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TEST PROBLEMS

IN

ALGEBRA

PREPARED TO ACCOMPANY RAY'S MATHEMATICAL SERIES

BY

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Teachers in the Cincinnati High Schools.

NEW-YORK ⁘ CINCINNATI ⁘ CHICAGO

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PREFACE.

THE TEST PROBLEMS IN ALGEBRA are intended to supplement any text-books in Algebra that may be in use, and not to take their place.

All the text-books on Algebra extant are, perhaps necessarily, deficient in the number of problems presented for solution; and, for years past, the need of a carefully-graded and classified supplementary list of problems has been keenly felt in those of our high schools and colleges where Algebra is thoroughly taught.

Instructors have been compelled, hitherto, to write out, laboriously, problems selected from various sources and adapted to the varied ability of their pupils. The problems so selected must be written on the board, or elsewhere, and copied by the pupils. Aside from the waste of time thus incurred by both master and pupil, there is another serious objection to this method. Even the most careful pupil occasionally makes mistakes in copying, and thus many hours are spent in trying to solve impossible problems. Only the parents of nervous, painstaking children, faithfully striving to do all

the work prescribed, can fully appreciate the loss of time, health, and spirits thus occasioned.

It is owing to the experience of these very difficulties that the authors have been led to publish this work. The manuscripts are simply the lists of supplementary problems that have been used for years in the Cincinnati High Schools, and have been found specially adapted to their purpose. Most of the problems are original with us, although various English, French, and German text-books have been freely consulted.

The problems are carefully classified and graded. In each class, the problems at the beginning are very easy; then more difficult ones, involving similar principles, are introduced; and, finally, a variety of the most difficult problems, for the best classes, finishes the list. Occasionally, a few problems are presented too difficult for class work, and meant only for those who intend to make mathematics a specialty. Teachers are expected to use their discretion in assigning these various grades of problems to their pupils.

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CINCINNATI,
Oct., 1882.

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TEST PROBLEMS

IN

ALGEBRA.

I. DEFINITIONS.

1. FACTORS.—15.*

Name the prime factors and the divisors in the following:

1. 24.	3. $3ac$.	5. $\frac{1}{3}ax$.	7. $\frac{1}{3}a^3b^2$.
2. ab.	4. $2bc$.	6. a^2.	8. $2^3b^2c^3$.

2. CO-EFFICIENTS.—18.

(1) *Name the co-efficients in the following:*

1. $12ab$.	3. ab.	5. $\frac{4}{7}axy$.	7. $\frac{5}{11}m^2n^4$.
2. $\frac{1}{3}ax$.	4. $\frac{2}{5}x^3z^3$.	6. $\frac{2}{3}a^2x^3$.	8. $\frac{7}{12}m^4n^2$.

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* These numbers refer to the corresponding articles in RAY'S NEW HIGHER ALGEBRA.

(2) *Name the co-efficient of:*

1. $2a$, in $6abc$.
2. $\frac{1}{3}x$, in $\frac{1}{3}x^2y$.
3. $2ab$, in abc.
4. a, in $4abc$.
5. $4c$, in $4abc$.
6. $4ab$, in $4abc$.
7. 4, in $4abc$.
8. $4abc$, in $4abc$.

(3) *What result is obtained by giving:*

1. $5b$ the co-efficient $4c$.
2. $2x$ the co-efficient $5yz$.
3. $\frac{1}{2}x$ the co-efficient $3b$.
4. $2b$ the co-efficient $2y$.
5. $\frac{2}{3}abc^2$ the co-efficient $3ab^2$.
6. $\frac{1}{6}mn$ the co-efficient $3mn$.
7. $\frac{4ay}{5}$ the co-efficient $5ay$.
8. $\frac{7}{15m^2x}$ the co-efficient $5x$.
9. $\frac{3xyz}{10}$ the co-efficient $\frac{5xyz}{9}$.
10. $\frac{4}{11a^2b^2}$ the co-efficient $11ab$.

3. NOTATION.

(1) **Powers.—19.**

Write:

1. The 4th power of 2.
2. The 3d power of x.
3. The square of m.
4. The cube of z.
5. The mth power of x.
6. The $a+b$ power of y.
7. The $\frac{1}{2}$ power of c.
8. The cube of a^2.
9. The square of a^3.
10. The $x+y+z$ power of a.

(2) Roots.—20.

Write:

1. The 4th root of a.
2. The 3d root of x.
3. The square root of c.
4. The cube root of r.
5. The mth root of b.
6. The $a+b$ root of y.
7. The y root of $a+b$.
8. The $\frac{1}{2}$ root of c.
9. The c root of $\frac{1}{2}$.
10. The cube root of a^2.

(3) Polynomials.—24.

Write:

1. The cube root of x, plus the square root of y.
2. The fourth root of a, plus the square root of ab.
3. The square root of a divided by the fifth power of b, plus the cube of b square.
4. Three times x, plus twice the square root of ab.
5. One half the cube of x, plus one third the square root of x.
6. Twice the square of a, minus the square root of a.
7. The xth root of a, plus the xth power of a.
8. Three times the cube root of a^3, plus x^2.
9. The square root of a, plus the square root of b, minus c^2.
10. $\frac{1}{2}$ the mth root of a, plus the cube of a, minus d^2.

4. CLASSIFICATION.—23, 24, 25, 26.

(1) *Classify the following expressions according to the number of terms:*

1. axy.
2. $a \times x$.
3. $a + xy$.
4. $a + x - y$.
5. $\frac{3x}{2y}$.
6. $2a \div 3c$.
7. $2a \times 6b$.
8. $3a \div 2b + 5c$.
9. $3a - \frac{b-c}{d}$.
10. $\frac{d}{b-c} + y \div x$.
11. $a - 2c \times d + 5$.
12. $a - 3b \div 2c \times 4d$.
13. $a \div b \times c + c \div a \times b$.
14. $\frac{a}{b} - \frac{b}{c} \times \frac{c}{d} \div \frac{d}{c} + 3$.
15. $\frac{2a}{3} + b - \frac{1}{2}c \times d$.
16. $\frac{a-x}{b} \times \frac{b}{a-x} \div \frac{b^2}{a^2-b^2}$.
17. $\frac{a-b}{c+d} + \frac{d-c}{x-y}$.
18. $3a \div b - \frac{c-x}{b} \times y$.
19. $a - b - c \div \frac{x-y}{4c}$.

(2) *Use of the parenthesis.*—32. *Name the terms in:*

1. $(a) - b$.
2. $(a - b)$.
3. $(a - b)c$.
4. $(a + b) + c$.
5. $4a\,(x - y) + 2$.
6. $(a + b)\,(a - b)$.
7. $(a + b)\,a - b$.
8. $a + (b \times a) - b$.
9. $3a - 2b\,(x + y)$.
10. $a + b \times a - b \times \overline{c - d}$.
11. $(a + b)\,(a - b)\,(c - d)$.
12. $2a - 3\,(a - \overline{2b + c})$
13. $5c\,(x - z) + \overline{3c \div x + z}$.
14. $4[(a - b)\,(c - d) + 3]$.
15. How many terms in the bracket in the 14th?
16. How many terms in the first parenthesis?
17. How many terms in the second parenthesis?
18. How many terms in this expression,

$$a - \{b - [c + (d - x) + y] + z\}\,?$$

19. How many terms in the brace in the 18th? How many in the bracket? In the parenthesis?
20. Classify $5a\,(a - b)\,(a + b) - 2\,[3x - (5y + z)]$.

5. DEGREE OF TERMS, ETC.—29, 30, 31, 34.

1. Write a binomial with one of its terms of the first degree.

2. Write a trinomial with its terms of different degrees.

3. Write a homogeneous polynomial of the 4th degree.

4. Write a homogeneous polynomial having 4 terms.

5. Write a homogeneous trinomial of the 2d degree.

6. Write a homogeneous binomial of the nth degree.

7. Arrange the following, first according to the powers of a, and then of b: $a^6+b^6+6ab^5+15a^4b^2+6a^5b+21a^3b^3+15a^2b^4$.

8. What is the form of the quantity in the 7th? Of what degree is it?

9. What is the difference in meaning between $5(a+b)$ and $5a+b$? If $a=3$, and $b=2$, what is the difference in value?

10. Give the difference in meaning and value between $(c+d)(c-d)$, and $c+d\times c-d$, if $c=10$, and $d=5$.

11. Between $a+3\sqrt{b+c}$ and $(a+3)\sqrt{b+c}$, if $a=4$, $b=6$, and $c=10$.

12. Write the residual of a and b; of a^2 and b^2.

13. Write the residual of a^2 and $\sqrt{a}$; of y and x.

14. Write the reciprocal of each quantity in the 12th and 13th.

15. Write the reciprocals of the answers to the 12th and 13th.

II. NUMERICAL VALUES.

1. EXPRESSIONS WITHOUT RADICALS.—28.

Given $a=4$, $b=3$, $c=2$, *and* $d=1$, *find the values of the following expressions:*

1. $5a+3b-7d$.
2. $3(a+d)+2(b+c)$.
3. $ab+2bc+d$.
4. $5(a-b)+3(2b-3d)$.
5. $ab+10a-7(2b-3d)$.
6. $a-\{b-[c-d]\}$.
7. $5a+2\{b-2(c-d)\}$.
8. $(a+b)(c-d)$.
9. $a^2+b^2+c^2+d^2$.
10. $\frac{ab}{c}+\frac{ac}{d}-\frac{abc}{cd}$.
11. $(a+b)^2(c+d)^2$.
12. $\frac{4b^2-5c^2}{2d+a}$.
13. $\frac{b}{a}+\frac{1}{c+d}$.
14. $\frac{a^3-b^3}{a-b}+\frac{c^2-d^2}{c+d}$.
15. $\frac{(a+c)(b+d)}{2a+d}$.
16. $\frac{b^4}{d^4}\div\frac{(a+5)^2}{c}$.
17. $\left(\frac{b}{d}\right)^4\div\left(\frac{a+5}{c}\right)^2$.
18. $4a-\{a+b+[a+b+c-(a+c-b)]\}$.
19. $2(a^2+b^2)-[(c+d)^2-(c-d)^2]-abcd$.
20. $\frac{bc}{3a}(a^2-b^2)\div(a+b)-\frac{ad}{c(b+d)}$.

2. EXPRESSIONS WITH RADICALS.—28.

Given $a=2$, $b=3$, $c=4$, $d=5$, $y=6$, *and* $x=8$, *find the values of the following quantities:*

1. $\sqrt[3]{\frac{abc}{3}}+\sqrt[5]{\frac{2ab}{3c}}$.

2. $\frac{ab(c-a)}{y-c} - \sqrt{aby}.$
3. $(a+d)\sqrt{c^2-5b} + b\sqrt{3c^2+\frac{1}{3}d}.$
4. $2\sqrt[3]{9b} + 3\sqrt{c} \times \sqrt{\frac{cx}{2}}.$
5. $c \times b\sqrt{4b+2a} - (c-b)\sqrt[3]{3c-4}.$
6. $(2c-b)(\sqrt{4c}+a) + \sqrt{(2c-b)(2a+1)}.$
7. $(b-a)\{\sqrt{6ab}+d^2\} + \sqrt{(3b-5)a^2}.$
8. $\sqrt[4]{x(b-a)+x^2(a+b-c)+b^2}.$
9. $\{(a-\sqrt{c})^2 + (a+\sqrt{c})^2\}^2.$
10. $\sqrt[3]{(b+a)^2(c+1)} + \sqrt[3]{(a+b-d)(2a-2b+d)}.$

3. PROVING EQUATIONS BY MEANS OF NUMERICAL VALUES.

Substitute any values for the letters in the following quantities, and prove that:

1. $a(m+n)(m-n) = am^2 - an^2.$
2. $\frac{x^3-y^3}{x-y} = x^2+xy+y^2.$
3. $(a+b+c)(a+b-c) = a^2+b^2-c^2+2ab.$
4. $x^4+x^2y^2+y^4 = (x^2+xy+y^2)(x^2-xy+y^2).$
5. $\frac{x^3-y^3}{x^2+xy+y^2} = x-y.$
6. $\frac{x^2-x-30}{x+5} = x-6.$
7. $\frac{x^3+y^3+3xy-1}{x^2-x(y-1)+y(y+1)+1} = x+y-1.$
8. $(a-b)^3+b^3-a^3 = 3ab(b-a).$
9. $(x^2+y^2)(m^2+n^2) = (mx+ny)^2 + (nx-my)^2.$

10. $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.
11. $(x+y+z)^3-(x^3+y^3+z^3)=3(x+y)(x+z)(y+z)$.
12. $4ab(a^2+b^2)=(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.

III. ADDITION.

(1) Addition of Simple Terms.—41.

1. $5a-2bc+3cb-7a+2c-7bc-2c+2a$.
2. $10ab+c^2-abd-3ab+3c^2-7ab+5abd-8ab-2c^2+6c^2-4abd$.
3. $a^3+3a-2a^2+a^2+a+1+4a^3+5a^2+2a+3-3a^2-2a-5$.
4. $10x^2y-12x^3yz-15y^2z^4+10-4x^2y+8x^3yz-4-10y^2z^4+20y^2z^4-3-3x^2y-3x^3yz+12x^3yz+5y^2z^4+2x^2y$.
5. $a^2+b^2+c^2-2ab+2ac-2bc+a^2+b^2+c^2+2ab-2ac-2bc+a^2-2b^2-2c^2+4bc$.
6. $\frac{1}{2}a^{\frac{1}{2}}+\frac{2}{3}b^{\frac{2}{3}}+\frac{1}{2}a^{\frac{1}{2}}+b^{\frac{1}{2}}-\frac{2}{3}b^{\frac{1}{2}}-\frac{2}{3}b^{\frac{2}{3}}$.
7. $a+a^2+a^3-3a^3-2a+a^6-3-a^3+2a^2$.
8. $ac^2+a^2c-c^2a-ac^2-ca^2+c^2a-2ac$.
9. $a^3-3a^2b+3ab^2-b^3+a^3+b^3+3a^2b+3ab^2$.
10. $x^3+3x^2y-y^3-6x^2y+2x^3+3xy^2-3y^2x+3yx^2$.
11. $\frac{1}{2}a+\frac{2}{3}b-\frac{3}{4}c-\frac{2}{3}a-\frac{3}{4}b+\frac{1}{2}c+2a-3b-\frac{1}{8}c-\frac{11}{6}a+\frac{2}{4}b+\frac{3}{8}c$.
12. $2x^{\frac{1}{2}}+3x^{\frac{1}{3}}-5x^{\frac{1}{2}}+6x^{\frac{2}{3}}-2x^{\frac{1}{3}}-5x^{\frac{2}{3}}$.
13. $2ab^{\frac{1}{2}}+3a^{\frac{1}{2}}b-2a^{\frac{1}{2}}b^{\frac{1}{2}}+6b^{\frac{1}{2}}a-5ba^{\frac{1}{2}}$.
14. $2\sqrt{a}+3\sqrt[3]{b}-2\sqrt{a}+3\sqrt[3]{a}-\sqrt[3]{b}-3\sqrt[3]{a}$.
15. $3a+\frac{2}{3}a+\frac{1}{3}a+a^{\frac{1}{2}}+a^{\frac{2}{3}}$.
16. $2a^x+3b^{2x}-2a^{2x}-2a^x-3b^{2x}+b^x$.

17. $\frac{ab}{5} - \frac{3xy}{5} + \frac{ab}{2} - \frac{9xy}{10} + \frac{3ab}{10} - \frac{xy}{2} + ab + 2xy.$
18. $2abc + 3abd + 3acd + 3bcd - 3bca - 2bad - 2cbd - 3dac.$
19. $x^4 + 4x^3 + 6x^2 + 4x + 1 + x^3 + 3x^2 + 3x + 1 + x^2 + 2x + 1 + x + 1.$
20. $x^m - 2x^{m-1} + 1^m - 2x^m + 3x^{m-1} + 1 + 3x^m - x^{m-1} - 1^m.$

(2) Addition of Compound Terms.

1. $(a + b) + 2(a + b).$
2. $3(x + y) - 4(x + y) + 5(x + y).$
3. $4(x + z) + 5(x + z) - 7(x + z).$
4. $2(x + y) + c(x + y) - 3(x + y).$
5. $axy + 3xy.$
6. $a(x + y) + b(x + y).$
7. $c(x^2 - y) + d(x^2 - y).$
8. $a(x - y + z) - b(x - y + z).$
9. Combine into two terms, $ma + my + ca + cy.$
10. Combine into one term, $m(a + y) + c(a + y).$
11. Combine into two terms, $ac + a + bc + b.$
12. Combine into one term, $a(c + 1) + b(c + 1).$

Combine the following expressions into one term:

1. $27ab + 6b + 9ac + 2c.$
2. $4ad + 6ab + 8cd + 12bc.$
3. $x^3 + x + 6x^2 + 6.$
4. $3x^2 + 6yz + 2xy + 9xz.$
5. $a^{\frac{1}{2}}b^{\frac{1}{2}} + 2c^{\frac{1}{2}} + b^{\frac{1}{2}}c^{\frac{1}{2}} + 2a^{\frac{1}{2}}.$
6. $15x^3 + 35x^2 + 3x + 7.$
7. $27x^4 + 63x^3 + 12x^2 + 28x.$
8. $ac + 2bx + 2ax + bc.$
9. $x^3 + xy^2 + x^2y + y^3.$
10. $x^3 + x^2 + x + 1.$
11. $x^2 + 3x - bx - 3b.$
12. $2a^2 + 4ac - 3ad - 6cd.$
13. $a^2xy + aby^2 - abx^2 - b^2xy.$

14. $6a^2xy - 3aby + 2acx - bc.$
15. $bdy^2 - acdy + abxy - a^2cx.$
16. $2bdy^2 - 3cd^2y + 4bcxy - 6c^2dx.$
17. $3x^3 + x + 1 + 3x^2.$
18. $2bdy^2 + 3cd^2y - 4bcxy - 6c^2dx.$
19. $\frac{2}{ab} - \frac{2}{b} - \frac{b}{a} + b.$
20. $\frac{ax}{by} - \frac{bx}{cy} - \frac{a^2}{bx} + \frac{ab}{cx}.$
21. $a^5 + a^4 + a^3 + a^2 + a + 1.$
22. $10xz + 4yz + 2z + 5mx + 2my + m.$
23. $x^4 - x^3 + 2x^2 - x + 1.$ (N. B. $2x^2 = x^2 + x^2$).
24. $a^3 + 3ab - ab^2 + a^2b + 3b^2 - b^3.$
25. $a^{-2}c^{-1} - c^{-1}b^{-3} + d^{-4}a^{-2} - b^{-3}d^{-4}.$
26. $\frac{1}{abcd} - \frac{ab}{cd} + \frac{cd}{ab} - abcd.$
27. $2ax + 4az - 6ay - 4bx - 8bz + 12by.$
28. $(a-b)x + (3c-d)y + (b-2a)x - (3c+d)y + ax - (d-c)y.$

IV. SUBTRACTION.

(1) Subtraction of Simple Terms.—45.

1. From $a+b+c$ take $b+c.$
2. From $a+b-c$ take $b-c.$
3. From $a+b-c$ take $c-b.$
4. From $a-b+c$ take $-b-c.$
5. From $a+b+c$ take $-a-b-c.$
6. From $2x+y-z$ take $x-z.$
7. From $2x+y-z$ take $x+z.$
8. From $2x-y+z$ take $x-y.$

9. From $2x+y-z$ take $x+z+y$.
10. From $2x+y-z$ take $2x+y+z$.
11. From $2x+y+z$ take $-2x-y-z$.
12. From $2x+y+z$ take $-2x+y-z$.
13. From $x^2+ax+3a^2$ take a^2-ax-x^2.
14. From $\frac{1}{2}x+\frac{2}{3}y-\frac{3}{4}z$ take $\frac{1}{2}x-\frac{2}{3}y-\frac{1}{2}z$.
15. From $\frac{x}{2}+\frac{y}{3}-\frac{z}{4}$ take $-\frac{3x}{2}+\frac{5y}{3}-\frac{3z}{8}$.
16. From x^4+3x^2-2x-5 take $-x^4-3x^3-12$.
17. From x^4+2y^3 take $2x^2y+x^2y^2$.
18. From $x^3-3x^2y+3x^2y^2$ take $-x^2y+5x^2y^2-y^3$.
19. From $2x^3-y^3$ take $x^3+x^2y-xy^2$.
20. From the sum of $3x^2+6xy-2y^2$ and x^2+3y^2 take the difference between $2x^2-5x-y$ and $x^2-6x-7y$.
21. From the difference between $7a-2b$ and $2b-4c$, take the difference between $5c-7a$ and $2a-3b$.

(2) Subtraction of Compound Terms.

1. From $ax+by+cx$ take $(1+a)x+by$.
2. From $(a+b)x$ take $(a-b)x$.
3. From $(a+b)x$ take $(a+b)y$.
4. From $(2a-b)(x+y)$ take $(a-2b)(x+y)$.
5. From $(2a-b)(x+y)$ take $(2a-b)(x-2y)$.
6. From $(a+b-c)d$ take $(2a-b+c)d$.
7. From $(3a-c)(x^2-y)+(2a+c)(x^2-y)$ take $(4a-c)(x^2-y)$.
8. From $(a+b)\sqrt{x+y}$ take $(2a-3b)\sqrt{x+y}$.
9. From $(a^2+ab+b^2)x^2-(a-b)x+1$ take $(a^2-ab+b^2)x^2+(a+b)x-1$.
10. From $(x^2+x+1)(a+b)+3(x+y)-2$ take $(x^2-2x+1)(a+b)-7(x+y)-1$.

11. From $(x^2 - y^2)\,4x + (x + y)\,(x^2 - y^2)$ take $(x^2 + y^2)\,4x - (x + y)\,(x^2 + y^2)$.
12. From $(3b + 2c - 1)\,m$ take $(b - 2c + 4)\,m$.
13. From $(a + x)\,m^2$ take $(a + x)\,n^2 - 1$.
14. From $(a^{\frac{1}{2}} - b)\,(m^x - y^x)$ take $(a^{\frac{1}{2}} - b)\,(m^x + y^x)$.
15. From $7\,(x - y)$ take $4\,(y - x)$.
16. From $(4a + 2b)\,(2m - n)$ take $(n - 2m)\,(3a - b)$.
17. From $(2a - b)\,(m - 2n)$ take $-(b - 2a)\,(3m - 2n)$.
18. Simplify $(3a - 2b)\,(x - y) - (5a + 3b)\,(y - x) + (2x + y)\,(8a + b)$.
19. Simplify $(5x - 3y)\,(m - 2n) - (3y - 5x)\,(-3m - n) + (2x + 3y)\,(2m + 3n)$.
20. Simplify $(3a - 2b)\,(2m + n) + (3n - m)\,(2b - 3a) + (3m - 2n)\,(a + b)$.
21. Simplify $(2x - 3y)\,(a + b) + (3y - 2x)\,(2b - 3a) + (b - 4a)\,(x + y)$.
22. Simplify $(x - 2y)\,(a + b) + (2y - x)\,(3a - b) - 2\,(a - b)\,(2y - x)$.
23. Simplify $(m - 3n)\,(2x - a) - (3m - 9n)\,(x + a) + 2\,(4a + x)\,(m + n)$.
24. Simplify $3\,(x - y)\,(3 + a) + (2y - 2x)\,(2 - a) - (3x + 2y)\,(1 + a)$.

