	e'	P. L. 2.0173
40° 40′	cosec. 0.1860	70°	cosec. 0.0270
60°	sin. 9.9375		9.9375
1° 25′ 7″	0.3253	1′ 53″	1.9818

approx. $\delta m = \frac{1}{2}(1°25'7''-1'53'') = \frac{1}{2}(1°23'14'') = 20'48'' = 1248''$

$\delta b = 42'54'' = 2574''$

$\delta b + \delta m = 3822$	3.5823
$\delta b - \delta m = 1326$	3.1225
$1''$	sin. 4.6856
$60°$	cotan. 9.7614
corr. $\delta m = 7'' = \frac{1}{2}(14'')$	1.1518

$\delta m = 20'48'' + 7'' = 20'55''$.

11. Compute the value of the auxiliary angle m, when the moon's apparent altitude is 25° 30′, the horizontal parallax 60′, and the star's apparent altitude 10°.

Ans. 60° 14′ 3″.

12. Find the correction of Table XX, when the distance is 25°, the sun's altitude 10°, and the moon's altitude 25°.

Solution. We should find, in this case,

$\delta b = 50'\ 6''$ $\qquad \delta a = 5'\ 6''$

$\delta'' E = -27'22''$ $\qquad \delta' E = -3'15''$

$\delta b - \delta'' E = 1°17'28'' = 4648''$ $\qquad \delta a - \delta' E = 8'21'' = 501''$

$\delta b + \delta'' E = 22'44''$ $\qquad \delta a + \delta' E = 1'51'' = 111''$

$22'44'' =$	P. L. 0.8986		0.899
$1°17'28'' = 4648''$	(ar. co.) 6.3327	P. L.	0.366
$25°$	tang. 9.6687	2 sin.	9.252
$1''$	cosec. 5.3144	$1''$ 2 cosec.	0.629
$1'6'' = 66''$	2.2144	$501''$ (ar. co.)	7.300
$\frac{1}{2}\ 66'') = 33''$		$111''$ (ar. co.)	7.955
			2)6.401
$24'' =$	$18'' + 6''$		3.200

$57''$ = corr. Table XX.

13. Calculate the correction of Table XX, when the distance is 120°, the sun's altitude 20°, and the moon's altitude 10°.

Ans. 10″.

14. Calculate the corrections of Tables XLVIII, XLIX, and L, when the apparent distance is 28°, the moon's apparent altitude 38°, the planet's apparent altitude 18°, and its horizontal parallax 16″.

Solution.

57′ 30″	P. L. 0.4956		0.4956
18°	cosec. 0.5100	38°	cosec. 0.2107
28°	sin. 9.6716		tang. 9.7257
5° — 1st. cor. = 4° 22′ 9″	0.6772	5° + 2d cor. = 6° 6′ 34″	0.4320

6° 6′ 34″ moon's par. in alt. = 45′

28° moon's approx. alt. = 38° 45′

28° 29′ = approx. dist.

18°		45′ + 29′ = 74′ = 4440″ ar. co.	6.3526
38° 45′		45′ — 29′ = 16′ P. L.	1.0512
28° 22′ = ½ sum	tang. 9.73235	28° tang.	9.7257
10° 22′ = ½ diff.	cotan. 0.73771	1″ cosec.	5.3144
½ (28°) = 14°	tang. 9.39677	2)39″	2.4439
A = 36° 21′	tang. 9.86683	20″	
1st ang. = 22° 21′	tang. 9.6140		9.614
18°	cotan. 0.4882		0.488
By Table XII 2′ 54″	P. L. 1.7929	Table X, A. 33″ P. L.	2.515
2′ 18″	1.8951	25″ = cor. Table XIX	2.617
2d ang. = 50° 21′	tang. 0.0816	$\frac{16}{35}$ × 25″ = 11″ = cor. Table L.	
38°	cotan. 0.1072		
By Table XII 1′ 13″	P. L. 2.1701		
47″	2.3589		

Cor. Table XLVIII = 2′ 18″ — 47″ + 20″ = 1′ 51″.

15. Calculate the corrections of Tables XLVIII, XLIX, and L, when the apparent distance is 60°, the moon's apparent altitude 50°, the planet's apparent altitude 30°, and its horizontal parallax 30″.

Ans. Cor. Table XLVIII = 1′ 25″
XLIX = — 21″
L = — 18″.

16. Find the correction of the Table [B., p. 245] for the interval of $2^h\ 30^m$, and the difference of the Proportional Logarithms equal to 88.

Ans. 15ˢ.

17. If the observed distance were 45° 34′ 10″, the moon's apparent altitude 22° 19′, its horizontal parallax 60′ 19″, the planet's apparent altitude 42° 12′, its horizontal parallax 15″,3; what is the true distance?

Solution. I. In this case $m = 60°\ 12'\ 28''$

$a = 42°\ 12'$ $\quad \delta a = 51''$ $\quad a' = 42°\ 11'\ 9''$

$b = 22°\ 19'$ $\quad \delta b = 53'\ 31''$ $\quad b' = 23°\ 12'\ 31''$

$a' + b' =$	65° 23′ 40″	— N. cos. =	— 0.41637	$E = 45°\ 34'\ 10''$
$E + m =$	105° 46′ 38″	N. cos. =	— 0.27189	
$a + b + m =$	124° 43′ 28″	N. cos. =	— 0.56963	
			— 1.25789	
$E - m =$	— 14° 38′ 18″	N. cos. =	0.96754	
$a + b - m =$	4° 18′ 32″	N. cos. =	0.99717	
$E' =$	45° 1′ 24″	N. cos. =	0.70682	

II.	$a - b + m =$	80° 5′ 28″	— N. cos. =	— 0.17208
	$a - b - m =$	— 40 19 28	— N. cos. =	— 0.76239
	$E + m =$	105 46 38	N. cos. =	— 0.27189
				— 1.20636
	$a - b' =$	18° 58′ 38″	N. cos. =	0.94565
	$E - m =$	— 14 38 18	N. cos. =	0.96754
	$E' =$	45 1 21	N. cos. =	0.70683

III. $s = \frac{1}{2}(a + b + E) = 55° 2' 35''$ sec. 0.2419

$E = 45° 34' 10''$ sin. 9.8538 | 9.8538

$s - a = 12\ 50\ 35$ cosec. 0.6531 | 0.6531

$s - E = 9\ 28\ 25$ sec. 0.0060 6′ 11″. T. XIX. 0.1920

59′ 8″. Table XVII | 1.8907 20′ 38″ | P. L. 0.9408

43″. | P. L. 2.4036 | 32″ Table XX.

59′ 51″ | 27′ 21″

$E' = 45° 34' 10'' + 59' 51'' + 27' 21'' - 2° = 45° 1' 22''.$

IV. $Z = 47° 48'$ | $z = 67° 41'$

$s_1 = 80° 31' 35''$ cosec. 0.0060

$E = 45° 34' 10''$ sin. 9.8538

9.6990

9.5588 | 9.5588

$Z = 47° 48'$ sin. 9.8697 | $z = 67° 41'$ sin. 9.9662

$s_1 - z = 12° 50' 35''$ cosec. 0.6531

$s_1 - Z = 32° 43' 35''$ | cosec. 0.2671

$\delta a =$ 51″ P. L. 2.3259 | $\delta b = 53' 31''$ P. L. 0.5268

1st cor. = 42″ P. L. 2.4075 | 2d cor. 1° 26′ 22″ 0.3189

$\delta b = 53' 31''$ | $\delta a =$ 51″

$E' = 54' 13'' + 45° 34' 10'' + 31'' - 18'' - 1° 27' 13'' = 45° 1' 23''.$

V. $\frac{1}{2}(a + b) = 32° 15' 30''$ tang. 9.80014

$\frac{1}{2}(a - b) = 9\ 56\ 30$ cotan. 0.75627

$\frac{1}{2}E = 22\ 47\ 5$ tang. 9.62330

$A = 123\ 28\ 14$ tang. 0.17971

1st ang. = 100° 41′ 9″ tang. 0.7242

2d ang. = 146° 15′ 19″ | tang. 9.8248

$a = 42° 12'$ cotan. 0.0425 | $b = 22° 19'$ cotan. 0.3867

$\delta a =$ 51″ P. L. 2.3259 | $\delta b = 53' 31''$ P. L. 0.5268

1st cor. = − 9″ P. L. 3.0926

2d cor. = − 32′ 53″ | P. L. 0.7383

$E' = 45° 34' 10'' - 9'' - 32' 53'' + 31'' - 18'' = 45° 1' 21''.$

VI.	60′ 19″ P. L. 0.4748			0.4748
$a = 42° 12'$	cosec. 0.1728	$b = 22° 19'$		cosec. 0.4205
$E = 45° 34' 10''$	sin. 9.8538			tang. 0.0086
1st cor. = 4° 3′ 16″	0.5014	2d cor. = 5° 22′ 28″		0.9039

Cor. Table XLVIII, XLIX, and L = 1′ 31″

$E' = 45° 34' 10'' + 4° 3' 16'' + 5° 22' 28'' + 1' 31'' - 10° = 45° 1' 25''.$

VII.

$a =$	42° 12′	N. sin.	0.67172	
$b + E =$	67° 53′ 10″,	½ N. sin.	— 0.46322	60′ 19″ P. L. 0.4748
$b - E =$	— 23° 15′ 10″,	½ N. sin.	0.19739	
			0.40589	ar. co. 0.3916
$E =$	45° 34′ 10″,			sin. 9.8538
Cor. Table XLVIII, &c. = 1′ 31″			cor. = — 34′ 17″	0.7202

$E' = 45° 34' 10'' + 1' 31'' - 34' 17'' = 45° 1' 24''.$

18. The apparent distance of the sun and moon is 70° 50′ 33″, the moon's apparent altitude is 35° 45′ 4″, its horizontal parallax is 54′ 24″, the sun's apparent altitude is 70° 48′ 1″; what is the true distance?

In this example $m = 60° 17' 28''.$

Ans. 70° 8′ 47″.

19. The apparent distance of a star from the moon is 31° 13′ 26″, the moon's apparent altitude is 8° 26′ 13″, its horizontal parallax is 60′, the star's apparent altitude is 35° 40′; what is the true distance?

In this example $m = 60° 4' 16''.$

Ans. 30° 24′ 48″.

20. Find the Greenwich time, Oct. 3, 1839, when the moon's distance from the sun was 38° 12′ 9″.

Solution.

Distance 1839, Oct. 3, 15^h	38° 59′ 21″	P. L. 0.3180
	38 12 9	
18^h P. L. 3189	47′ 12″	P. L. 0.5813
3180 $T = 1^h\ 33^m\ 16^s$		P. L. 0.2633
9 cor. T. =	-3^s	
Greenwich time = $16^h\ 33^m\ 7^s$.		

21. Find the Greenwich time, Jan. 2, 1839, when the moon's distance from Aldebaran was 70° 45′ 13″.

1839. Jan. 2, 9^h Greenwich time,	Dist. = 69° 26′ 29″
	P. L. = 0.2852
12^h	P. L. = 0.2863
	Ans. $11^h\ 31^m\ 47^s$.

22. The correct distance of the moon from β Corvi, 1839, April 3d, $11^h\ 20^m$, in longitude 70° W. by account, was 54° 8′ 15″; what was the longitude?

Solution.	54° 8′ 15″	Gr. T. = $11^h\ 20^m + 4^h\ 40^m = 16^h$
☽'s Dec. =	26 48 52	by N. A. sec. 0.04940
✱'s Dec. =	22 30 11	sec. 0.03439
½ sum =	51° 43′ 39″	cos. 9.79198
Dist. ∼ ½ sum =	2 24 36	cos. 9.99961
		2)19.87538
	$3^h\ 59^m\ 43^s$	cos. 9.93769
✱'s R. A. =	12 25 56	

☽'s R. A. = $16^h\ 25^m\ 37^s$ at Greenw. time = 16^h

Long. = $16^h - 14^h\ 20^m = 4^h\ 40^m$ = 70°, as supposed.

23. The correct distance of the moon from Castor, 1839, Nov.

29^d 19^h, in longitude 45° W. by account, was 78° 3′; what was the longitude?

Greenwich, 1839,

Nov. 29^d 21^h,	☽'s R. A. = 12^h 15^m $16^s.5$,	Dec. = 3° 48′ 31″ S.
22^h,	☽'s R. A. = 12 17 2 .9,	Dec. = 4 2 39 S.
Castor's	R. A. = 7 24 24 .4,	Dec. = 32 14 2 N.

Ans. 44° 18′ W.

24. Find the distance of the moon from the sun, 1839, August 12^d, Greenwich time at mean noon.

☉'s R. A. = 9^h 25^m $51^s.72$, Dec. = 15° 7′ 51″.5 N.

☽'s R. A. = 11 42 23 .48, Dec. = 0 57 27 .9 N.

Ans. 36° 33′ 14″.

25. Find the distance of the moon from the sun, 1839, August 14^d, Greenwich time at mean noon.

☉'s R. A. = 9^h 33^m $24^s.57$, Dec. = 14° 31′ 28″.2 N.

☽'s R. A. = 13 8 27 .62, Dec. = 10 25 54 .5 S.

Ans. 58° 50′ 38″.

CHAPTER IX.

ABERRATION.

104. The apparent position of the stars is affected by two sources of optical deception, so that they are not in the direction in which they appear to be.

The first of these sources is the motion of the earth, and the corresponding correction is called *aberration.*

Aberration, like the earth's motion, is either *annual* or *diurnal.*

105. *Problem. To find the aberration of a star.*

Solution. The apparent direction of a star is obviously that of the telescope, through which the star is seen. Let S (fig. 47) be the star, and O the place of the observer at the instant of the observation; SO is the true direction of the star, or the path of the particle of light which proceeded from the star to the observer; and it would be the direction of the telescope if the observer were stationary. But if he is moving in the direction OP, the direction of the telescope OT must be such, that the end T was at the point R, in the line OS, at the same instant in which the particle of light was at this point. The length RT is, therefore, the distance gone by the observer while the light is describing the line OR.

If, then, we put

$V =$ the velocity of light,

$v =$ the earth's velocity,

$I = TOP = RTO,$

$\delta I = - ROT =$ the aberration from the true place,

$$m = \frac{v}{V \sin. 1''}, \qquad (647)$$

we have,

$$V : v = OR : TR = \sin. I : -\delta I \sin. 1''$$

$$\delta I = -m \sin. I. \qquad (648)$$

106. *Problem. To find the annual aberration in latitude and longitude.*

Solution. The earth is moving in the plane of the ecliptic at nearly right angles to the direction of the sun. Hence if TP (fig. 48) is the ecliptic, T the point toward which the earth is moving, S the true star, S' the apparent star,

$\odot$ = the sun's longitude,

$\mathcal{A}$ = the star's longitude, $\delta \mathcal{A}$ = the aberration in long.

L = the star's latitude, δL = the aberration in lat.

we have

$$ST = I, SP = L,$$

$$\text{long. of } T = \odot - 90°, PT = \odot - 90° - \mathcal{A} = \mathcal{A}_1$$

$$PP' = \delta \mathcal{A} = TP - TP', \delta L = S'P' - SP$$

$$\cos. T = \text{cotan.} I \text{ tang.} \mathcal{A}_1 = \text{cotan.} (I + \delta I) \text{ tang.} (\mathcal{A}_1 - \delta \mathcal{A}),$$

whence

$$\frac{\text{tang.} (\mathcal{A}_1 - \delta \mathcal{A})}{\text{tang.} \mathcal{A}_1} = \frac{\text{tang.} (I + \delta I)}{\text{tang.} I}, \qquad (649)$$

and, by (346 and 347),

$$\frac{\sin. \delta \mathcal{A}}{\sin. (2 \mathcal{A}_1 - \delta \mathcal{A})} = - \frac{\sin. \delta I}{\sin. (2 I + \delta I)}, \qquad (650)$$

or omitting $\delta \mathcal{A}$ and δI in the denominators, and reducing by means of (648),

$$\delta \mathcal{A} = - \frac{\sin. 2 \mathcal{A}_1}{\sin. 2 I} \delta I = - \frac{\sin. \mathcal{A}_1 \cos. \mathcal{A}_1}{\sin. I \cos. I.} \delta I$$

$$= m \frac{\sin. \mathcal{A}_1 \cos. \mathcal{A}_1}{\cos. I}. \qquad (651)$$

But

$$\cos. I = \cos. \mathcal{A}_1 \cos. L, \qquad (652)$$

whence

$$\delta \mathcal{A} = m \sin. \mathcal{A}_1 \sec. L \qquad (653)$$

$$= - m \cos. (\odot - \mathcal{A}) \sec. L.$$

We also have

$$\sin. T = \frac{\sin. L}{\sin. I} = \frac{\sin. (L + \delta L)}{\sin. (I + \delta I)}, \tag{654}$$

whence

$$\sin. L \sin. (I + \delta I) = \sin. I \sin. (L + \delta L,) \tag{655}$$

and

$$\delta L = \frac{\sin. L \cos. I}{\cos. L \sin. I} \delta I$$

$$= - m \text{ tang. } L \cos. I$$

$$= - m \cos. A_1 \sin. L$$

$$= - m \sin. (\odot - A) \sin. L. \tag{656}$$

107. *Problem. To find the annual aberration in distance and direction from the vernal equinox.*

Solution. Let A (fig. 48) be the vernal equinox, and let

$$M = SA, \quad \delta M = \text{aberration of } M$$
$$N = SAT, \delta N = \text{aberration of } N.$$

Now we have

$$\delta M = \delta I \cos. AST = \frac{\sin. \odot - \cos. M \cos. I}{\sin. M \sin. I} \delta I$$

$$= - m \frac{\sin. \odot - \cos. M \cos. I}{\sin. M}. \tag{657}$$

But

$$\cos. I = \sin. \odot \cos. M - \cos. \odot \sin. M \cos. N, \tag{658}$$

whence if we put

$$B = - m \sin. \odot \tag{659}$$
$$C = - m \cos. \odot, \tag{660}$$

we have

$$\delta M = B \sin. M + C \cos. M \cos. N.$$

Again; the triangles ASS' and ATS' give, by (302),

$$\sin. AST = \frac{\sin. M \,.\, \delta N}{\delta I} = - \frac{\cos. \odot \sin. N}{\sin. I} \tag{661}$$

$$\delta N = m \cos. \odot \frac{\sin. N}{\sin. M} = - \frac{C \sin. N}{\sin. M}. \tag{662}$$

108. *Problem. To find the annual aberration in right ascension and declination.*