(3) Removal of Parentheses.—46.

Remove the parentheses from the following quantities, and simplify the results:

1. $a + (b - c)$.
2. $a - (b + c)$.
3. $a - (b - c)$.
4. $a - [b + (c + d)]$.
5. $a - [b - (c - d)]$.
6. $a - [(b - c) - d]$.
7. $a + b - \{a - [-(a - b)]\}$.
8. $a - (x - a) - [x - (a - x)]$.

9. $4a^3b + \{a^2 + [-4a^2 + (2a - 3b)]\}$.
10. $a - b - \{a - b - [a - b - (a - b - \overline{a - b})]\}$.
11. $(x + y) + (x - y) - (x + \{y - [x - y]\})$.
12. $2a - \{2a - [2a - (2a - \overline{2a - a}) - a] - a\} - a$.
13. $ab - (cd + ab - [cd - ad] - ab) - ad$.
14. $x - \{y - [x - (x - \overline{y - x}) - y] - x\} - y$.
15. $2a - \{(3a - 2b) - [3a - 2b - (-b)] - 2a\}$.
16. $a - \{2b + [3c - (2b + \overline{2a + b})] - 3c\}$.
17. $x^4 - \{4x^3 - [6x^2 - (4x - 1)]\} - [x^4 + 4x^3 + 6x^2 - (4x - 1)]$.
18. $x - (y - z) + (2x - [3y - 2z]) - \{x - [y - (z - x)]\}$.
19. $a - [5b - \{a - (5c - \overline{2c - b}) - 4b + 2a - (a - \overline{2b + c})\}]$.
20. $a - \{a + b - (a + b - c) - [a + b + c + d] + d - (c + d)\} + c$.
21. $x - [3y - \{2x - (3x + 5y) + 2y\} - (3x + y)] - y$.
22. $-(2x^2 - 3x + 1) - (3x^2 - 3x - 1) - [2 - (x^2 - x - 5)]$.
23. $1 - (1 - a + [1 - (a - a^2)] - \{1 - a + \overline{a^2 - a^3}\})$.
24. $a - [2b + \{3c - 3a - (a + b)\} + 2a - (b + 3c)] - (a + b)$.
25. $a - \{3b - c + [a - (c - \overline{b - a}) + b] - a\} - \{-(a + b) - (a + 3b)\}$.
26. $2x - [-3y - \{4x - (5y - \overline{-3x - 6y}) + 3y\} - 5x]$.
27. $-(5x - 7y) - [-(2x - 3y)] - \{2x - (7y - 2x) + (3y - x)\}$.
28. $1 - (1 - a + [1 - (a - a^2)] - \{1 - a - \overline{a^2 - a}\})$.
29. $a - \{-b - [(6a - b) - (2a - 3b) - \overline{5a + 6b}]\}$.
30. $-(x - y) - \{-(x + y) + [(x - y) - (x - y)] - [(x - y) - 2x - \overline{3y + x}]\}$.

V. MISCELLANEOUS PROBLEMS.

Given:

$(d) = 2(a-2)x^4 - (a+2)x^3 + 3(-a+2)x^2.$
$(e) = -3(-a+1)x^4 + 2(-a+1)x^3 - 2(-a+3)x^2.$
$(f) = -(a+1)x^4 + (-a+8)x^3 + 2(-a+3)x^2.$
$(g) = -(a-1)x^4 - (a+1)x^3 - (-a-1)x^2.$

Substitute and simplify the following quantities:

1. $(d)+(e)$.
2. $(d)+(f)$.
3. $(d)+(e)+(f)$.
4. $(d)+(e)-(f)$.
5. $(d)-(f)$.
6. $(e)+(f)$.
7. $(e)+(f)-(d)$.
8. $(g)+(d)$.
9. $(g)+(e)$.
10. $(g)+(f)$.
11. $(g)+(e)+(f)$.
12. $(g)+(d)-(f)$.
13. $(g)-(d)$.
14. $(e)+(f)-(g)$.
15. $\{(d)+(e)\}-\{(f)+(g)\}$.
16. $\{(d)+(e)\}-\{(d)-(e)\}$.
17. $\{(e)+(f)\}-\{(d)-(g)\}$
18. $\{(e)+(g)\}-\{(d)+(f)\}$.
19. $(d)+(e)+(f)+(g)$.
20. $(d)-(e)-(f)-(g)$.

VI. MULTIPLICATION.

(1) A Monomial by a Monomial.—57

Simplify the following expressions:

1. $3x \times 3y \times 2a$.
2. $5a \times 3a^2$.
3. $6ab \times 2ac$.
4. $(5a^3b^3)(2a^{-2}b^5)$.

5. $(-2a^4)\,(-3a^2b)$.
6. $(-3a^2)\,(-5a^5b^3)$.
7. $(2a^3b)\,(-3a^2b^{-3})$.
8. $(-3^2)\,(-3^{-3})\,(3^{-1})$.
9. $-x^2y^3z^{-2} \times -x^{-4}y^{-5}z^4$.
10. $3a^{\frac{2}{3}} \times 2a^{\frac{1}{6}} \times 3\,a^{\frac{7}{6}}$.
11. $(a+b)^2\,(a+b)^3$.
12. $(x+y)^3\,(x+y)\,(x-y)\,(x-y)^2$.
13. $(-a^{x-1})\,(a^{y+1})\,(y^{a-1})\,(y^{2-a})$.
14. $(x^{a+b})\,(-x^{a-b})$.
15. $-3\,(x+y)^2 \times -5\,a\,(x+y)$.
16. $(a+b)\,(x^2-y^2)\,(a+b)^2\,(x^2-y^2)^3$.
17. $-5\,(a^2+b^2)\,(x+3) \times -2\,(a^2+b^2)\,(x+3)$.
18. $a^{-2}\,(ax-by)^2 \times a^3\,(ax-by)^3$.
19. $3a^2b\,(2a-r) \times -4a^3b^2\,(2a-r)$.
20. $\frac{a}{x}(4+3b)\,\frac{x}{a}(3b+4)$.
21. $\frac{a^2}{x^2}(3+2b-a)\,\frac{x^3}{a^3}(2b-a+3)$.
22. $\frac{a^{-2}}{b^{-3}}(x^2-y^3)\frac{a^3}{b^4}(x^2-y^3)^2$.
23. $a\,(x+z)\,a^{2m}\,(x+z)^m$.
24. $a^{m-1}\,(x^2-y)^n a^{m+2}\,(x^2-y)^{1-n}$.
25. $x^2y^{\frac{1}{3}}\,(a+b-c)^{\frac{1}{3}}\,x^{-1}\,y^{\frac{2}{3}}\,(a+b-c)^{\frac{2}{3}}$.
26. $3^{\frac{1}{2}}a^2b\,(x-4+y)^3 \times 3^{\frac{3}{2}}a^{-3}b\,(x+y-4)^{-2}$.
27. $(a-3b^2)\,3\,(x-y)^{\frac{1}{2}}\,(a-3b^2)\,4\,(x-y)^{\frac{5}{2}}$.
28. $(x+y)^{\frac{1}{5}}\,(x^3-y^4)^{\frac{1}{4}}\,5\,(x+y)^{\frac{3}{10}}\,(x^3-y^4)^{\frac{1}{2}}$.
29. $(a-2b)^2\,(x-y) \times (y-x)^3\,(2b-a)$.
30. $(x-y)^3\,(2a-3b) \times (2x-2y)\,(3b-2a)$.
31. $5\,(x-y)^{-2}\,(m+n)^{\frac{1}{3}} \times 3\,(x-y)^3\,(m+n)^{-\frac{4}{3}}$.
32. $a^{m-2}.\ b^{x-y}.\ 3a^{2m+3}.\ b^{2x+2y}$.
33. $5a^{x+y-z}.\ b^{m-n}.\ 2a^{y+z-x}.\ b^{n-m+1}$.
34. $2\,(x-y).\ (m-n)^2.\ (y-x)^3.\ (n-m)$.

(2) A Polynomial by a Monomial.—58.

Expand the following expressions:

1. $(x+y)\,x$.
2. $(x+y-z)\,x^2$.
3. $(a-2b-3c)\,(-4c)$.
4. $(x^{-2}-2x+3)2x^3$.
5. $-(a-b-c)\,(-2abc)$.
6. $(x^{\frac{1}{2}}-2x^{\frac{1}{3}}-1)\,3x^{\frac{1}{2}}$.
7. $(\frac{1}{2}x^2-3x^{-2}+3)\,6x^{-4}$.
8. $[a-(b-c)]\,(-ac)$.

Expand and simplify:

9. $(a-b)\,a-a^2+ab$.
10. $5\,(x^2-y^2)-3\,(x^2+2y^2)+9y^2$.
11. $3\,(x^2+y^2)-[(x^2+2xy+y^2)-(2xy-y^2-x^2)]$.
12. $2\,(a-3b)-3\,(a-4b)+a-6b$.
13. $3a\,(a-2b)-2\,\{a^2-4ab+2\,[a^2-ab-(a^2-2ab)]\}$.
14. $2a\,(a^2-2ab+b^2)-a^3-2a^2b-2b^3+a\,[a^2+2b\,(3a-b)]$.
15. $2a-2\,\{a-2\,(a-2-\overline{a-b})\}$.

(3) A Polynomial by a Polynomial.—61.

Expand:

1. $(x+y)\,(x+y)$.
2. $(x+2y)\,(x+3y)$.
3. $(x+y)\,(x-y)$.
4. $(a+5b)\,(a-3b)$.
5. $(2a+b)\,(2a-b)$.
6. $(3a+5b)\,(2a+7b)$.
7. $(5x^2-3x)\,(5x^2+3x)$.
8. $(a^2+ab+b^2)\,(a-b)$.
9. $(a^2-ab+b^2)\,(a+b)$.
10. $(1-x^2+x^4)\,(1+x^2)$.
11. $(x^2-2x+1)\,(x^2+2x+1)$.
12. $(a^3+3a^2b+3ab^2+b^3)\,(a^3-3a^2b+3ab^2-b^3)$.

13. $(x^2 - xy + 2y^2)\,(x^2 - 2y^2 + xy)$.
14. $(x^3 + 2x^2 + 4x + 8)\,(x - 2)$.
15. $(x^2 + y^2 + z^2 + xy + xz - yz)\,(y + z - x)$.
16. $(9a^2 + b^2 + 3ab - 6a + 4 + 2b)\,(3a + 2 - b)$.
17. $(x^2 + y^2 + z^2 - xy - xz - yz)\,(x + y + z)$.
18. $(a^3 + 3a - 2a^2 + 5)\,(a^2 - 2 + 2a)$.
19. $(a^3 - 4a^2 - 24 + 11a)\,(5 + a^2 + 4a)$.
20. $(1 + 2x + 3x^2 + 4x^3 + 5x^4)\,(1 - 2x + x^2)$.
21. $(x^5 + x^4 + x^3 + x^2 + x + 1)\,(x - 1)$.
22. $(x^3 + 3x^2y + 9xy^2 + 27y^3)\,(x - 3y)$.
23. $(1 - 3x + 3x^2 - x^3)\,(1 + 2x + x^2)$.
24. $(x^2 + y^2 - xy + x + y)\,(x + y - 1)$.
25. $[a^2 - x(a - b) + b^2]\,(a + b + x)$.
26. $(x - 3)\,(x + 4)\,(x + 3)\,(x - 4)$.
27. $(x^{\frac{1}{2}} + y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}})\,(x^{\frac{1}{2}} + y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}})$.
28. $(c^{4n} - c^{2n})\,(c^{2n} + 1)$.
29. $(a^{2n} - 2a^n b^n + b^{2n})\,(a^n - b^n)$.
30. $(a^{2m} - 2a^m b^m + b^{2m})\,(a^{2m} + b^{2m})$.
31. $(x^{2n} + 4x^n + 4)\,(x^{2n} - 3x^n + 3)$.
32. $(x^n - 1)\,(x^n + 1)\,(x^{2n} + px^n + q)$.
33. $(\frac{2}{3}a^2 + 2ab + \frac{1}{2}b^2)\,(3a^2 - ab + 2b^2)$.
34. $(4 - 5x^2 + \frac{1}{4}x)\,(2 - x + \frac{x^2}{2})$.
35. $(x + a)\,(x + b)\,(x + c)$.
36. $(x + y + z)\,(x + y - z)\,(x - y + z)\,(y + z - x)$.
37. $(a + b)\,(b + c) - (c + d)\,(d + a) - (a + c)\,(b - d)$.
38. Prove that $x(x + 1)\,(x + 2)\,(x + 3) + 1 = (x^2 + 3x + 1)^2$.
39. Prove that $(a + b + c)\,(a^3 + b^3 + c^3 + abc) - (ab + bc + ac)\,(a^2 + b^2 + c^2) = a^4 + b^4 + c^4$.
40. Prove that $(a^{-4} - a^{-2} + 1)\,(a^{-4} - 1) + a^{-2}(a^{-2} - 1)\,(a^{-2} + 1) = (a^{-2} - 1)\,(a^{-6} + a^{-4} + a^{-2} + 1)$.

(4) Binomials having one term alike in each.

Expand:

1. $(x+5)\,(x+1)$.
2. $(x+2)\,(x-3)$.
3. $(x-3)\,(x-4)$.
4. $(x+7)\,(x+7)$.
5. $(x-5)\,(x-5)$.
6. $(x+13)\,(x-1)$.
7. $(x+1)\,(x-13)$.
8. $(x-3)\,(x+3)$.
9. $(x+y)\,(x+y)$.
10. $(a-b)\,(a-b)$.
11. $(x+y)\,(x-y)$.
12. $(3x-y)\,(3x-y)$.
13. $(2a-7)\,(2a-7)$.
14. $(3a-b)^2$.
15. $(3a+b)^2$.
16. $(3a+b)\,(3a-b)$.
17. $(2x+3)\,(2x+5)$.
18. $(2a-7)\,(2a-3)$.
19. $(1+b)\,(1-b)$.
20. $(2-b)\,(2+3b)$.
21. $(7+2b)\,(7-2a)$.
22. $(2a+3b)\,(2a+5b)$.
23. $(a-3b)\,(5a-3b)$.
24. $\left(x+\frac{1}{x}\right)^2$.
25. $\left(2x-\frac{x}{2}\right)\left(2x+\frac{x}{3}\right)$.
26. $(a^3-2ab)\,(a^3+2ab)$.
27. $(a^{\frac{1}{2}}-b^{\frac{1}{2}})\,(a^{\frac{1}{2}}-3b^{\frac{1}{2}})$.
28. $(a^{\frac{2}{3}}+3b)\,(a^{\frac{2}{3}}-3b)$.
29. $(a^x-3b^2)\,(a^x+4b^2)$.
30. $(x-\frac{1}{3}b)\,(x+\frac{2}{3}b)$.
31. $\{(a-b)\,(a+b)\}^2$.
32. $\{(a-b)^2+(a+b)^2\}^2$.
33. $\{(a-b)^2-(a+b)^2\}^2$.
34. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$.
35. $\left(\frac{x}{2}-\frac{2}{x}\right)^2$.
36. $(x^{-3}+x^2)^2$.
37. $(\frac{1}{2}x^2-2x^{-3})^2$.
38. $(a^x-b^y)\,(a^x+b^y)$.
39. $(a^x+b^y)^2$.
40. $(a^m+b^r)\,(a^m-b^r)$.
41. $(2x^{\frac{1}{2}}-3y^{\frac{1}{3}})^2$.
42. $(2^{-1}+\frac{1}{3})^2$.
43. $(3^{-2}-2^{-3})^2$.
44. $(a-3^{\frac{1}{2}})\,(a+3^{\frac{1}{2}})$.
45. $\left(\frac{x^{-3}}{2}-\frac{x^{-2}}{3}\right)^2$.
46. $(x+2)^2-(x-2)^2$.
47. $(x+3)\,(x+4)-(x+3)^2=?$
48. $(a-5)\,(a-6)+(2a-4)\,(2a+4)-(3a-5)\,(3a-5)=?$

49. $(x-1)(x-2)+(x-2)(x-3)-(x-7)(x-1)=?$
50. $[(a+b)^2+(a-b)^2][(a+b)^2-(a-b)^2]=?$
51. $[4xy^2(x+2y)-\{(x+y)^2-(x+y)(x-y)\}^2]^2=?$
52. $[(a+b)^2+(a-b)^2][(a+b)^2+(a-b)^2]=?$
53. From $(a+b+c)(a+b-c)$ take $a^2-\{b^2-[2b^2-(-2ab+c^2)]\}$.

(5) Powers of Monomials.—172, I.

Expand:

1. $(a^2)^4$; $(-2a^2)^3$; $(-a^4)^5$; $(-\frac{1}{2}x^6)^3$; $(-a^2)^4$.
2. $(2a^2b^{\frac{1}{3}})^3$; $\{2(-c^{\frac{1}{3}})^3\}^2$; $(a^{\frac{2}{3}}.\ a^{\frac{3}{4}})^{12}$; $(a^{\frac{2}{3}}.\ a^{-\frac{4}{5}})^6$.
3. $(a^{\frac{6}{7}}.\ a^{\frac{2}{7}})^{\frac{7}{8}}$; $(x^{\frac{2}{3}}.\ x^{\frac{4}{7}})^{\frac{14}{15}}$; $(a^{\frac{1}{2}}.\ b^{\frac{1}{3}})^{\frac{2}{3}}$.
4. $(a^{-\frac{1}{2}}.\ 3a^4b^{\frac{1}{2}})^2$; $(x^{\frac{4}{5}}.\ x^{-\frac{3}{5}}.\ x)^{10}$; $(a^{\frac{1}{m}}.\ b^{\frac{1}{m}})^{3m}$.
5. $2(a^3b^2c^{-2})^{-2}$; $\frac{1}{2}(a^{\frac{1}{2}}b^2c^{-3})^{-3}$; $(3x^2)^3\,2y$.
6. $-2(-2x^2y)^4(-3xy^2)^3$; $4(2x)^4(3y)^2$; $4(2a)^3(3x)$.
7. $\{[2(3a)^2]^2(a)\}^3$; $\left\{\left[\left(\frac{a^{-m}}{b^{-n}}\right)^2\right]^{\frac{p}{m}}\right\}^{\frac{n}{p}}$.

(6) Squares of Polynomials.

Find the squares of the following polynomials:

1. $a+b+c$.
2. $x+y+1$.
3. a^2+2a+1.
4. $2x+3y-z$.
5. $a^2-1-\dfrac{1}{a^2}$.
6. $2a-3b-4c+d$.
7. $2a^{-2}+3a^{-1}b^{-1}+b^{-2}$.
8. $3a^{-1}+a-2a^2$.
9. $\frac{1}{2}a^{\frac{1}{2}}+\frac{2}{3}b^{\frac{1}{2}}-3c^{\frac{1}{3}}$.
10. $a^x+b^y+c^z$.
11. $a^{-x}+b^{-2x}+c^{-y}$.
12. $\frac{3}{2}h^3-5h^2+\frac{1}{4}h+9$.
13. $a^m+b^p-2c^m$.

14. $2^3-3^2+4^{\frac{1}{2}}$.

15. $\frac{a}{2}+\frac{2}{a}-1$.

16. $\frac{a^{-2}}{z^{-1}}+\frac{z^{-1}}{a^{-2}}+\frac{1}{2}$.

17. $[\frac{1}{2}(a-b)+2(a+b)+c]$.

18. $x^{\frac{1}{2}}-y^{\frac{1}{2}}+z^{\frac{1}{2}}$.

19. $x^{\frac{1}{3}}-x^{\frac{2}{3}}-x$.

20. $a+a^m+a^{2m}$.

21. $\frac{2}{a}+1+2a$.

22. $2a^{-2}-2+a$.

23. $a^{-\frac{2}{3}}+a^{\frac{2}{3}}-a^2$.

24. $1+x^3y^{-2}-x^3y$.

25. Expand and simplify

(A) $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)$.

(B) $(a+b+c+d)^2+(a-b-c+d)^2+(a+b-c-d)^2+(a-b+c-d)^2$.

26. Prove that $4xy(x^2+y^2)=(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$.

27. Prove that $4xy(x^2-y^2)=(x^2+xy-y^2)^2-(x^2-xy-y^2)^2$.

(7) Higher Powers of Polynomials.—172, IV.

Expand the following polynomials:

1. $(a+b)^3$.
2. $(a-b)^4$.
3. $(x-y)^5$.
4. $(x+y)^7$.
5. $(x^2-y^3)^3$.
6. $(x^2-1)^4$.
7. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^4$.
8. $(x^{\frac{1}{2}}y^{\frac{1}{2}}-1)^5$.
9. $(x^{\frac{2}{3}}+y^{\frac{5}{3}})^3$.
10. $\left(\frac{1}{x}-\frac{1}{y}\right)^4$.
11. $(a^{-1}+b^{-2})^4$.
12. $(a^{-2}+c^{-4})^6$.
13. $(x+2y)^{-3}$.
14. $(2x-3a)^{-4}$.
15. $\left(3b-\frac{a}{3}\right)^3$.
16. $\left(\frac{x}{2}-\frac{3y}{4}\right)^3$.
17. $(2a+3b^{-2})^3$.
18. $(x+y+z)^3$.
19. $(x-y+z)^3$.

20. $(2x+y-1)^3$.

21. $\left(2x^2+\frac{x}{2}+1\right)^4$.

22. $(a+b-c-d)^2$.

23. $(a+b)^{-4}$.

24. $(2a+3b)^{-3}$.

25. $(x+y)^3-(x-y)^3=?$

26. $(x+y)^4-4xy(x+y)^2+2x^2y^2=?$

27. Substitute $y+3$ for x in $x^4+2x^2-x^3-3$.

28. Expand $\{(x+y)^3+(x-y)^3\}^2-\{(x+y)^3-(x-y)^3\}^2$.

29. Prove that $\{(a-b)^2+(b-c)^2+(c-a)^2\}^2=2\{(a-b)^4+(b-c)^4+(c-a)^4\}$.

30. Prove that $(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3+a^3+b^3+c^3=6abc$.

31. Prove that $x^3+y^3+z^3+3xyz=(x+y+z)^3-\{3x^2(y+z)+3x(y^2+z^2)+3yz(x+y+z)\}$.

32. Prove that $(x+y+z)^3-(x^3+y^3+z^3)=3(y+z)(x+z)(x+y)$.

33. Show that, in the expansion of $(a+b)^n$, the sum of the co-efficients of the *odd* terms is equal to the sum of the co-efficients of the *even* terms, and that each sum is 2^{n-1}.

34. In the expansion of $(2a+3b)^n$, show that the sum of the co-efficients of the *odd* terms is $\frac{5^n+(-1)^n}{2}$, and of the *even* terms is $\frac{5^n-(-1)^n}{2}$.

35. In the expansion of $(a+3b+2c)^n$, show that the sum of the co-efficients of terms containing *odd* powers of b is equal to the sum of the co-efficients of the remaining terms.

36. In the expansion of $(a+b)^n$, show that the sum of the exponents of a in the various terms is $\frac{n(n+1)}{2}$.

VII. DIVISION.

(1) A Monomial by a Monomial.—71.

Perform the operations indicated in the following expressions:

1. $75a^2b^4 \div 15ab^4$.
2. $28a^6b^3 \div 4a^5b^2$.
3. $6a^2bc \div 3ab$.
4. $28a^6b^3 \div -4a^3b^3$.
5. $\frac{1}{2}a^3b^3c \div \frac{1}{4}ab^2c$.
6. $-m^2n^2 \div -mn^3$.
7. $\frac{a^2b^3d^{-2}}{2} \div -2a^2d^3$.
8. $2x^{\frac{1}{2}}y^{\frac{2}{3}}c \div \frac{x^{-\frac{1}{2}}y^{-\frac{1}{3}}c}{3}$.
9. $\frac{1}{4}m^{-2}n^{-3} \div \frac{1}{2}m^{-3}n^{-4}$.
10. $\frac{2c^4d^{\frac{1}{2}}}{3} \div \frac{5c^{-1}d^{-\frac{1}{2}}}{6}$.
11. $\frac{5a^{\frac{2}{3}}b^{\frac{3}{5}}c}{6} \div \frac{a^{-\frac{1}{3}}b^{-\frac{2}{5}}}{6}$.
12. $5(a+b)^3 \div 2(a+b)^2$.
13. $12(a+b)^2(x-y)^3 \div 2(a+b)(x-y)^2$.
14. $\frac{6(a+b)^3(a-b)^2(x+y)}{2(a+b)(a-b)(x+y)}$.
15. $\frac{3(a+b)^2(a-b)(x-y)^3}{4(a+b)^4(a-b)(x-y)^3}$.
16. $x^2(a-2b)^4 \div x(a-2b)^2$.
17. $x^{-3}(a+3)^{-2} \div x^{-4}(a+3)^{-3}$.
18. $(a-c)^{-3}(a+b)^{-2} \div (a-c)^{-5}(a+b)^{-3}$.
19. $(a+x)^2\,3(a-x)^2 \div (a+x)^{-3}\,6(a-x)^{-1}$.
20. $3x^0y^ab^3 \div xy^{-a}b^{-3}$.
21. $a^{n+1}b^{n-1} \div a^{n-1}b^{n+1}$.
22. $ax^{m-2}y^2 \div a^2x^{2-m}y^{-2}$.
23. $(x-2a)(x+2a)^2 \div (x-2a)^{-1}$.
24. $\frac{m(x^2-y)^3(x+y)^2}{n(x^2-y)^{\frac{1}{2}}(x+y)^4}$.

25. $\dfrac{x^{-3}(m^2-n^2)^4(a+b)^{-\frac{1}{3}}}{\frac{1}{3}x^{-5}(m^2-n^2)(a+b)^{-\frac{4}{3}}}$.

26. $x^{2m}y^n \div x^m y^{2n} + x^m y^n \div x^{2m}y^{-n}$.

27. $[(a^2-b^2)^{-2}4(a-b)^{-1}] \div (a^2-b^2)\,2(a-b)^2$.

28. $3^2(2a-b)^3 x^{-1} \div 3^{-1}(2a-b)^{-2}x^{-3}$.

(2) A Polynomial by a Monomial.—74.

Divide:

1. $3a^2-6a^2y-3ay^2$ by $3a$.
2. $-5x^3-15x^2+10x^4$ by $-5x^2$.
3. a^2b-ab by $-ab$.
4. $x^5-x^4-x^3+x^2-x$ by $\dfrac{x}{3}$.
5. $3x^2by-6bxy^2-9x^2b^2y^2$ by $3bxy$.
6. $20a^2bc^2-30a^3b^2c^3-40a^4b^3c^4$ by $-10abc$.
7. $2x^{-3}-3x^{-2}+x^4-x^{-3}$ by x^{-3}.
8. $\frac{1}{2}x^2-3x^{-2}+x^4$ by $\frac{1}{6}x^{-2}$.
9. $3(a+b)^3-2(x+y)(a+b)^2-(a+b)$ by $a+b$.
10. $(x+1)^3-2(x+1)^2+x+1$ by $x+1$.
11. $(m-n)^2xy+(m-n)x^2y-m+n$ by $m-n$.
12. $ma(x-y)^2-m^2b(x-y)^3-abm(x-y)$ by $b^2m(x-y)^2$.
13. $3(x-y)^3(a-b)^2-4(x-y)(a-b)+(x-y)^{-3}$ by $(x-y)(a-b)^2$.
14. $(x+y)^{\frac{1}{2}}-(x+y)mn+x+y$ by $(x+y)^{-\frac{1}{2}}$.
15. $(a+b+c)^m-x(a+b+c)^{m-n}-y(a+b+c)^2$ by $(a+b+c)^m$.
16. $12(x^2+y^2)^{-m}-18(x^2+y^2)^{-3m}$ by $6(x^2+y^2)^{-4m}$.

(3) A Polynomial by a Polynomial.—76.

Divide:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $x^2 + 2x + 1$ by $x + 1$.
3. $m^2 - 2mn + n^2$ by $m - n$.
4. $a^2 - b^2$ by $a + b$.
5. $4a^2 - 9b^2$ by $2a - 3b$.
6. $x^4 - 16y^4$ by $x^3 + 2x^2y + 4xy^2 + 8y^3$.
7. $27a^3 + b^3$ by $9a^2 - 3ab + b^2$.
8. $64x^6 - y^6$ by $2x - y$.
9. $x^3 + 8$ by $x + 2$.
10. $x^4 + 64$ by $x^2 + 4x + 8$.
11. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
12. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
13. $a^3 - 2ay^2 + y^3$ by $a - y$.
14. $81x^4 - y^4$ by $3x - y$.
15. $a^5 + b^5$ by $a + b$.
16. $12x^4 - 23x^2 + 34x + 7$ by $3x^2 + 6x + 1$.
17. $x^3 - 40x - 63$ by $x - 7$.
18. $x^4 + x^3 - 4x^2 + 5x - 3$ by $x^2 + 2x - 3$.
19. $x^6 - y^6$ by $x^3 - 2x^2y + 2xy^2 - y^3$.
20. $x^4 + 24x + 55$ by $x^2 + 4x + 5$.
21. $x^5 + 151x - 264$ by $x^2 - 4x + 11$.
22. $x^6 + 1008x + 720$ by $x^3 - 6x^2 + 12x + 12$.
23. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$ by $x^4 + 2x^3 + 3x^2 + 2x + 1$.
24. $x^3 + y^3 + 3xy - 2x - 2y + 1$ by $x + y - 1$.
25. $7x^3 - 24x + 58x^{-1} - 21x^{-3}$ by $7x - 3x^{-1}$.
26. $x^4 + x^3 - 9x^2 - 16x - 4$ by $x^2 + 4x + 4$.
27. $32a^4 + 54ab^3 - 81b^4$ by $2a + 3b$.
28. $x^5 - 5x^4 + 4x^3 + 3x^2 - 12x$ by $x^2 - 4x$.
29. $a^6 - y^6 - 3a^2y^2 - 1$ by $a^2 - y^2 - 1$.

30. $a^4-4a^3-19a^2+106a-120$ by a^2-5a+6.
31. $5xy^4-5x^4y+10x^3y^2-10x^2y^3+x^5-y^5$ by x^2+y^2-2xy.
32. $3x^4+2x^3+x^6-4x^5+4x^2-15$ by x^3-x^2-3.
33. $x^4-x^3+x^6+x^2+1$ by x^2+x+1.
34. $4x^3-32x^2+3x^4+49$ by $2x-3x^2+7$.
35. $x^3y^2+x^5-x^4y+xy^4-y^5-x^2y^3$ by x^2+xy+y^2.
36. $x^4+2x^2y^2+9y^4$ by $x^2-2xy+3y^2$.
37. $x^3+x^2y+x^2z-xyz-y^2z-yz^2$ by x^2-yz.
38. $x^3-8y^3+125z^3+30xyz$ by $x-2y+5z$.
39. $(a+b-c)(b+c-a)(a-b+c)$ by $a^2-b^2-c^2+2bc$.
40. $\frac{11ab}{3}-\frac{5a^2}{9}-\frac{10ac}{3}+\frac{15b^2}{4}+25bc$ by $5b-\frac{2a}{3}$.
41. $7x^3y^{-3}-2x^6y^{-6}-9+3x^{-3}y^3+9x^{-6}y^6$ by $2-x^3y^{-3}-3x^{-3}y^3$.
42. $x^4-px^3+(q-1)x^2+px-q$ by x^2-1.
43. $x^4-qx^3+(p-1)x^2+qx-p$ by x^2-qx+p.
44. $abx^3+(ac-bd)x^2-(ak+cd)x+dk$ by $ax-d$.
45. $(x^3-1)a^3-(x^3-x^2)a^2+(2x^2+3x+4)a+3x+3$ by $(x-1)a^2-(x-1)a+3$.
46. $(a+b+c)(ab+bc+ac)-abc$ by $a+b$.
47. $(a^2-bc)^3+8b^3c^3$ by a^2+bc.
48. $(a-b)x^3+(b^3-a^3)x+ab(a^2-b^2)$ by $(a-b)x+a^2-b^2$.
49. $ax^2-ab^2+b^2x-x^3$ by $(x+b)(a-x)$.
50. $(x^3+y^3)(x^2-y^2)$ by $(x^2-xy+y^2)(x-y)$.
51. $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2$ by x^2+y^2.
52. $a^3+b^3-1+3ab$ by $a-1+b$.
53. $x^{-\frac{3n}{2}}-x^{\frac{3n}{2}}$ by $x^{\frac{n}{2}}-x^{-\frac{n}{2}}$.
54. $(x^2-xy+y^2)^3+(x^2+xy+y^2)^3$ by $2x^2+2y^2$.
55. $(x^2+y^2)(m^2+n^2)$ by $(mx+ny)^2+(xn-my)^2$.

VIII. FACTORING.

1. BINOMIALS.—94, 2d and 5th.

Factor the following binomials:

1. $a^2 - b^2$.
2. $4a^2 - 9b^2$.
3. $a^4 - 9b^4$.
4. $12x^2 - 3$.
5. $11a^5b - 176ab^3$.
6. $9x^4y - y^3$.
7. $162x^4 - 2b^2$.
8. $5x^6 - 45y^4$.
9. $9x^2 - 16$.
10. $4x^2 - 9b^6$.
11. $x^4 - 1$.
12. $25x^4 - 1$.
13. $25 - 16x^8$.
14. $x^3 - 1$.
15. $x^3 + b^3$.
16. $x^6 - 1$.
17. $m^6 - n^6$.
18. $9 - 25x^6$.
19. $x^2y^4 - 1$.
20. $9x^2y^2 - 16a^4b^2c^6$.
21. $1 - x^8$.
22. $x^6 - xy^5$.
23. $27 - x^3$.
24. $a^3 + 64$.
25. $a^5 - 32b^5$.
26. $64 - m^{12}$.
27. $a^3m - b^6m$.
28. $1 - (a + b + c)^5$.
29. $(x + y)^2 - z^2$.
30. $(2x + 3)^4 - (2x - 3)^2$.
31. $(a + b)^2 - (x - y)^2$.
32. $a^3 - (x - y)^6$.
33. $a^{5x} + c^{15}$.
34. $x^{2n} - b^2$.
35. $x^{4a} - y^{10}$.
36. $32 - \frac{a^3}{128}$.
37. $4a^2 - \frac{b^2}{4}$.
38. $(a + b)^4 - (a - b)^2$.
39. $(a - b)^3 + (x + y)^6$.
40. $(a + b + c)^2 - d^2$.
41. $(x^4 + 2x^2y^2 + y^4) - m^2$.
42. $\frac{1}{16} - \frac{81}{a^4}$.
43. $a^{-3} + b^{-6}$.
44. $x^{-5} - \frac{y^{10}}{32}$.
45. $a^3 + a^{-3}$.
46. $x^{-3} - 27y^{-6}$.
47. $(x^2 + y^2 - 2xy) - (x + y)^2$.

48. $16(a+b)^{-4}-81x^{-8}$.

49. $(a-b)^{-8}-16$.

50. $\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}$.

51. $\dfrac{\dfrac{1}{x^2}}{y^2}-\dfrac{\dfrac{1}{y^2}}{x^2}$.

52. $\dfrac{(x+y)^2}{(x-y)^2}-\dfrac{(x-y)^4}{(x+y)^4}$.

2. TRINOMIALS.

(1) Squares.—94, 1st.

Factor:

1. $a^2 \pm 2ab + b^2$.
2. $4a^2 \pm 4ab + b^2$.
3. $9a^3 - 6a^2b + ab^2$.
4. $(a+b)^2 - 2(a+b)c^2 + c^4$.
5. $(a+2b)^2 - 4(a^2-4b^2) + 4(a-2b)^2$.
6. $(a^2+2ab+b^2) - 2(ac+bc) + c^2$.
7. $(x^2+xy+y^2)^2 - 6(x^3-y^3) + 9(x-y)^2$.
8. $(x^2-xy+y^2)^2 - 4(x^3+y^3) + 4(x+y)^2$.
9. $a^{\frac{2}{3}} - a^{\frac{1}{3}}b + \dfrac{b^2}{4}$.
10. $a^{-2} - 3a^{-1}b^{\frac{1}{2}} + \dfrac{9b}{4}$.
11. $a^{2m} - 2a^m b^{-c} + b^{-2c}$.
12. $\left(\dfrac{x}{y}\right)^4 - 4\left(\dfrac{x}{y}\right)^2 + 4$.
13. $(a-b)^{\frac{1}{2}} - 2(a-b)^{\frac{1}{4}} + 1$.
14. $(a^2-b^2)^2 - 2(a^4-b^4) + (a^2+b^2)^2$.
15. $a^9 - 6a^6b^{-\frac{2}{3}} + 9a^3b^{-\frac{4}{3}}$.
16. $(a-b)^{2x} - \frac{2}{3}(a-b)^x c^{-2} + \frac{1}{9}c^{-4}$.
17. $2^{2x} - 2^{x+2} + 4$.
18. $25^x - 5^{x+1} + \frac{25}{4}$.
19. $(2^{2x} + 6^x + 3^{2x})^2 - 2(8^x - 27^x) + (2^x - 3^x)^2$.

(2) Quadratics having Binomial Factors.—94, a.

a. The first term having the co-efficient 1.

Factor:

1. $x^2+9x+20$.
2. $x^2-15x+50$.
3. x^2-x-30.
4. x^2+x-30.
5. $x^2-11x+30$.
6. $a^2-9ab+20b^2$.
7. $x^2+2bcx-3b^2c^2$.
8. $x^2-2bcx-3b^2c^2$.
9. $a^2x^2+5abdx+4b^2d^2$.
10. $5x^2-75+10x$.
11. $18x^2-3x^4-15x^3$.
12. $162a^3-108a^4+18a^5$.
13. $9x^4+360-117x^2$.
14. x^6-18x^3+77.
15. $x^8+54+15x^4$.
16. $7ax^3-14ax^2-21ax$.
17. $5abx^2-5abx-150ab$.
18. $6ac^3+48ac^2+90ac$.

b. The co-efficient of the first term a square other than 1.

Factor:

1. $4a^2-8ab-21b^2$.
2. $9a^2+3ac-20c^2$.
3. $25x^4+5x^2y-12y^2$.
4. $9a^2m^2-6acm-35c^2$.
5. $4x^2-8adx-5a^2d^2$.
6. $4a^2b^2c^2-4abcd-15d^2$.
7. $36x^2-6x-2$.
8. $49x^2+21x-40$.
9. $25x^2-25x-24$.
10. $9x^2-72x+143$.
11. $4x^2+2(b-3)x-3b$.
12. $4x^2-2(a+5)x+5a$.
13. $25a^2b^2+5(3d-2c)ab-6cd$.
14. $4a^2-2a(3x-5y)-15xy$.
15. $9x^2-3(2bx-4cx)-8bc$.
16. $36x^2-x(3y-4z)-\frac{1}{3}yz$.

c. The co-efficient of the first term not a square.

Factor:

1. $6x^2-13x+5$.
2. $3x^2+11x-4$.
3. $10x^2+21x-13$.
4. $3x^2-7x-20$.

5. $6x^2 + 17xy + 5y^2$.
6. $6x^2 - 29x + 28$.
7. $12x^2 + 4x - 5$.
8. $3x^2 - 16x - 35$.
9. $6x^2 + x - 15$.
10. $6x^2 + 29x - 57$.
11. $3x^2 - 25x + 28$.
12. $3x^4 - x^2 - 10$.
13. $6x^4 - x^2(3a - 2b) - ab$.
14. $6x^{-4} - 11x^{-2} - 10$.
15. $2a^2x^2 - ax(2b - c) - bc$.
16. $2(a+b)^2 + (a+b) - 6$.
17. $\frac{x^2}{12} - 2x + 12$.
18. $\frac{x^2}{2} - \frac{3x}{2} - 20$.
19. $\frac{x^2}{2y^2} - \frac{5x}{y} + 12$.
20. $2x^{-4}y^{-2} - 5x^{-2}y^{-1} - 12$.
21. $(2x)^{-2} - (2x)^{-1} - 30$.
22. $\frac{(a-b)^2}{6} + 2(a-b) - 48$.
23. $2(a-b)^{-2} + 2(a-b)^{-1} - 12$.
24. $2x + 2x^{\frac{1}{2}} - 12$.
25. $3x + 5\sqrt{x} - 2$.
26. $\frac{2}{x^2} - \frac{6}{x} - 8$.
27. $\frac{10}{9x^2} - \frac{7}{x} - 10$.
28. $\frac{x^2}{ab} + x - 2ab$.

(3) Trinomials having Trinomial Factors.

a. Exercises in completing the square.

What must be added to each of the following quantities to make it a square?

1. $a^2 \pm 2ab + (\ )$.
2. $a^2 + 5ab + (\ )$.
3. $a^4 + 6a^2b + (\ )$.
4. $a^2 \pm (\ ) + b^2$.
5. $4a^2 \pm (\ ) + b^2$.
6. $(a-b)^2 + 2(a-b) + (\ )$.
7. $(a-b)^2 - 6(a-b) + (\ )$.
8. $4x^{2n} - 12x^n y^r + (\ )$.
9. $(a-b)^4 \pm (\ ) + (c+d)^4$.
10. $a^2 + 6a + 8$.
11. $4a^2 + 12a + 8$.
12. $x^4 + 2x^2 + 9$.
13. $2^{2x} + 2^x + (\ )$.
14. $16^x + 2^{2x+1} + (\ )$.

b. Finding factors by completing the square.

Factor:

1. $x^4 + 4y^4$.
2. $4x^4 + 1$.
3. $x^4 + x^2y^2 + y^4$.
4. $x^8 + x^4y^2 + y^4$.
5. $x^4 - 7x^2y^2 + y^4$.
6. $x^4 - 23x^2y^2 + y^4$.
7. $x^4 - 12x^2y^2 + 4y^4$.
8. $x^4 - 21x^2y^2 + 4y^4$.
9. $x^4 + 5x^2y^2 + 9y^4$.
10. $x^4 + 2x^2y^2 + 9y^4$.
11. $x^4 - 3x^2y^2 + 9y^4$.
12. $x^4 + 4x^2 + 16$.
13. $x^4 - x^2 + 16$.
14. $4x^4 + 11x^2 + 9$.
15. $9x^4 + 29x^2 + 25$.
16. $16x^4 + 15x^2 + 9$.
17. $(a+b)^4 - 3(a+b)^2 + 9$.
18. $(a+b)^4 - 19(a+b)^2 + 9$.
19. $4(x-y)^4 + 8(x-y)^2 + 9$.
20. $9x^4y^4 + 21x^2y^2 + 25$.

21. Show that the sum of two squares multiplied by the sum of two squares produces the sum of two squares, and also the sum of two other squares.

3. POLYNOMIALS.

(1) Squares, or difference of Squares.

Factor:

1. $x^2 - y^2 + 4y - 4$.
2. $x^2 + 2xz - 2xy + y^2 - 2yz + z^2$.
3. $a^2 - 2ab + 4 + 4a - 4b + b^2$.
4. $9x^4 - 4x^2 + 4x - 1$.
5. $x^2y^2 - 1 - x^2 + 2x$.
6. $m^2n^2 + 4ab - 4a^2 - b^2$.
7. $x^2 - y^2 - 2xz + 4y + z^2 - 4$.
8. $a^4 + b^2 - 2a^2b + 2a^2c + c^2 - 2bc$.
9. $\dfrac{1}{x^2} - \dfrac{1}{y^4} + \dfrac{4}{y^2} - 4$.
10. $\dfrac{4(a-b)^{-1}}{(c-d)^2} - 4 + \dfrac{4}{(c-d)^4} + \dfrac{1}{(a-b)^2}$.

(2) Polynomials having Binomial Factors.