Solution. If AT (fig. 48) were the equator, we should have

$$D = SP, R = AP,$$

and if we put

$$N_1 = SAP, \omega = \text{obliquity of ecliptic},$$

we have

$$N_1 = N + \omega,$$

and the triangles ASP, $AS'P'$ give

$$\sin. D = \sin. M \sin. N_1 \quad (663)$$

$$\sin. (D - \delta D) = \sin. (M - \delta M) \sin. (N_1 - \delta N) \quad (664)$$

$$\cos. D\, \delta D = \sin. M \cos. N_1\, \delta N + \cos. M \sin. N_1\, \delta M \quad (665)$$

$$= B \sin. M \cos. M \sin. N_1$$

$$- C (\sin. N \cos. N_1 - \cos.^2 M \sin. N_1 \cos. N),$$

and if we put

$$A = C \cos. \omega \quad (666)$$

$$b' = \sin. M \cos. M \sin. N_1 \sec. D \quad (667)$$

$$a' = -(\sin. N \cos. N_1 - \cos.^2 M \sin. N_1 \cos. N) \sec. D \sec. \omega, \quad (668)$$

we have

$$\cos. M = \cos. D \cos. R \quad (669)$$

$$\text{cotan. } N_1 = \sin. R \text{ cotan. } D \quad (670)$$

$$\sin. M \cos. N_1 = \frac{\sin. D \cos. N_1}{\sin. N_1} = \sin. D \text{ cotan. } N_1 \quad (671)$$

$$= \sin. D \sin. R \text{ cotan. } D = \sin. R \cos. D$$

$$b' = \sin. D \cos. R \cos. D \sec. D = \sin. D \cos. R \quad (672)$$

$$a' = -[\sin. (N - N_1) + \sin.^2 M \sin. N_1 \cos. N] \sec. D \sec. \omega$$

$$= [\sin. \omega - \sin.^2 M \sin.^2 N_1 \sin. \omega] \sec. D \sec. \omega$$

$$- \sin.^2 M \sin. N_1 \cos. N_1 \cos. \omega \sec. D \sec. \omega$$

$$= (1 - \sin.^2 D) \sin. \omega \sec. D \sec. \omega - \sin. M \sin. D \cos. N_1 \sec. D$$

$$= \cos. D \tan. \omega - \sin. R \sin. D \quad (673)$$

$$\delta D = A a' + B b'. \quad (674)$$

Again, we have

$$\cos. M = \cos. R \cos. D \qquad (675)$$

$$\cos. (M + \delta M) = \cos. (R + \delta R) \cos. (D + \delta D)$$

$$\cos. D \sin. R \delta R = \sin. M \delta M - \cos. R \sin. D \delta D$$

$$= B (\sin.^2 M - b' \cos. R \sin. D)$$

$$+ A (\sin. M \cos. M \cos. N \sec. \omega - a' \cos. R \sin. D), \qquad (676)$$

and if we put

$$a = (\sin. M \cos. M \cos. N \sec. \omega - a' \cos. R \sin. D) \sec. D \operatorname{cosec.} R$$

$$b = (\sin.^2 M - b' \cos. R \sin. D) \sec. D \operatorname{cosec.} R,$$

we have

$$a \cos. D \sin. R = \sin. M \cos. M \cos. N_1 + \sin. R \cos. R \sin.^2 D$$

$$+ (\sin. M \cos. M \sin N_1 - \cos. R \sin. D \cos. D) \tan. \omega$$

$$= \sin. R \cos. R (\cos.^2 D + \sin.^2 D)$$

$$+ (\sin. M \sin. N_1 \cos. R \cos. D - \cos. R \sin. M \sin. N_1 \cos. D) \tan. \omega$$

$$= \sin. R \cos. R$$

$$a = \cos. R \sec. D \qquad (677)$$

$$b \cos. D \sin. R = 1 - \cos.^2 M - \sin.^2 D \cos.^2 R$$

$$= 1 - \cos.^2 D \cos.^2 R - \sin.^2 D \cos.^2 R$$

$$= 1 - \cos.^2 R = \sin.^2 R$$

$$b = \sin. R \sec. D \qquad (678)$$

$$\delta R = A a + B b, \qquad (679)$$

and formulas (659, 660, 672, 673, 674, 677, 678, 679) agree with those given in the Nautical Almanac for finding the annual aberration.

109. *Corollary.* The value of m, which is used in the Nautical Almanac, is

$$m = 20''.3600,$$

which gives

$$m \cos. \omega = 20''.3600 \cos. 23° \, 27' \, 36''.98 = 18''.6768.$$

110. *Scholium.* In the values of the aberration in right ascension and declination, each term consists of two factors, one of which is the

same each instant for all the stars, and the other is the same for each star, during several years.

111. *Corollary.* If in (674) and (679) we put

$$i = A \tan.\ \omega \qquad (680)$$

$$B = h \cos.\ H \qquad (681)$$

$$A = h \sin.\ H; \qquad (682)$$

they become

$$\delta\, D = i \cos.\ D - h \sin.\ H \sin.\ R \sin.\ D + h \cos.\ H \cos.\ R \sin.\ D$$

$$= i \cos.\ D + h \cos.\ (H + R) \sin.\ D \qquad (683)$$

$$\delta\, R = h \sin.\ H \cos.\ R \sec.\ D + h \cos.\ H \sin.\ R \sec.\ D$$

$$= h \sin.\ (H + R) \sec.\ D, \qquad (684)$$

which agree with the formulas in the Nautical Almanac.

112. We have from (659 - 679)

$$\delta\, R = \sec.\ D\, [- m \cos.\ \omega \cos.\ \odot \cos.\ R - m \sin.\ \odot \sin.\ R] \quad (685)$$

$$= \sec.\ D\, [- \tfrac{1}{2} m\, (\cos.\ \omega + 1)\, (\cos.\ \odot \cos.\ R + \sin.\ \odot \sin.\ R)$$

$$+ \tfrac{1}{2} m\, (1 - \cos.\ \omega)\, (\cos.\ \odot \cos.\ R - \sin.\ \odot \sin.\ R)]$$

$$= \sec.\ D\, [- m \cos.^2 \tfrac{1}{2} \omega \cos.\ (R - \odot) + m \sin.^2 \tfrac{1}{2} \omega \cos.\ (R + \odot)],$$

and if we put

$$Q = R - \odot,\ Q' = R + \odot \qquad (686)$$

$$n = - m \cos.^2 \tfrac{1}{2}\, \omega,\ n' = m \sin.^2 \tfrac{1}{2}\, \omega, \qquad (687)$$

(645) becomes

$$\delta\, R = \sec.\ D\, (n \cos.\ Q + n' \cos.\ Q'), \qquad (688)$$

and the values of n cos Q and n' cos. Q' may be put in tables like Parts I and II of Table XLII of the Navigator.

Again, we have

$$\delta\, D = \sin.\ D\, (m \cos.\ \omega \sin.\ R \cos.\ \odot - m \cos.\ R \sin.\ \odot)$$

$$- m \sin.\ \omega \cos.\ \odot \cos.\ D$$

$$= \sin. D\,[\tfrac{1}{2} m\,(\cos. \omega + 1) \sin. Q - \tfrac{1}{2} m\,(1 - \cos. \omega) \sin. Q']$$
$$- \tfrac{1}{2} m \sin. \omega\,[\cos. (\odot + D) + \cos. (\odot - D)]$$
$$= \sin. D\,[- m \cos.^2 \tfrac{1}{2} \omega \cos. (Q + 90°) + m \sin.^2 \tfrac{1}{2} \omega \cos. (Q' + 90°)]$$
$$- \tfrac{1}{2} m \sin. \omega\,[\cos. (\odot + D) + \cos. (\odot - D)]$$
$$= \sin. D\,[n \cos. (Q + 90°) + n' \cos. (Q' + 90°)]$$
$$- \tfrac{1}{2} m \sin. \omega\,[\cos. (\odot + D) + \cos. (\odot - D)], \qquad (689)$$

and the values of

$$- \tfrac{1}{2} m \sin. \omega \cos. (\odot + D) \text{ and } - \tfrac{1}{2} m \sin. \omega \cos. (\odot - D)$$

may be put in a table like Part III of Table XLII. The rules for finding the variations in right ascension and declination are then the same as in the explanation of this table.

113. In constructing Table XLII, the values of m and ω were taken

$$m = 20'', \omega = 23° 27' 28'', \qquad (690)$$

whence

$$n = - 19''.173, n' = 0''.827, \qquad (691)$$
$$- \tfrac{1}{2} m \sin. \omega = - 3''.9841. \qquad (692)$$

114. By putting

$$\odot - \varLambda = P, \qquad (693)$$

we have, by (653 and 656),

$$\delta L = - m \cos. (P - 90°) \sin. L \qquad (694)$$
$$\delta \varLambda = - m \cos. P \sec. L, \qquad (695)$$

so that if the values of

$$- m \cos. P$$

are inserted in tables like Table XLI of the Navigator, the variations of latitude and longitude are found by the rule given in the explanation of this table.

115. If the star is nearly in the ecliptic, the aberration in latitude may be neglected, and the aberration in longitude will be by (695)

$$\delta \varLambda = - m \cos. P. \qquad (696)$$

116. *Problem. To find the diurnal aberration in right ascension and declination.*

Solution. Let

v' = the velocity of a point of the equator, arising from the earth's rotation,

$$m' = \frac{v'}{V \sin. 1''}. \tag{697}$$

The velocity of the observer is evidently in proportion to the circumference which he describes in a day, that is, to the radius of this circumference, or to the cosine of the latitude.

The velocity of the observer $= v' \cos.$ lat.

Now, the diurnal motion is parallel to the equator, whence the formulas (653) and (656) may be referred at once to the present case by putting

Z = the right ascension of the zenith,

and changing m into $m' \cos.$ lat., $\odot - A$ into $Z - R$, and L into D; whence the diurnal aberrations in right ascension and declination are

$$\delta' R = - m' \cos. (Z - R) \sec. D \cos. \text{lat.} \tag{698}$$

$$\delta' D = - m' \sin. (Z - R) \sin. D \cos. \text{lat.} \tag{699}$$

117. The value of m' is nearly

$$m' = 0''.31. \tag{700}$$

118. *Problem. To find the aberration which arises from the motion of a planet.*

Solution. The most important planets revolve about the sun almost uniformly in circles, and in the plane of the ecliptic. At the instant, then, of the light's reaching the earth, the planet has advanced in its orbit by a distance proportioned to its velocity, and to the time which the light takes in reaching the earth. Let then S (fig. 49) be the sun, and $O_1 O_1'$ perpendicular to $O_1 S$ the path of the planet; and put

v_1 = the velocity of the planet,

$$m_1 = \frac{v_1}{V \sin 1''}, \quad P_1 = OO_1S,$$

$$r = OS, \ r_1 = O_1S,$$

we have

$$\delta_1 A = -O_1OO_1' = -\frac{O_1O_1' \cos. P_1}{OO_1 \sin. 1''} = -m_1 \cos. P_1. \tag{701}$$

But it will be shown in Theoretical Astronomy that

$$v^2 : v_1^2 = r_1 : r;$$

hence

$$m^2 : m_1^2 = v^2 : v_1^2 = r_1 : r$$

$$m : m_1 = \sqrt{r_1} : \sqrt{r}$$

$$m_1 = m\sqrt{\frac{r}{r_1}} \tag{702}$$

$$\delta_1 A = -m\sqrt{\frac{r}{r_1}} \cos. P_1; \tag{703}$$

and this aberration being combined with (696) gives the whole aberration in longitude, from which a table, like Table XXXIX of the Navigator, may be constructed.

119. Examples.

1. Find the values of log. A, log. B, h, H, and i for May 1, 1839, when $\odot = 40° 52' 56''$.

Ans. log. $A = 1.1498_n$
log. $B = 1.1248_n$
$h = 19''.42$
$H = 226° 40'$
$i = -6''.13$

2. Find the values of log. a, log. b, log. a', log. b' for Altair in the year 1839.

Solution.

$R = 19^h 42^m 55^s$		cos. 9.63760		sin. 9.95466_n
$D = 8° 26' 52''$		sec. 0.00474		sec. 0.00474
		log. a = 9.64234		log. b = 9.95940_n
R	cos. 9.63760		sin. 9.95466_n	
D	sin. 9.16704		sin. 9.16704	cos. 9.99526
	log. b' = 8.80464	0.13234	9.12170_n	co tan. 9.63747
		0.42927		9.63273
		a' = 0.56161		log. a' = 9.74944

3. Find the values of log. a, log. b, log. a', log. b', for Regulus in the year 1839; for which,

$$R = 9^h\ 59^m\ 48^s,\quad D = 12^\circ\ 45'\ 7''.$$

Ans. log. $a = 9.94816_n$

log. $b = 9.71048$

log. $a' = 9.49516$

log. $b' = 9.28122_n$

4. Find the numbers of the different parts of Table XLII for the argument $7^s\ 20^\circ = 230^\circ$.

Ans. 12''.32 for Part I,

— 0''.53 for Part II,

2''.56 for Part III.

5. Find the number of Table XLI for $7^s\ 20^\circ$.

Ans. 12''.9.

6. Find the aberration in right ascension and declination of Altair for May 1, 1839.

Solution. I.

	A 1.1498_n		1.1498_n
	a 9.6423		a' 9.7494
— 6''.20	0.7921_n	— 7''.93	0.8992_n
	B 1.1248_n		1.1248_n
	b 9.9594_n		b' 8.8046
12''.14	1.0842	— 0''.85	9.9294_n

$\delta R = 5''.9 = 0^s.39$ $\delta D = -8''.78$

II.

$H + \alpha = 162° \; 44' + 360°$	sin. 9.4725		cos. 9.9300$_n$
$h = 19''.42$	1.2882		1.2882
$D = 8° \; 27'$	sec. 0.0047		sin. 9.1670
$\delta R = 5''.83 = 0^s.39$	0.7654	$-2''.72$	0.4352$_n$
	i cos. $D =$	$-6''.06$	
	$\delta D =$	$-8''.78$	

III.

$R - \odot = 255° \; 40' = 8^s \; 15° \; 50'$	P. I. $= 4''.75$	
$R + \odot = 76° + 360 = 2^s \; 16° + 12^s$	P. II. $= 0''.20$	
	$4''.95$	0.6946
	D	sec. 0.0047
$\delta R = 5'' = 0^s.33$		0.6993
$8^s \; 15° \; 40' + 3^s = 11^s \; 15° \; 40'$	P. I. $- 18''.57$	
$2^s \; 16° + 3^s = 5^s \; 16°$	P. II. $- \; 0''.80$	
	$-19''.37$	1.2871$_n$
	D	sin. 9.1670
	$-2''.85$	0.4541$_n$
$\odot + D = 48° = 1^s \; 18°$	$-2''.66$	
$\odot - D = 32° = 1^s \; \; 2°$	$-3''.38$	
$\delta D =$	$-8''.89$	

7. Find the aberration in right ascension and declination of Regulus for May 1, 1839.

Ans. By Naut. Alm. $\delta R = \; 0^s.38$

$\delta D = -1''.87$

By the Navigator $\delta R = \; 0^s.38$

$\delta D = -1''.91$

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8. Find the aberration of Regulus in latitude and longitude for May 1, 1839.

Ans. $\delta A = 6''.5$
$\delta D = 0''.15.$

9. Find the aberration of Venus in longitude, when the difference of longitude of Venus and the sun is 45°.

Solution.	r	0.0000	0.0000
	r_1	ar. co. 0.1407	$\frac{1}{2}$ (ar. co.) 0.0703
$P = 45°$		sin. 9.8495	20″ log. 1.3010
$P_1 =$		sin. 9.9902	cos. 9.3214
			0.6927

— 5″ when P_1 is acute, + 5″ when P_1 is obtuse,
— 14″ from Table XLI, — 14″

$\delta A =$ — 19″ when P_1 is ac., = — 9″ when P_1 is obtuse.

10. Find the aberration of each of the planets in longitude, when the difference of longitude of the sun and planet is 15°. The mean values of log. r_1 for the several planets are:

for Mercury	9.5878
Venus	9.8593
The Earth	0.0000
Mars	0.1829
Jupiter	0.7161
Saturn	0.9795
Uranus	1.2829
Neptune	1.4756.

Ans. For Mercury — 43″ when P_1 is acute,
4″ when P_1 is obtuse,
Venus — 41″ when P_1 is acute,
3″ when P_1 is obtuse,
Mars — 35″
Jupiter — 28″
Saturn — 26″
Uranus — 24″
Neptune — 23″.

11. Find the diurnal aberration of right ascension and declination of Polaris for Jan. 1, 1839, and latitude 45°, when the hour angle is $0^h\ 30^m$.

Solution	0″.31	9.4914		9.4914
	45°	cos. 9.8495		9.8495
$D = 88°\ 27'$		sec. 1.5678		sin. 9.9998
$0^h\ 30^m$		cos. 9.9963		sin. 9.1157
$\delta' R = -8''.04 = -0^s.53$		0.9050	$\delta' D = 0''.03$	8.4564

12. Find the diurnal aberration of δ Ursæ Minoris in right ascension and declination for Jan. 1, 1839, and latitude 0°, when the star is upon the meridian.

Dec. of δ Ursæ Minoris = 86° 35′.

Ans. $\delta' R = -0^s.35$

$\delta' D = 0.$

CHAPTER X.

REFRACTION.

120. LIGHT proceeds in exactly straight lines only in the void spaces of the heavens; but when it enters the atmosphere of a planet, it is sensibly bent from its original direction according to known optical laws, and its path becomes curved. This change of direction is called *refraction;* and the corresponding change in the position of each star is the *refraction of that star.*

121. *Problem. To find the refraction of a star.*

Solution. Let O (fig. 50) be the earth's centre, A the position of the observer, AOK the section of the surface formed by a vertical plane passing through the star. It is then a law of optics that

Astronomical Refraction takes place in vertical planes, so as to increase the altitude of each star without affecting its azimuth.

Let, now, ZIH be the section of the upper surface of the upper atmosphere formed by the vertical plane, SI the direction of the ray of light which comes to the eye of the observer. This ray begins to be bent at I and describes the curve IA, which is such that the direction AC is that at which it enters the eye. Let, now,

$\varphi = ZAC =$ the ✱'s apparent zenith distance,

$r =$ the refraction,

$\quad = $ the diff. of directions of AC and IS,

$\quad = SIL - S'CL$

$u = COZ$,

and we have

$$LCS' = \varphi - u,$$
$$SIL = \varphi - u + r.$$

Again, it is a law of optics that the *ratio of the sines of the two angles LIS and ZAS' is constant for all heights and dependent upon the refractive power of the air at the observer.*

Denote this ratio by n, and we have

$$\frac{\sin. (\varphi - u + r)}{\sin. \varphi} = n, \tag{704}$$

and if

U and $R =$ the values of u and r at the horizon,

we have

$$\frac{\sin. (\varphi - u + r)}{\sin. \varphi} = n = \cos. (U - R), \tag{705}$$

whence

$$\frac{\sin. \varphi - \sin. (\varphi - u + r)}{\sin. \varphi + \sin. (\varphi - u + r)} = \frac{1 - \cos. (U - R)}{1 + \cos. (U - R)} \tag{706}$$

$$\frac{\text{tang. } \frac{1}{2} (u - r)}{\text{tang. } [\varphi - \frac{1}{2} (u - r)]} = \text{tang.}^2 \tfrac{1}{2} (U - R) = N, \tag{707}$$

and since $\frac{1}{2} (u - r)$ is small,

$$\tfrac{1}{2} (u - r) = N \text{ tang. } [\varphi - \tfrac{1}{2} (u - r)]. \tag{708}$$

Again, to find u, the triangle COA gives

$$\frac{\sin. (\varphi - u)}{\sin. \varphi} = \frac{OA}{OC}. \tag{709}$$

Now the point C is at different heights for different zenith distances of the star; but this difference in the values of OC is small, and may be neglected in this approximation; so that

$$\frac{\sin. (\varphi - u)}{\sin. \varphi} = \cos. U = \frac{OA}{OK}, \tag{710}$$

$$\frac{\sin. \varphi - \sin. (\varphi - u)}{\sin. \varphi + \sin. (\varphi - u)} = \frac{1 - \cos. U}{1 + \cos. U}, \tag{711}$$

$$\text{tang. } \tfrac{1}{2} u = \text{tang}^2 \tfrac{1}{2} U \text{ tang. } (\varphi - \tfrac{1}{2} u); \tag{712}$$

and since u is small,

$$\tfrac{1}{2} u = \text{tang.}^2 \tfrac{1}{2} U \text{ tang. } (\zeta - \tfrac{1}{2} u), \qquad (713)$$

which compared with this rough value of $\frac{1}{2}(u-r)$ from (708),

$$\tfrac{1}{2}(u - r) = N \text{ tang. } (\zeta - \tfrac{1}{2} u), \qquad (714)$$

gives

$$u = \frac{r}{1 - N \cot.^2 \tfrac{1}{2} U} = N r, \qquad (715)$$

and if we put

$$m = \frac{2 N}{N - 1} \qquad (716)$$

$$p = \tfrac{1}{2} (N - 1), \qquad (717)$$

we have, by (708),

$$\tfrac{1}{2} (u - r) = p\, r \qquad (718)$$

$$r = m \text{ tang. } (\zeta - p\, r), \qquad (719)$$

and the values of m and p must be determined by observation; and their mean values, as found by Bradley, and adopted in the Navigator, are

$$m = 57''.035, \; p = 3, \qquad (720)$$

by which Table XII is calculated.