Factor:

1. $x^3 - 6x^2 + 11x - 6$.
2. $x^4 - 2x^3 - 2x^2 + 3x - 108$.
3. $x^4 - 2x^3 + x - 30$.
4. $x^4 - 8x^3 + 10x^2 + 24x + 5$.
5. $x^4 + 2x^3 - 7x^2 - 8x + 12$.
6. $x^4 - 10x^3 + 35x^2 - 50x + 24$.
7. $x^4 - 5x^2 + 10x - 6$.
8. $x^4 - 5x^3 - 7x^2 + 29x + 30$.
9. $6x^5 - 54x^4 + 54x^3 + 246x^2 - 252x$.
10. $x^6 - 2x^5 - 22x^4 + 83x^3 - 78x^2$.
11. $5x^3 - 35x - 30$.
12. $6x^4 + 13x^3 - 18x^2 - 7x + 6$.
13. $x^6 - 21x^2 - 20$.
14. $x^{\frac{3}{2}} - 73x^{\frac{1}{2}} + 72$.
15. $2x^3 - 11x^2 + 17x - 6$.
16. $(x^2+1)^2 - 4(x^3 - x) + 4x^2(2x+1)$.
17. $3x^{-3} - 2x^{-2} - 6x^{-1} + 4$.
18. $\frac{8}{x^3} - \frac{26}{x^2} + \frac{11}{x} + 10$.
19. $x^2 - 9x^{\frac{3}{2}} + \frac{45}{4}x + 27x^{\frac{1}{2}} - \frac{81}{4}$.

(3) Polynomials having Trinomial Factors.

Factor:

1. $x^4 - 13x^2 + 34x + 2x^3 - 15$.
2. $x^4 + x^3 - 4x^2 + 11x - 3$.
3. $x^4 - 4x^3 - 20x^2 + 53x + 120$.
4. $x^4 + x^3 - 5x^2 - 27x - 30$.
5. $x^4 + 5x^3 + 17x^2 + 27x + 30$.
6. $x^4 + x^3 + 10x^2 + 8x + 64$.
7. $x^4 + 3x^3 + 9x^2 + 10x + 12$.
8. $x^4 + 5x^3 + 15x^2 + 22x + 20$.
9. $5x^4 + 17x^3 + 12x^2 + 5x + 1$.
10. $6x^4 + x^3 + 3x^2 - 3x - 2$.
11. $2x^4 - 6x^3 + 9x^2 - 6x + 2$.
12. $x^4 - 5x^3 - 2x^2 + 21x + 15$.

(4) Polynomials having several Factors.

Resolve:

1. $a^{15}-1$ into 4 factors.
2. $a^{15}+1$ into 4 factors.
3. $a^{21}+1$ into 4 factors.
4. $a^{12}+64$ into 4 factors.
5. $a^{30}-b^{30}$ into 8 factors.
6. $a^{14}+a^7+1$ into 2 factors.
7. $a^{33}-b^{33}$ into 4 factors.
8. $1+c^{35}$ into 4 factors.
9. $1+b^{30}$ into 4 factors.
10. $27a^6+1$ into 3 factors.
11. $64a^{12}+1$ into 4 factors.
12. $(a+b)^{10}-(a+b)^5+1$ into 2 factors.

IX. MISCELLANEOUS PROBLEMS.

1. $\{mn^2(x-a)^{c-1}-m^2n(x-a)^{c+1}\}m^{-2}n^{-2}(x-a)^{1-c}=?$
2. $\{x^{\frac{1}{2}}+y^{\frac{1}{2}}+x^{-\frac{1}{2}}y\}(xy^{-\frac{1}{2}}-x^{\frac{1}{2}}+y^{\frac{1}{2}})=?$
3. $(x^2-2x^{\frac{3}{2}}+x^{-\frac{1}{2}})^3=?$
4. $\{a^{-3}b^{-3}(m-n)^{p-1}-a^{-3}b^3(m-n)^{1+p}\}a^{-3}b^{-3}\times(m-n)^{1-p}=?$
5. $(4a^{x-2y}b^{2x-y})^{x+y}=?$
6. $(-3x^ny^2)^{2n}=?$ n being any integer.
7. $(-2x^ay^{-b})^{2n+1}=?$ n being any integer.
8. Factor $ax^4-bx^4-2ax^3+2bx^3+ax^2-bx^2$.
9. Factor $a(a-b)^3+b(b-a)^3+a(a-b)^2-b(b-a)^2$.
10. Express and simplify the following: From the difference between $(m+n-p)$ and minus the quantity $-m-n+p$ take their sum.
11. Simplify $\dfrac{(a^{-1}-b^{-1})^{-2}}{a^{-2}-b^{-2}}$.
12. Simplify $\dfrac{a^{-\frac{1}{2}}b^{\frac{1}{2}}}{(ab^{-1})^{-\frac{1}{2}}+(a^{-1}b^{-\frac{1}{2}})^{-\frac{1}{2}}}$.

13. Find the numerical value of the quantity,
$(a-c)(a+c)+am^2-2(a+c+m)^2-c^2+a\times m+c$,
if $a=5$, $c=-3$, and $m=-2$.

14. Find the numerical value of
$(c+d)^2-(c-d)^2+(c-d)(c+d)(c+d)^0+(c-d)^{-2}$,
if $c=5$ and $d=3$.

15. Find the numerical value of
$(a-x)^3-(a^2-b^2+ab+c^2-\overline{a+b}\times 2+b\div a)$,
if $a=2$, $b=-4$, $c=5$, and $x=3$.

16. Divide $1+x^3y^{-\frac{1}{2}}-x^{-3}y-y^{\frac{1}{2}}$ by x^3-y.

17. Divide x^4-y^4 by $x^{-1}-y^{-1}$.

18. Factor $x^4-xy^3-x^3y+y^4$.

19. Factor $x^4+xy^3-x^3y-y^4$.

20. Divide x^3-y^6 by $x^{-\frac{1}{2}}+y^{-1}$.

21. Divide $a^{\frac{3}{2}}-2a^{\frac{1}{2}}+2a^{-1}-a^{-\frac{3}{2}}$ by $a^{\frac{1}{2}}+1-a^{-\frac{1}{2}}$.

22. Find the value of
$(c+d)^2-(c-d)^2+(c-d)(c+d)-(c+d)^0+(c-d)^{-2}$,
when $c=2$ and $d=-3$.

23. Factor a^3+a^2-1-a.

24. From $(m+nx)^{a+1}$ take $-(m-nx)^{a+1}$.

25. Simplify $ac^3-3m^3(x-y)+2n-(4+d)c^3$
$+2m^2(x-y)-3n+(a+d)c^3+m(x-y)+n$.

26. Factor $a^3x^3+3a^2x^2-ax-3$.

27. Divide $x^{4a-4b}-x^{4b-4a}$ by $x^{a-b}-x^{b-a}$.

28. Find the sum, difference, product, and quotient of $3c(x-cd)^{c-(d-2)}$ and $c(x-cd)^{2-(d-c)}$.

29. Simplify $(x+y)^2-(x-y)^2-[-(x+y)^2$
$+(x+y)(x-y)+2(-y)^2]^2$.

30. Factor a^3-4a^2+a+6.

31. Find the value of $(a+c)^2+c\times b-[a-c\times 2$
$-(a-c-b)^2-(b^2-a^2)^{-1}\times 21(c^2-b^2)^{-1}]$,
when $a=2$, $b=3$, $c=4$.

32. Factor $a^6 + 27$; also, factor $a^9 + 64a$.

33. Divide $x^{\frac{4}{5}} - y^{\frac{4}{5}}$ by $x^{-\frac{1}{5}} + y^{-\frac{1}{5}}$.

34. Simplify $(a+1)^2 + (-1)^3 - [-(a-c)(a+c) + (-c)^2 + (a-c)^2 + 2(-c^{\frac{2}{3}})^3]^2$.

35. Divide $2a^3 + 2b^3 + 2c^3 - 6abc$ by $(a-b)^2 + (b-c)^2 + (c-a)^2$.

36. Factor $x^2 + y^2 + 6x + 2xy + 6y$.

37. Simplify $bx - a(c+d) + bc + (b+d)x + dx - (b-x)c$.

38. Find the sum, difference, product, and quotient of $3a(a+xy)^{(x+1)(x-1)}$ and $a^2(a+xy)^{2x^2-[(x-1)^2-2(-x)]}$.

39. $6 + 4 \times 0 + 2 = ?$ $6 + 4(0+2) = ?$

40. Find the sum, difference, and product of $x^{\frac{2}{3}} + (a+b)x^{\frac{1}{3}} + ab$ and $x^{\frac{2}{3}} - (a-b)x^{\frac{1}{3}} - ab$.

41. Factor $4x^2 + y^2 + 4xy + 16x + 8y$.

42. Factor $a^{-4}b^2 + c^{-2} - 2a^{-2}bc^{-1} + 2a^{-2}b - 2c^{-1}$.

43. Factor $x^2 - xy - 6y^2 - 4x + 12y$.

44. Find the value of $ab^2 + bc \div ab - (c+d) - (a+c)^3$, when $a=8$, $b=3$, $c=-4$, $d=3$.

45. Divide $(ax + x^2) - (x + 2a + 2x) + (2 + a^2) - (a - ax)$ by $a - (1-x)$.

46. $\left[a^{\frac{n+1}{n}} + b\left(ab^{\frac{1-n}{n}} + a^{\frac{1}{n}}\right) + b^{\frac{n+1}{n}}\right] \div (a+b) = ?$

47. Find the value of $x\left\{x - y - \dfrac{b(a-b)^2}{bx^2}\right\}^3 + (a+b)(a-b)^2$, when $a=8$, $b=3$, $x=5$, $y=5$.

48. Divide $(x - ay^2)^{n+1}$ by $(ay^2z - xz)^{n-1}$.

49. From the sum of $3(a-b)^x - 3(a+b)y^2 - (a-b)y^4$ and $2(a-b)^x + 2(a-b)y^2 - by^4$, take their difference.

50. Factor $x^4 - x^3 - 6x^2 + 4x + 8$.

51. Find the value of $a+c\times a-c-\{ac^2-3a\div c^2\div 2a\times c\div 2\frac{2}{3}\}\times 5\overline{c-a}-a^2c$, when $a=4$, and $c=2$.

52. $\{x^3-a\times x+a(x-a)^2\div 5\frac{1}{3}-5a\div 2\}a^3\div 2a^2\div 2=?$ when $a=4$, $x=6$.

53. $\sqrt{-a-2\{-a-2b+c[-a+3b+b(2a-3b)]\}}=?$ if $a=5$, $b=4$, $c=2$.

54. $\sqrt[3]{a-b\times c+2a^{\frac{1}{2}}-3b^{\frac{2}{3}}+4c^{-\frac{2}{4}}-b^{-\frac{1}{3}}}=?$ when $a=9$, $b=8$, $c=16$.

Factor each of the following:

55. $3x^3-12xy^2-6yx^2+24y^3$.

56. $5a^2-5y^2-3a+3y$.

57. $6a-5ax+6b-5bx+ax^2+bx^2$.

58. $a^8-2a^6-7a^4-8a^2+16$ into six factors.

59. x^5+5x^3+6 and x^6+5x^3+6.

60. $(x^2-xy+y^2)^2-6(x^4+x^2y^2+y^4)+9(x^2+xy+y^2)^2$.

61. $(x^2-2xy+2y^2)^2-8(x^4+4y^4)+16(x^2+2xy+2y^2)^2$.

62. $(x^2-2x+3)^2-b^2(x^4+2x^2+9)+\frac{b^4}{4}(x^2+2x+3)^2$.

63. $5^{2x}-2\times 5^x+1$.

64. $4^x-2^{x+1}+1$.

65. $36^x-6^{x+2}+180$.

66. $9^x-6^x+4^{x-1}$.

67. $(2^x-1)^2-2(8^x-1)+(4^x+2^x+1)^2$.

68. $2x^4-3x^3-x^2+3x-4$.

69. $3x^4-27x^3+46x^2-13x+1$.

70. $16^x+2^{x+1}+3(4^x+1)$.

71. $2^{2x+1}-2^{2x-1}-3(2^{x-1}+1)$.

72. $2^{12x}+27$.

73. $3^{12x}+64$.

X. GREATEST COMMON DIVISOR.—96 et seq.

Find the G. C. D. of the following polynomials:

1. x^2-4x+3 and x^2-x-6.
2. $5x^2-15x-50$ and x^2-6x+5.
3. $2x^2-4x-16$ and $2x^2-10x+8$.
4. $3x^2-24x+45$ and $2x^2+2x-24$.
5. $x^3-39x+70$ and $x^2-3x-70$.
6. x^3-4x^2+5 and x^3+1.
7. $x^4+10x^3+35x^2+50x+24$ and $x^3+9x^2+26x+24$.
8. $2x^3+9x^2+7x-3$ and $3x^3+5x^2-15x+4$.
9. $3x^3-16x^2+23x-6$ and $2x^3-11x^2+17x-6$.
10. x^3-x^2-5x-3 and x^3-3x^2-x+3.
11. x^3+x^2-3x-2, x^3-3x^2+x+2, and $x^4-x^3-5x^2+4x+4$.
12. $x^2-(a+b)x+ab$, $x^2-(b+c)x+bc$, and $x^2-2bx+b^2$.
13. x^3-1, $3x^2-5x+2$, and $4x^3-4x^2-x+1$.
14. $10x^3+5x^2-14x-1$ and $2x^2+x-3$.
15. $6x^3-3x^2-2x+1$ and $2x^2+x-1$.
16. $15x^3-25x^2-2x+8$ and $3x^3-5x^2-x+2$.
17. $15x^3-24x^2-6x+9$ and $3x^3-5x^2-x+2$.
18. $10x^3+11x^2-54x-16$ and $2x^3+2x^2-11x-2$.
19. $2x^4+2x^3-19x^2-12x-9$ and $x^3+x^2-10x-6$.
20. $x^4+3x^3-x^2-8x+5$ and $x^4+2x^3-5x^2-10x+12$.
21. $6x^4-23x^3+11x^2+11x+22$ and $2x^4-8x^3+5x^2+3x+6$.
22. $3x^4+10x^3-x^2+4x+9$ and $x^4+3x^3-x^2+2x+2$.
23. $6x^4-17x^3-31x^2+9x+12$ and $3x^4-9x^3-14x^2+7x+4$.

24. $4x^4-22x^3+35x^2-11x-12$ and
$4x^4-24x^3+47x^2-34x+3$.
25. $2x^2+3x-2$ and $2x^3+3x^2+4x-3$.
26. $10x^4-15x^3-53x^2+35x+2$ and
$10x^4-20x^3-43x^2+59x-30$.
27. $4x^3+2x^2-1$ and $4x^4+6x^3-2x^2+x-1$.
28. $10x^4+5x^3-80x^2-70x-165$ and
$2x^4+6x^3-x^2-9x-18$.
29. $4x^4-8x^3-20x^2+24x+20$ and
$3x^3+6x^2-24x-45$.
30. $4x^4+28x^3+35x^2-88x+21$ and
$2x^4+9x^3-44x+33$.
31. $x^5+2x^4-x^3+x^2-3$ and $x^4-x^3+2x^2+x-3$.
32. $x^4+16x^3y-50x^2y^2-31xy^3+30y^4$ and
$x^3+16x^2y-52xy^2-15y^3$.
33. $2x^4-5x^3-2x^2-17x-10$ and
$x^4-4x^3-x^2+8x+4$.
34. $6x^5-16x^3-8x^2-42x-54$ and
$6x^5-6x^4-10x^3-8x^2-24x$.
35. $16x^4-15x^3+7x^2+9x-8$ and
$16x^3-27x^2+27x-11$.
36. $18x^4-17x^3+50x^2+6x-9$ and
$18x^3-25x^2+61x-21$.
37. $30x^4-64x^3-17x^2+49x+2$ and
$10x^3-23x^2-x+14$.
38. $6x^4+3x^3-32x^2+18x+5$ and
$3x^3+x^2-16x+12$.
39. $24x^4+6x^3-2x^2+26x+4$ and
$24x^4+18x^3-x^2+25x+17$.
40. $x^4-x^3-x^2+9x-2$ and $x^3-5x^2-19x-10$.
41. $2ax^4+12a^3x^2+11a^4x-3a^5$ and
$2ax^3+6a^3x+8a^4$.
42. $x^4-3x^3+7x^2-7x-6$ and $x^4+x^3-8x^2+7x-6$.

43. $10x^4 + 24x^3 - 12x^2 - 43x - 6$ and
$10x^3 + 9x^2 - 25x - 6$.

44. $x^4 - 5x^3 + 8x^2 - 7x + 3$ and $2x^3 - 9x^2 + 10x - 3$.

45. $2x^4 - 3x^3 + 3x^2 - 3x + 1$ and $6x^3 - 7x^2 + 4x - 1$.

46. $18x^4 - 9x^3 - 59x^2 + 12x + 54$ and
$18x^4 - 39x^3 + 6x^2 + 22x - 6$.

47. $6x^4 - 11x^3y - 7x^2y^2 + 8xy^3 + 4y^4$ and
$3x^4 + 2x^3y - 18x^2y^2 + 3xy^3 + 10y^4$.

48. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ and
$3x^3 - 7ax^2 + 3a^2x - 2a^3$.

49. $2x^3 + (2a - 9)x^2 - (9a + 6)x + 27$ and
$2x^2 - 13x + 18$.

50. $a^4 - ba^3 + (c - 1)a^2 + ba - c$ and
$a^4 - b - ca^3 + ca + (b - 1)a^2$.

51. $x^3 + (a - 1)x^2 + (b - a)x - b$ and
$x^3 + (b - 1)x^2 + (a - b)x - a$.

52. $x^4 + 2x^3y - a^2x^2 + x^2y^2 - 2axy^2 - y^4$ and
$x^3 + ax^2 - axy - y^3$.

53. $6ax^4 + (6a + 3)x^3 - (6a^3 - 3)x^2 - (18a^3 - 3a + 3a^2)x - 6a^2(1 - 2a^2)$ and $2ax^4 + 2ax^3 - 2a^3x^2 - 6a^3x + 4a^4$.

54. $a^3 + b^3 + c^3 - 3abc$ and $(a - b)^2 + (b - c)^2 + (c - a)^2$.

55. $x^6 - x^4y^2 - x^2y^4 + y^6$ and $x^6 - x^5y + x^4y^2 - x^2y^4 + xy^5 - y^6$.

56. $3x^3 + x^2(4n - 2m) + x(n^2 + 3m - 2mn) + mn - 2m^2$ and $3x^2 - 2x(m + n) + 2mn - n^2$.

57. $2a^2 - 9ac - 5ab + 4c^2 - 8bc - 12b^2$ and
$ab + 2a^2 - 3b^2 + 4bc + ac - c^2$.

58. $16^x + 4^x + 1$ and $8^x + 1$.

59. $16^x - 9(4^x + 2^{x+1} + 1)$ and $4^{3x} + 27$.

60. $5^{4x} - 10^{2x} + 4^{2x-1}$ and $25^{3x} - 8^{2x-1}$.

61. $[2(5 - b)^2]^6 - 1$ and $[2(5 - b)^2]^9 - 1$.

XI. LEAST COMMON MULTIPLE.—109 et seq.

Find the L. C. M. of the following polynomials:

1. x^2-y^2 and $x^2-2xy+y^2$.
2. x^2-y^2 and $x^2+2xy+y^2$.
3. x^3-1 and x^2-2x+1.
4. $2x-y$, $4x^2-y^2$ and $4x^2+y^2$.
5. a^3-a, $a+1$ and a^3+1.
6. x^2-2x-3, x^2-9 and $x+1$.
7. x^2+xy, $xy-y^2$ and x^2-y^2.
8. $1+y$, $1-y^2$ and $1+y^3$.
9. $1+y$, $(1-y)^2$ and $(1+y)^3$.
10. x^4-y^4, x^3-y^3, x^2-y^2, $x-y$ and $x+y$.
11. $x^2-7x+12$, x^2-6x+8 and x^2-5x+6.
12. $(x+5)(x+1)$, $(1-x^2)$ and $(x+5)(x-1)$.
13. $5(a-b)$ and $5(b-a)$.
14. $a+b$, $a-b$, $a^2+2ab+b^2$, $a^2-2ab+b^2$ and a^2-b^2.
15. $3x^2-12y^2$, $6(x-2y)^2$ and $12(x+2y)^2$.
16. x^5-5x^3+4x and x^3-2x^2-x+2.
17. $(x+y)(x^2-y^2)$ and x^4-y^4.
18. $6x^2+7x-3$, $3x^2+5x-2$ and $2x^2+7x+6$.
19. $21a^2-26a+8$ and $7a^3-4a^2-21a+12$.
20. x^3+x^2-4x+6 and x^3-5x^2+8x-6.
21. $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$ and $x^3-8x^2+19x-12$.
22. x^4-5x^2+4, x^3-x^2-4x+4 and x^3+2x^2-x-2.
23. $1+x$, $1+x+x^2$ and $1+x+x^2+x^3$.
24. $x^2-(a+b)x+ab$, $x^2-(a+c)x+ac$ and $x^2-(b+c)x+bc$.
25. x^3+x^2-4x-4 and $x^3+6x^2+11x+6$.
26. $ac+bc-ax-bx$ and $ad+bd-af-bf$.

27. $2ax-4ay+3bx-6by$ and $4ax+3by+6bx+2ay$.
28. $2x^3+(2a-3b)x^2-(2b^2+3ab)x+3b^3$ and $2x^2-(3b-2c)x-3bc$.
29. $a^4-10a^3+35a^2-50a+24$, a^4-5a^2+4 and $a^4+5a^3+5a^2-5a-6$.
30. $x^4+10x^3+35x^2+50x+24$, x^4-5x^2+4 and $x^4+7x^3+17x^2+17x+6$.
31. $x^4+2x^3-x^2-2x$, $x^4-3x^3-4x^2+12x$ and $x^4+4x^3+x^2-6x$.
32. $x^4+6x^3+11x^2+6x$, x^4-7x^2-6x and x^3+x^2-9x-9.
33. $x^4-2x^3-x^2+2x$, $x^4-4x^3+5x^2-2x$ and x^3-x^2-4x+4.
34. $(a-b)(a-c)$, $(b-c)(b-a)$ and $(b-c)(c-a)$.
35. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$ and $2a^3-2ab(a-b)-2b^3$.
36. $(a+b)^2+(a-b)^2+(a^2+b^2)^2$ and $a^4+2(a+b)(a-b)-b^4$.
37. $x^3+(a+c-d)x^2+(ac-ad-cd)x-acd$ and $x^2+cx+dx+cd$.

XII. FRACTIONS.

1. REDUCTION.

(1) To Lowest Terms.—119.

Reduce the following fractions to their lowest terms:

1. $\dfrac{18xy^4+3x^2y^2z}{6xyz}$.
2. $\dfrac{15ax-10ay}{18bx-12by}$.
3. $\dfrac{12m^2-12mn}{4m^2-4n^2}$.
4. $\dfrac{m^2-n^2}{mn+n^2}$.