122. The variation in the values of m and p for different altitudes of the star can only be determined from a knowledge of the curve which the ray of light describes. But this curve depends upon the law of the refractive power of the air at different heights; and this law is not known, so that the variations of m and p must be determined by observation. At altitudes greater than 12 degrees, the mean values of m and p are found to be nearly constant, and observations at lower altitudes are rarely to be used.

123. The mean values of m and p, which are given in (720), correspond to

the height of the barometer	$= 29.6$ inches	(721)
the thermometer	$= 50°$ Fahrenheit.	(722)

Now the refraction is proportional to the density of the air; but, at the same temperature, the density of the air is proportional to its elastic power, that is, to the height of the barometer. If then

$h =$ the height of the barometer in inches,

$r =$ the refraction of Table XII,

$\delta r =$ the correction for the barometer;

we have

$$r : r + \delta r = 29.6 : h \tag{723}$$

$$29.6\, \delta r = (h - 29.6)\, r \tag{724}$$

$$\delta r = \frac{(h - 29.6)}{29.6} r, \tag{725}$$

whence the corresponding correction of Table XXXVI is calculated.

Again, the density of the air, for the same elastic force, increases by one four-hundredth part for every depression of 1° of Fahrenheit; hence the refraction increases at the same rate, so that if

$\delta' r =$ the correction for the thermometer,

$f =$ the temperature in degrees of Fahrenheit,

we have

$$\delta' r = \frac{50 - f}{400} r, \tag{726}$$

whence the corresponding correction of Table XXXVI is calculated.

124. Examples.

1. Find the refraction, when the altitude of the star is 14°, and the corrections for this altitude, when the barometer is 31.32 inches, and the thermometer 72° Fahrenheit.

Solution.

	$57''.035$ log. 1.75614		
	$76°$ tan. 0.60323		
1st app. $r = 228''.7 = 3'\ 48''.7$	2.35937		
	$57''.035$ 1.75614		
$76° - 3r = 75°\ 48'\ 34''$	tan. 0.59711		
2d app. $r = 226'' = 3'\ 46''$	2.35325		2.353
$31.32 - 29.6 = 1.72$	0.235	$50 - 72 = -22$	1.342_n
29.6	ar. co. 8.529	400 ar. co.	7.398
$\delta r = 13''$	1.117	$\delta' r = -12''$	1.093_n

2. Find the refraction, when the altitude of the star is 50°, and the corrections for this altitude, when the barometer is 31.66 inches, and the thermometer 36°.

Ans. The refraction $= 48''$
Correction for barometer $= 3''$
Correction for thermom. $= 2''$.

3. Find the refraction, when the altitude of the star is 10°, and the corrections for this altitude, when the barometer is 27.80 inches, and the thermometer 32°.

Ans. The refraction $= 5'\ 15''$
Correction for barometer $= -19''$
Correction for thermom. $= 15''$.

125. *Problem. To find the radius of curvature of the path of the ray of light in the earth's atmosphere.*

Solution. By the *radius of curvature* is meant the radius of the circular arc which most nearly coincides with the curve. Now this radius may be found with sufficient accuracy by regarding the whole curve AI as the arc of a circle; and if we put

$$r_1 = \text{the radius of curvature,}$$

$$R_1 = OA = \text{the earth's radius,}$$

we have

$$AC : R_1 = \sin. u : \sin. (\varphi - u), \tag{727}$$

or, nearly,

$$AI : R_1 = u \sin. 1'' : \sin. \varphi$$

$$AI = \frac{R_1\, u \sin. 1''}{\sin. \varphi}. \tag{728}$$

Again, the radii of the arc AI which are drawn to the points A and I are perpendicular to the tangents AS' and IS, so that the angle which they make with each other is

$$S'AS = r;$$

that is, r is the angle at the centre, which is measured by the arc AI, consequently

$$AI = r_1 \sin. r = r_1 r \sin. 1'', \tag{729}$$

whence

$$r_1 = \frac{u R_1}{r \sin. \varphi}. \tag{730}$$

But, by (718),

$$u = 7 r, \tag{731}$$

whence

$$r_1 = \frac{7 R_1}{\sin. \varphi}, \tag{732}$$

so that at the horizon

$$r_1 = 7 R_1, \tag{733}$$

as in (284, 285).

126. *Problem. To find the dip of the horizon.*

Solution. The dip of the horizon is the error of supposing the apparent horizon to be only 90° from the zenith, whereas it is more than 90°. If O (fig. 51) is the centre of the earth, B the position of the observer at the height AB above the surface, O' the centre of curvature of the visual ray BT, which just touches the earth's surface at T, BT' perpendicular to $O'B$, is the direction of the apparent horizon, and

$$\delta H = HBT' = OBO' = \text{the dip.}$$

The triangle BOO' gives

$$BO' : OO' = \sin. BOO' : \sin. \delta H = \sin. BOT : \sin. \delta H,$$

or, since $BO' = 7\, BO$ nearly, and $OO' = 6\, BO$,

and δH and BOT are small,

$$7 : 6 = BOT : \delta H$$

$$\delta H = \tfrac{6}{7} BOT = \tfrac{6}{7} \frac{AT}{AO \sin. 1''}. \tag{734}$$

But, by (285), we have, if we put

$$R = AO,\ h = AB,$$

$$\tfrac{6}{7} AT = \tfrac{6}{7} \surd (\tfrac{7}{3} R h)$$

$$= 2 \surd (\tfrac{3}{7} R h), \tag{735}$$

whence

$$\delta H = \frac{2}{\sqrt{(\frac{7}{3} R)} \sin. 1''} \sqrt{h}, \tag{736}$$

and

$$\log. \delta H = \log. 2 - \log. (\sqrt{\tfrac{7}{3} R}) - \log. \sin. 1'' + \tfrac{1}{2} \log. h$$

$$= 1.77128 + \tfrac{1}{2} \log. h, \tag{737}$$

which is the same with the formula, given in the preface to the Navigator, for calculating Table XIII.

127. *Problem. To find the dip of the sea at different distances from the observer.*

Solution. Let O (fig. 52) be the centre of the earth, B the observer at the height

$$h = AB \text{ (in feet)}$$

above the sea, and A' the point of the sea which is observed at the distance

$$d = AA' \text{ (in sea miles)} = AOA'$$

from B; and let

$$M = \text{the length of a sea mile in feet.}$$

If the radius OA' is produced to B', so that

$$A'B' = AB,$$

the point B' will be elevated by refraction nearly as much as the point A'. But the visual ray BB' will, from the equal heights of B and B', be perpendicular to the radius OC, which is half way between B and B', so that the dip of B' is, by (734),

$$\delta B = \tfrac{6}{7} BOC = \tfrac{3}{7} AOA' = \tfrac{3}{7} d. \tag{738}$$

The dip of the point A' will be greater than B' by the angle

$$i = B'BA,$$

which it subtends at B, and which is found with sufficient accuracy by the formula

$$\sin. i = \frac{A'B'}{A'B} = \frac{h}{Md} = i \sin. 1' \tag{739}$$

$$i = \frac{h}{M \text{ sin. } 1' d}. \tag{740}$$

But, by (286,)

$$M = \frac{\pi R}{10800'} \tag{741}$$

$$\frac{1}{M \text{ sin. } 1'} = \frac{10800'}{\pi R \text{ sin. } 1'} = 0.56514, \tag{742}$$

so that the dip of A' is

$$\delta A = \tfrac{3}{7} d + 0.56514 \frac{h}{d}, \tag{743}$$

which is the same with the formula, given in the preface to the Navigator, for calculating Table XVI.

128. Refraction, by elevating the stars in the horizon, will affect the times of their rising and setting; and the star will not set until its zenith distance is

$$90^\circ + \text{horizontal refraction,}$$

and the corresponding hour angle is easily found by solving the triangle PZB (fig. 35).

129. Another astronomical phenomenon, connected with the atmosphere, and dependent upon the combination of reflection and refraction, is the *twilight*, or the light before and after sunset, which arises from the illuminated atmosphere in the horizon. This light begins and ends when the sun is about 18° below the horizon; so that the time of its beginning or ending is easily calculated from the triangle PZB (fig. 35).

130. EXAMPLES.

1. Find the dip of the horizon, when the height of the eye is twenty feet.

Ans. $264'' = 4' \, 24''$.

2. Find the dip of the sea at the distance of 3 miles, when the height of the eye is thirty feet.

Solution.

$$\tfrac{3}{7} \times 3 = \tfrac{9}{7} = 1'.3$$

$$0.56514 \times \tfrac{30}{3} = 5'.6$$

$$\text{dip} \doteq 7'.$$

3. Find the dip of the sea at the distance of $2\frac{1}{2}$ miles, when the height of the eye is forty feet.

Ans. 10′

4. Find the dip of the sea at the distance of $\frac{1}{4}$ of a mile, when the height of the eye is thirty feet.

Ans. 68′.

CHAPTER XI.

PARALLAX.

131. THE fixed stars are at such immense distances from the earth that their apparent positions are the same for all observers. But this is not the case with the sun, moon, and planets; so that, in order to compare together observations taken at different places, they must be reduced to some one point of observation. The point of observation which has been adopted for this purpose is the earth's centre; and the difference between the apparent positions of a heavenly body, as seen from the surface and from the centre of the earth, is called its *parallax*.

132. *Problem. To find the parallax of a star.*

Solution. Let O (fig. 53) be the earth's centre, A the observer, S the star, and OSA, being the difference of the directions of the visual rays drawn to the observer and to the earth's centre, is the parallax. Now since SAZ is the apparent zenith distance of the star, and SOZ is its distance from the same zenith to an observer at O, the parallax

$$OSA = p$$

is the excess of the apparent zenith distance above the true zenith distance. If, then,

$z = SAZ$, $R = OA =$ the earth's radius,

$r = OS =$ the distance of the star from the earth's centre,

we have $$r : R = \sin. z : \sin. p,$$

or $$\sin. p = \frac{R \sin. z}{r}, \tag{744}$$

or $$p = \frac{R \sin. z}{r \sin. 1''}. \tag{745}$$

133. *Corollary.* If P is the horizontal parallax, we have

$$\sin. P = \frac{R}{r}, \tag{746}$$

or

$$P = \frac{R}{r \sin. 1''}; \tag{747}$$

whence

$$\sin. p = \sin. P . \sin. z, \tag{748}$$

or

$$p = P . \sin. z, \tag{749}$$

which agrees with (604), and Tables X. A., XIV, and XXIX are computed by this formula, combined, in the last Table, with the refraction of Table XII.

134. *Corollary.* In common cases, the value of the horizontal parallax can be taken from the Nautical Almanac; but, in eclipses and occultations, regard must be had to the length of the earth's radius, which is different for different places. The earth is not a sphere, but *a spheroid slightly compressed at the poles; the polar radius being less than the equatorial one by about $\frac{1}{300}$th part.* The spheroid may be obtained from the sphere by such a compression over the whole surface parallel to the polar axis *that each place is brought nearer to the plane of the equator by $\frac{1}{300}$th part.*

Thus, if $OEAP$ (fig. 54) is a section of the earth through the polar axis OP, and $OEA'P'$ the section of the sphere of which the equatorial semidiameter OE is the radius; and if $A'AM$, $B'BN$, &c. are drawn parallel to OP, each of the distances $A'A$, $B'B$, $P'P$, &c. will be $\frac{1}{300}$th part of the distances $A'M$, $B'N$, $P'O$, &c.

135. *Problem. To find the reduction of parallax.*

The horizontal parallax is, by (747), proportional to the earth's radius, so that it diminishes at the same rate from the equatorial value which is given in the Nautical Almanac. Hence, if AR is drawn perpendicular to OA, and if

$$L'' = A'OL,$$

$\delta R =$ the diminution of R for the latitude L,

$\delta P =$ that of P,

$R =$ the radius at the equator,

$P =$ the parallax at the equator,

$m = \frac{1}{300}$,

we have

$$A'M = OA' \text{ sin. } A'OM = R \text{ sin. } L''$$
$$AA' = m\, R \text{ sin. } L'' = m\, R \text{ sin. } L \text{ nearly}$$
$$\delta R = A'R \text{ nearly}$$
$$= AA' \text{ sin. } A'AR = m\, R \text{ sin.}^2 L$$
$$= \tfrac{1}{300} R \text{ sin.}^2 L$$
$$= \tfrac{1}{600} R (1 - \text{cos. } 2\, L) \qquad (750)$$
$$\delta P = \tfrac{1}{300} P \text{ sin.}^2 L$$
$$= \tfrac{1}{600} P (1 - \text{cos. } 2\, L), \qquad (751)$$

and if P is expressed in minutes, while δP is expressed in seconds, (751) becomes

$$\delta P \text{ in seconds} = \tfrac{1}{10} (P \text{ in minutes}) (1 - \text{cos. } 2\, L), \qquad (752)$$

which agrees with the formulas for calculating the reduction of parallax given in the explanation to Table XXXVIII of the Navigator.

136. In reducing delicate observations to the centre of the earth, it must be observed, that the centre is not exactly in the direction of the vertical. Thus, if A is the observer, Z the zenith, ZAL the vertical, Z' the point where the radius OA produced meets the celestial sphere, Z' is called the *true zenith*, and Z the *apparent zenith*. The angle ZAZ', which is the difference between the polar distances of the true and the apparent zenith, is called the *reduction of the latitude* and must be subtracted from the angle ALE, or the latitude, to obtain the angle AOE, or the direction of the observer from the earth's centre. The angle AOE is called the *reduced latitude* and is to be substituted for the latitude in reducing delicate observations to the centre of the earth.

137. *Problem. To find the reduction of the latitude.*

Solution. Draw AC and $A'C'$ (fig. 54) parallel to OE; and since AB is perpendicular to AL, the angle

$$L = ALE = CBA.$$

Let $$L' = AOE,$$

and $$\delta L = L - L'$$

is the reduction of the latitude.

Let also $$x = OM,\ x' = ON$$

$$y = A'M,\ y' = B'N$$

$$n = 1 - m = \tfrac{299}{300} \qquad (753)$$

so that $$AM = n\,y,\ BN = n\,y',$$

and we have

$$\text{tang. } L'' = \frac{A'C'}{B'C'} = \frac{x - x'}{y - y'} = \frac{A'M}{MO} = \frac{y}{x} \qquad (754)$$

$$\text{tang. } L = \frac{AC}{BC} = \frac{x - x'}{n\,(y - y')} = \frac{1}{n}\,\text{tang. } L'' \qquad (755)$$

$$\text{tang. } L' = \frac{AM}{MO} = \frac{n\,y}{x} = n\,\text{tang. } L'' \qquad (756)$$

$$\frac{\text{tang. } L}{\text{tang. } L'} = \frac{1}{n^2} = \left(\frac{300}{299}\right)^2 = 1.0067001$$

$$= \frac{\text{tang. } L}{\text{tang. } (L - \delta L)}, \qquad (757)$$

which agrees with the formula given in the explanation of Table XXXVIII in the Navigator, and which must be computed by means of tables of 7 places of decimals.

138. *Corollary.* By applying (346 and 347) to (757), we obtain

$$\frac{\text{sin. } \delta L}{\text{sin. } (2\,L - \delta L)} = \frac{1 - n^2}{1 + n^2} = \frac{2\,m - m^2}{2 - 2\,m + m^2} = m + \tfrac{1}{2}\,m^2 + \&c. = m'$$

$$= .0033389$$

$$\doteq m \text{ nearly}$$

$$\delta L = \frac{m'}{\text{sin. } 1''}\,\text{sin. } (2\,L - \delta L) \qquad (759)$$

$$= \frac{m'}{\sin. 1''} \sin. 2\,L \text{ nearly}$$

$$= \frac{1}{300 \sin. 1''} \sin. 2\,L = \frac{1}{5 \sin. 1'} \sin. 2\,L \text{ nearly} \quad (760)$$

$$= \frac{\sin. 2\,L}{\sin. 5'} \text{ nearly.}$$

139. *Problem. To find the parallax in latitude and longitude.*

Solution. Let Z (fig 55) be the zenith, P the pole of the ecliptic, M' the apparent place of the body whose parallax is sought, and M its true place. Let also

$B = PZ =$ the zenith distance of the pole,
$=$ the altitude of the nonagesimal,
$A = 90° - ZM' =$ the apparent altitude,
$A' = 90° - ZM =$ the true altitude,
$D = 90° - PM =$ the true latitude of the body,
$h = ZPM$ $=$ the true diff. of long. of the body and the zenith,
$P =$ the horizontal parallax,
$p = P \cos. A = MM' =$ the parallax in altitude,
$\delta h = ZPM' - ZPM =$ the parallax in longitude,
$\delta D = PM' - PM =$ the parallax in latitude,
$D' = D - \delta D.$

The triangles PMM' and ZPM' give

$$\delta h = \frac{p \sin. M'}{\cos. D} = \frac{p \sin. B \sin. (h + \delta h)}{\cos. A \cos. D}$$

$$= P \sin. B \sec. D \sin. (h + \delta h). \quad (761)$$

Draw PN to bisect the angle MPM', draw MH and $M'H'$ perpendicular to PN, join ZH and ZH', and we have

$$\delta D = HH' = HN + H'N'$$
$$= MN \cos. N + M'N \cos. N$$
$$= (MN + M'N) \cos. N = MM' \cos. N$$
$$= P \cos. A \cos. N. \quad (762)$$

But the triangle $ZH'M'$ gives, by putting

$$N' = ZH'H,\ ZH' = 90^\circ - A'',$$

since $$H'M'N = 90^\circ - N,$$

$$\cos. A'' : \cos. N = \cos. A : \cos. N';$$

whence $$\cos. A \cos. N = \cos. A'' \cos. N'$$

and $$\delta D = P \cos. A'' \cos. N'.$$

Produce $H'Z$ and $H'P$ to E and C,

making $$90^\circ = H'E = H'C.$$

The right triangle ZEC will give

$$EC = N',\ ZE = 90^\circ - ZH' = A''$$

$$\cos. ZC = \cos. ZE \cos. EC = \cos. A'' \cos. N',$$

whence $$\delta D = P \cos. ZC; \qquad (763)$$

and the triangle ZPC gives

$$PC = 90^\circ - PH' = D' \text{ nearly},$$

$$ZPC = 180^\circ - ZPH' = 180^\circ - (h + \tfrac{1}{2}\delta h),$$

whence, by (307),

$$\cos. ZC = \cos. B \cos. D' - \sin. B \sin. D' \cos. (h + \tfrac{1}{2}\delta h)$$

$$\delta D = P \cos. B \cos. D' - P \sin. B \sin. D' \cos. (h + \tfrac{1}{2}\delta h), \qquad (764)$$

and formulas (761) and (764) agree with the rule in the Navigator [B., p. 406.]