5. $\dfrac{12x^2-3}{20x-10}$.

6. $\dfrac{x^2-8x+12}{x^2-4}$.

7. $\dfrac{x^2-4x+3}{x^2-2x-3}$.

8. $\dfrac{ax^3-5ax^2+3ax}{bx^2-5bx+3b}$.

9. $\dfrac{x^3+2x-3x^2}{x^3-4x}$.

10. $\dfrac{x^4+x^3-x-1}{x^4-x^3-x+1}$.

11. $\dfrac{a^2-b^2}{(a+b)^2}$.

12. $\dfrac{(a-b)^2}{a^3-b^3}$.

13. $\dfrac{3x^2y+3xy^2}{x^4+xy^3}$.

14. $\dfrac{1+a^2+a^4}{1-a+a^2}$.

15. $\dfrac{(x-y)^5-x^5+y^5}{(x-y)^4-x^4+y^4}$.

16. $\dfrac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}$.

17. $\dfrac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$.

18. $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^2x^2}$.

19. $\dfrac{a^3+(a+b)ax+bx^2}{a^4-b^2x^2}$.

20. $\dfrac{x^4-21x^2y^2+4y^4}{3x^2+6y^2-15xy}$.

21. $\dfrac{(x+y+z)^2-(x+y-z)^2}{(x-y+z)^2-(y+z-x)^2}$.

22. $\dfrac{x^4-5x^3+8x^2-7x+3}{2x^3-9x^2+10x-3}$.

23. $\dfrac{x^4-4x^3+3x^2+4x-12}{2x^3-x^2-18x+9}$.

24. $\dfrac{2x^7-5x^5y+2x^2y^5-5y^6}{2x^6-5x^4y-4x^2y^2+10y^3}$.

25. $\dfrac{(x-y)^2+5(x^2-y^2)+6(x+y)^2}{(x-y)^2-9(x+y)^2}$.

26. $\dfrac{(a^2-b^2)^2-10(a^4-b^4)+9(a^2+b^2)^2}{(a^2-b^2)^2+2(a^4-b^4)-3(a^2+b^2)^2}$.

(2) Improper Fractions and Mixed Numbers.—121.

Reduce the following quantities to improper fractions, or the converse:

1. $x^3+xy+y^2+\dfrac{x^4-x^2y^2+y^4}{x^2-xy+y^2}$.

2. $1+y+y^2+y^3+\dfrac{-y^2+y^4}{1-y}$.

3. $x^4+x^2y^2+y^4-\dfrac{x^6-y^6}{x^2-y^2}$.

4. $(a-b)^2+\dfrac{2b\,(3a^2+b^2)}{a-b}$.

5. $(x+2)^2-\dfrac{x^2-4}{5}$.

6. $2a-6c-\dfrac{2a^2-9c^2}{a+3c}$.

7. $x^2-x+1+\dfrac{1-x^3}{x+1}$.

8. $4x-3-\dfrac{2}{x+3}$.

9. $\dfrac{x^2-8x+12}{x^2-4}$.

10. $\dfrac{x^2-4x+3}{x^2-2x+3}$.

11. $\dfrac{x^4-y^4}{(x+y)\,(x^2-y^2)}$.

12. $\dfrac{x^4-5x^2+4}{x^3-x^2-4x+4}$.

13. $x^3+xy^2+\dfrac{y^4\,(x-y)}{x^2-y^2}$.

14. $\dfrac{15a^2-21ab+3b^2}{5a}$.

15. $\dfrac{x^2-5x+5}{x-5}$.

16. $\dfrac{x^4-x^3-x^2+9x-2}{x^3-5x^2-19x-10}$.

17. $\dfrac{x^3-6x^2+11x-6}{x^3-9x^2+26x-24}$.

18. $\dfrac{x^4+2x^3+2x-1}{x^3-x^2-7x+3}$.

19. $\dfrac{60x^3-17x^2-4x+1}{5x^2+9x-2}$.

20. $\dfrac{2x^3+ax^2+a^2x-4a^3}{x^3-3a^2x+2a^3}$.

21. $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$. | 22. $\dfrac{(x+a)^2-(b+c)^2}{(x+b)^2-(a+c)^2}$.

23. $x^2+8-4x-\dfrac{x^4+64}{x^2+4x+8}$.

24. $\dfrac{12x^4-18x^3+18x^2-18x+6}{6x^3-7x^2+4x-1}$.

2. ADDITION AND SUBTRACTION.–130; see also 124.

Perform the operations indicated in the following:

1. $\dfrac{x^2}{y^2}+2+\dfrac{y^2}{x^2}$. | 2. $\dfrac{x+6}{x-3}-\dfrac{x+4}{x-2}$.

3. $\dfrac{a}{2a-2b}+\dfrac{b}{2b-2a}$.

4. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-.20x}{4x^2-1}$.

5. $\dfrac{2x-y}{c-x}+\dfrac{x-y}{c+x}+\dfrac{3c(y-x)}{c^2-x^2}$.

6. $\dfrac{x^2-9}{x^2+x-12}-\dfrac{x^2+x-20}{x^2-25}$.

7. $\dfrac{x}{(1+x)(x+y)}+\dfrac{y}{(1-y)(x+y)}$.

8. $\dfrac{x^2+4x+3}{x^2+x-6}+\dfrac{x^2+8x+15}{x^2-25}$.

9. $\dfrac{x+1}{x^2-5x+4}-\dfrac{x-1}{x^2-3x-4}$.

10. $\dfrac{x-y}{xy}+\dfrac{z-x}{xz}+\dfrac{y-z}{yz}$.

11. $x^2-\dfrac{x^2y}{x+y}+\dfrac{x^2y}{x-y}-\dfrac{2x^2y^2}{x^2-y^2}$.

12. $\dfrac{4}{3(x-2)}-\dfrac{1}{2(x-1)}+\dfrac{1}{6(x+1)}$.

13. $\dfrac{m}{x(x-m)}-\dfrac{x}{m(m-x)}-\dfrac{2}{x-m}$.

14. $\dfrac{1}{1-x}-\dfrac{1}{(1-x)^2}+\dfrac{x-1}{(1-x)^3}$.

15. $\dfrac{b}{a}+\dfrac{b}{a+b}+\dfrac{a^2}{a^2+ab}$.

16. $\dfrac{1}{x^2-4}-\dfrac{1}{x^3-8}+\dfrac{1}{x^3+8}$.

17. $\dfrac{x^3+x^2+x+1}{x^3+x^2-x-1}+\dfrac{x^3-x^2-x+1}{x^3-x^2+x-1}$.

18. $\dfrac{x-3}{x+5}+\dfrac{x+3}{x-4}-\dfrac{x^2+2x+57}{x^2+x-20}$.

19. $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}$.

20. $\dfrac{3a}{a^2+3ab+2b^2}-\dfrac{a+3b}{a^2+5ab+6b^2}$.

21. $\dfrac{1}{x^2-y^2}+\dfrac{1}{(x+y)^2}-\dfrac{1}{(x-y)^2}+\dfrac{4xy}{(x^2-y^2)^2}$.

22. $\dfrac{x^2+x-5}{2x^2-11x+12}-\dfrac{x^2+x-1}{2x^2+5x-12}$.

23. $\dfrac{1}{x-3a}-\dfrac{1}{x+3a}+\dfrac{3}{x+a}-\dfrac{3}{x-a}-\dfrac{48ax^2}{x^4-10a^2x^2+9a^4}$.

24. $\dfrac{(a^2+b^2)^2}{ab(a-b)^2}-\dfrac{a}{b}-\dfrac{b}{a}-2$.

25. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}$.

26. $\dfrac{b}{(a-b)(a-c)}+\dfrac{a}{(b-a)(b-c)}$.

27. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

28. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2}{(c+a)(c+b)}$.

29. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

30. $\dfrac{a^2-b}{(a-b)(a-1)}+\dfrac{b^2+a}{(b+1)(b-a)}+\dfrac{1+ab}{(1-a)(1+b)}$.

31. $\dfrac{bc}{(c-a)(a-b)}+\dfrac{ca}{(a-b)(b-c)}+\dfrac{ab}{(b-c)(c-a)}$.

32. $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}$.

33. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}-\dfrac{1}{abc}$.

34. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}$

$+\dfrac{1}{c(c-a)(c-b)}$.

35. $\dfrac{1}{(a-b)(a-c)(x-a)} + \dfrac{1}{(b-a)(b-c)(x-b)} + \dfrac{1}{(c-a)(c-b)(x-c)}$.

36. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-a)(b-c)} + \dfrac{c^2}{(c-a)(c-b)}$.

37. $\dfrac{a^2+a+1}{(a-b)(a-c)} + \dfrac{b^2+b+1}{(b-a)(b-c)} + \dfrac{c^2+c+1}{(c-a)(c-b)}$.

38. $\dfrac{2}{a-b} + \dfrac{2}{b-c} + \dfrac{2}{c-a} + \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$.

39. $\dfrac{a}{(a-b)(a-c)(x-a)} + \dfrac{b}{(b-a)(b-c)(x-b)} + \dfrac{c}{(c-a)(c-b)(x-c)}$.

40. $\dfrac{x^{3n}}{x^n-1} - \dfrac{x^{2n}}{x^n+1} - \dfrac{1}{x^n-1} + \dfrac{1}{x^n+1}$.

3. MULTIPLICATION AND DIVISION.—131.

Perform the indicated operations in the following:

1. $\dfrac{x^2-9}{x^2-4} \times \dfrac{x+2}{x+3}$.

2. $\dfrac{x+1}{x-1} \times \dfrac{x+2}{x^2-1} \div \dfrac{x^2+4x+4}{x-1}$.

3. $\dfrac{x^2-4x+3}{x^2+5x+6} \times \dfrac{x^2+8x+15}{x^2-5x+4} \div \dfrac{x^2+2x-15}{x^2-2x-8}$.

4. $\dfrac{ax-x^2}{(a+x)^2} \div \dfrac{x^2}{a^2-x^2}$.

5. $(x^2-x+1)\left(\frac{1}{x^2}+\frac{1}{x}+1\right)$.

6. $\frac{a^3-b^3}{a^3+b^3}\times\frac{a+b}{a-b}\div\left(\frac{a^2+ab+b^2}{a^2-ab+b^2}\right)^2$.

7. $\left(\frac{1-y^2}{x+x^2}\right)\left(1+\frac{x}{1-x}\right)\div\frac{1+y}{1-x^2}$.

8. $\frac{3ax}{4by}\times\frac{a^2-x^2}{c^2-x^2}\times\frac{bc+bx}{a^2+ax}\times\frac{c-x}{a-x}$.

9. $\frac{x^2+xy}{x^2+y^2}\times\frac{x^3-y^3}{x^2y+xy^2}\times\frac{x^2y+y^3}{x^4+x^2y^2+y^4}$.

10. $\frac{ax-x^2}{a^2+2ax+x^2}\div\frac{a^2-2ax+x^2}{a^2+ax}$.

11. $\left(b+\frac{a^2}{b}\right)\left(a-\frac{b^2}{a}\right)$.

12. $\frac{xa}{x+a}\left(\frac{x}{a}-\frac{a}{x}\right)$.

13. $\frac{a^2-a-6}{a^2+a-30}\times\frac{a^2-a-42}{a^2-7a+12}\div\frac{a^2-5a-14}{a^2-9a+20}$.

14. $\frac{a^2+ax+x^2}{a^3-ax(a-x)-x^3}\times\frac{a^2-ax+x^2}{a+x}$.

15. $\frac{2x^3+ax^2-8xy-4ay}{2x^2+3bx-6xy-9by}\times\frac{2x^4+3bx^3-54xy^3-81by^3}{2x^3-ax^2-8xy+4ay}$.

16. $\frac{a^4+a^2x^2+x^4}{a^6-x^6}\times\frac{a^3-x^3}{a^2-ax+x^2}\div\frac{a^2+ax+x^2}{a+x}$.

17. $\frac{bx+3cx+2ab+6ac}{ax+3cx+2ab+6bc}\times\frac{x^2-4b^2}{b^2-9c^2}\times\frac{a+3c}{x+2a}$.

18. $\left(\frac{4a^2-16b^2}{5a-10b} \div \frac{4a^2+16ab+16b^2}{5b}\right)(a-2b).$

19. $\left(\frac{a}{a-x}-1\right) \div \left(1-\frac{a}{a+x}\right).$

20. $\frac{x^4-y^4}{x^2-2xy+y^2} \div \left(x+\frac{2xy}{x-y}\right).$

21. $\left(1+\frac{b}{a}\right)\left(1+\frac{c}{a}\right) \div \left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right).$

22. $\frac{(a+b-c)\,(a-b+c)}{a-b-c} \times \frac{c+b-a}{a^2-b^2-c^2+2bc}.$

23. $\left(a-\frac{x^2}{a}\right)\left(\frac{a}{x}+\frac{x}{a}\right) \div \frac{a-x}{ax}.$

24. $\left(1+\frac{x+1}{x-1}\right)\left(1+\frac{x-1}{x+1}\right) \div \left(1-\frac{x+1}{x-1}\right)\left(1-\frac{x-1}{x+1}\right).$

25. $\left(x-1+\frac{6}{x-6}\right) \div \left(x-2+\frac{3}{x-6}\right).$

26. $\frac{3}{x+y} \div \left[(2x-y) \div \left(x^2+\frac{xy}{2}-\frac{y^2}{2}\right)\right].$

27. $\left(\frac{x+2y}{y}+\frac{2y}{x}\right) \div \left(\frac{x+2y}{y}-\frac{x}{x+y}\right).$

28. $\frac{a^2-b^2-c^2+2bc}{a^2+c^2-b^2+2ac} \div \frac{a+b-c}{a+b+c}.$

29. $\left(x^2+ax-18a^2+\frac{36a^3}{x+3a}\right) \div \left(3x-6a-\frac{2x^2}{x+3a}\right).$

30. $\frac{x^3+3x^2+5x+3}{x^3-4x^2-x+4} \div \frac{x^3+x^2+x-3}{x^3-2x^2-7x-4}.$

31. $\left(x+\frac{2x}{x-3}\right)\div\left(x-\frac{2x}{x-3}\right)$.

32. $\frac{x^6-3x^4y^2+3x^2y^4-y^6}{(x^2-y^2)^4}\div\frac{(x+y)^3}{(x-y)^4}$.

33. $\left(\frac{1}{1+x}+\frac{x}{1-x}\right)\div\frac{1}{1-x}+\frac{2x}{1+x}$.

34. $\frac{1}{1+x}+\frac{x}{1-x}\div\frac{1}{1-x}-\frac{x}{1+x}$.

35. $\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

36. $\frac{(a+b)^m}{x^m}\times\frac{(ax-bx)^m}{(a^2-b^2)^m}$.

37. $\left(\frac{a^2x^3}{bd}+\frac{abx^2}{c^2d}-\frac{acx^2}{d^2}-\frac{b^2x}{cd^2}+\frac{a^2x}{bc}-\frac{a}{d}\right)\div\left(\frac{ax}{c}-\frac{b}{d}\right)$.

38. $\left(\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}-1\right)\div\left[2-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right]$.

39. $\left(\frac{a+1}{a}+\frac{b+1}{b}-\frac{c+1}{c}-\frac{d+1}{d}\right)\div\left(\frac{cd}{c+d}-\frac{ab}{a+b}\right)$.

40. $\frac{a^2+(2ac-b^2)x^2+c^2x^4}{a^2+2abx+(2ac+b^2)x^2+2bcx^3+c^2x^4}$

$\div\frac{a^2+(ac-b^2)x^2+bcx^3}{a^2+(ac-b^2)x^2-bcx^3}$.

41. $\frac{6^{2x}-6^{x+1}+9}{6^{2x}-3^x.2^{x+2}+3}\times\frac{6^{2x}-2^x.3^{x+1}+2}{9[9^{x-1}.4^x-1]}$.

42. $\frac{(3^{3x-3}+2^{3x}+1-6^x)\,(3^{x-1}+2^x+1)^{-2}}{9^{x-1}+4^x+1-3^{x-1}.2^x-3^{x-1}-2^x}$.

4. COMPLEX FRACTIONS.—133.

Reduce the following to simple fractions:

1. $$\frac{\dfrac{a^3+3a^2x+3ax^2+x^3}{x^3+y^3}}{\dfrac{(a+x)^2}{x^2-xy+y^2}}.$$

2. $$\frac{\dfrac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}}{\dfrac{x^2-a^2}{x^2-b^2}}.$$

3. $$\frac{\dfrac{x^2-3x+2}{x^2-6x+9}}{\dfrac{x^2-5x+6}{x^2-2x+1}}.$$

4. $$\frac{\dfrac{a^2+b^2+2ab-c^2}{c^2-a^2-b^2+2ab}}{\dfrac{a+b+c}{c+b-a}}.$$

5. $$\frac{a^2-b^2-c^2-2bc}{\dfrac{a+b+c}{a+b-c}}.$$

6. $$\frac{\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2}}{\dfrac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)}}.$$

7. $$\left(\frac{\dfrac{x^2-4x+3}{x^2-6x+8}}{\dfrac{x^2+x-12}{x^2+x-20}}\right) \div \frac{x^2+3x-10}{x^2+3x-4}.$$

8. $$\frac{\dfrac{x}{a}+\dfrac{a}{x}-2}{x-a}+\frac{\dfrac{x}{a}+\dfrac{a}{x}+2}{x+a}.$$

9. $$\frac{\dfrac{x}{a}-\dfrac{a}{x}+\dfrac{y}{b}-\dfrac{b}{y}}{\dfrac{bx^2y+a^2by+axy^2+ab^2x}{abx}}.$$

10. $\dfrac{\frac{3x}{2}+\frac{x-1}{3}}{\frac{1}{6}^{3}(x+1)-\frac{x}{3}-2\frac{1}{2}}$.

11. $\dfrac{\frac{x^2+3(a+b)x+9ab}{x^3-4x}}{\frac{x^2+(3a-2b)x-6ab}{x-2}}$.

Substitute for x and y in the following as indicated, and reduce:

12. $\dfrac{4ab}{a+b}$ for x in $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$.

13. $\dfrac{a^2}{a-b}$ for x in $\dfrac{x-a}{b}-\dfrac{x-b}{a}$.

14. $\dfrac{3x}{4}$ for y in $\dfrac{x}{x+y}+\dfrac{y}{x-y}-\dfrac{y^2}{x^2-y^2}$.

15. $\dfrac{a+b}{2}$ for x in $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$

16. $\dfrac{a+1}{ab+1}$ for x, and $\dfrac{ab+a}{ab+1}$ for y in $\dfrac{x+y-1}{x-y+1}$.

Reduce to simple fractions:

17. $1+\dfrac{x}{1+x+\frac{2x^2}{1-x}}$.

18. $\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}}$.

19. $\dfrac{\frac{m^2+n^2}{n}-m}{\frac{1}{n}-\frac{1}{m}}\cdot\dfrac{m^2-n^2}{m^3+n^3}$.

20. $\dfrac{\frac{a+bx}{a-bx}+\frac{b+ax}{b-ax}}{\frac{a+bx}{a-bx}-\frac{b+ax}{b-ax}}$.

21. $$\frac{\frac{1}{x+2}-\frac{1}{x+3}}{\frac{1}{x+3}-\frac{1}{x+4}} \div \frac{\frac{1}{x+5}-\frac{1}{x+6}}{\frac{1}{x+6}-\frac{1}{x+7}}.$$

22. $$\frac{\frac{x+3}{x+4}-\frac{x+2}{x+3}}{\frac{x+2}{x+6}-\frac{x+1}{x+5}} \div \frac{\frac{x+4}{x+2}-\frac{x+5}{x+3}}{\frac{x+4}{x+8}-\frac{x+2}{x+6}}.$$

23. $$\frac{3abc}{bc+ac-ab} - \frac{\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}.$$

24. $$\frac{\frac{a^2}{b^2}+\frac{b^2}{a^2}+2}{\frac{a^3}{b^3}-\frac{b^3}{a^3}-3\left(\frac{a}{b}-\frac{b}{a}\right)} \div \frac{\frac{a}{b}+\frac{b}{a}}{\frac{a^2}{b^2}+\frac{b^2}{a^2}-2}.$$

25. $$\frac{x}{x-a}-\frac{x}{x+a}-\frac{\frac{x+a}{x-a}-\frac{x-a}{x+a}}{\frac{x+a}{x-a}+\frac{x-a}{x+a}}.$$

26. $$\frac{\left[\frac{1}{a}+\frac{1}{b+c}\right]\left[1+\frac{b^2+c^2-a^2}{2bc}\right]\times 4b^2c^2}{\left[\frac{1}{a}-\frac{1}{b+c}\right]\left[a^2+b^2+c(c+2a)+2b(a+c)\right]}.$$

27. $$\frac{\left(\frac{a}{b}-\frac{b}{a}\right)^4+\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^4}{\left(\frac{a}{b}-\frac{b}{a}\right)^4-\left(\frac{2b^2}{a^2}+2\right)^2}.$$

XIII. SIMPLE EQUATIONS.

1. PROBLEMS CONTAINING ONE UNKNOWN QUANTITY.

(1) Abstract Problems.—153.

Find the value of x in the following equations:

1. $5x+50=4x+56.$
2. $16x=-3(4-x)+38.$
3. $7(x-18)=3(x-14).$
4. $7(x-2)=5(x+1)-9.$
5. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{x}{5}+\frac{x}{6}+13.$
6. $\frac{2x}{3}=\frac{x}{5}+\frac{2x-9}{3}.$
7. $a-\frac{ax-x^2}{x}=10.$
8. $\frac{x}{6}+\frac{x}{9}-\frac{x}{5}=7.$
9. $\frac{6}{x}+\frac{9}{x}-\frac{5}{x}=2.$
10. $\frac{3}{x}+\frac{2}{x}+\frac{1}{x}=\frac{1}{3}.$
11. $\frac{x+6}{5}-\frac{x-6}{6}=3.$
12. $5ax-c=b-3ax.$
13. $(x+1)(x+2)=(x^2-3x+8).$
14. $(x-4)(x+5)=(x-3)(x+3).$
15. $2(x+b)+3(x+2b)=5c-2b.$
16. $6x-2\{8x-3(x-\overline{b-x})\}=0.$
17. $a-\{a-(a-\overline{a-x})\}=2x-5.$
18. $(x-1)(x-2)+(x-2)(x-3)-(x-7)(x-1)=x^2+x.$
19. $3(x-3)-2(x-2)+x-1=x+3+2(x+2)+3(x+1).$
20. $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14.$
21. $5(5x-6)-4(4x-5)+3(3x-2)-2x-16=0.$

22. $\frac{3x-1}{2}-\frac{2x-5}{3}+\frac{x-3}{4}-\frac{x}{6}=x+1.$

23. $\frac{x-1}{4}-\frac{x-5}{32}+\frac{15-2x}{40}=\frac{9-x}{2}-\frac{7}{8}.$

24. $9-x-2(x-1)(x+2)=(x-3)(5-2x).$

25. $\frac{3x-1}{5}-\frac{13-x}{2}=\frac{7x}{3}-\frac{11(x+3)}{6}.$

26. $x-\frac{x-2}{3}=\frac{x+23}{4}-\frac{10+x}{5}.$

27. $x-\frac{x+23}{4}-\frac{x-2}{3}=-\frac{7-x}{5}.$

28. $\frac{\frac{1}{2}x-3}{5}+\frac{1}{2}(\frac{3}{4}x-10)+\frac{4-x}{4}=\frac{10-x}{6}.$

29. $\frac{x-4}{3}+(x-1)(x-2)=x^2-2x-4.$

30. $(3x-1)^2+(4x-2)^2=(5x-3)^2.$

31. $\frac{3x^2-2x-8}{5}=\frac{(7x-2)(3x-6)}{35}.$

32. $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=\frac{17a}{60}.$

33. $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a-4b}{21}=x.$

34. $\frac{7x+9}{4}-\left(x-\frac{2x-1}{9}\right)=7.$

35. $\frac{x+7}{11}-\frac{2x-16}{3}+\frac{2x+5}{4}=5\frac{1}{3}+\frac{3x+7}{12}.$

36. $\frac{1}{2}\left(\frac{2x}{3}+4\right)-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left(\frac{6}{x}-1\right).$

37. $a^2x+2ac-c^2x=a^2+c^2$.

38. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$.

39. $\frac{3+2x}{1+2x}-\frac{5+2x}{7+2x}=1-\frac{4x^2-1}{7+16x+4x^2}$.

40. $\frac{3-2x}{1-2x}-\frac{2x-5}{2x-7}=1-\frac{4x^2-1}{7-16x+4x^2}$.

41. $4bx-2a=3ab-6b^2x$.

42. $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}$.

43. $\frac{x}{2}-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\left(\frac{x^2+2}{3x-2}\right)$.

44. $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{3x+1}{21}-\frac{2x-2\frac{1}{5}}{6}+\frac{1}{105}$.

45. $\frac{66x+1}{1.5x+1}+\frac{4x+5}{.5x-1}=52$.

46. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x+\frac{1}{x^2-1}$.

47. $(a-b)(x-c)-(b-c)(x-a)=(c-a)(x-b)$.

48. $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{a(c-x)}$.

49. $(x+1)(x+2)(x+3)=(x-1)(x-2)(x-3)$
$+3(4x-2)(x+1)$.

50. $(x-3)^3-3(x-2)^3+3(x-1)^3=x^3-3-x$.

51. $(x-1)(x-5)(x-7)(x-9)=$
$(x-10)(x-6)(x-4)(x-2)$.

52. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}$.

53. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2.$

54. $\frac{1}{x-a}-\frac{1}{x-a+c}=\frac{1}{x-b-c}-\frac{1}{x-b}.$

55. $\frac{2x-a}{3}+\frac{1}{4}(3x+a+2b)-\frac{4b+a}{3}=a+2b.$

56. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{10\frac{1}{4}}{14}.$

57. $4.8x-\frac{.72x-.05}{.5}=1.6x+8.9.$

58. $\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax+b}{px+q}.$

59. $x^2+\frac{1}{x^2}+2=(x+1)\left(x+\frac{1}{x}\right).$

60. $\frac{x+2a}{x+a}+\frac{x}{a}=2x\left(\frac{x+2a}{a}-\frac{x}{x+a}\right).$

61. $\frac{x-\frac{1}{6}}{x-\frac{2}{3}}-\frac{x-\frac{2}{3}}{x-\frac{7}{6}}=\frac{x-\frac{1}{3}}{x-\frac{5}{6}}-\frac{x-\frac{5}{6}}{x-\frac{4}{3}}.$

62. $x^2-3ax+3a^2-\frac{9a^3}{x+3a}=x\left(3x-6a-\frac{2x^2}{x+3a}\right).$

63. $\frac{x-b-c}{a}+\frac{x-a-c}{b}+\frac{x-a-b}{c}=3.$

64. $\frac{1}{x+6a}+\frac{2}{x-3a}+\frac{3}{x+2a}=\frac{6}{x+a}.$

65. $\left(\frac{x-a}{x+b}\right)^3=\frac{x-2a-b}{x+a+2b}.$

66. $\frac{3x-\frac{2}{3}(1+x)}{4}+\frac{1-\frac{1}{5}x}{5\frac{1}{2}}=\frac{2\frac{2}{5}+\frac{1}{25}(x-1)}{2\frac{1}{5}}.$

67. $(x-a)^3+(x-b)^3+(x-c)^3=$
$3(x-a)(x-b)(x-c)$.

68. $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

69. $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}-\frac{1}{x}=\frac{abc}{x(x-a)(x-b)(x-c)}$.

(2) Concrete Problems.

1. A has a certain number of dollars, B has $\frac{1}{3}$ as many, and C $\frac{1}{4}$ as many; B and C together have \$21. How many has A?

2. In a certain granary there are 4 bins of grain. In the second bin there are $\frac{1}{3}$ as many bushels as in the first; in the third $\frac{1}{3}$ as many; and in the fourth $\frac{1}{12}$ as many. The second, third, and fourth contain 39 bushels. How many bushels in the first?

3. A fruiterer has apples, oranges, and lemons; he has $\frac{1}{18}$ as many oranges, and $\frac{1}{7}$ as many lemons, as apples. The lemons are 6 more than the oranges. How many apples has he? 91

4. A man has a horse, a cow, and a calf; the cow is worth $\frac{2}{3}$, and the calf $\frac{5}{48}$, as much as the horse. If the cow be worth \$81 more than the calf, what is the value of the horse?

5. Out of one of two equal loads of wheat, 22 bushels were taken; out of the other, 47 bushels; there were then left twice as many bushels in one load as in the other. How many bushels in each at first? 72

6. Divide 270 cents between two men so that $\frac{1}{5}$ of what one receives shall equal $\frac{1}{4}$ of what the other receives. 150

7. A man divided $33 between two laborers in such a way that 4 times what one received was equal to 7 times what the other received. How much did he give to each? 21

8. Five dollars more than $\frac{1}{2}$ of A's money is $20 less than $\frac{3}{4}$ of B's. If both have the same sum, how much has each? 150

9. A, B, and C gave $70 in charity; B gave $5 more than twice as much as A, and C as much as both. How much did each give? 10

10. The difference between 56 and $\frac{3}{4}$ of a certain number, is equal to the difference between 48 and $\frac{5}{8}$ of the same number. Find the number. $x = 64$

11. The sum of $\frac{1}{3}$ and $\frac{1}{5}$ of John's money is $7 less than all of his money. How much has he? 15

12. Divide 60 into two such parts that one part divided by 2 shall equal the other multiplied by 2.

13. A and B together have $125; $\frac{1}{2}$ of A's money is $25 more than twice B's. How much has each?

14. John and James together have $28; $\frac{1}{2}$ of John's equals $\frac{2}{3}$ of James's. How much has each?

15. A whale's head is 12 feet long; his tail is as long as his head plus $\frac{1}{2}$ the length of his body; his body is as long as his head and tail together. How long is the whale?

16. Four places, A, B, C, and D, are in the same straight line. The distance from A to B is $\frac{3}{5}$ of the distance from B to C; the distance from C to D is $\frac{1}{6}$ of the distance from A to D. From B to D is 80 miles. How far from A to D?

17. In a certain church there are 45 children. The number of men in the church equals the number of women and children, and the number of women

equals the number of children plus $\frac{1}{2}$ the number of men. How many of each?

18. A, B, and C together received \$1080. A received 5 times as much as C, while B received $\frac{1}{2}$ as much as A and C together. How much did each receive?

19. Divide \$720 among A, B, and C so that A shall have twice as much as C, and B \$120 more than A.

20. What sum of money may be divided among A, B, and C so that A shall get \$9 less than $\frac{1}{3}$ of it, B \$2 more than $\frac{1}{4}$, and C \$9 more than $\frac{1}{8}$?

21. If a certain amount of gold be put into a silver cup, the value of both will be \$130; if a quantity of silver be put into the same cup, the value of both will be \$60. The gold is worth \$15 more than twice the silver. Find the value of the gold, the silver, and the cup.

22. A gold watch and its key are worth \$360. A silver watch and the same key are worth \$60. Find the value of each watch and of the key, if all are worth \$410.

23. Divide 24 into two such parts that their sum shall be to their difference as 3 to 2.

24. Divide 175 into four such parts that the first increased by 2, the second diminished by 3, the third multiplied by 4, and the fourth divided by 5, shall be equal to each other.

25. Divide 25 into two such parts that the difference of their squares shall be 375.

26. The numerator of a fraction is 1 more than the denominator. If twice the fraction be added to thrice its reciprocal, the sum will be 5. Find the fraction.

27. Divide $4.20 into two such parts that in one part there shall be as many quarter dollars as there are dimes in the other.

28. The difference between the squares of two consecutive numbers is 17. Find the numbers.

29. After losing $\frac{1}{5}$ of my money and $\frac{1}{4}$ of what was left, I gained $\frac{1}{3}$ of the remainder, and had then $80. How much had I at first?

30. In a mixture of various grains amounting to 144 pounds, there were 3 pounds of barley to every 4 pounds of wheat and 5 of rye. What is the value of the mixture, if a pound of wheat is worth 2 cents, a pound of rye 1 cent, and a pound of barley $1\frac{1}{2}$ cents?

31. If from each of two loads of wheat 20 bushels be taken, the remainders will be in the ratio of 8 to 13; if 30 bushels more be then taken from each, the number of bushels remaining will be in the ratio of 1 to 2. How many bushels were there in each load at first?

32. Divide 40 into two such parts that if 6 be taken from the first, and 5 be added to the second, the results shall have the ratio of 4 to 9.

33. From a box of oranges and a box of lemons were taken numbers in the ratio of 5 to 7; if 6 fewer oranges and 7 more lemons had been taken, the ratio would have been 2 to 5. How many of each were taken?

34. A and B were rival candidates at an election. A beat B by $\frac{1}{2}$ of B's vote. But, if the same number of votes had been cast, and B had received 16 more, the vote would have been a tie. How many votes did each receive?

35. The weights in two scale-pans are as 4 to 5. If 3 pounds be taken from the lighter and put into the heavier pan, the latter will be twice as heavy as the former. What is the weight in each?

36. A's money is to B's as 6:7; A gains \$2, and B loses \$3, when their amounts are as 10:9. How much had each at first?

37. A and B have sums of money in the ratio of 4:5; A's money is \$7 less than $\frac{3}{5}$ of the whole sum. What has each?

38. A and B have sums of money in the ratio of 3:4; $\frac{2}{7}$ of A's exceeds $\frac{1}{7}$ of B's by \$6. How much has each?

39. A's age is to B's as 3:4; in 15 years it will be to B's as 6:7. How old is each?

40. Divide 60 into two such parts that the product of the two parts shall be three times the square of the less.

41. A and B had equal sums of money; A gave B \$20, and then had only $\frac{1}{2}$ as much as B. How much had each at first?

42. A and B began to play with equal sums of money. A won \$30, then lost $\frac{1}{2}$ of what he had, and afterward won \$15. He then had $\frac{5}{8}$ as much as B. How much had each at first?

43. A is 4 times as old as B; in 15 years he will be $2\frac{1}{2}$ times as old. How old is each?

44. A man is now 4 times as old as his son; 5 years ago he was 7 times as old. In how many years will he be twice as old?

45. A and B began to play with equal sums; A won \$10 and then had $1\frac{1}{2}$ times as much as B. How much had each to begin with?

46. If a lease is given for 60 years, and $\frac{3}{4}$ of the time it has already run equals $\frac{2}{5}$ of the time it has yet to run, how many years has it yet to run?

47. What o'clock is it, if $\frac{2}{5}$ of the time past noon is equal to twice the time to midnight?

48. A watch gains as much in an hour as a clock loses, and 1799 hours are accomplished by the clock in the same time as 1801 hours by the watch. What is the hourly gain of the watch?

49. A can do a piece of work in 10 days; A and B can do the same work in 7 days. How long will it take B?

50. A does $\frac{2}{5}$ of a piece of work in 24 days; then, with the aid of B, he finishes it in $10\frac{2}{7}$ days. How long would it take each to do it alone?

51. A and B together, in $2\frac{1}{3}$ days, do a piece of work which would require A $3\frac{1}{4}$ days to do alone. How long would it take B to do it alone?

52. A can do a piece of work in $\frac{1}{3}$ of a day, B in $\frac{1}{4}$ of a day, and C in $\frac{2}{5}$ of a day. In what time can they do it working together?

53. A can do a piece of work in 12 days; B and C together, in 7 days. They all work together 4 days, after which C finishes it in 2 days. How long would it take each to do it alone?

54. A and B do $\frac{1}{4}$ of a piece of work in 2 days; B alone can do $\frac{1}{3}$ of it in 6 days. How long would it take A to do $\frac{1}{2}$ of it?

55. A cistern has two inlet pipes and one outlet pipe; the first inlet pipe can fill the cistern in $4\frac{1}{2}$ hours, and the second in 6 hours, while the outlet pipe can empty it in 5 hours. If all run together, how long will it take to fill the cistern?

56. A can do a piece of work in $\frac{1}{2}$ the time that B can; B in $\frac{2}{3}$ the time C can. Working together, they can do it in 6 days. How long would it take each of them alone?

57. A can do $\frac{1}{3}$ as much work in a day as B; B $\frac{1}{3}$ as much as C; together, they can finish a piece of work in $\frac{1}{8}$ of a day. How long will it take each?

58. A, B, and C can do a piece of work in 40 days. A does 8 times as much in a day as B, and B $\frac{1}{3}$ as much as C. How long will it take each alone?

59. Two trains remove an embankment containing 54720 cubic yards of earth in 24 days of 12 hours each. The smaller train carries $\frac{2}{5}$ as much as the larger, and makes a trip each hour, while the larger makes 3 trips every 4 hours. How many cubic yards does each carry at a trip?

60. A, B, and C engaged to do a piece of work for \$16.80, to be paid in proportion to the amount of work done by each. A could have done the whole work in 6 days, B in 12 days, and C in 24 days. A works alone 2 days, is then joined by B, and at the end of the third day C joins them, and the three complete the job. What is each man's share?

61. A, B, and C engaged to do a piece of work for \$8.40. A could have done the work in $\frac{1}{2}$ of the time it would have taken B, and B in $\frac{1}{2}$ of the time it would have taken C. A worked one day, and was then joined by B for another day; C then joined them, and the three completed the task in $\frac{5}{7}$ of a day more. What did each receive if they were paid in proportion to their work?

62. A and B agree to do a piece of work for \$17. They work together for 2 days, when A leaves and

B finishes in $2\frac{1}{3}$ days. If B had left, A could have finished it in $3\frac{1}{2}$ days. What is each entitled to, if paid in proportion to the amount of work done?

63. A was hired to do a certain work, agreeing that for each working day he should receive 50 cents, and for each idle day he should pay $18\frac{3}{4}$ cents. He worked twice as many days as he was idle, and received \$19.50. How many days did he work?

64. A engaged to deliver $3\frac{7}{12}$ dozen of eggs to B. He was to receive 5 cents for every 4 eggs that he delivered unbroken, and to pay 2 cents for every 7 broken. When he settled, nothing was coming to him. How many did he break?

65. A gambler engaged to throw dice for an ace, for 100 consecutive throws. He was to get 3 cents for each time he succeeded, and to pay $\frac{1}{2}$ cent for each time he failed. On finishing the throws he received \$1.60. How many times did he fail?

66. G was hired for a days, at b cents for each day he worked, and at a forfeit of c cents for each day he was idle. At the end of the time he received d cents. How many days was he idle?

67. A farmer's boy engaged for a year, for \$340 in money and a colt. He worked for 7 months, when he left, receiving for his wages \$190 and the colt. What was the colt worth?

68. A rents a farm for \$80 in money and a certain fixed number of bushels of wheat. When wheat is worth \$1.00 a bushel, the rental is \$8 an acre; when wheat is \$1.25 a bushel, the rental is \$9 an acre. How many acres in the farm?

69. A paid B \$25 in half-dollars and dimes; if there were 170 pieces in all, how many of each kind?

70. A man rents 50 acres of land for \$76,—one portion at \$2 an acre, and the remainder at \$1.25 an acre. How many acres of each kind?

71. A farmer sold cows and calves at an average price of \$19.80. Each cow sold at \$25; and there were 40 calves at \$12 each. How many cows did he sell?

72. A bought apples at 15 cents a dozen; B bought 3 less for the same total price, and found he had paid $1\frac{7}{8}$ cents a dozen more than A. How many did each buy?

73. A man gave \$6.35 in charity among a number of men, women, and children. To the men he gave 25 cents each, to the women 20 cents each, and to the children 10 cents each. There were 5 more women than men, and 10 more children than women. How many of each?

74. I can buy note-paper at the rate of 15 sheets for 5 cents, and cap-paper at the rate of 6 sheets for $4\frac{1}{2}$ cents. I buy a quire of the mixed paper for 13 cents. How many sheets of each kind do I buy?

75. A man bought a lot of potatoes at \$1.10 a bushel. He sold $\frac{1}{3}$ of them at \$1.25; $\frac{1}{4}$ of them at \$1; 6 bushels, which had been damaged, at 75 cents; and the balance at first cost. He lost \$1.90; how many bushels did he buy?

76. A man bought a number of hogs for \$559; for $\frac{1}{3}$ of them he paid \$7 apiece; for $\frac{1}{4}$ of them \$5 apiece; for $\frac{1}{10}$ of them \$6 apiece; and for the balance \$5$\frac{1}{2}$ apiece. How many did he buy in all?

77. Bought apples at 3 for a dime; sold $\frac{1}{3}$ of them at 4 cents apiece, and the balance at 2 for 7 cents, clearing a dime on the lot. How many did I buy?

78. In the division of a prize captured by a privateer, the officers received \$7560, and the 27 men of the crew divided the remainder equally. If there had been 25 men in the crew, and each had received the same share as before, there would have been \$9560 left for the officers. What was the value of the prize?

79. The income of a R. R. Co. is 6 per cent on its entire stock; \$40,000, preferred stock, draws 7½ per cent; this leaves a 5 per cent dividend on the common stock. How much common stock was there?

80. A and B divide equally the value of 175 shares of R. R. stock, A taking 95 shares and paying B \$375. Find the value of each share.

81. Eggs sold the day before Christmas at a certain price per dozen, and a man bought 8 eggs at that price. The day after Christmas he bought 8 more, and found that he could buy 3 times as many eggs for 2 dimes on the second day as he could buy dozens for a dollar on the first day. The 16 eggs that he bought cost him 18 cents. What was the price per dozen each day?

82. A bought an equal number of yards of cloth at \$4 and \$5 a yard respectively. B bought the same kinds of cloth for the same total amount of money, but laid out his money equally between the two kinds. By so doing he obtained 1 yard more than A did. How many yards did each buy?

83. Two farmers, who owned equal shares in a flock of sheep, agreed to divide the flock. A took 72 sheep, while B took 92 sheep and paid A \$35. What was the value of the flock?

84. Two men received \$43 for their work. A

worked 16 days and B 12 days; now 8 of A's days' work were worth \$9 more than 4 of B's. What was the daily wages of each?

85. A dealer sold a certain number of oranges and apples for \$6.60, there being 100 more oranges than apples. He sold the apples at the rate of 2 for 5 cents; now 30 oranges cost 85 cents more than 20 apples. How many were there of each kind?

86. A debt of \$500 was paid in eagles and dollars. $\frac{1}{2}$ the number of dollars exceeded $\frac{2}{3}$ the number of eagles by 80. How many of each kind of coin?

87. A debt of \$500 was paid in eagles and dollars. $\frac{1}{2}$ the amount paid in dollars exceeded $\frac{2}{3}$ the amount paid in eagles by \$110. How many of each kind of coin were used?

88. A waterman rows 40 miles and back in 13 hours, rowing 16 miles with the current for 10 miles against it. What is his time down the stream?

89. In a horse-race, one horse runs at an average rate of 11 yards per second; the other runs the first half of the distance at the rate of 10 yards per second; and the other half at the rate of 11 yards per second, and reaches the post 8 seconds behind the first. What is the length of the course?

90. How far can a man walk at the rate of $2\frac{1}{3}$ miles an hour so as to return to the place of starting in 5 hours, walking back at the rate of $3\frac{1}{2}$ miles an hour?

91. A man having a hours at his disposal, walked a certain distance at the rate of c miles an hour, and rode back at the rate of d miles an hour. How far did he walk?

92. A body of troops stationed 44 miles in front

of an enemy, retreats at an average rate of 16 miles a day. The pursuers, starting 2 days afterwards, move at the rate of 25 miles a day, but on two different occasions are obliged to halt a day at a time. How far does the retreating army march before being overtaken?

93. A passenger train, running 15 miles an hour, leaves Hamilton for Cincinnati at the same time that a freight train, running $8\frac{1}{3}$ miles an hour, leaves Cincinnati for Hamilton. The passenger train stops 40 minutes in Cincinnati, and then returns, arriving at Hamilton 1 hour after the freight. Find the distance between Hamilton and Cincinnati.

94. A sets out at noon to travel from C to D at the rate of $3\frac{1}{2}$ miles an hour. At 20 minutes to 1 o'clock, B sets out from D towards C at the rate of $4\frac{1}{2}$ miles per hour. They pass each other at a point which is half a mile nearer C than D. What distance does B travel before he meets A?

95. The express train, which travels at the rate of 80 miles in 3 hours, makes the trip from Wyoming to Cincinnati in 23 minutes less than the accommodation train, which runs $14\frac{2}{5}$ miles an hour. How far is it from Wyoming to Cincinnati?

96. A hare takes 4 leaps to a greyhound's 3, but 2 of the greyhound's leaps are equivalent to 3 of the hare's; the hare has a start of 50 leaps; how many leaps must the greyhound take to catch the hare?

97. A courier, A, starts 820 of his own steps ahead of a courier, B. A takes 5 steps while B takes 4; if 3 of B's steps be equal to 4 of A's, how many steps must B take to overtake A?