140. *Corollary.* By putting

$$k = P \sin. B \sec. D, \qquad (765)$$

(761) becomes

$$\delta h = k \sin. (h + \delta h)$$
$$= k \sin. h \cos. \delta h + k \cos. h\, \delta h.$$

Hence, if

$$n = k \cos. h \qquad (766)$$

$$(1 - n)\, \delta h = k \sin. h \cos. \delta h$$

$$\delta h = \frac{k \sin. h}{(1 - n) \sec. \delta h} = \frac{P \sin. B \sec. D \sin. h}{(1 - n) \sec. \delta h}. \qquad (767)$$

The logarithm of the reciprocal of $1 - n$ is called the correction for n and is found from Table I, at the end of this volume, where it is placed opposite to the log. n.

141. *Corollary*. Another process for computing δD may be obtained from (762). This equation gives

$$\begin{aligned}\delta D &= P \cos. N \cos. (A' - p)\\ &= P \cos. N \cos. A' \cos. p + P p \cos. N \sin. A'\\ &= P \cos. N \cos. A' \cos. p + P . P \cos. A \cos. N \sin. A'\\ &= P \cos. N \cos. A' \cos. p + P\, \delta D \sin. A'. \qquad (768)\end{aligned}$$

Let

$$n' = P \sin. A', \qquad (769)$$

and (768) gives

$$(1 - n')\, \delta D = P \cos. N \cos. A' \cos. p$$

$$\delta D = \frac{P \cos. N \cos. A'}{(1 - n') \sec. p}. \qquad (770)$$

The triangle ZMH gives, by putting

$$N'' = ZHH', \quad ZH = 90^\circ - A''',$$

since

$$HMZ = 90^\circ + N,$$

$$\cos. A''' : \cos. N = \cos. A' : \cos. N'';$$

whence

$$\cos. A''' \cos. N'' = \cos. N \cos. A'$$

$$\delta D = \frac{P \cos. A''' \cos. N''}{(1 - n') \sec. p}, \qquad (771)$$

and A''' and N'' can be deduced by direct solution of the triangle ZHP, in which

$$ZPH = h + \tfrac{1}{2}\, \delta h, \quad PH = PM = 90^\circ - D \text{ nearly},$$

and A''' may be substituted for A' in determining the value of the small quantity n' by means of (769), and sec. δD may be substituted for sec. p.

142. *Problem. To find the parallax in right ascension and declination.*

Solution. Formulas (761 – 771) may be applied immediately to this case, by putting

$B =$ the altitude of the equator $=$ the co-latitude,

$D =$ the true declination,

$D' =$ the apparent declination,

$h =$ the right ascension of the body diminished by that of the zenith $=$ the hour angle of the body,

$\delta D =$ the parallax in declination,

$\delta h =$ the parallax in right ascension.

And formulas (761, 767, 771) correspond to those given by Woodhouse, in his method of calculating eclipses and occultations, in the British Nautical Almanac for 1826. The mean value of sec. δD and sec. p, which is 0.00006, is there substituted for them.

143. The *apparent diameter* of a heavenly body is the angle which its disc subtends.

144. *Problem. To find the apparent semidiameter of a heavenly body.*

Solution. Let O' (fig. 56) be the centre of the heavenly body, A the observer, and AT the tangent to the disc of the body. The angle TAO' is the apparent semidiameter. Let

$$R_1 = OT$$

$$\sigma = O'AT$$

$$r = AO',$$

we have

$$\sin.\ \sigma = \frac{O'T}{AO'} = \frac{R_1}{r}. \tag{772}$$

Hence, by (fig. 53), if A is the apparent altitude of the body, A' the true altitude,

$$\sin.\ \sigma = \frac{R_1 \sin.\ p}{R \cos.\ (A + p)} = \frac{R_1 \sin.\ p}{R \cos.\ A'} \tag{773}$$

$$\sigma = \frac{R_1}{R} p \sec. (A + p)$$

$$= \frac{R_1 P}{R} \frac{\cos. A}{\cos. A'}. \qquad (774)$$

But if Σ is the horizontal semidiameter, we have

$$\Sigma = \frac{R_1 P}{R}, \qquad (775)$$

which is also the semidiameter, as seen from the earth's centre; whence (774) becomes

$$\sigma = \Sigma \frac{\cos. A}{\cos. A'} = \Sigma \frac{\cos. (A' - p)}{\cos. A'}$$

$$= \Sigma \frac{\cos. A' + p \sin. A'}{\cos. A'} = \Sigma (1 + P \sin. A'), \qquad (776)$$

or, by (769),

$$\sigma = \Sigma (1 + n') \qquad (777)$$

$$= \frac{\Sigma}{1 - n'} \text{ (nearly)} \qquad (778)$$

$$= P \frac{R_1}{R} \cdot \frac{1}{1 - n'}.$$

145. *Corollary.* We have, for the moon,

$$R_1 = 0.2725 \ R \qquad (779)$$

$$R = 3.67 \quad R_1 \qquad (780)$$

whence $\log. \frac{R_1}{R} = 9.43537, \text{ (ar. co.)} = 0.5646, \qquad (781)$

so that formula (775) agrees with [B., p. 443 ; No. 10 of the Rule].

146. *Corollary.* If $\delta \sigma$ is the augmentation of the semidiameter for the altitude A, we have, by (776),

$$\delta \sigma = \Sigma . P \sin. A' = \Sigma P \sin. A$$

$$= \frac{R_1}{R} P^2 \sin. A, \qquad (782)$$

or, in order to express $\delta\,\sigma$ and P in seconds,

$$\delta\,\sigma = \frac{R_1}{R}\,P^2 \sin.\,1'' \sin.\,A. \tag{783}$$

Now for the mean horizontal parallax of 57′ 30″, we have

$$\log.\,\frac{R_1}{R}\,P^2 \sin.\,1'' = 1.19658 \tag{784}$$

$$\frac{R_1}{R}\,P^2 \sin.\,1'' = 15.72, \tag{785}$$

agreeing very nearly with the explanation to Table XV of the Navigator.

147. *Corollary.* The augmentation can also be calculated without determining the altitude. Thus, from (774)

$$\delta\,\sigma = \Sigma\left(\frac{\cos.\,A}{\cos.\,A'} - 1\right). \tag{786}$$

But from (fig. 55) and (761)

$$\cos.\,A = \sin.\,ZM' = \frac{\sin.\,(h + \delta\,h)\,.\,\cos.\,(D - \delta\,D)}{\sin.\,Z} \tag{787}$$

$$\cos.\,A' = \sin.\,ZM = \frac{\sin.\,h \cos.\,D}{\sin.\,Z} \tag{788}$$

$$\begin{aligned}\frac{\cos.\,A}{\cos.\,A'} - 1 &= \frac{\sin.\,(h + \delta\,h)\,.\,\cos.\,(D - \delta\,D)}{\sin.\,h \cos.\,D} - 1 \\ &= \frac{\cos h \cos.\,(D - \delta\,D)\,\delta\,h}{\cos.\,D \sin.\,h} + \frac{\cos.\,(D - \delta\,D)}{\cos.\,D} - 1 \\ &= \frac{P.\cos.\,h\,.\sin.\,B \sin.\,(h + \delta\,h)}{\cos.\,D \sin.\,h} + \frac{\cos.\,(D - \delta\,D)}{\cos.\,D} - 1\end{aligned} \tag{789}$$

Now the latitude of the moon is so small that, in the first term, we may put

$$\cos.\,D = 1, \tag{790}$$

which gives by (786), and putting

$$H = \Sigma\,P\,.\,\cos.\,h \sin.\,B \tag{791}$$

$$H' = \Sigma\left(\frac{\cos.\,(D - \delta\,D)}{\cos.\,D} - 1\right) \tag{792}$$

$$\delta\,\sigma = H + H \cot. h\;\delta\,h + H'$$
$$= H + H\,.\,P\,.\cos. h \sin. B + H'$$
$$= H + \frac{H^2}{\Sigma} + H'. \qquad (793)$$

Now we have, by (791) and (792),

$$H = \tfrac{1}{2}\,\Sigma\,.\,P\,.\,[\sin. (B+h) + \sin. (B-h)] \qquad (794)$$
$$H' = \Sigma\,(\text{tang. } D\,.\,\delta\,D + \cos. \delta\,D - 1), \qquad (795)$$

and formulas (793 to 795) agree with the method of calculating the augmentation of the semidiameter given in Table XLIV of the Navigator. The three first parts of this table are calculated for the value of Σ,

$$\Sigma = 16' = 960''$$

whence $\tfrac{1}{2}\,\Sigma\,.\,P = 8''.18.$

The fourth part of the table is the correction which arises from the difference between the actual and value of Σ that assumed in the three former parts. If we put

$$\delta'\,\sigma = \text{the value of } \delta\,\sigma \text{ for } \Sigma = 16',$$

we have, by (782) and (795),

$$\delta\sigma : \delta'\sigma = \Sigma^2 : (16')^2 \qquad (796)$$
$$\delta\,\sigma = \frac{\Sigma^2}{256}\,\delta'\,\sigma$$
$$= \delta'\,\sigma + \left(\frac{\Sigma^2}{256} - 1\right)\delta'\,\sigma$$
$$= \delta'\,\sigma + \frac{\Sigma^2 - 256}{256}\,\delta'\,\sigma$$
$$= \delta'\,\sigma + \frac{(\Sigma+16)\,(\Sigma-16)}{256}\,\delta'\,\sigma, \qquad (797)$$

as in the explanation of this table.

148. Examples.

1. Find a planet's parallax in altitude, when its horizontal parallax is 25″, and its altitude 30°.

Ans. 22″.

2. Calculate the reduction of parallax for parallax 61′, and latitude 82°.

Solution. We have, in (752), $\frac{1}{10} P = 6.1$

$2\,L = 164°$, $\cos. 2\,L = -.961$, $1 - \cos. 2\,L = 1.961$

$\delta p = 12''.0$

3. Calculate the reduction of parallax for parallax 57′, and latitude 22°.

Ans. 1″.6.

4. Calculate the reduction of parallax for parallax 53′, and latitude 68°.

Ans. 7″.9.

5. Calculate the reduction of latitude for latitude 70°.

Solution. We have by (759),

	$\frac{m'}{\sin. 1''}$		cos.	2.83804
$2\,L =$	$140°$		sin.	9.80807
1st. app. $\delta L =$	$0°\ 7'\ 23''$	=	$443''$	2.64611
$2\,L - \delta L =$	$139°\ 52'\ 37''$		sin.	9.80918
$\delta L =$	$7'\ 23''.8$	=	$443''.8$	2.64722

6. Calculate the reduction of latitude for latitude 20°.

Ans. 7′ 21″.5.

7. Calculate the reduction of latitude for latitude 50°.

Ans. 11′ 18″.6.

8. Find the moon's parallax in latitude and longitude, when her horizontal parallax is 59′ 10″.3, her latitude 3° 7′ 19″ S., her longitude 44° 36′ 16″, the altitude of the nonagesimal 37° 56′ 14″, its longitude 25° 27′ 16″, the latitude of the place 43° 17′ 18″ N.

Solution. By (761) and (764),

Reduced parallax $= 59' \, 10''.3 - 5''.3 = 59' \, 5'' = 3545''$

Reduced latitude $= 43° \, 17' \, 18'' - 11' \, 27'' = 43° \, 5' \, 51''$

$h = 44° \, 36' \, 16'' - 25° \, 27' \, 16'' = 19° \, 9'$

3545		3.54962			3.54962		3.550
37° 56′ 14″	sin.	9.78873		cos.	9.89691	sin.	9.789
3° 7′ 19″	sec.	0.00064	3° 7′ 19″	cos.	9.99936		
		3.33899	46′ 32″		3.44589		
19° 9′	sin.	9.51593	3° 53′ 51″	cos.	9.99899		
12′		2.85492	46′ 30″		3.44552		
19° 21′	sin.	9.52027	3° 53′ 49″			sin.	8.831
$\delta h = 12' 3''$		2.85926			19° 15′	cos.	9.975
19° 21′ 3″			− 2′ 20″				2.145
			$\delta D = 44' 10''$				

9. Find the moon's parallax in latitude and longitude, when the horizontal parallax is 60′ 6″.2; her latitude 1° 30′ 12″ N., her longitude 130° 17′; the altitude of the nonagesimal 85° 14′, its longitude 125° 17′, the latitude of the place 46° 11′ 28″.4 N.

Ans. Parallax in longitude = 5′ 18″
Parallax in latitude = 4′ 30″.5.

10. Calculate the parts of Table XLIV, when the argument of the first part is $3^s \, 19° = 109°$, that of the second 12″.4, the moon's true latitude 1° 20′ N., the moon's parallax in latitude 50′, the sum of the three first parts 13″, and the moon's horizontal semidiameter 14′ 50.″

Solution. $8''.1845 \sin. 109° = 7''.74 =$ Part I.

$$\text{Part II} = \frac{(12''.4)^2}{960''} = 0''.16.$$

$$\text{Part III} = 960'' \, [\sin. 50' \tan. 1° \, 20' - 1 + \cos. 50']$$

$$= 960'' \, [\sin. 50' \tan. 1° \, 20' - 2 \sin.^2 25']$$

$$= 960'' \, [0.00023] = 0''.22.$$

$$\text{Part IV} = -13'' \times \frac{30'\,50'' \times 1'\,10''}{256'} = -\frac{13'' \times 30.83 \times 1.17}{256}$$

$$= -1''.83.$$

11. Calculate the parts of Table XLIV, when the argument of the first part is $2^s\ 16°$, that of the second $15''.5$, the moon's true latitude 3° S., the moon's parallax in latitude 30′, the sum of the first three parts 11″, and the moon's horizontal semidiameter 15′ 20′.

Ans. Part I = 7″.94
Part II = 0 .25
Part III = — 0 .48
Part IV = — 0 .90

12. Calculate the number of Table XV, when the altitude is 45°.

Ans. 11″.

13. Calculate the augmentation of the moon's semidiameter, in Example 8, when the horizontal semidiameter is 16′ 50″.

Solution. Part I = 6″.87 + 2″.58 = 9″.45
Part II = 0 .09
Part III = — 0 .75

sum = 8″.79
Part IV = 0 .92

augmentation = 9″.71

14. Calculate the augmentation of the moon's semidiameter, in Example 9, when the horizontal semidiameter is 15′ 30″.

Ans. 15″.54.

15. Calculate the moon's parallax in right ascension and declination and her augmented semidiameter, for the Cambridge Observatory, when her hour angle is 57° 46′ 48″, declination 21° 42′ 55″ S., and horizontal parallax 61′ 16″.9.

Solution.

$P =$ the reduced parallax $= 61' \, 16''.9 - 5''.6 = 61' 11''.3 = 3671''.3$

$90° - B =$ reduced latitude $= 42° \, 22' \, 48'' - 11' \, 26'' = 42° \, 11' \, 22''$

P	3.56482	$h =$	57° 46′ 48″		
sin. B	9.86978	$\frac{1}{2}\,\delta h$	20 48	tang. B.	0.04268
sec. D	0.03197	$h + \frac{1}{2}\,\delta h$	58° 7′ 36″	cos.	9.72268
k	3.46657		3.46657	tang. θ	9.76536
h cos.	9.72687	sin.	9.92737	$\theta =$	30° 13′ 30″
n	3.19344	corr.	330	$D =$	−21° 42′ 55″
sec. δh			3	$\theta' =$	8° 30′ 35″
$\delta h = 2495''.8$			3.39721	sin. θ	9.70191
θ'		tang.	9.17500	sec.	0.00481
				tang. $(h + \frac{1}{2}\,\delta h)$	0.20635
N''		cos.	9.88863	tang.	9.91307
A'''		tang.	9.06363	sin.	9.06074
A'''		cos.	9.99711	P	3.56482
		P	3.56482	n'	2.62556
	n' corr.		89		89
	sec. δD		6	hor. par.	3.36548
$\delta D = 2827''.4$			3.45139	const.	9.43537
				$s = 1004''.0$	3.00179

16. Calculate the moon's parallax in right ascension and declination and her augmented semidiameter, for Providence, when her hour angle is 58° 0′ 18″, declination 21° 42′ 52″ S., and horizontal parallax 61′ 16″.2.

The latitude of Providence is 41° 49′ 22″ N.

Ans. The parallax in right ascension = 2523″.2
" " declination = 2803″.9
The augmented semidiameter = 1003″.8

17. Calculate the moon's parallax in right ascension and declination and her augmented semidiameter, for Mount Joy Observatory, Portland, when her hour angle is 58° 15′ 54″, declination 21° 42′ 52″ S., and horizontal parallax 61′ 16″.2.

The latitude of Mount Joy Observatory is 43° 39′ 52″ N.

Ans. The parallax in right ascension = 2426″.0
" " declination = 2864″.2
The augmented semidiameter = 1003″.8

18. Calculate the moon's parallax in right ascension and declination and her augmented semidiameter, for Mr. Bond's observatory, in Dorchester, when her hour angle is 60° 38′ 34″, declination 22° 42′ 8″ N., and horizontal parallax 56′ 14″.4.

The latitude of Mr. Bond's observatory is 42° 19′ 10″.

Ans. The parallax in right ascension = 2375″.3
" " declination = 1632″.9
The augmented semidiameter = 928″.5

CHAPTER XII.

ECLIPSES.

149. A SOLAR *eclipse* is an obscuration of the sun, arising from the moon's coming between the sun and the earth; and it occurs therefore at the time of new moon.

It is central to an observer when the centre of the moon passes over the sun's centre. It is *total* when the moon's apparent disc is larger than the sun's and totally hides the sun. It is *annular* when the moon's apparent disc is smaller than the sun's, but is wholly projected upon the sun's disc.

A *lunar eclipse* is an obscuration of the moon by the earth's shadow; and it occurs at the time of full moon.

The *phase* of an eclipse is its state as to magnitude.

150. An *occultation of a star or planet* is an eclipse of this star or planet by the moon.

A *transit* of Venus or Mercury is an eclipse of the sun by one of these planets.

151. *Problem. To find when a solar eclipse will take place.*

Solution. Let O (fig. 57) be the sun's centre, and O_1 the moon's centre at the time of new moon, and let

$$\beta = \text{the latitude of the moon at new moon} = OO_1.$$

Let ON be the ecliptic, and N the moon's node, so that NO_1 is the moon's path. Let

$$N = \text{the inclination of the moon's orbit to the ecliptic;}$$

Draw OP perpendicular to the moon's orbit, and if, when the moon arrives at P, the sun arrives at O', the least distance of the centres of sun and moon is nearly equal to $O'P$. Now the triangle OPO_1 gives

$$OP = \beta \cos. N = \beta - \beta\,(1 - \cos. N)$$
$$= \beta - 2\,\beta \sin.^2 \tfrac{1}{2} N = \beta - \tfrac{1}{2}\,\beta \sin.^2 N;$$

and if

$n =$ ratio of sun's mean motion to moon's $= \frac{1}{12}$ nearly, (798)

we have $$OO' = n \times O_1 P = n\,\beta \sin. N.$$

Draw $O'B$ perpendicular to OP, and we have nearly

$$OB = OP - O'P = OO' \sin. N$$
$$= n\,\beta \sin.^2 N.$$

Hence

$$O'P = \beta - (\tfrac{1}{2} + n)\,\beta \sin.^2 N = \beta - \tfrac{7}{12}\,\beta \sin.^2 N. \qquad (799)$$

The apparent distance of the centres of the sun and moon is affected by parallax, and the true distance is diminished as much as possible for that observer who sees the sun and moon in the horizon and OP vertical, in which case the diminution is equal to the difference of the horizontal parallaxes of the sun and moon. Let, then,

$\pi =$ the moon's horizontal parallax,

$\Pi =$ the sun's horizontal parallax,

$\Delta =$ the apparent distance of the centres,

we have

$$\text{the least apparent dist.} = OP - (\pi - \Pi)$$
$$= \beta - \tfrac{7}{12}\,\beta \sin.^2 N - \pi + \Pi. \qquad (800)$$

Now, an eclipse will take place, when this least apparent distance of the centres is less than the sum of the semidiameters of the sun and moon. Thus, let

$s =$ the moon's semidiameter,

$\sigma =$ the sun's semidiameter.