98. From one of two places, 505 miles apart, A

sets out at the rate of $10\frac{1}{2}$ miles per hour. 10 hours afterwards, B sets out to meet A, at the rate of $9\frac{1}{2}$ miles per hour. How far from A's starting-point will they meet?

99. A walks at the rate of $3\frac{3}{4}$ miles per hour, and starts 18 minutes before B. At what rate per hour must B walk to overtake A at the 9th mile-stone?

100. A and B start at the same time and from the same point to go 39 miles. A, by traveling $\frac{1}{4}$ of a mile an hour slower than B, lacks 3 miles of finishing the journey when B reaches the end. What are their rates of traveling?

101. Three boats are running from Cincinnati to Louisville,—A at the rate of 12, B 8, and C 6, miles an hour. C starts at 10 o'clock A. M., and A and B, each, at noon. At what o'clock will A be just half way between B and C?

102. A sets out from a certain place at the rate of 4 miles an hour. $1\frac{1}{2}$ hours afterwards, B, who travels at the rate of $4\frac{3}{4}$ miles an hour, starts from the same place to overtake A. When, and how far from the starting-place, will he overtake him?

103. A traveled from C to D at the rate of 9 miles in 2 hours; 6 hours after A set out, B started from C, and, by traveling at the rate of 36 miles in 5 hours, reached D at the same time as A. How far from C to D?

104. From two places, 62 miles apart, A and B set out at the same time to meet. A traveled at the rate of 7 miles in 2 hours, and B at the rate of 5 miles in 3 hours. Find the time and the distance each traveled.

105. A boatman rows with the tide 36 miles in 4

hours, and returns against a tide $\frac{4}{5}$ as strong in 18 hours. Find the rate of the tide in each case.

106. A party of 10 men wished to go a distance of 20 miles, and for this purpose hired a coach which would hold 5 men. The entire party set out at the same time, 5 riding in the coach at the rate of 5 miles an hour, and the other 5 walking at the rate of 3 miles an hour. How far may the first party ride, so that, while they walk from the point where they leave the coach to the end of the trip at the rate of 3 miles per hour, the coach may return, take up the second party, and reach the end of the trip at the same time?

107. A and B set out at the same time and travel towards each other, A at the rate of 3 miles an hour, and B at the rate of 4 miles an hour. At the same time, C sets out with A at the rate of 5 miles an hour, travels till he meets B, then turns about, and, in 10 hours after setting out, meets A. How far apart were A and B at first?

108. A waterman rows a distance of 40 miles and back again in 24 hours; he finds he can row 10 miles with the current to 2 miles against it. Find his time *down* and *up*, his *rate of rowing*, and *the rate of the current.*

109. A and B start to run a race to a certain post and back again. A, running the faster, arrives at the post, and returning meets B 90 yards from it, and reaches the starting-place 3 minutes before B. Had he then turned back, he would have met B again at $\frac{1}{8}$ the distance from the first post to the second. Find the distance between the posts, the rates of running, and the time of the race.

110. A farmer mixes rye worth 75 cents a bushel with other rye worth 90 cents a bushel, and makes a mixture of 50 bushels, worth $42.00. How many bushels of each kind does he take?

111. A grocer mixes 50 pounds of sugar worth 9 cents a pound with 10 pounds of flour worth 3 cents a pound, and sells the mixture at 10 cents a pound. What per cent profit does he make?

112. Twenty gallons of molasses worth 60 cents a gallon are mixed with a keg of molasses worth 90 cents a gallon. If the mixture be worth 65 cents a gallon, how many gallons does the keg hold?

113. How many pounds of tea at 70 cents must be mixed with each pound of tea at $1.20, so that, by selling the mixture at 99 cents a pound, a merchant may make 10 per cent?

114. When, between 10 and 11 o'clock, will the hour and minute-hands of a clock be 10 minutes apart?

115. When, between 5 and 6 o'clock, will they be at right angles? (*Two solutions.*)

116. When, between 9 and 10 o'clock, will they be together? Opposite to each other?

117. It is between 11 and 12 o'clock, and it is observed that the number of minute spaces between the hour and minute-hands of a clock is $\frac{2}{3}$ of what it was 10 minutes ago. Find the time.

118. When, between 3 and 4 o'clock, will the hour and minute-hands be 20 minutes apart?

119. At what time after 3 o'clock will the second-hand of a watch be opposite the minute-hand, if both turn upon the same pivot?

120. At what time after 4 o'clock will the second-hand be opposite the hour-hand?

121. A gas meter has two pointers turning upon the same center; the faster pointer makes a revolution every 12 minutes, and the slower, every 16 minutes. How long will it take the faster to gain one revolution?

122. When, after 2 o'clock, will the second-hand of a watch be at right angles to the minute-hand, if both turn upon the same center?

123. All the hands of a watch turn upon the same pivot. At what time between noon and 1 o'clock will the second-hand be half way between the other two? The minute-hand half way between the other two? The hour-hand?

124. When, between 4 and 5 o'clock, will the figure 7 on the dial be half way between the hour-hand and minute-hand?

125. When, between 2 and 3, will the minute-hand be as far past 8 as the hour-hand is past 1?

126. When, between 3 and 4, will the hour-hand be as far past 2 as the minute-hand lacks of being at 8?

127. In the population of a city, the number of men was 5000 more than $\frac{1}{10}$ of the whole; the number of women, 2000 more than $\frac{1}{8}$ of the whole; and the number of children, 6000 more than the whole number of men and women. What was the total population?

128. From two casks of equal size, were drawn quantities in the ratio of 6:7. If 16 gallons less had been drawn from the one which then contained the less, only $\frac{1}{2}$ as much would have been drawn from it as from the other. How many gallons were drawn from each?

129. In a certain weight of borax, the sodium was 7 pounds more than $\frac{1}{6}$ of the whole, the oxygen was 5 pounds less than $\frac{1}{2}$ of the whole, and the boron was 20 pounds more then $\frac{1}{2}$ of the oxygen. What was the weight of the borax?

130. A boy gave away 20 more than $\frac{1}{6}$ of his marbles, and had left 10 more than $\frac{1}{6}$ of them. How many had he at first?

131. A and B have equal incomes. A's expenses exceed his income by $\frac{1}{6}$ of it, while B lives on $\frac{3}{4}$ of his. At the end of 3 years, B lends A enough to pay his debts, and has $150 left. What is the income of each?

132. A man spends $100 a year for clothing; $\frac{2}{3}$ of what is left of his salary in the support of his family; $50 for cigars; $\frac{1}{5}$ of what is left for amusements, and has $200 left. What is his income?

133. Out of a basket of oranges, John was given 4 and $\frac{1}{4}$ of the remainder; James was then given 4 and $\frac{1}{2}$ of what was left. John and James then had the same number. How many did the basket hold?

134. Two boys were reading a book aloud. The first read 4 pages and $\frac{1}{5}$ of what remained; the second read the next 5 pages and $\frac{1}{5}$ of what remained. They found that each had read the same number of pages. How many pages in the book?

135. A man went to market and spent, at one stand, $\frac{1}{3}$ of his money and $6 more; at a second stand, he spent $\frac{1}{2}$ of what he then had and $10 more; he then had $2 left. How much did he start with?

136. A merchant adds $1,700 to his capital the first year; during the second year he further increased it by a sum equal to $\frac{1}{10}$ of his original capital; during

the third year he lost 40 per cent of what he had at the end of the second year, and found that he then has just what he began with. What was his original capital?

137. A farmer bought a number of chickens. For each of three years afterwards, he sold 3, but the remainder was increased each year by $\frac{1}{2}$ of itself. At the end of the third year, he had three times as many as he first bought. How many did he buy at first?

138. A gambler lost $\frac{2}{7}$ of his money, and then won \$10; he then lost $\frac{3}{8}$ of what he had, and afterwards lost \$50; he next won $\frac{1}{2}$ of what he had, but found that his total loss amounted to \$135. What did he have at first?

139. A merchant increased his capital the first year by 10 per cent of itself; the second year he gained 20 per cent; the third year he lost 25 per cent. He then had \$100 less than at first. What was his original capital?

140. From a basket of apples, 10 more than $\frac{1}{2}$ were taken; from the remainder, 20 more than $\frac{1}{3}$; and from what was left, 30 more than $\frac{1}{4}$. There were then none left. How many were in the basket at first?

141. From a bin of wheat, 20 bushels more than $\frac{1}{3}$ were taken; from what was left, 10 bushels more than $\frac{1}{4}$; and from the remainder, 5 bushels more than $\frac{1}{5}$. There were left 11 bushels. How many bushels were in the bin at first?

142. A, B, C, and D divided a gift as follows: A took $\frac{1}{3}$ of it, lacking \$3; B took $\frac{1}{3}$ of what was left, lacking \$3; C, $\frac{1}{3}$ of what was then left, lacking \$3; and D's share was \$17. What was the share of each?

143. From a certain sum, I took a $\frac{1}{3}$ part and put

back \$50; from this amount I took $\frac{1}{4}$ and put back \$30; from this I took $\frac{1}{5}$ and put back \$9. I then found the same amount left as at first. What was the original sum?

144. The epitaph of Diophantus states that he passed one sixth of his life in childhood, one twelfth of it in the state of youth; that, after an interval of 5 years more than $\frac{1}{7}$ of his life, he had a son who died when he had attained to $\frac{1}{2}$ the father's age at death, and that the father survived the son 4 years. How old was Diophantus?

145. Divide a sum of money among a certain number of persons so that the first shall get \$1,000 and $\frac{1}{7}$ of what is left; the second, \$2,000 and $\frac{1}{7}$ of what is left; the third, \$3,000 and $\frac{1}{7}$ of what is left, and so on. Each gets an equal portion. What is the amount of money, and what the number of persons?

146. A country paid $4\frac{1}{2}$ per cent interest on its debt. A war increased the amount of debt by $\frac{1}{4}$. During a peace which followed, the debt was diminished by \$25,000,000, and the rate was reduced to 4 per cent. The annual interest was then the same as at first. What was the indebtedness of the country before the war?

147. A and B played cards with equal sums of money. B lost the first game, and gave A one shilling less than $\frac{1}{2}$ of his money. A lost the second game, and gave B one shilling less than $\frac{1}{2}$ of what he (A) then had. B now has two shillings more than A. What did each have at first?

148. If one side of a square field be 2 rods longer than one side of another, and its contents 12 square rods more, how many square rods in each field?

149. A general can draw up his army in a solid square, and have 284 men left. By increasing the side of the square by 1 man, he lacks 25 men of having enough for the complete square. How many men are there in the army?

150. A regiment of 656 men can be formed into a hollow square 6 men deep, and there will be left one extra file containing the same number of men as one side of the square. How many in each outside file?

151. An officer can form his men into a hollow square 4 deep, and also into a hollow square 8 deep. The latter formation contains 16 men fewer in the front rank than the former. How many men has he?

152. A school-room contains 40 pupils, the ratio of boys to girls being 5 : 3. How many boys must withdraw, so that for every 5 girls there shall be 7 boys?

153. A mixture of sand and sugar, weighing 30 pounds, contains 3 pounds of sand. How much sugar must be added, so that for every 16 pounds of sugar there shall be one pound of sand?

154. A piece of gold, alloyed with silver, is 14 carats fine, and weighs 72 pennyweights. How much gold must be added to make it 18 carats fine? How much silver, to reduce it to 12 carats?

155. A mixture of black and green teas weighs 16 pounds, and for every 5 pounds of black, there are 3 pounds of green, tea. How much green tea must be added, so that for every 2 pounds of black tea in the new mixture there shall be 3 pounds of green?

156. Two purses, A and B, contain mixtures of gold and silver coins, A in the ratio of 2 : 3, and B in the

ratio of 3:1. A contains 50 pieces. How many must B contain, so that when the two purses are emptied together the number of gold coins shall be the same as the number of silver coins?

157 A cask, A, contains 12 gallons of wine and 18 gallons of water; another cask, B, contains 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each so as to produce, by their mixture, 7 gallons of wine and 7 gallons of water?

158. Each of two bins, A and B, contains a mixture of wheat and rye: A, in the ratio of 3 of wheat to 5 of rye; B, in the ratio of 2:9. What quantity must be taken from each to form a mixture consisting of 14 bushels of wheat and 29 of rye?

159. A man borrowed money at 6 per cent. By securing a lower rate, he was able to borrow $\frac{1}{8}$ as much more and still save $\frac{1}{2}$ of his interest. What was his last rate of interest?

160. A man borrowed enough money at 6 per cent to pay for a house, and also for repairs amounting to 2 per cent of the purchase money. The house was idle for a year, and during that time he had to pay $34.60 taxes. At the end of the year he sold it for $4,600 and found his net loss to be $6\frac{2}{3}$ per cent of the purchase price. What did the house cost?

161. A ship set out upon a voyage of 60 days, with sufficient provisions to allow each of her crew one pound of bread per day. After being out 20 days, she encountered a storm, in consequence of which she lost 5 men, and her voyage was lengthened 24 days. She then had provisions enough to allow each man $\frac{5}{7}$ of a pound of bread per day for the rest of the voyage. How many were in the original crew?

162. An emigrant party had potatoes enough to last 30 days, allowing to each 1 quart daily. At the end of 3 days, 5 died, and the allowance was increased to 2 quarts each per day. Five days after, 27 newcomers joined the party and remained the rest of the 30 days. After their arrival, the allowance was limited to 1 pint each per day. What was the original number?

2. SIMPLE EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES,

(1) Abstract Problems.—156, 157, 158.

Solve the following equations:

1. $\begin{cases} 2x+3y=16 \\ x-2y=1 \end{cases}$

2. $\begin{cases} 4x-\frac{y}{2}=17 \\ 6x-5y=0 \end{cases}$

3. $\begin{cases} 2x-7y=0 \\ 3x+5y=31 \end{cases}$

4. $\begin{cases} 4x-2=6y \\ 3x+2y=21 \end{cases}$

5. $\begin{cases} 4x+5y=7 \\ 5x-4y=-22 \end{cases}$

6. $\begin{cases} 3x=2y \\ 6x-5y=-3\frac{1}{2} \end{cases}$

7. $\begin{cases} \frac{x}{2}+\frac{y}{3}=6 \\ \frac{x}{3}-\frac{y}{9}=1 \end{cases}$

8. $\begin{cases} \frac{x}{7}+\frac{y}{2}=\frac{9}{14} \\ \frac{2x}{7}-\frac{7y}{2}=0 \end{cases}$

9. $\begin{cases} \frac{x}{4}-\frac{3y}{8}=0 \\ \frac{x}{6}-\frac{5y}{8}=3 \end{cases}$

10. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{5}{12} \\ \frac{1}{x}-\frac{1}{y}=-\frac{1}{12} \end{cases}$

11. $\begin{cases} 2x^{-1}+3y^{-1}=\frac{9}{20} \\ 3x^{-1}+2y^{-1}=\frac{7}{15} \end{cases}$

12. $\begin{cases} \frac{6}{2x}-\frac{4}{3y}=3\frac{1}{3} \\ \frac{2x^{-1}}{3}+\frac{2y^{-1}}{5}=2\frac{2}{15} \end{cases}$

13. $\begin{cases} 2x^{-1}-3y^{-1}=-\frac{1}{2} \\ \frac{3x^{-1}}{2}+\frac{4y^{-1}}{3}=\frac{59}{72} \end{cases}$

14. $\begin{cases} \frac{3x}{4}-\frac{y}{2}=\frac{3(x-y)}{2}+1 \\ \frac{x}{6}+\frac{y}{5}=\frac{1}{7}(2x+4) \end{cases}$

15. $\begin{cases} \frac{x}{5}+5y=51 \\ 5x+\frac{y}{5}=27 \end{cases}$

16. $\begin{cases} \frac{.3x+.4y}{.11}=\frac{.4x+.3y}{.1} \\ \frac{.5x-.1y}{.3}=\frac{\frac{1}{5}x-\frac{8}{10}}{.16} \end{cases}$

17. $\begin{cases} \frac{x+y}{3}+\frac{x-y}{4}=13 \\ \frac{x}{5}-\frac{x+y}{12}=1 \end{cases}$

18. $\begin{cases} \frac{x-y}{x+y}=\frac{3}{5} \\ \frac{x-6}{x+2y+4}=\frac{1}{3} \end{cases}$

19. $\begin{cases} ax=by \\ x+y=c \end{cases}$

20. $\begin{cases} c^2x+a^2y=2ac \\ ax+cy=\frac{a^3+c^3}{ac} \end{cases}$

21. $\begin{cases} \frac{x+\frac{1}{2}}{y+\frac{2}{3}}=\frac{x+4\frac{1}{2}}{y+3\frac{2}{3}} \\ \frac{x-\frac{1}{2}}{y-\frac{1}{3}}=\frac{2x+2}{2y+1\frac{1}{3}} \end{cases}$

22. $\begin{cases} \frac{x+2}{y+3}=\frac{x-2}{y-3} \\ \frac{3x-5}{2y-2}=\frac{6x-14}{4y-2} \end{cases}$

23. $\begin{cases} 3x-2y+z=5 \\ 5x+3y-5z=2 \\ 2x-5y+4z=7 \end{cases}$

24. $\begin{cases} \frac{1}{5}(x+z)+\frac{1}{5}y=3 \\ \frac{1}{3}x+\frac{2}{3}(y+z)=4 \\ \frac{2}{11}(x+y)+\frac{3}{4}z=5 \end{cases}$

25. $$\begin{cases} \frac{5x}{4} - \frac{3y-z}{3} = 1 \\ \frac{x-y}{3} + \frac{2z-3y}{2} = 1 \\ \frac{x-y+z}{12} = 1 \end{cases}$$

26. $$\begin{cases} 5x-4y+3z = 7\frac{1}{2} \\ 4x+3y-2z = 10 \\ 3x-2y+z = 4\frac{1}{2} \end{cases}$$

27. $$\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \\ \frac{1}{2x} + \frac{1}{3y} + \frac{8}{3z} = 1 \\ \frac{2}{3x} + \frac{4}{3y} + \frac{4}{3z} = 1 \end{cases}$$

28. $$\begin{cases} \frac{x}{a} + \frac{y}{b} = m \\ \frac{x}{a} + \frac{z}{c} = m \\ \frac{y}{b} + \frac{z}{c} = m \end{cases}$$

29. $$\begin{cases} \frac{a}{x} + \frac{b}{y} = 5 \\ \frac{a}{x} + \frac{c}{z} = 5 \\ \frac{b}{y} + \frac{c}{z} = 5 \end{cases}$$

30. $$\begin{cases} x + \frac{y}{2} + \frac{z}{3} = 32 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 22 \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 17 \end{cases}$$

31. $$\begin{cases} \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{23}{24} \\ \frac{2}{x} + \frac{3}{y} + \frac{4}{z} = \frac{3}{2} \\ \frac{3}{x} + \frac{4}{y} + \frac{7}{z} = \frac{55}{24} \end{cases}$$

32. $$\begin{cases} x-y+2z = 2(2c-a) \\ y-z+2x = 2(2a-b) \\ z-x+2y = 2(2b-c) \end{cases}$$

33. $$\begin{cases} 5xy = 6(x+y) \\ 3xz = 4(x+z) \\ 7yz = 12(y+z) \end{cases}$$

34. $$\begin{cases} 2x-y+3z = 4a \\ 4x-u = 4a \\ y+z = 4a \\ 4y-2u = 4a \end{cases}$$

35. $$\begin{cases} 3z+4t-3u = 10 \\ z-t+2u = 9 \\ x+y-2z+u = 2 \\ 3x+2y = 7 \\ 2x-3y+4z = 8 \end{cases}$$

36. $$\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{3}{2z}=1 \\ \frac{1}{x}+\frac{1}{y}+\frac{2}{u}=1 \\ \frac{1}{x}+\frac{1}{z}+\frac{8}{3u}=1 \\ \frac{1}{y}+\frac{1}{z}+\frac{14}{3u}=1 \end{cases}$$

37. $$\begin{cases} 6x-2y^{\frac{1}{2}}+5z^{\frac{1}{3}}=34 \\ 4x+3y^{\frac{1}{2}}-2z^{\frac{1}{3}}=25 \\ x-y^{\frac{1}{2}}+z^{\frac{1}{3}}=4 \end{cases}$$

38. $$\begin{cases} x-ay+a^2z=a^3 \\ x-by+b^2z=b^3 \\ x-cy+c^2z=c^3 \end{cases}$$

39. $$\begin{cases} \frac{1.5x+.3y}{1.2}-\frac{.7x+.6y}{.9}=10 \\ \frac{.07x+.04y}{.05}-\frac{.06x+.08y}{.13}=30 \end{cases}$$

40. $$\begin{cases} x+\frac{y}{a-b}=a(2a+b) \\ (a^2+b^2)x-(a+b)y=ab(b^2-a^2) \end{cases}$$

41. $$\begin{cases} \frac{6x^2-6y^2+1}{2x-2y-1}=3x+3y-2 \\ x+\frac{3y-2x}{x+3}=x+4 \end{cases}$$

42. $$\begin{cases} 1+\frac{a}{x-3a}=2-\frac{y-4b}{y-3b} \\ \frac{x+2a}{a}=1+\frac{3b}{y+3b} \end{cases}$$

43. $$\begin{cases} x+y+10-z=4(y+z-x) \\ x+30=1-(29-z-y) \\ x+y+z=34+8(z-y) \end{cases}$$

44. $$\begin{cases} \dfrac{2x+3y+4z}{5} = \dfrac{x+7z-y}{3} \\ 3x+2y+5z = 4x+6y-2z-2 \\ \dfrac{3x+y+3z}{7} + \dfrac{5x+2y+2z}{6} = \dfrac{6x+4y+2z}{5} \end{cases}$$

45. $$\begin{cases} x-y+z=0 \\ (a+b)x-(a+c)y+(b+c)z=0 \\ abx-acy+bcz=(a-b)(a-c)(b-c) \end{cases}$$

46. $$\begin{cases} x+y+z=a+b-c \\ ax+by+cz=a^2+b^2-c^2 \\ bx+cy+az=b^2+c^2+a^2-2ac \end{cases}$$

47. $$\begin{cases} mx+ny+rz=rn+rm+mn \\ mnx+nry+mrz=rn^2+r^2m+m^2n \\ m^2x+n^2y+r^2z=3mnr \end{cases}$$

(2) Concrete Problems.

1. Four horses and 3 cows cost \$700, and 3 horses and 11 cows cost \$700. What did a cow cost?

2. A and B sat down to play cards with \$1,500. When they finished it was found that B had won $\frac{1}{2}$ of A's money, and that A then had just $\frac{1}{4}$ as much as B. How many dollars had each at first?