In case of an eclipse, we must have

$$\beta - \tfrac{7}{12}\,\beta \sin.^2 N - \pi + \Pi < s + \sigma, \qquad (801)$$

or $$\beta < \pi - \Pi + s + \sigma + \tfrac{7}{12}\,\beta \sin.^2 N. \qquad (802)$$

152. *Corollary.* We have, by observation,

the greatest value of π	$=$	$61' \, 32''$,
the least value	$=$	$52' \, 50''$,
the mean value	$=$	$57' \, 11''$,
the greatest value of Π	$=$	$9''$,
the least value	$=$	$8''$,
the greatest value of s	$=$	$16' \, 46''$,
the least value	$=$	$14' \, 24''$,
the mean value	$=$	$15' \, 35''$,
the greatest value of σ	$=$	$16' \, 18''$,
the least value	$=$	$15' \, 45''$,
the mean value	$=$	$16' \, 1''$,
the greatest value of N	$=$	$5° \, 20' \, 6''$,
the least value	$=$	$4° \, 57' \, 22''$,
the mean value	$=$	$5° \, 8' \, 44''$.

Now, in the last term of (802) we may put for N its mean value and for β its mean value obtained by supposing it equal to the preceding terms, which gives

$$\beta = \pi - \Pi + s + \sigma = 88' \, 38'' = 5318'' \qquad (803)$$

$$\tfrac{7}{12} \beta = 3102''$$

$$\sin. N = \sin. 5° \, 8' \, 44'' = 0.09, \quad \sin.^2 N = 0.008$$

$$\tfrac{7}{12} \beta \sin.^2 N = 25'', \qquad (804)$$

whence (802) becomes

$$\beta < \pi - \Pi + s + \sigma + 25''. \qquad (805)$$

153. *Corollary.* If, in (805), the greatest values of π, s, and σ, and the least value of Π are substituted, the limit

$$\beta < 1° \, 34' \, 52''$$

is the greatest limit of the moon's latitude at the time of new moon for which an eclipse can occur.

154. *Corollary.* If, in (805), the least values of π, s, and σ, and the greatest value of Π are substituted, the limit

$$\beta < 1^\circ\ 23'\ 15''$$

is the least limit of the moon's latitude at the time of new moon for which an eclipse can fail to occur.

155. *Problem. To find when a lunar eclipse will happen.*

Solution. The solution is the same as in § 151, except that the semidiameter of the earth's shadow at the distance of the moon is to be substituted for that of the sun; and the change in the position and apparent magnitude of the moon from parallax may be neglected, because when the earth's shadow falls upon the moon, the moon is eclipsed to all who can see it. Now if S (fig. 63) is the sun, E the earth, GF the semidiameter of the earth's shadow at the moon, we have

$$\begin{aligned} \text{the app. semi.} &= FEG = EFL - EIF = \pi - EIF \\ &= \pi - (KES - EKI) \\ &= \pi - \sigma + \Pi, \end{aligned}$$

or rather, this would be the apparent semidiameter, if it were not for the earth's atmosphere, which increases the breadth of the shadow about $\frac{1}{60}$th part; so that

$$\text{the app. semidiam.} = \tfrac{61}{60}(\pi - \sigma + \Pi),$$

and therefore, in order that an eclipse must happen, we must have, by (802),

$$\beta = \text{the latitude at the time of full moon,}$$

$$\beta < \tfrac{61}{60}(\pi + \Pi - \sigma) + s + \tfrac{7}{12}\,\beta \sin.^2 N. \qquad (806)$$

156. *Corollary.* In the last term of (806), we may put for N its mean value and for β its mean value obtained by supposing it equal to the preceding terms, which gives

$$\beta = 57'\ 35'' = 3455'',\quad \tfrac{7}{12}\,\beta = 2015''$$

$$\sin.^2 N = 0.008,\quad \tfrac{7}{12}\,\beta \sin.^2 N = 16'',$$

whence (806) becomes

$$\beta < \tfrac{61}{60}(\pi + \Pi - \sigma) + s + 16''. \qquad (807)$$

157. *Corollary.* If, in (807), the greatest values of π, Π, and s are substituted, and the least value of σ, the limit

$$\beta < 63' \, 45''$$

is the greatest limit of the moon's latitude at the time of full moon for which an eclipse can occur.

158. *Corollary.* If, in (807), the least values of π, Π, and s are substituted, and the greatest value of σ, the limit

$$\beta < 51' \, 57''$$

is the least limit at which an eclipse can fail to occur.

159. *Problem. To calculate when a given phase of a lunar eclipse will occur.*

Solution. If, in fig. 57, NPO_1 is the path of the moon relatively to the centre of the earth's shadow which is at O, the required computation consists, simply, in finding the instant when the moon's distance from O is that which corresponds to the required phase. The indefiniteness of the outline of the earth's shadow renders an accurate calculation superfluous, and it is sufficient to regard $O_1 ON$ as a plane triangle.

160. *Corollary.* At the beginning or end of the lunar eclipse, we have

$\Delta =$ the distance of the centres of the moon and the shadow

$$= \tfrac{61}{60} (\pi + \Pi - \sigma) \pm s, \qquad (808)$$

in which the upper sign corresponds to the first and last contacts with the shadow, and the lower sign to the beginning and end of the total phase.

161. *Problem. To compute the general circumstances of a solar eclipse.*

Solution. This problem will be found to subdivide itself naturally into several others, but the general mode of solution may be developed in a preliminary view of the whole question. The method here given is substantially *Bessel's.*

The moon's shadow upon the earth is the geometrical intersection of a right cone which is in contact with the sun and moon, and for every point within this shadow there is a *total* eclipse of the sun. If, however, this shadow does not reach the earth, there will still be, within the limits of the *umbral* cone produced beyond its vertex, an eclipse of a portion of the sun equal to the apparent size of the moon, and this dark portion, surrounded by the bright ring of the uneclipsed portion of the sun, consitutes an *annular* eclipse. But there is also an eclipse, beyond the limits of this cone, of all that portion of the sun which is hidden by the moon, and, therefore, for every place included within the *penumbral* cone which is drawn in contact with the sun and moon, and which has its vertex between these two bodies; but this is a *partial* eclipse.

A plane may now be supposed to be drawn through the earth's centre, perpendicular to the line which joins the centres of the sun and moon. The moon's shadow and penumbra upon this plane are concentric circles, and the path of their common centre upon this plane may be computed and described. Any point of the earth may be referred to this plane by a line drawn from the vertex of the cone through the point, and the relative position of the common intersection of this line with the plane and the moving shadow or penumbra of the moon, will show the successive phases of an eclipse at that point.

It will promote perspicuity to carve the problem into several subdivisions.

162. *Problem. To find the position of the line which is drawn through the earth's centre parallel to the line joining the centres of the sun and moon.*

Solution. In the triangle formed by joining the centres of the sun, moon, and earth, the angle at the earth is the apparent angular distance of the sun and moon, and the angle at the sun is the angle which the required line makes with the line drawn to the sun.

Let $\gamma =$ the angular distance of the sun and moon,

$g =$ the angle at the sun;

$r' =$ the distance of the moon from the earth,

$r =$ the distance of the sun;

we have

$$\sin. g = \frac{r'}{r} \sin \gamma, \qquad (809)$$

or on account of the smallness of g

$$g = \frac{r'}{r} \gamma. \qquad (810)$$

Since the line which is drawn from the earth's centre parallel to that which joins the centres of the sun and moon is in the plane of the above triangle, it cuts the surface of the celestial sphere at a point (F) which is in the arc of the great circle joining the sun and moon, and produced on the side of the sun by a distance equal to g.

163. *Corollary.* By putting

$\Pi =$ the sun's equatorial horizontal parallax

$\pi =$ the moon's equatorial horizontal parallax

$\Pi' =$ the mean value of $\Pi = 8''.5776$

$q = \frac{r'}{r}$

we have the following form in the computation of q,

$$q = \frac{\sin. \Pi}{\sin. \pi} = \frac{\sin. \Pi'}{r \sin. \pi}$$

$$\begin{aligned} \log. q &= \log. \sin. \Pi' - \log. r - \log. \sin. \pi \\ &= 5.6189 \quad - \log. r - \log. \sin. \pi \qquad (811) \end{aligned}$$

in which r is expressed in units of the sun's mean distance.

164. *Corollary.* The right ascension and declination of the point (F) may easily be computed from the sun's right ascension and declination. Let

$\alpha =$ the sun's right ascension

$a =$ the right ascension of F

$\delta =$ the sun's declination

$d =$ the declination of F

$l =$ the sun's longitude

$\lambda_1 =$ the (moon's — sun's) longitude

$\beta_1 =$ the (moon's — sun's) latitude

$O =$ the obliquity of the ecliptic

$u =$ the angle which γ makes with the circle of latitude drawn through the sun

$\omega =$ the angle which γ makes with the circle of declination drawn through the sun.

If then (fig. 59) S is the sun's place in the ecliptic, M the moon's relative place, N the pole of the ecliptic, Z that of the equator, we have

$$MSN = u, \qquad MSZ = \omega,$$

$$MS = \gamma \qquad ZN = O,$$

$$\text{tang. } u = \text{sin. } \lambda_1 \text{ cot. } \beta_1 \tag{812}$$

$$\text{tang. } \gamma = \text{tang. } \lambda_1 \text{ cosec. } u, \tag{813}$$

γ being taken between 0° and 90°; and from these formulæ u and γ may be computed, and the substitution of (813) in (809) gives by (811)

$$g = q \text{ sin. } \gamma \text{ cosec. } 1''. \tag{814}$$

The sun's place in the ecliptic gives

$$\text{cosin. } l = \text{cot. } O \text{ . tang. } ZSN = \text{cot. } O \text{ tang. } (u - \omega)$$

$$\text{tang. } (u - \omega) = \text{cos. } l \text{ tang. } O, \tag{815}$$

from which $u - \omega$ may be computed, and thence ω. We then have obviously, taking ω of the same sign as λ_1,

$$\delta - d = g \text{ cos. } \omega, \tag{816}$$

$$\alpha - a = g \text{ sin. } \omega \text{ sec. } \delta. \tag{817}$$

165. *Problem. To find the path of the centre of the moon's shadow upon the plane which passes through the earth's centre perpendicular to the line which joins the centres of the sun and moon.*

Solution. The angle which the line drawn from the moon to the earth makes with that drawn to the centre of the shadow, which is simply the continuation of the line drawn from the sun, is $g + \gamma$.

Hence if ϱ is the distance from the earth's centre to the centre of the shadow, we have

$$\varrho = r' \sin. (g + \gamma) = \frac{1}{\sin. \pi} \sin. (g + \gamma), \qquad (818)$$

in which r' and ϱ are expressed in units of the earth's equatorial radius.

The direction of the line ϱ, may be conveniently referred to the intersection of the plane of reference with the circle of declination drawn through the earth's centre and the point (F). Let

ω' = the angle which ϱ makes with the line of intersection,

and ω' is evidently the inclination of the arc g to the circle of declination drawn through (F). It differs, therefore, very little from ω, and the difference may be found from the triangle formed with g and the circles of declination passing through (F) and the sun to be

$$\omega - \omega' = g \sin. \omega \tan. \delta = (\alpha - a) \sin. \delta. \qquad (819)$$

166. *Corollary.* Let x be the distance towards the east of the centre of the shadow from the above line of intersection, and y the elevation towards the north of the foot of the perpendicular let fall from the centre of the shadow upon this line of intersection above the earth's centre, and we have

$$x = \varrho \sin. \omega', \qquad (820)$$

$$y = \varrho \cos. \omega'. \qquad (821)$$

167. *Problem. To find the umbral and penumbral radii upon the plane of reference of the preceding problem.*

Solution. Either of these radii is plainly equal to the product of the distance of the vertex of the cone from the plane by the tangent of the angle of the cone. If then

H = the apparent semidiameter of the sun at his mean distance $= 959''.788$

K = the ratio of the moon's radius to that of the earth $= 0.27227$

f = the angle of the cone

$S =$ the distance of the vertex of the cone from the plane of reference
$s' =$ the distance of the centres of the sun and moon,

we have, since g is very small,

$$\begin{aligned} s' &= r \cos. g - r' \cos. (g + \gamma) \\ &= r - r' \cos. (g + \gamma) \\ &= r [1 - q \cos. (g + \gamma)] \end{aligned} \tag{822}$$

the sun's radius $= \sin. H$
the earth's radius $= \sin. \Pi$
the moon's radius $= K \sin. \Pi$

$$\sin. f = \frac{\sin. H \mp K \sin. \Pi}{s'}, \tag{823}$$

in which the upper sign corresponds to the umbral and the lower to the penumbral cone and s' is expressed in units of the sun's mean distance; we have, moreover, by taking the earth's equatorial radius as the unit,

$$\begin{aligned} z &= \text{the moon's distance from the plane of reference} \\ &= r' \cos. (g + \gamma) = \frac{1}{\sin. \pi} \cdot \cos. (g + \gamma) \end{aligned} \tag{824}$$

$$S = z \mp \frac{K}{\sin. f} \tag{825}$$

$$\begin{aligned} \rho' &= \text{the radius of the shadow} \\ &= S \tan. f \\ &= z \tan. f \mp K \sec. f. \end{aligned} \tag{826}$$

168. *Corollary.* We have in (823)

$$\log. (\sin. H - K \sin. \Pi) = 7.66669 \tag{827}$$

$$\log. (\sin. H + K \sin. \Pi) = 7.66880. \tag{828}$$

169. *Corollary.* For any plane which is drawn parallel to the plane of reference, and at a distance z' from it towards the vertex of the cone, the radius of the shadow will be diminished by

$$z' \tan. f, \tag{829}$$

and the relative position of the centre of the shadow and of the point of intersection with the line drawn through the centre of the earth parallel to the axis of the cone will remain unchanged.

170. *Problem. To find the position of any point of the earth's surface with reference to the axis of the shadow.*

Solution. Let (fig. 35) $NESW$ represent the plane of reference drawn through the centre of the earth, P the north pole, Z the point in which the line drawn from the earth's centre parallel to the axis of the cone cuts the surface, and B the place. Let

$\varphi =$ the latitude of the place,
$\varphi' =$ its reduced latitude,
$\lambda =$ its west longitude,
$R =$ its distance from the earth's centre,
$\mu_1 = BPC$,
$d = PN$,

and let now that plane of reference be adopted which is drawn through B parallel to the original plane. The line of intersection of this plane with the plane of the meridian NZS corresponds to the line NS in the original plane. If BC is drawn perpendicular to NZS, we have

$x_1 =$ the distance of B towards the east from this line
$= R \sin. BC$
$$= R \cos. \varphi' \sin. \mu_1 \qquad (830)$$
$y_1 =$ the distance by which the foot of the perpendicular from B upon this line is north of the intersection of the plane with the line from the centre to Z
$$= R \cos. BN = R \sin. \varphi' \cos. d - R \cos. \varphi' \sin. d \cos. \mu_1 \qquad (831)$$
$z_1 =$ the height of B above the original plane
$$= R \cos. BZ = R \sin. \varphi' \sin. d + R \cos. \varphi' \cos. d \cos. \mu_1. \qquad (832)$$

171. *Corollary.* The radius of the shadow or penumbra for this plane is

$$\varrho' \pm z_1 \tan. f, \qquad (833)$$

the upper sign being for the shadow in a total eclipse, and the lower for the other cases.

172. *Corollary.* The distance Δ of the place B from the axis of the shadow is obviously given by the equation

$$\Delta^2 = (x - x_1)^2 + (y - y_1)^2. \quad (834)$$

173. *Problem. To investigate the condition of the commencement or termination of an eclipse.*

Solution. At either of these phases of an eclipse, the distance Δ is exactly equal to the radius of the shadow, or by (833 and 834)

$$(\varrho' \pm z_1 \tan. f)^2 = (x - x_1)^2 + (y - y_1)^2, \quad (835)$$

or by transposition, if we put

$$a = x - x_1,$$
$$a^2 = (\varrho' \pm z_1 \tan. f)^2 - (y - y_1)^2. \quad (836)$$

The second member of this equation, being the difference of two squares, may be separated into the two factors

$$b = (\varrho' \pm z_1 \tan. f) + (y - y_1) \quad (837)$$
$$c = (\varrho' \pm z_1 \tan. f) - (y - y_1), \quad (838)$$

or by (831 and 832)

$$b = \varrho' + y - R \sin. \varphi' (\cos. d \mp \sin. d \tan. f)$$
$$+ R \cos. \varphi' \cos. \mu_1 (\sin. d \pm \cos. d \tan. f) \quad (839)$$
$$c = \varrho' - y + R \sin. \varphi' (\cos. d \pm \sin. d \tan. f)$$
$$- R \cos. \varphi' \cos. \mu_1 (\sin. d \mp \cos. d \tan. f). \quad (840)$$

Hence, if we put

$$A = x$$
$$B = \varrho' + y \quad (841)$$
$$C = -\varrho' + y \quad (842)$$
$$E = \cos. d + \sin. d \tan. f = \cos. (d - f) \sec. f \quad (843)$$
$$F = \cos. d - \sin. d \tan. f = \cos. (d + f) \sec. f \quad (844)$$
$$G = \sin. d - \cos. d \tan. f = \sin. (d - f) \sec. f \quad (845)$$
$$H = \sin. d + \cos. d \tan. f = \sin. (d + f) \sec. f, \quad (846)$$

we have, by a modification of Bessel's formulæ suggested by Mr. Truman Henry Safford, Jr., for the penumbra

$$b = \quad B - ER \sin. \varphi' + GR \cos. \varphi' \cos. \mu_1 \tag{847}$$

$$c = -C + FR \sin. \varphi' - HR \cos. \varphi' \cos. \mu_1, \tag{848}$$

and, at the time of commencement or termination,

$$a^2 = b\ c. \tag{849}$$

The formulæ for the shadow in a total eclipse are obtained from (847 and 848) by interchanging E with F and G with H, or if we make ϱ' negative in this case, the formulæ may remain unchanged.

174. The values of A, B, C, E, F, G, and H are independent of the place; and the "American Ephemeris and Nautical Almanac" gives their values for every eclipse, computed at intervals of five minutes of time during the period of the eclipse and arranged in a tabular form.

The value of R may be found from (750).

The value of μ_1 changes for different places with the longitude; so that if

$\mu =$ the value of μ_1 for the first meridian,

$= R.\ A.$ of the first meridian $- R.\ A.$ of the point (F) of § 162,

we have

$$\mu_1 = \mu - \lambda; \tag{850}$$

and the value of μ is given in the Ephemeris at intervals of five minutes.

175. *Problem. To find the time of the beginning or ending of an eclipse at any place.*

Solution. If for any time the value of bc is found to differ but little from a^2, the instant of the required phase may be computed by the following process of approximation. Let for the assumed time

$$m^2 = b\ c.$$

Let also

$A' =$ the change of A in one second

$B' =$ that of B

$C' =$ that of C

$\mu' =$ that of μ, expressed, as in (16), in terms of its arc in the circle of which the radius is unity.

The changes of ϱ', E, F, G and H are so small that they may be neglected, and B' may for the present be regarded as equal to C'. Hence, if a', b', c', and m' are the changes of a, b, c, and m in one second, we have very nearly

$$b' = \quad B' - GR \cos.\ \varphi' \sin.\ \mu_1 \,.\, \mu' \qquad (851)$$

$$c' = - B' + HR \cos.\ \varphi \ \sin.\ \mu_1 \,.\, \mu' \qquad (852)$$

$$2\,mm' = bc' + cb' = (c - b)\, B' - \mu' (G c - H b)\, R \cos.\varphi' \sin.\mu_1 \quad (853)$$

$$a' = A' - R \cos.\ \varphi' \cos.\ \mu_1 \,.\, \mu'. \qquad (854)$$

Owing to the smallness of tan. f we may by (845) and (846) put in (853)

$$G = H = \sin.\ d, \qquad (855)$$

whence (853) becomes

$$2\ mm' = (c - b)\ (B' - \mu' R \cos.\varphi' \sin. d \sin.\mu_1) = b' (c - b). \quad (856)$$

If now we put

$$\tan.\ \tfrac{1}{2}\,\psi = \frac{c}{m} = \frac{m}{b}, \qquad (857)$$

we have

$$\frac{c - b}{2\ m} = \tfrac{1}{2} \tan.\ \tfrac{1}{2}\ \psi - \tfrac{1}{2} \cot.\ \tfrac{1}{2}\ \psi$$

$$= \frac{\sin.^2 \tfrac{1}{2}\ \psi - \cos.^2 \tfrac{1}{2}\ \psi}{2 \sin.\ \tfrac{1}{2}\ \psi \cos.\ \tfrac{1}{2}\ \psi} = - \frac{\cos.\ \psi}{\sin.\ \psi} = - \cot.\ \psi = - \frac{y - y_1}{x - x_1}$$

$$m' = - \cot.\ \psi\ (B' - \mu' R \cos.\ \varphi' \sin.\ d \sin \mu_1)$$

$$= - b' \cot.\ \psi. \qquad (858)$$

The change of $m - a$ in one second is then $m' - a'$, so that the number of seconds in which it will decrease by the whole amount of difference $m - a$ is

$$t = \frac{m - a}{a' - m'} = \frac{m - a}{a' + b' \cot.\ \psi}. \qquad (859)$$

176. *Corollary.* It is easy to see that ψ is the angle which the line joining the centre of the shadow with the place makes with the line of reference ; it is nearly *the angle from the north point of the sun measured towards the east to the point of contact at the end of the eclipse.*

177. It is sufficiently accurate in this solution to put

$$\log.\ \mu' = 5.8617 = \log.\ \sin.\ 15''. \qquad (860)$$

The values of A', B', and C', multiplied by 1000000, are given in the American Ephemeris at intervals of fifteen minutes of time.

The value of μ' there assumed is also 1000000 times the true value. Hence 1000000 must be multiplied into the numerator of (859), if a', b', and m' are calculated from the Ephemeris.

178. *Problem.* *To find the limits within which the eclipse is seen in the horizon.*

Solution. In this case the place is nearly in the plane of reference which passes through the earth's centre, and the deviation from this plane may be neglected without much error. If then (fig. 62) S is the earth's centre, AB the path of the centre of the shadow, M the position of the centre at the instant when the eclipse is seen in the horizon at m, the sides of the triangle $MS\,m$ are

$$\varrho = SM, \; \varrho' = M\,m, \; R = S\,m,$$

Let

$$\eta = M\,S\,m,$$

and we have by (152)

$$\sin.\ \tfrac{1}{2}\,\eta = \pm \sqrt{\left(\frac{(\varrho' - \varrho + R)\,(\varrho' + \varrho - R)}{4\,\varrho\,R}\right)}, \tag{861}$$

in the first computation of which we may suppose R to be the earth's mean radius, or that for the latitude of 45°.

If, then, SC is the line of reference already adopted, we have

$$CSM = \omega'$$

$$CS\,m = \omega' \pm \eta = \eta', \tag{862}$$

in which both signs must be used for the two different solutions of the problem.

If now in fig. 35, L represents the place m of fig. 62, we have by the right triangle LNP

$$\sin.\ \varphi' = \quad \cos.\ \eta' \cos.\ d \tag{863}$$

$$\cot.\ \mu_1 = - \cot.\ \eta' \sin.\ d \tag{864}$$

$$\lambda = \mu - \mu_1. \tag{865}$$

179. *Corollary.* The beginning or ending of the eclipse upon the earth corresponds to the cases of

$$\varrho = \varrho' \pm R, \; \eta = 0. \tag{866}$$

180.* *Problem. To find the northern and southern limits of the eclipse upon the earth.*

Solution. For this problem, it is accurate enough to regard the shadow upon the primitive plane of reference as being of uniform width and the path of its centre as a straight line. If fig. 62 represents a plane of reference at any height

$$z = R \sin. \zeta, \tag{867}$$

above the original plane, and if AB is one of the bounding lines of the shadow which is drawn parallel to the path of the centre at the distance

$$\varrho' - z \tan. f \tag{868}$$

from this path. If FS is the perpendicular let fall upon AB, let

$$P' = FS$$
$$\chi = FSC,$$

FSC being counted from F to C, we have

$$\tan. \chi = \frac{B'}{A'} \tag{869}$$

$$P = \text{the perpendicular upon the path}$$
$$= \varrho \cos. (\omega' + \chi) \tag{870}$$
$$P' = P - \varrho' + z \tan. f. \tag{871}$$

Let $\qquad CSB = \eta',$

and we have

$$SB = R \cos. \zeta$$

$$\cos. (\chi + \eta') = \frac{P'}{R \cos. \zeta}. \tag{872}$$

We have, then, (fig. 35)

$$BZP = \eta'$$
$$BL = \zeta$$

whence the triangle BZP gives by Napier's analogies

$$\tan. \tfrac{1}{2} (B - \mu_1) = \sin. \tfrac{1}{2} (\zeta - d) \sec. \tfrac{1}{2} (\zeta + d) \cot. \tfrac{1}{2} \eta' \tag{873}$$
$$\tan. \tfrac{1}{2} (B - \mu_1) = \sin. \tfrac{1}{2} (\zeta - d) \operatorname{cosec}. \tfrac{1}{2} (\zeta + d) \cot. \tfrac{1}{2} \eta' \tag{874}$$
$$\cos. \varphi' = \sin. \eta' \cos. \zeta \operatorname{cosec}. \mu_1. \tag{875}$$

The value of μ may be found from the value of A by inspection of the table, and the value of A is given by the equation

$$A = R \cos. \zeta \sin. \eta' - (\varrho' - z \tan. f) \sin. \chi. \qquad (876)$$

This solution may be corrected by introducing the actual motion of the shadow's centre at the instant and the motion of the point upon the earth's surface, which is effected by substituting in (869) for B' and A' the motion of $y - y_1$ and $x - x_1$, by which it becomes

$$\tan. \chi = \frac{b'}{a'}. \qquad (877)$$

181. *Corollary.* The phenomena of the central eclipse may be determined by putting

$$\varrho' = 0 \qquad (878)$$

in the various equations.

182. *Problem. To find the instant and amount of greatest obscuration.*

Solution. The instant of greatest obscuration must be when the motion of the axis of the shadow and of the place are neither towards nor from each other, but in parallel lines. In this case the relative motion of the centre of the shadow on the plane of reference is perpendicular to the line drawn from the place, or in other words we then have

$$\chi = -\psi. \qquad (879)$$

Now when χ has been found for a time near that of greatest conjunction, it changes so slowly, that it is only necessary to find when $-\psi$ has this same value.

But if for any time we have $-\psi$ different from χ, and denote by P the perpendicular upon the relative path of the centre, we have

$$P = m \cos. (\psi + \chi), \qquad (880)$$

and the distance by which the centre must approach the line of reference before it arrives at the point of nearest approach is

$$m \sin. \psi + m \cos. (\psi + \chi) \sin. \chi = m \sin. (\psi + \chi) \cos. \chi. \qquad (881)$$

Hence the interval of time required for this approach is

$$t = \frac{m \sin. (\psi + \chi) \cos. \chi}{a'}. \tag{882}$$

The amount of obscuration is proportioned to the distance by which the place is immersed within the penumbra, and is denoted by 12 digits when it is total; that is, when it is immersed by the distance

$$\varrho' \text{ for penumbra} - \varrho' \text{ for shadow} = M. \tag{883}$$

When it is therefore immersed, as in this case, by the quantity

$$\varrho' \text{ for penumbra} - P = N, \tag{884}$$

we have

$$n = \text{the number of digits eclipsed}$$

$$= 12 \cdot \frac{N}{M}. \tag{885}$$

183. *Corollary.* In the case of an annular eclipse ϱ' in the second term of M must be taken negatively.

184. *Corollary.* In the case of the first or last instant of contact, when

$$m = \varrho' \text{ for penumbra}, \tag{886}$$

we have

$$N = m\,[1 - \cos. (\psi + \chi)] = 2\, m \sin.^2 \tfrac{1}{2} (\psi + \chi) \tag{887}$$

and by putting

$$e = \frac{\varrho' \text{ for shadow}}{\varrho' \text{ for penumbra}} \tag{888}$$

we have

$$n = \frac{24}{1 - e} \sin.^2 \tfrac{1}{2} (\psi + \chi). \tag{889}$$

185. *Corollary.* In the case of occultations we have

$$f = 0$$

$$\varrho' = .27227. \tag{890}$$

186. *Problem. To compute the longitude of a place from an observed eclipse.*

Solution. By means of an assumed longitude find the approximate Greenwich time of the observation; and compute the eclipse for this time by art. 175. The principal effect of an error in the assumed longitude is to change the Greenwich time, and does not materially affect the value of μ_1. If, then, in computing the correction of the time, μ' is supposed to be zero, the correction obtained becomes one of longitude, to be added to the western longitude.

187. *Problem. To compute the effect of an increase of one second of arc in the moon's relative longitude upon the computed time of an eclipse.*

Solution. By this change in longitude, the moon's shadow is carried back upon the plane of reference by a quantity

$$s = r' \sin. 1'',$$

in a direction which is inclined by an angle $90° - (u - \omega)$ to the line of reference, so that

$$A \text{ and } a \text{ are decreased by } r' \sin. 1'' \cos. (u - \omega),$$

and

$$y, B, \text{ and } C \text{ are increased by } r' \sin. 1'' \sin. (u - \omega).$$

Hence m^2 will be increased by

$$(c - b)\, r' \sin. 1'',$$

and m by

$$\frac{c - b}{2\, m}\, r' \sin. 1'' = -\cot. \psi\, r' \sin. 1'' \sin. (u - \omega), \qquad (891)$$

and $m - a$ by

$$-[\cot. \psi \sin. (u - \omega) + \cos. (u - \omega)]\, r' \sin. 1''$$

$$= -\frac{\sin. (\psi + u - \omega)}{\sin. \psi}\, r' \sin. 1'', \qquad (892)$$

and the corresponding change of time is found from (859).

188. *Problem. To compute the effect of an increase of one second in the moon's relative latitude upon the computed time of an eclipse.*

Solution. By this increase of latitude the values of A and a are increased by

$$- r' \sin. 1'' \sin. (u - \omega),$$

and y, B and C are increased by

$$r' \sin. 1'' \cos. (u - \omega).$$

Hence m is increased by

$$- \cot. \psi\, r' \sin. 1'' \cos. (u - \omega),$$

and $m - a$ by

$$- \cos. (\psi + u - \omega) \operatorname{cosec.} \psi\, r' \sin. 1'', \tag{893}$$

and the change of time is found by (859).

189. *Problem. To compute the effect of an increase in the moon's semidiameter upon the time of an eclipse.*

Solution. An increase of δs in the moon's semidiameter increases ϱ' for the total shadow and penumbra, and decreases it for the annular phase by about this same amount. Hence m^2 is increased by

$$(b + c)\, \delta s$$

and m is increased by

$$\frac{b+c}{2\,m}\, \delta s = (\tfrac{1}{2} \tan. \tfrac{1}{2} \psi + \tfrac{1}{2} \cot. \tfrac{1}{2} \psi)\, \delta s = \operatorname{cosec.} \psi\, \delta s, \tag{894}$$

and the change of the time is computed by (859).

190. *Problem. To compute the effect of an increase of the moon's parallax upon the time of an eclipse.*

Solution. By an increase of a fractional part $\delta \pi$ in the moon's parallax, the quantities x, y, z are proportionally diminished, the moon's distance from the earth is proportionally diminished, and, to preserve the same apparent semidiameter of the moon, K must be proportionally diminished, and therefore also ϱ'.

Hence B is diminished by

$$(\varrho' + y)\, \delta\, \pi,$$

C is diminished by

$$(y - \varrho')\, \delta\, \pi,$$

m^2 is diminished by

$$c\, (\varrho' + y)\, \delta\, \pi + b\, (\varrho' - y)\, \delta\, \pi, \tag{895}$$

m is diminished by

$$\frac{c+b}{2\, m}\, \varrho'\, \delta\, \pi + \frac{c-b}{2\, m}\, y\, \delta\, \pi = \text{cosec.}\, \psi\, \varrho'\, \delta\, \pi - \text{cot.}\, \psi\, y\, \delta\, \pi, \tag{896}$$

and a is diminished by

$$x\, \delta\, \pi.$$

Hence $m - a$ is diminished by

$$\begin{aligned} &\text{cosec.}\, \psi\, \varrho'\, \delta\, \pi - \text{cot.}\, \psi\, y\, \delta\, \pi - x\, \delta\, \pi \\ &= \text{cosec.}\, \psi\, \varrho'\, \delta\, \pi - \varrho\, \text{cot.}\, \psi\, \text{cos.}\, \omega'\, \delta\, \pi - \varrho\, \text{sin.}\, \omega'\, \delta\, \pi \\ &= \text{cosec.}\, \psi\, \delta\, \pi\, [\varrho' - \varrho\, \text{cos.}\, (\psi - \omega')]. \end{aligned} \tag{897}$$

The effect upon the time is computed by (859).

191. Examples.

1. In the solar eclipse of July 28, 1851, to find the position of the line which is drawn through the earth's centre parallel to the line joining the centres of the sun and moon.

Solution. The following data are taken from the Nautical Almanac and Airy's Lunar Tables with Longstreth's corrections.

Greenw. m. s. t.	☽'s — ☉'s long. $= \lambda_1$	☽'s — ☉'s lat. $= \beta_1$	☉'s long. $= l$
0^h	$-1° 32' 55''.6$	$0° 37' 16''.0$	$124° 45' 14''.2$
1	$-0\ 58\ 24\ .0$	$0\ 40\ 39\ .2$	$124\ 47\ 37\ .7$
2	$-0\ 23\ 51\ .0$	$0\ 44\ 2\ .2$	$124\ 50\ 1\ .2$
3	$0\ 10\ 43\ .4$	$0\ 47\ 24\ .9$	$124\ 52\ 24\ .6$
4	$0\ 45\ 19\ .1$	$0\ 50\ 47\ .4$	$124\ 54\ 48\ .1$
5	$1\ 19\ 56\ .1$	$0\ 54\ 9\ .7$	$124\ 57\ 11\ .6$

Greenw. m. s. t.	☉'s R. A. $= \alpha$	☉'s Dec. $= \delta$	☽'s hor. par. $= \pi$
0^h	127° 6′ 5″.0	19° 5′ 24″.7	60′ 30″.6
1	127 8 32 .6	19 4 50 .2	60 31 .7
2	127 11 0 .0	19 4 15 .7	60 32 .8
3	127 13 27 .3	19 3 41 .2	60 33 .8
4	127 15 54 .6	19 3 6 .6	60 34 .8
5	127 18 21 .8	19 2 32 .1	60 35 .9

$O =$ obliquity of the ecliptic $= 23° 27' 27''.1$

log. $r =$ log. of dist. from sun to earth $= 0.00657$

sidereal time of mean noon $= 8^h 22^m 13^s.27$

q is thus computed for 0^h from (811)

const.	5.6189
log. r	0.0066
log. sin. π	8.2455
log. q	7.3668

a and d are found from (812 – 817)

λ_1 sin.	8.43181_n	tan.	8.43197_n
β_1 cot.	1.96494		
u tan.	0.39675_n	cosec.	0.03240
$\gamma =$	1° 40′ 7″.2	tan.	8.46437
$u =$	— 68° 8′ 40″	γ cos.	9.99982
		q	7.3668
		cosec. 1″	5.3144
$g =$	14″.0	g	1.1454
$g + \gamma =$	1° 40′ 21″.2	cos. l	9.75599_n
		tan. O	9.63742
$u - \omega =$	— 13° 53′ 38″	tan.	9.39333
$\omega =$	— 54° 15′ 2″	sin.	9.90933_n
cos. ω	9.76659	sec. δ	0.02457
g	1.1454		1.1454
$\delta - d$	0.9120	$\alpha - a$	1.0793_n
$\delta - d$	8″.2	$\alpha - a$	— 12″.0
$\delta =$	19° 5′ 24″.7	$\alpha =$	127° 6′ 5″.0
$d =$	19° 5′ 16″.5	$a =$	127° 6′ 17″.0

Similar computations for the other dates give

Gr. m. s. t.	a	d	ω	$g+\gamma$
0^h	127° 6′ 17″.0	19° 5′ 16″.5	—54° 15′ 2″	100′ 21″.2
1	127 8 39 .5	19 4 42 .8	—41 15 5	71 19 .2
2	127 11 1 .8	19 4 9 .0	—14 31 4	50 11 .8
3	127 13 24 .0	19 3 35 .1	+26 40 36	48 43 .6
4	127 15 46 .3	19 3 1 .2	+55 41 10	68 13 .5
5	127 18 8 .5	19 2 27 .6	+69 50 6	96 46 .4

2. In the solar eclipse of July 28, 1851, to find the path of the centre of the moon's shadow upon the plane which passes through the earth's centre perpendicular to the line which joins the centres of the sun and moon.

Solution. We have for 0^h by (818 - 821)

	g	1.1454
	sin. ω	9.90933_n
	tan. δ	9.53919
$\omega - \omega' = -3''.9$	$\omega - \omega'$	0.5939_n
$\omega' = -54° 14' 58''$	cosec. π	1.75447
	sin. $(\gamma + g)$	8.46520
	ρ	0.21967
	sin. ω'	9.90933_n
	cos. ω'	9.76661
$x = -1.34586$		0.12900_n
$y = .96890$		9.98628

The same computation for the other dates gives

Gr. m. s. t.	x	y
0^h	—1.34586	0.96890
1	—0.77694	0.88585
2	—0.20786	0.80260
3	+0.36121	0.71897
4	0.93018	0.63487
5	1.49895	0.55048

3. In the solar eclipse of July 28, 1851, to find the umbral and penumbral radii upon the plane of reference of the preceding example.

Solution. Equations (822–828) give for 0^h

		g	7.3668
		$\cos.(g+\gamma)$	9.99982
$q\cos.(g+\gamma)$	$=.002326$		7.3666
$1-q\cos.(g+\gamma)$	$=.997674$		9.99899
		r	0.00657
		s'	0.00556
		$\log.(\sin.H - K\sin.\Pi)$	7.66669
		$\log.(\sin.H + K\sin.\Pi)$	7.66880
for shadow		$\sin.f = \tan.f$	7.66113
for penumbra		$\sin.f = \tan.f$	7.66324
		$\text{cosec.}\,\pi$	1.75447
		$\cos.(g+\gamma)$	9.99982
		z	1.75429
for shadow	$z\tan.f = .26027$		9.41542
for penumbra	$z\tan.f = .26154$		9.41753
	$K = K\sec.f = .27227$		
for shadow	$\varrho' = .01200$		
for penumbra	$\varrho' = .53381$		

Similar computations give for the other dates

Gr. m. s. t.	ϱ' for shadow.	ϱ' for penumbra.
0^h	0.01200	0.53381
1	0.01203	0.53378
2	0.01207	0.53373
3	0.01214	0.53366
4	0.01225	0.53356
5	0.01237	0.53343

f does not perceptibly change its values.

4. To find the elements for determining the beginning or end of the solar eclipse of July 28, 1851, for any place.

Solution. The value of $A = x$ is already determined. The values of B, C, E, F, G and H are computed for 0^h from equations (841–846).

	For shadow	For penumbra.
y	0.96890	0.96890
ϱ	— 0.01200	0.53381
B	0.95670	1.50271
C	0.98090	0.43509
$d+f$	19° 21′ 1″.8	19° 21′ 6″.4
$d-f$	18° 49′ 31″.2	18° 49′ 26″.6
sec. f	0.00000	0.00000
log. E	9.97612	9.97613
log. F	9.97475	9.97475
log. G	9.50878	9.50874
log. H	9.52028	9.52031

Right ascension of Greenwich meridian 0^h = $8^h\ 22^m\ 13^s.27$

= 125° 33′ 19″.0

Right ascension of $(F) = a$ of Ex. 1 = 127 6 17 0

$\mu =$ — 1 32 58 0

These values may be computed in the same way for other dates, and being interpolated by differences for every five minutes, may be arranged in a table as follows.

For penumbra

Greenw. m. s. t.	A	B	C	log. E 9.97	log. F 9.97	log. G 9.50	log. H 9.5	μ
$0^h\ 0^m$	—1.34586	1.50271	.43509	613	475	874	2031	—1° 32′ 58″.0
5	—1.29845	1.49580	.42818	13	75	872	2029	0 17 57 .6
10	—1.25104	1.48889	.42127	13	75	871	2027	+0 57 2 .8
15	—1.20363	1.48197	.41436	13	76	869	2026	2 12 3 .3
20	—1.15622	1.47505	.40744	14	76	867	2024	3 27 3 .7
25	—1.10881	1.46813	.40052	14	76	866	2022	4 42 4 .1
30	—1.06140	1.46121	.39360	14	76	864	2021	5 57 4 .9

Greenw. m. s. t.	A	B	C	log. E 9.97	log. F 9.97	log. G 9.50	log. H 9.5	μ
0ʰ 35ᵐ	—1.01399	1.45429	.38668	614	476	863	2019	7° 12′ 5″.0
40	—0.96658	1.44737	.37976	15	76	861	2017	8 27 5 .5
45	—0.91917	1.44044	.37284	15	77	859	2016	9 42 5 .9
50	—0.87176	1.43351	.36592	15	77	857	2014	10 57 6 .4
55	—0.82435	1.42658	.35900	16	77	855	2012	12 12 6 .8
1 0	—0.77694	1.41965	.35207	16	77	853	2011	13 27 7 .3
5	—0.72952	1.41272	.34514	16	77	851	2009	14 42 7 .8
10	—0.68210	1.40578	.33821	16	77	850	2007	15 57 8 .2
15	—0.63468	1.39884	.33128	17	78	848	2006	17 12 8 .7
20	—0.58726	1.39190	.32435	17	78	846	2004	18 27 9 .2
25	—0.53983	1.38496	.31742	17	78	845	2002	19 42 9 .6
30	—0.49241	1.37802	.31049	17	78	843	2001	20 57 10 .1
35	—0.44499	1.37108	.30356	17	78	842	1999	22 12 10 .5
40	—0.39756	1.36413	.29663	17	78	840	1997	23 27 11 .0
45	—0.35014	1.35718	.28969	18	79	838	1996	24 42 11 .5
50	—0.30272	1.35023	.28275	18	79	837	1994	25 57 12 .0
55	—0.25529	1.34328	.27581	18	79	835	1992	27 12 12 .4
2 0	—0.20786	1.33633	.26887	18	79	833	1990	28 27 12 .9
5	—0.16043	1.32937	.26192	18	79	831	1988	29 42 13 .4
10	—0.11300	1.32241	.25497	18	79	830	1987	30 57 13 .8
15	—0.06558	1.31545	.24802	19	80	828	1985	32 12 14 .3
20	—0.01816	1.30848	.24106	19	80	826	1983	33 27 14 .8
25	+0.02927	1.30151	.23410	19	80	825	1982	34 42 15 .2
30	0.07669	1.29454	.22714	19	80	823	1980	35 57 15 .7
35	0.12411	1.28757	.22018	19	81	821	1978	37 12 16 .2
40	0.17153	1.28059	.21323	19	81	820	1977	38 27 16 .6
45	0.21895	1.27361	.20624	20	81	818	1975	39 42 17 .1
50	0.26637	1.26663	.19927	20	81	816	1973	40 57 17 .6
55	0.31379	1.25965	.19229	20	82	814	1972	42 12 18 .0
3 0	0.36121	1.25267	.18531	20	82	812	1970	43 27 18 .5
5	0.40863	1.24568	.17832	20	82	810	1968	44 42 19 .0
10	0.45605	1.23868	.17133	20	82	809	1967	45 57 19 .4
15	0.50347	1.23169	.16434	21	83	807	1965	47 12 19 .9
20	0.55088	1.22467	.15734	621	483	805	1963	48 27 20 .3

Greenw. m. s. t.	A	B	C	log. E 9.97	log. F 9.97	log. G 9.50	log. H 9.5	μ
3h 25m	0.59830	1.21764	.15034	621	483	804	1962	49° 42′ 20″.8
30	0.64571	1.21061	.14334	21	83	802	1960	50 57 21 .3
35	0.69312	1.20358	.13634	21	84	800	1958	52 12 21 .7
40	0.74054	1.19655	.12934	21	84	799	1957	53 27 22 .2
45	0.78795	1.18952	.12234	22	84	797	1955	54 42 22 .6
50	0.83536	1.18249	.11533	22	84	795	1953	55 57 23 .1
55	0.88277	1.17546	.10832	22	85	793	1952	57 12 23 .6
4 0	0.93018	1.16843	.10131	22	85	791	1950	58 27 24 .0
5	0.97758	1.16140	.09430	22	85	789	1948	59 42 24 .5
10	1.02498	1.15436	.08729	23	85	787	1947	60 57 24 .9
15	1.07238	1.14732	.08027	23	85	786	1945	62 12 25 .4
20	1.11978	1.14028	.07325	23	86	784	1943	63 27 25 .8
25	1.16718	1.13324	.06623	23	86	782	1942	64 42 26 .3
30	1.21458	1.12620	.05921	23	86	780	1940	65 57 26 .7
35	1.26198	1.11916	.05219	24	86	779	1938	67 12 27 .2
40	1.30937	1.11211	.04517	24	86	777	1937	68 27 27 .6
45	1.35687	1.10506	.03814	24	87	775	1935	69 42 28 .1
50	1.40416	1.09801	.03111	24	87	774	1933	70 57 28 .5
55	1.45156	1.09096	.02408	25	87	772	1932	72 12 29 .0
5 0	1.49895	1.08391	.01705	625	487	770	1930	73 27 29. 4

$$A' = \quad 0.0001580$$
$$B' = -0.0000232$$

Gr. m. s. t.	For shadow. B	C
1h 25m	.83914	.86322
30	.83220	.85629
35	.82526	.84936
40	.81832	.84243
45	.81138	.83549
50	.80443	.82856
55	.79748	.82151
2 0	.79053	.81467
5	.78358	.80772

Gr. m. s. t.	For shadow. B	C
2^h 10^m	.77662	.80077
15	.76966	.79382
20	.76269	.78686
25	.75572	.77990
30	.74874	.77294
35	.74176	.76598
40	.73477	.75901
45	.72778	.75204
50	.72079	.74507
55	.71379	.73809
3 0	.70679	.73111
5	.69979	.72413
10	.69279	.71715
15	.68579	.71016
20	.67879	.70317
25	.67178	.69618
30	.66477	.68919
35	.63776	.68219
40	.65075	.67519
45	.64374	.66019

A and μ are the same, and log. E and log. F are sensibly the same, for shadow and for penumbra, while log. G and log. H for shadow are obtained from the corresponding values for penumbra by increasing log. G by 0.00003, and decreasing log. H by the same quantity.

5. To compute the phases of the eclipse of July 28, 1851, for Dantzic.

Solution. For Dantzic

$$\varphi = \text{the latitude} = 54° \ 20' \ 18''$$

$$\lambda = \text{the longitude} = -1^h \ 14^m \ 41^s.5$$

$$= -18° \ 40' \ 22''.5$$

Reduction of latitude = 10′ 53″

$\varphi' = 54° \ 9' \ 25''$

then by (750)

		$\sin.^2 \varphi$	9.8196
		$\frac{1}{300}$	7.5229
δR	= 0.002201		7.3425
R	= .997799		9.99904
$\sin. \varphi'$			9.90882
$\cos. \varphi'$			9.76758
$R \sin. \varphi' = k$			9.90786
$R \cos. \varphi' = h$			9.76642

I. For the beginning, computing now for $2^h \ 15^m$ by the equations (830 - 859) we have

		μ	32° 12′ 14″.3
		$-\lambda$	18° 40′ 22″.5
		μ_1	50° 52′ 36″.8
k	9.90786		9.90786
E	9.97619	F	9.97480
$E k$	9.88405	$F k$	9.88266
h	9.76662		9.76662
G	9.50828	H	9.51985
$\cos. \mu_1$	9.80002		9.80002
$G h \cos. \mu_1$	9.07492	$H h \cos. \mu_1$	9.08649
B	= 1.31545	$-C$	= − .24820
$-E k$	= − .76568	$F k$	= .76323
$G h \cos. \mu_1$	= .11883	$-H h \cos. \mu_1$	= − .12204
b	= .66860	c	= .39317
		c	9.59458
		b	9.82517
		$b c$	9.41975
$\dot{m}$	= − .51271		9.70987_n

		h	9.76662
m	$= -.51271$	sin. μ_1	9.88974
x_1	$= .45327$		9.65636
x	$= -.06558$		
a	$= -.51885$	h cos. μ_1	9.5666
$m - a$	$= .00614$	const.	5.8617
$\mu' h$ cos. μ_1	$= .0000268$		5.4283
A'	$= .0001580$	x_1	9.6564
a'	$= .0001312$	const.	5.8617
B'	$= -.0000232$	sin. d	9.5142
$\mu' x_1$ sin. d	$= .0000108$		5.0323
b'	$= -.0000340$		5.5315_n
$\frac{1}{2}\psi$	$= -37^\circ 28' 58''$	tan. $\frac{1}{2}\psi$	9.88471_n
ψ	$= -74^\circ 57' 56''$	$-$cot. ψ	9.4291
m'	$= -.0000091$		4.9606_n
$a' - m'$	$= .0001403$		6.1471
$m - a$			7.7882
t	$= 43''.8$		1.6411
Gr. time of beg.	$= 2^h 15' 43''.8$		
long.	$= 1\ 14\ 41\ .5$		
Dant. time of beg.	$= 3\ 30\ 25\ .3$		

II. For the end, computing at $4^h 17^m$.

μ	$62^\circ 42' 25''.6$
$-\lambda$	$18\ 40\ 22\ .5$
μ_1	$81\ 22\ 48\ .1$

k	9.90786		9.90786
E	9.97623	F	9.97485
$E k$	9.88409	$F k$	9.88271
h	9.76662		9.76662
G	9.50785	H	9.51944
cos. μ_1	9.17575		9.17575
$G h$ cos. μ_1	8.45022	$H h$ cos. μ_1	8.46181

B	$= 1.14450$	$-C$	$= -.07746$
$-Ek$	$= -.76575$	Fk	$= .76332$
$G\,h$ cos. μ_1	$= .02820$	$-H\,h$ cos. μ_1	$= -.02896$
b	$= .40695$	c	$= .65690$
		c	9.81750
		b	9.60955
		$b\,c$	9.42705
m	$= .51704$		9.71352
		h	9.76662
		sin. μ_1	9.99507
x_1	$= .57769$		9.76169
x	$= 1.09134$		
a	$= .51365$	h cos. μ_1	8.9424
$m - a$	$= .00339$	const.	5.8617
$\mu'\,h$ cos. μ_1	$= .0000064$		4.8041
A'	$= .0001580$	x_1	9.7617
a'	$= .0001516$	sin. 15″ sin. d	5.3759
$\mu'\,x_1$ sin. d	$= .0000137$		5.1376
B'	$= -.0000232$		
b'	$= -.0000369$		5.5670_n
$\frac{1}{2}\psi$	$= 51°\ 47'\ 40''$	tan. $\frac{1}{2}\psi$	0.10398
ψ	$= 103\ \ 35\ \ 20$	$-$cot. ψ	9.3833
m'	$= -.0000089$		4.9503_n
$a' - m'$	$= .0001605$		6.2054
$m - a$			7.5302
t	$= 21''.1$		1.3248

Greenw. time of end. $4^h\ 17^m\ 21^s.1$

long. 1 14 41 .5

Dantzic time of end. 5 32 2 .6

III. For beginning of total phase, computing at $3^h\ 17^m$.

		μ	47° 42′ 20″·1
		$-\lambda$	18 40 22.5
		μ_1	66 22 42.6
k	9.90786		9.90786
E	9.97621	F	9.97483
Ek	9.88407	Fk	9.88269
h	9.76662		9.76662
G	9.50809	H	9.51961
cos. μ_1	9.60281		9.60281
$G\ h$ cos. μ_1	8.87752	$H\ h$ cos. μ_1	8.88904
B	= .68299	$-C$	= — .70736
$-Ek$	= — .76572	Fk	= .76328
$G\ h$ cos. μ_1	= .07542	$-Hh$ cos. μ_1	= — .07745
b	= — .00731	c	= — .02153
		c	8.33304_n
		b	7.86392_n
		bc	6.19696
m	= — .01255		8.09848_n
		h	9.76662
x	= .52243	sin. μ_1	9.96199
x_1	= .53531		9.72861
a	= — .01288	h cos. μ_1	9.3694
$m-a$	= .00033	const.	5.8617
$\mu'\ h$ cos. μ_1	= .0000170		5.2311
A'	= .0001580	sin. 15″ sin. d	5.3759
a'	= .0001410	x_1	9.7286
$\mu'\ x_1$ sin. d	= .0000127		5.1045
B'	= — .0000232		
b'	= — .0000359		

b'	$= -$ .0000359		5.5551_n
$\frac{1}{2}\psi$	$=$ 59° 46′ 14″	tan. $\frac{1}{2}\psi$	0.23456
ψ	$=$ 119° 32′ 28″	$-$ cot. ψ	9.7534
m'	$= -$ .0000203		5.3085_n
$a' - m'$	$=$ .0001613		6.2076
$m - a$			6.5185
t	$=$ 2^s.0		0.3109

Greenwich time of beg.	3^h 17^m 2^s.0
long.	1 14 41 .5
Dantzic time of beg.	4 31 43 .5

IV. For end of total phase, computing at 3^h 20^m.

μ	48° 27′ 20″.3
$-\lambda$	18 40 22 .5
	67 7 42 .8

k	9.90786		9.90786
E	9.97621	F	9.97483
Ek	9.88407	Fk	9.88269
h	9.76662		9.76662
G	9.50808	H	9.51960
cos. μ_1	9.58958		9.58958
$G\,h$ cos. μ_1	8.86428	$H\,h$ cos. μ_1	8.87580

B	$=$ .67879	$-C$	$= -$ .70317
$-Ek$	$= -$ .76572	Fk	$=$ .76328
$G\,h$ cos. μ_1	$=$ .07316	$-H\,h$ cos. μ_1	$= -$ .07513
b	$= -$ .01377	c	$= -$ .01502

		c	8.17667_n
		b	8.13893_n
		$b\,c$	6.31560
m	$=$ .01438		8.15780

m	=	.01438	h	9.76662
x	=	.55088	sin. μ_1	9.96443
x_1	=	.53833		9.73105
a	=	.01255	h cos. μ_1	9.3562
$m - a$	=	.00183	const.	5.8617
$\mu' h$ cos. μ_1	=	.0000338		5.2179
A'	=	.0001580	sin. 15″ sin. d	5.3759
a'	=	.0001242	x_1	9.7311
$\mu' x_1$ sin. d	=	.0000128		5.1070
B'	=	— .0000232		
b'	=	— .0000360		5.5563_n
$\frac{1}{2}\psi$	=	133° 45′ 21″	tan. $\frac{1}{2}\psi$	0.01887_n
ψ	=	267 30 42	— cot. ψ	8.6381_n
m'	=	.0000016		4.1944
$a' - m'$	=	.0001226		6.0885
$m - a$				7.2625
t		14^s.9		1.1760

Greenwich time of end.	3^h 20^m $14^s.9$
long.	1 14 41 .5
Dantzic time of end.	4 34 56 .4

6. To compute the places for which the solar eclipse of July 28, 1851, is visible in the horizon at 2^h 0^m, Greenwich mean solar time.

Solution. By formulas (861 - 866) we find for lat. 45°

$R =$	.99833	(ar. co.)	0.00073
$\varrho' =$	.53373	log. $\frac{1}{4}$	9.39794
$\varrho =$	.82910	(ar. co.)	0.08139
$\varrho' - \varrho + R =$	.70296		9.84693
$\varrho' + \varrho - R =$	.36450		9.56170
		sin.^{2} $\frac{1}{2}\eta$	8.88869
$\frac{1}{2}\eta =$	16° 9′ 9″	sin.	9.44434
$\eta =$	32 18 18		
$\omega' =$	— 14 31 4		

1st	$\eta' =$	17° 47′ 14″	cos. 9.97873	cot. 0.49374
	$d =$	19 4 9	cos. 9.97548	sin. 9.51411
	$\varphi' =$	64 9	sin. 9.95421	
	$\mu_1 =$	135 31		cot. 0.00785_n
	$\mu =$	28 27		
	$\lambda =$	—107 4		
	$\varphi =$	64 18		
2d	$\eta' =$	— 46 49 22	cos. 9.83522	cot. 9.97235_n
	$d =$	19 4 9	cos. 9.97548	sin. 9.51411
	$\varphi' =$	40 18	sin. 9.81070	
	$\mu_1 =$	—107 2		cot. 9.48646
	$\mu =$	28 27		
	$\lambda =$	135 29		
	$\varphi =$	40 29		

7. To find a place upon the southern limits of the eclipse of July 28, 1851, at an angular height of 40° from the plane of reference.

Solution. By equations (867 – 877) we find

B'			5.3655_n
A'			6.1987
χ	$= -$ 8° 21′	tan.	9.1668_n
at 2^h ω'	$= -$14 31	ϱ	9.9186
$\omega' + \chi$	$= -$22 52	cos.	9.9644
P	$=$ .7638		9.8830
ϱ'	$=$ .5337	R for 45°	9.9993
		sin. 40°	9.8081
		tan. f	7.6632
z tan. f	$=$ .0030		7.4706
P'	$=$ .2331		

P'	$=$ .2331			9.3676
d	$= 19° \; 4'$		R (ar. co.)	0.0007
ζ	$= 40$		sec.	0.1157
$\chi + \eta'$	$= 72 \; 15$		cos.	9.4840
η'	$= 80 \; 36$			
$\frac{1}{2} \eta'$	$= 40 \; 18$	cot. 0.0716	cot.	0.0716
$\frac{1}{2} (\zeta + d)$	$= 29 \; 32$	sec. 0.0604	cosec.	0.3072
$\frac{1}{2} (\zeta - d)$	$= 10 \; 28$	sin. 9.2593	cos.	9.9927
$\frac{1}{2} (B - \mu_1)$	$= 13 \; 50$	tan. 9.3913		
$\frac{1}{2} (B + \mu_1)$	$= 66 \; 58$		tan.	0.3715
μ_1	$= 53 \; 8$		cosec.	0.0969
ζ	$= 40$		cos.	9.8843
η'	$= 80 \; 36$		sin.	9.9941
φ'	$= 19 \; 6$		cos.	9.9753
$R \cos. \zeta \sin. \eta' =$	.7546			9.8777
			$\varrho' - z \tan. f$	9.7249
			$\sin. \chi$	9.1620
$(\varrho' - z \tan. f) \sin. \chi =$	.0771			8.8869
A	$=$ .6775			
whence μ	$= 52° \; 36'$			
λ	$= - \; 0 \; 32$			

Whence a second approximation may be made with greater accuracy for $3^h \; 30^m$.

	R	9.9999		R	9.9999
	$\cos. \varphi'$	9.9753		$\cos. \varphi'$	9.9753
	$\cos. \mu_1$	9.7781		$\sin. \mu_1$	9.7031
	$\sin. 15''$	5.8617		$\sin. 15''$	5.8617
$\mu' h \cos. \mu_1 =$ .0000412		5.6150		$\sin. d$	9.5141
$A' =$ .0001580			$\mu' x_1 \sin. d =$	.0000113	5.0541
$a' =$ .0001168			B'	$= -$.0000232	
			b'	$= -$.0000345	5.5378
			a'		6.0674
$\chi = -16° \; 27'$				tan.	9.4704_n

χ	$= -16^\circ\ 27'$			
ω'	$= 43\ \ 39$		ϱ	9.9710
$\omega' + \chi$	$= 27\ \ 12$		cos.	9.9491
P	$= .8320$			9.9201
ϱ'	$= .5336$			
z tan. f	$= .0030$			
P'	$= .3014$			9.4791
			$(R \text{ cos. } \zeta)^{-1}$	0.1158
$\chi + \eta'$	$= 66^\circ\ 50'$		cos.	9.5949
η'	$= 83\ \ 17$			
$\frac{1}{2}\eta'$	$= 41\ \ 38$	cot. 0.0512	cot.	0.0512
$\frac{1}{2}(\zeta + d)$	$= 29\ \ 32$	sec. 0.0604	cosec.	0.3072
$\frac{1}{2}(\zeta - d)$	$= 10\ \ 28$	sin. 9.2593	cos.	9.9927
$\frac{1}{2}(B - \mu_1)$	$= 13\ \ 13$	tan. 9.3709		
$\frac{1}{2}(B + \mu_1)$	$= 65\ \ 59$		tan.	0.3511
μ_1	$= 52\ \ 46$		cosec.	0.0990
ζ	$= 40$		cos.	9.8843
η'			sin.	9.9970
φ'	$= 17^\circ\ \ 8'$		cos.	9.9803
φ	$= 17\ \ 14$			

R cos. ζ sin. η'	$= .7607$		9.8812
		$\varrho' - z$ tan. f	9.7248
		sin. χ	9.4521
$(\varrho' - z \text{ tan. } f)$ sin. χ	$= .1503$		9.1769
A	$= .6104$		
μ	$= 50^\circ\ \ 1'$		
λ	$= -2\ \ 45$		

8. To find the instant and amount of greatest obscuration in the total eclipse of July 28, 1851, for Dantzic.

Solution. From the computation for $3^h\ 17^m$ by (879 – 889) we have

b'				5.5551_n	
a'				6.1045	
χ	$= -$	15° 46′	tan.	9.4506_n	
ψ	=	119 32			
$\psi_{\prime} + \chi$	=	103 46	sin.	9.9873	cos. 9.3765
m				8.0985	8.0985
χ			cos.	9.9833	
a'			(ar. co.)	3.8955	
t	=	$92^s.2$		1.9646	
Gr. t. of gr. obs.	=	$3^h\ 18^m\ 32\ .2$			
long.	=	1 14 41 .5			
Dantzic t.	=	4 33 13 .7			

ϱ' for penumbra	= .5337	
ϱ' for shadow	= .0121	
M	= .5216	9.7173
P	= .0030	7.4750
N	= .5307	9.7248
12		1.0792
digits eclipsed = 12.2		1.0867

9. To compute the longitude of the Cambridge Observatory from the solar eclipse of July 28, 1851, the beginning of which was observed at $19^h\ 49^m\ 35^s.3$.

Solution. The longitude of Cambridge being about $4^h\ 44^m\ 30^s$, the Greenwich time is not far from $0^h\ 34^m$, for which time the following computation is made.

The latitude	= 42° 22′ 49″	
assumed longitude	= 71 7 30	
reduction of lat.	= 11 25	
φ'	= 42 11 24	
sin.² lat.		9.6574
$\frac{1}{300}$		7.5229
$\delta R =$	.001515	7.1803

R	.998485		9.99933
sin. φ'			9.82711
cos. φ'			9.86977
k			9.82644
h			9.86910
		μ	6° 57′ 5″.0
		λ	71 7 30
		μ_1	—64 10 25
k	9.82644		9.82644
E	9.97614	F	9.97476
Ek	9.80258	Fk	9.80120
h	9.86910		9.86910
G	9.50863	H	9.52019
cos. μ_1	9.63913		9.63913
$G\,h$ cos. μ_1	9.01686	$H\,h$ cos. μ_1	9.02842
B =	1.45567	$-C$ =	— .38806
$-Ek$ =	— .63471	Fk =	.63270
$G\,h$ cos. μ_1 =	.10396	$-H\,h$ cos. μ_1 =	— .10676
b =	.92492	c =	.13788
		c	9.13950
		b	9.96610
		$b\,c$	9.10560
m =	— .35711		9.55280_n
		h	9.86910
$x = A$ =	—1.02347	sin. μ_1	9.95430_n
x_1 =	— .66589		9.82340_n
a =	— .35758		
$m - a$ =	.00047		6.6721
$\frac{1}{2}\psi$ =	—21° 6′ 42″	tan. $\frac{1}{2}\psi$	9.58670_n
ψ =	—42 13 24	—cot. ψ	0.0422
A' =	.0001580	B'	5.3655_n
$-B'$ cot. ψ =	— .0000256		5.4077_n
$A' - B'$ cot. ψ	.0001324		6.1219
corr. of long.	$3^s.5$		0.5502

corr. of long.	=	$3^s.5$
assumed long.	=	$4^h\,44^m\,30$
computed	=	$4\;\;44\;\;33.5$

10. To compute the effect of changes in the moon's relative longitude and latitude, semidiameter and horizontal parallax, upon the time of the beginning of the eclipse of July 28, 1851, for Dantzic.

Solution. By equations (892 – 897) we find

ψ	$= -74°\,57'.9$ cosec.	0.0151_n		
$a'-m'$		6.1471		
$\dfrac{\text{cosec.}\,\psi}{a'-m'}$	$= -7378^s$	3.8680_n		
r'		1.7544		
sin. $1''$		4.6856		
$\dfrac{\text{cosec.}\,\psi}{a'-m'}\,r'\,\text{sin.}\,1''$		0.3080_n		0.3080_n
$u-\omega$	$= -13°\,55'.6$			
$\psi+u-\omega$	$= -88\;\;53.3$ sin.	9.9999_n		cos. 8.2866
$-\dfrac{(892)}{a'-m'}$	$= \;\;2^s.032$	0.3079	$-\dfrac{(893)}{a'-m'} = -0^s.039$	8.5946_n
ω'	$= -\;4°\,48'.7$ ϱ	9.8946		
$\psi-\omega'$	$= -70\;\;9.2$ cos.	9.5308		
ϱ cos. $(\psi-\omega')$	$= .2663$	9.4254		
ϱ'	$= .5337$		$\dfrac{\text{cosec.}\,\psi}{a'-m'}$	3.8680_n
$\varrho'-\varrho$ cos. $(\psi-\omega')$	$= .2674$			9.4272
$\dfrac{(897)}{(a'-m')\,\delta\,\pi}$	$= -1973^s$			3.2952_n

Hence the changes of the time of beginning for a change of $\delta\,\lambda$ and $\delta\,\beta$ expressed in seconds of arc in the moon's relative longitude and latitude, a change δs in the moon's semidiameter, and a change of a fractional part $\delta \pi$ in the moon's horizontal parallax are respectively

$$-2^s.032\ \delta \lambda$$
$$0^s.039\ \delta \beta$$
$$-7378^s\ \delta s$$
$$-1973^s\ \delta \pi.$$

11. To compute the beginning and end of the solar eclipse of July 28, 1851, for Washington, Paris, Göttingen, Rome and Königsberg.

The latitudes and longitudes of these places are as follows : —

	Latitude.	Longitude.
Washington	38° 53′ 34″	$18^h\ 51^m\ 48^s$
Paris	48 50 13	0 9 21 .5
Göttingen	51 31 48	0 39 46 .5
Rome	41 53 52	0 49 54 .7
Königsberg	54 42 50	1 22 0 .5

Ans. The times of beginning and end of the general eclipse are as follows : —

	Beginning.	End.
Washington	$19^h\ 21^m\ 16^s.5$	$20^h\ 50^m\ 24^s.7$
Paris	2 21 0 .4	4 30 52 .4
Göttingen	2 53 42 .3	5 0 14 .4
Rome	3 24 27 .3	3 24 32 .7
Königsberg	3 38 20 .1	5 38 48 .3

For the total phase we have

	Beginning.	End.
Königsberg	$4^h\ 39^m\ 10^s.9$	$4^h\ 42^m\ 0^s.8$.

TABLE I.

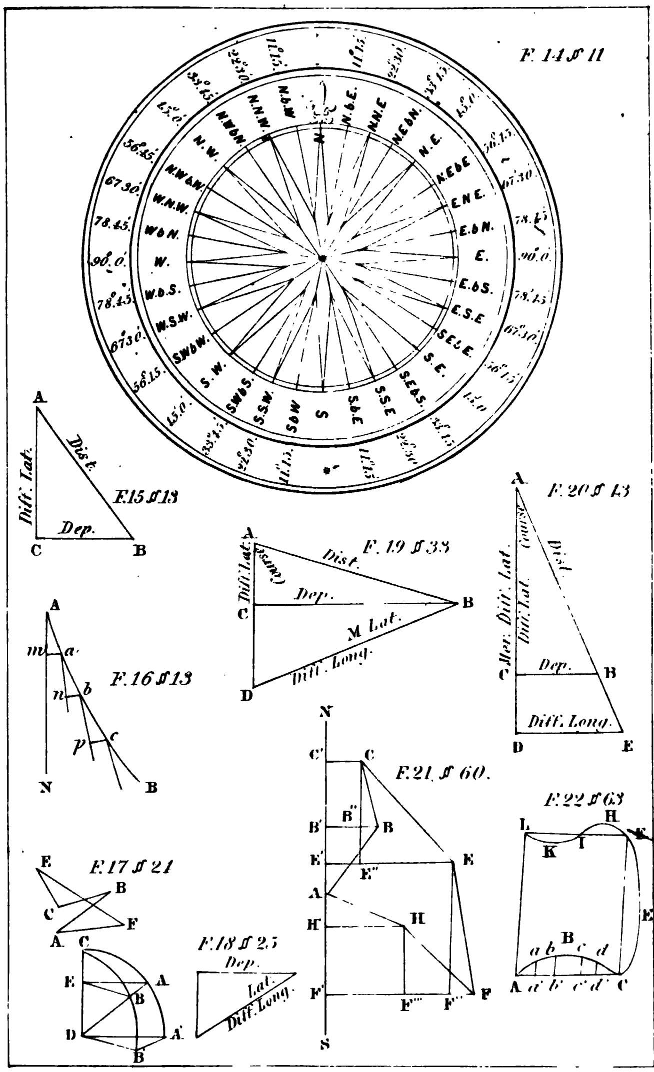
Log. n	Corr.	Log. n	Corr.	Log. n	Corr.	Log. n	Corr.	Log. n	Corr.
0.00	0	2.360	49	2.660	97	2.960	192	3.260	384
0.10	0	2.370	50	2.670	99	2.970	197	3.270	394
0.20	0	2.380	51	2.680	101	2.980	202	3.280	403
0.30	0	2.390	52	2.690	103	2.990	207	3.290	413
0.40	0	2.400	53	2.700	106	3.000	211	3.300	423
0.50	1	2.410	54	2.710	108	3.010	216	3.310	433
0.60	1	2.420	55	2.720	111	3.020	221	3.320	443
0.70	1	2.430	56	2.730	113	3.030	227	3.330	453
0.80	1	2.440	58	2.740	116	3.040	232	3.340	464
0.90	2	2.450	59	2.750	119	3.050	238	3.350	475
1.00	2	2.460	61	2.760	122	3.060	243	3.360	486
1.10	2	2.470	62	2.770	124	3.070	249	3.370	497
1.20	3	2.480	64	2.780	127	3.080	255	3.380	508
1.30	4	2.490	65	2.790	130	3.090	261	3.390	520
1.40	6	2.500	67	2.800	133	3.100	267	3.400	532
1.50	7	2.510	69	2.810	136	3.110	273	3.410	545
1.60	9	2.520	70	2.820	139	3.120	279	3.420	558
1.70	11	2.530	72	2.830	143	3.130	286	3.430	571
1.80	14	2.540	73	2.840	146	3.140	292	3.440	584
1.90	17	2.550	75	2.850	150	3.150	299	3.450	598
2.00	21	2.560	77	2.860	153	3.160	306	3.460	612
2.10	26	2.570	78	2.870	157	3.170	312	3.470	626
2.20	33	2.580	80	2.880	160	3.180	320	3.480	641
2.30	42	2.590	82	2.890	164	3.190	328	3.490	657
2.300	42	2.600	84	2.900	167	3.200	335	3.500	672
2.310	43	2.610	86	2.910	171	3.210	343	3.510	688
2.320	44	2.620	88	2.920	175	3.220	351	3.520	704
2.330	45	2.630	90	2.930	179	3.230	360	3.530	720
2.340	47	2.640	92	2.940	184	3.240	368	3.540	737
2.350	48	2.650	94	2.950	189	3.250	376	3.550	754

TABLE II.

D	$\frac{1}{100} R$												
	0	2	4	6	8	10	12	14	16	18	20	22	24
0°	0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0
1	0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.2	.2	.2
2	0	.0	.0	.0	.1	.1	.1	.2	.2	.3	.3	.4	.5
3	0	.0	.0	.0	.1	.1	.2	.2	.3	.4	.5	.6	.7
4	0	.0	.0	.1	.1	.2	.2	.3	.4	.5	.7	.8	1.0
5	0	.0	.0	.1	.1	.2	.3	.4	.5	.7	.8	1.0	1.2
6	0	.0	.0	.1	.2	.3	.4	.5	.6	.8	1.0	1.2	1.5
7	0	.0	.0	.1	.2	.3	.4	.6	.8	1.0	1.2	1.4	1.7
8	0	.0	.1	.1	.2	.3	.5	.7	.9	1.1	1.3	1·6	1.9
9	0	.0	.1	.1	.2	.4	.5	.7	.9	1.2	1.5	1.8	2.2
10	0	.0	.1	.1	.3	.4	.6	.8	1.0	1.3	1.7	2.0	2.4
11	0	.0	.1	.2	.3	.5	.7	.9	1.2	1.5	1.8	2.2	2.6
12	0	.0	.1	.2	.3	.5	.7	1.0	1.3	1.6	2.0	2.4	2.8
13	0	.0	.1	.2	.3	.5	.8	1.1	1.4	1.7	2.1	2.5	3.0
14	0	.0	.1	.2	.4	.6	.8	1.1	1.4	1.8	2.3	2.8	3.3
15	0	.0	.1	.2	.4	.6	.9	1.2	1.5	1.9	2.4	2.9	3.5
16	0	.0	.1	.2	.4	.6	.9	1.3	1.7	2.1	2.6	3.1	3.7
17	0	.0	.1	.2	.4	.7	1.0	1.4	1.7	2.2	2.7	3.3	3.9
18	0	.0	.1	.3	.5	.7	1.0	1.4	1.8	2.3	2.8	3.4	4.1
19	0	.0	.1	.3	.5	.8	1.1	1.5	1.9	2.4	3.0	3.6	4.3
20	0	.0	.1	.3	.5	.8	1.1	1.5	2.0	2.5	3.1	3.8	4.5
21	0	.0	.1	.3	.5	.8	1.2	1.6	2.1	2.6	3.2	3.9	4.7
22	0	.0	.1	.3	.5	.8	1.2	1.7	2.2	2.7	3.4	4.1	4.8
23	0	.0	.1	.3	.6	.9	1 3	1.7	2.2	2.8	3.5	4.2	5.0
24	0	.0	.1	.3	.6	.9	1.3	1.8	2.3	2.9	3.6	4.4	5.2
25	0	.0	.1	.3	.6	.9	1.3	1.8	2.4	3.0	3.7	4.5	5.4
26	0	.0	.1	.3	.6	1.0	1.4	1.9	2 5	3.1	3.8	4.6	5.5

TABLE III.

δh or δD	
Log.	Sec.
0.0	0.00000
1.0	0.00000
2.0	0.00000
3.0	0.00000
3.1	0.00001
3.2	0.00001
3.3	0.00002
3.4	0.00003
3.5	0.00005
3.6	0.00008
3.7	0.00013
3.8	0.00020


F. 14 ∫ 11
N. N.b.E. N.N.E. N.E.b.N. N.E. N.E.b.E. E.N.E. E.b.N. E. E.b.S. E.S.E. S.E.b.E. S.E. S.E.b.S. S.S.E. S.b.E. S S.b.W. S.S.W. S.W.b.S. S.W. S.W.b.W. W.S.W. W.b.S. W. W.b.N. W.N.W. N.W.b.W. N.W. N.W.b.N. N.N.W. N.b.W.
11.15. 22.30. 33.45. 45.0. 56.15. 67.30. 78.45. 90.0.
F. 15 ∫ 13
Diff. Lat.
Dist.
Dep.
F. 16 ∫ 13
F. 17 ∫ 21
F. 18 ∫ 25
Dep.
Lat.
Diff. Long.
F. 19 ∫ 38
Diff. Lat.
Course
Dist.
Dep.
M. Lat.
Diff. Long.
F. 20 ∫ 43
Mer. Diff. Lat.
Diff. Lat.
Course
Dist.
Dep.
Diff. Long.
F. 21 ∫ 60.
F. 22 ∫ 63