3. There are 2 bins containing wheat. If 5 bushels be transferred from the first bin to the second, the second will then contain twice as much as the first; but if 5 bushels be transferred from the second to the first, they will both contain the same number. How many bushels are there in each bin?

4. Six times A's money is equal to 7 times B's, and, if A give B $1, he will then have $1 more than B. How many dollars has each?

5. A bought a number of cows and sheep for $650, paying $25 for each cow, and $5 for each sheep. He sold $\frac{1}{5}$ of his cows and $\frac{1}{8}$ of his sheep at first cost for $125. How many of each did he buy?

6. Find four numbers, such that the first shall equal $\frac{1}{2}$ the sum of the second and third; the second, $\frac{1}{4}$ the sum of the third and fourth; the third, $\frac{5}{11}$ of the sum of the first and fourth; and the fourth, the sum of the other three minus 20.

7. A man can buy 5 horses and 2 cows for $400; if the price of cows decreases 20 per cent, and that of horses increases $16\frac{2}{3}$ per cent, he can buy 4 horses and 3 cows for the same money. What is the first price of one of each?

8. When meal is 60 cents a bushel and bran 50 cents, a certain number of bushels of each costs $4.00. If the meal rise in value 10 per cent and the bran 6 per cent, the same number of bushels will cost $4.36. How many bushels of each are there?

9. In an election, A beat B by 6 votes less than $\frac{1}{2}$ of the number that B received. If the same number of votes had been cast as before, but B had received $\frac{1}{8}$ as many more votes, the result would have been a tie. How many votes did each get?

10. A gave B $6, and B then had twice as many as A; had B given A $8, A would have had twice as many as B. What amount had each at first?

11. Five years ago, A was $2\frac{1}{2}$ times as old as B; in 15 years from now, he will be $1\frac{1}{2}$ times as old. How old is each now?

12. A and B have a bet of $50. If A win, he will have twice as much as B; if he lose, they will have equal amounts. How much has each?

13. Find a fraction, whose numerator being doubled and the denominator increased by 1, its value becomes 1; and, if the denominator be doubled and the numerator be increased by 1, its value will be $\frac{2}{5}$.

14. Find a fraction whose numerator is 3 less than its denominator, and if its numerator be diminished by 2, the value of the fraction will be diminished by $\frac{1}{4}$.

15. Find a proper fraction such that, if 3 be subtracted from each term, the result will equal $\frac{2}{3}$; and if the sum of the numerator and denominator be divided by their difference, the result will be 8.

16. A number of two digits is reversed by subtracting 36; the sum of the number and its reverse is 110. Find the number.

17. If 27 be added to twice a number of two digits, the result will equal 27 less than twice the reverse of the number. The same result will be obtained by adding 12 more than $\frac{5}{4}$ of the number to $\frac{2}{3}$ of its reverse. Find the number.

18. The tens' digit of a number exceeds the units' digit by 5. The number being reversed, the first number is to its reverse as 8:3. Find the number.

19. Find a number of two figures, which, added to its reverse, equals 132, and, subtracted from its reverse, equals 36.

20. Find a number of two places which equals 21 times the difference of its digits, and if 36 be taken from it, the number will be reversed.

21. A number of two places, when divided by 9, gives a quotient equal to the tens' digit, and a re-

mainder equal to the units' digit plus 2. When divided by 19, the quotient equals the units' digit, and the remainder equals the tens'. Find the number.

22. Find a number of three figures, such that the sum of its digits equals 7; the right-hand digit equals the middle digit plus twice the left-hand one; and, when 297 is added to the number, it is reversed.

23. A number of three places is equal to 70 times the sum of its digits; if the left-hand digit be dropped and 3 subtracted from the result, the remainder will be equal to $\frac{1}{80}$ of the original number; and if the right-hand digit be dropped, the resulting number will be $\frac{1}{10}$ of the original number. Find the number.

24. A number consists of 3 digits, and equals 48 times the sum of its digits. The difference between its left-hand and middle digits is the same as the difference between the middle and right-hand ones; and, if 198 be taken from the number, the digits will all be the same as the former tens' digit. Find the number.

25. A number of four figures is equal to 207 times the sum of its digits, and will have the order of its digits reversed by adding 2547; the sum of its two left-hand digits exceeds the sum of the two right-hand ones by 2, and the sum of its two middle digits is equal to that of the two outside ones. Find the number.

26. Find a number of three digits, from which, 297 being subtracted, the order of the digits will be reversed. If the number denoted by the two left-hand digits be divided by the number denoted by the two right-hand digits, the result will be 2; and the num-

ber diminished by 1 will equal 35 times the reverse of the number denoted by the 2 right-hand digits.

27. A owes \$3,000 and B owes \$2,000; if A get $\frac{1}{6}$ of B's fortune, he can pay his debts. B could pay his by borrowing $\frac{2}{7}$ of A's fortune. What amount has each?

28. A banker has money invested in both 4 per cent and 5 per cent bonds, which yields an annual income of \$290; if he were to exchange \$1,000 of 5 per cents for the same amount of 4 per cents, his annual income on the 4 per cents would be the same as that from the 5 per cents. How much has he invested in each?

29. A man spent \$2,800 buying 5 horses, 20 oxen, and 70 hogs; he paid $3\frac{1}{3}$ times as much for an ox as for a hog; and if \$120 be taken from the sum of the prices of a horse, an ox, and a hog, the result will equal the number of animals bought. How much did he pay apiece for each?

30. A farmer mixed good wheat, that cost \$1.20 a bushel, with poor wheat, that cost 80 cents a bushel, in such a way as to clear 20 per cent by selling it at \$1.14 a bushel. The whole number of bushels was 24. How many of each kind were there?

31. If 2 quarts of good wine be mixed with 3 of poor, the mixture will be worth 42 cents a quart; but if 7 quarts of the good be mixed with 8 of the poor, the mixture will be worth 44 cents a quart. What is a quart of each kind worth?

32. B gives A $\frac{1}{2}$ of his money; C gives B $\frac{1}{3}$ of his; D gives C $\frac{1}{4}$ of his, and E gives D $\frac{1}{5}$ of his; each now has \$30. How much had each at first?

33. A farmer makes up a herd of 100 cattle, put-

ting in 40 steers, worth $35 apiece; a number of cows, worth $30 each; and a number of calves, worth $10 each. The average value of the herd is $27 apiece. How many cows and calves are there?

34. A man bought 100 bushels of grain, paying $1 per bushel for wheat, 80 cents for barley, and 60 cents for rye, the average price being 84 cents a bushel. If he had bought the same amount of rye and $\frac{2}{3}$ as much wheat as before, he would have had to buy $12\frac{1}{2}$ bushels less of barley, so as not to exceed the former cost. How many bushels of each did he buy?

35. A boy bought apples, peaches, and pears, 24 in all. Each peach cost the price of 2 apples, and each pear cost the price of 2 peaches. The pears cost 20 cents, and the peaches and apples together cost 11 cents. If he had bought each apple for the price of a peach, and each peach for the price of an apple, he would have saved 1 cent. How many of each kind did he buy, and at what prices?

36. A laborer was engaged for a certain number of days, to receive $1.50 for each day he worked, and to forfeit 50 cents for each day he was idle. On settling, he was entitled to $25. If he had been idle 10 days more, and had received $2 for each working day, he would have been entitled to $10 at the end of the time. For how many days was he hired?

37. On a certain list, a teacher allows 3 credits on each problem which is solved correctly, and gives 2 demerits for each one that is wrong. On settling accounts, a certain pupil is entitled to 3 credits. If the teacher had allowed 5 credits instead of 3,

and the pupil had failed on one problem more, he would have received 14 credits. How many problems did he miss?

38. If a rectangular field were 10 rods longer and 4 rods narrower, it would contain 1 acre less; if it were 20 rods wider and 36 rods shorter, it would contain 1 acre more. What are the contents of the field?

39. A man divided apples among some boys. He observed that, if there had been 5 more boys, each would have received 2 less; but if there had been 3 fewer, each would have received 6 apples more. How many boys, and how many apples, were there?

40. A man bought a certain number of sheep; if he had bought 20 fewer for the same total price, each would have cost $1.50 apiece more; if he had bought 20 more for the same price, each would have cost $1 less. How many did he buy, and at what price?

41. A colonel has his regiment in a solid column. If there were 2 ranks less, each rank would have 10 men more. If there were 3 ranks more, each rank would have 10 men less. How many men are there in the regiment?

42. A man bought a certain number of sheep. If the price had been $5 less on each, he could have bought 6 more for the same money. If the price had been $3 more on each, he could have bought, for the same money, 2 fewer than he did. How many did he buy, and at what price?

43. A and B play cards for money, with the arrangement that the loser is to give each time to his opponent as many dollars as the winner then has.

They win and lose alternately for 4 games, when each has $16. How much had each at first?

44. If A were to receive from B and C each $3, he would have $1 less than both of them together. If B were to receive from A and C each $2, he would have $1 less than both the others; while, if C were to receive $1 each from A and B, he would have $1 less than both of them. How many dollars has each?

45. A, B, and C played cards for money. In the first game, B and C each won his own amount from A; in the second game, A and C won from B the amounts they had at the end of the first game; in the third game, A and B won from C the amounts they had at the end of the second game. Each then had $16. With how much did each begin?

46. A and B played cards for money, each beginning with the same sum. B lost the first game, and gave A 1 shilling less than $\frac{1}{2}$ of what he (B) had left. A lost the second game, and gave B 1 shilling less than $\frac{1}{2}$ of what he (A) had left. It was then found that B had 2 shillings more than A. How much had each at first?

47. A has a house insured for a certain amount, at a certain rate of premium. B has $500 more insurance, and pays $\frac{1}{2}$ per cent higher rate than A. C carries $1,000 more than B, and pays $\frac{1}{4}$ per cent higher rate than B. B's annual premium exceeds A's by $25, and C's exceeds B's by $35. What amount is insured by each, and at what rates?

48. A had a certain amount of money, with which he could buy, at one store, 50 pounds of coffee and 100 pounds of sugar. At another store, he buys 25

pounds of each, paying an average price of $\frac{7}{16}$ of a cent per pound more than at the first store. He finds that the price of coffee at the second store is 4 per cent higher than at the first, and that the total increase on the sugar is equal to $\frac{1}{2}$ the cost of a pound of sugar at the first store. He has $14.65 left. What had he at first?

49. A has a purse containing gold, silver, and copper coins, the gold being 50 per cent, and the copper 30 per cent, of the whole number of coins. B has a purse which contains only silver and copper. If B give A his purse, the silver will be 40 per cent, and the copper 35 per cent, of the whole. B's purse contains 20 coins. How many are copper and how many silver? (*Also, solve with one unknown quantity.*)

50. In two purses, each containing half-dollars, quarters, and dimes, the number of coins in each is 17, and the total value of each is the same. There are 3 half-dollars more in one than in the other; the value of all the half-dollars is $\frac{7}{16}$, and of the dimes $\frac{3}{16}$, of the value of all the coins. What number of each kind does each purse contain?

51. Sixteen bales of cotton and 9 casks of rum exactly fill a room; 4 bales of cotton and 6 casks of rum fill $\frac{1}{3}$ of it. How many of each will it hold?

52. A pasture will sustain 9 cows and 16 sheep; $\frac{7}{10}$ of it will sustain 6 cows and 12 sheep. How many of each kind alone will it sustain?

53. Two full meals and one lunch will cost $\frac{11}{24}$ of a man's money. One full meal and two lunches cost $\frac{5}{12}$ of it. How many of each kind can he buy for the amount he has?

54. A man has just money enough to pay for fill-

ing his coal-bin with Youghiogheny coal, at 16 cents, and Pomeroy, at 12 cents. If he buy 50 bushels of Youghiogheny, and fill the rest of his bin with Pomeroy, he will have $4 left; but, if he buy 125 bushels of Pomeroy, and lay out the rest of his money in Youghiogheny, he will lack 25 bushels of filling his bin. How much money has he, and how many bushels does his bin hold?

55. A and B can do a piece of work in 9 days; A and C in 8 days; and B and C in $14\frac{4}{7}$ days. How long would it take each separately to do it, and how long would it take all together?

56. Three pipes, A, B, and C, can fill a cistern in $7\frac{1}{5}$ hours. A and B carry $2\frac{3}{4}$ as much as C, while C can carry only $\frac{1}{2}$ as much as A. How long would it take each alone to fill the cistern?

57. A, B, and C have 3,600 silver dollars to count, which they do in 20 minutes. It would take A alone 15 minutes longer than C alone to count all of them. A and B together can count 20 more each minute than C. How many can each count in a minute?

58. A cistern could be filled by two pipes in 50 minutes. One stops running at the end of 35 minutes, and the other finishes filling the cistern in 27 minutes more. How long would it take each pipe alone to fill it?

59. Two clerks have 2,000 circulars to address. One quits at the end of 4 hours, and it takes the other 10 hours to finish. If the one who left had remained 2 hours longer, the other could have finished it in 5 hours. How many can each address in 1 hour?

60. A and B can finish a piece of work in 9 days;

but A loses 6 days and B 2 days while the work is going on, and it takes them 14 days to finish. How long would it take each alone to do it?

61. A, B, and C finish $\frac{1}{2}$ of a job of work in 2 days. A leaves, and B and C finish $\frac{1}{2}$ of what is left in 2 days; B then leaves, and it takes C 3 days to finish. How long would it take each alone to finish it?.

62. A and B have 15 acres to plow. At the end of $1\frac{1}{2}$ days A leaves, and B finishes in $3\frac{3}{4}$ days. If B had left instead of A, it would have taken A $2\frac{1}{2}$ days to finish. How long would it take each to plow the field alone?

63. In the last problem, if they be paid in proportion to the amount of work actually done by each, and if the whole pay be $20, how much will each receive?

64. A and B engage to do a piece of work in 10 days, A being $\frac{4}{5}$ as fast a worker as B. Finding, at the end of 5 days, that they can not finish it in time, they call in C, and thus finish it according to agreement. The time it would take A and C to do the whole work, is to the time it would take B and C to do it as 23 is to 22. How long would it take each to do it alone?

65. If one horse start 20 seconds before another, both can reach the end of a mile track at the same time. The time it takes the first to run $\frac{1}{2}$ a mile, plus the time it takes the second to run $\frac{1}{4}$ of a mile is 2 minutes, 10 seconds. What is the rate of each?

66. A and B each has 1,000 silver dollars to count. B counts 250 before A begins, but A finishes counting his, 1 minute before B. Now, if B had counted

4 minutes before A began, A would have had 200 to count when B got through. How long will it take each to count the $1,000?

67. A man goes from home to a village, 8 miles away, in 2 hours, walking half the distance and riding the other half. He rides back half-way at a rate $\frac{1}{3}$ less than in going, and walks the rest of the way at a rate $\frac{1}{3}$ greater than in going. He thus returns in 2 hours. At what rates did he ride and walk?

68. Two trains, 460 feet and 420 feet long, respectively, pass each other in $7\frac{1}{2}$ seconds when moving in opposite directions, and in 30 seconds, when moving in the same direction. At what rates are they moving?

69. At 10 o'clock A. M., a packet left Cincinnati and, sometime after, was followed by a mail-boat running $\frac{1}{3}$ faster. The mail-boat overtook the packet 96 miles from Cincinnati. If the packet had started at 10:30 A. M., and the mail-boat as before, the packet would have been overtaken 72 miles from Cincinnati. What was the rate of each, and at what time did the mail-boat start?

70. A man rides from M to N. At the end of 2 hours his horse loses a shoe. He stops an hour while the blacksmith replaces it, and, the job being poorly done, the horse goes only $\frac{3}{4}$ as fast as before, and in consequence the man arrives at N 2 hours late. If the shoe had been lost 8 miles nearer N he would have been only 1 hour, 40 minutes late. How far is it from M to N, and what is his rate?

71. A left M at the rate of 12 miles an hour, to travel towards N, which is 60 miles away. He over-

took B 24 miles from N. When A reached N, he rested $1\frac{1}{4}$ hours, and then started back to M. He met B at a point 15 miles from N. At what rate did B travel, and how long before A did he start?

72. A train from H to C injures its air-brake at G, and, having to diminish its speed $\frac{1}{5}$, it is 9 minutes late at C. If the accident had happened at W, which is 3 miles nearer C, it would have been only $7\frac{1}{5}$ minutes late. At what rate did the train run from H to G?

73. A left M 6 hours before B. When B overtook A, B increased his rate by $\frac{1}{6}$, and A decreased his by $\frac{1}{8}$. Five hours after B overtook A, he was 45 miles ahead of A. If their rates had remained as at first, B would have been 30 miles ahead of A. What were their rates?

74. A train runs from A to B at the rate of 12 miles an hour, including 4 stops. From B to C, no stops are made, and the rate is 15 miles an hour. From C to D, there are 6 stops, and the rate is $10\frac{1}{2}$ miles an hour. The distance from A to D is 90 miles, and the run is made in 7 hours. What is the distance between stations?

75. A line of railway is up-grade from A to B, a distance of 40 miles; it is level-grade from B to C, 30 miles; and is down-grade from C to D, 50 miles. A train runs from A to D in $8\frac{1}{2}$ hours. It returns at the same rates in 9 hours; and it can go from A to a point 8 miles beyond B and back to A, in 7 hours, 4 minutes. What are the rates of running up and down-grade and on a level?

76. A towboat with coal leaves C for L. She runs to M at the rate of 10 miles an hour. Leaving

some of her tow, she is able to run to L from M at the rate of 15 miles an hour. She starts back at the rate of 11¼ miles per hour, expecting to make the return trip to C in the same time as before. On reaching M, she is delayed by an accident 33 minutes, 20 seconds, in consequence of which she has to increase her speed to 12 miles an hour in order to finish in the time expected. How far is it from C to M, and how far from M to L?

XIV. INDETERMINATE EQUATIONS.—168.

1. Find integral values for x and y in $7x + 4y = 43$
2. Find integral values for x and y in $2x + 3y = 24$
3. Find integral values for x and y in $\frac{x}{7} + \frac{y}{5} = \frac{53}{35}$

4. Find two integers such that 5 times one plus ½ of the other shall equal 64.

5. In how many and in what ways may $45 be paid in dimes, quarters, and half-dollars so as to make 370 pieces in all?

6. Find an integral value for the least number, which, being divided by 7, leaves a remainder 1, and being divided by 9, leaves a remainder 4.

7. Find a number of two places such that if it be divided by 19, the quotient will equal the units' digit, and the remainder will equal the tens'.

8. What number of two places, being increased by its ⅘, will equal the number inverted plus 18?

9. In how many and in what ways may a man invest $100 in books, paying $3 apiece for some, and for the others $7 apiece?

10. The difference between a certain number and the sum of its digits is 99. If the number be divisible by 10, find it.

11. How many benches, holding 4 and 5 persons respectively, will be needed exactly to accomodate 50 persons?

12. In a school-room there are between 40 and 50 pupils. If they be seated on benches, each holding 6, there will be 5 over; if seated on benches holding 5, there will be 2 over. How many pupils are there in the room?

13. A man puts 1,000 head of cattle in cars holding 40 and 50 respectively; for the smaller cars he is charged $10 each, and, for the larger, 15 dollars each, but can not get over 12 of the smaller cars. What is his cheapest selection of cars?

14. With four weights, whose sum is 40 pounds, every pound from 1 to 40 can be weighed. What are the four weights?

15. A man bought cows, hogs, and sheep, 40 in all, for $181. The cows cost $20 each, the hogs $5 each, and the sheep $3 each. In how many and what ways might he have laid out his money?

XV. ROOTS.

1. OF NUMBERS.—173 et seq.

Find the square root of:

1. 8281
2. 254016
3. 36060025
4. 103041
5. 502681
6. 9042049
7. 1871424

8. 49.4209
9. 2.1609
10. .625 (4 places)
11. 5.76
12. .576 (4 places)
13. .9 (4 places)
14. 8.1 (4 places)
15. 7 (4 places)

Find the cube root of:

16. 6859
17. 91125
18. 753571
19. 54010152
20. 12326391
21. 731189187729
22. 167284151
23. 8120601
24. 1.728
25. .148877
26. 19.683
27. .091125
28. .005832
29. .004913
30. 12.167

Find the 4th root of:

31. 28561
32. 65536
33. 194481
34. 390625
35. 4304.6721

Find the 5th root of:

36. 1419857
37. 1048576
38. 33554432
39. 6436343
40. 1889568

Find the 6th root of:

41. 34012224
42. 148035889
43. 244140625
44. 6321363049
45. 1073741824

Find the 7th root of:

46. 1801088541
47. 19487171

2. OF ALGEBRAIC QUANTITIES.

(1) Square Root.—183.

Find the square root of the following polynomials:

1. $4x^4 - 12x^3 + x^2 + 12x + 4$.
2. $9x^2 + y^2 + 4z^2 - 6xy + 12xz - 4yz$.
3. $16x^2 + 4y^2 + z^4 + 16xy - 8xz^2 - 4yz^2$.

4. $x^4 - 2x^3 - x^2 + 2x + 1$.

5. $9x^2 - 30ax - 3a^2x + 25a^2 + 5a^3 + \frac{a^4}{4}$.

6. $a^4 + b^4 - c^4 - 2c^2\,(a^2 + b^2 - c^2) + 2a^2b^2$.

7. $a^2x^2 + b^2x^2 + 2abx^2 + a^2y^2 + b^2y^2 - 2aby^2 - 2a^2xy + 2b^2xy$.

8. $x^4y^{-2} - 2x^3y^{-3} + x^2y^{-4} + 2x^2y^{-1} - 2xy^{-2} + 1$.

9. $\frac{1}{4}x^2 + 4x^{-2} + 2 + 2x - 8x^{-1}$.

10. $\frac{9x^2}{16y^2} + \frac{36x^2}{49z^2} + \frac{4y^2}{9z^2} - \frac{9x^2}{7yz} - \frac{x}{z} + \frac{8xy}{7z^2}$.

11. $\frac{4x^2}{25y^2} + \frac{36x^2}{25z^2} + \frac{25y^2}{4z^2} + \frac{24x^2}{25yz} - \frac{2x}{z} - \frac{6xy}{z^2}$.

12. $x^{-6} - 2x^{-5} + 3x^{-4} - 2x^{-3} + x^{-2}$.

13. $x + y^{\frac{1}{2}} + z^{\frac{4}{3}} + 2x^{\frac{1}{2}}y^{\frac{1}{4}} - 2x^{\frac{1}{2}}z^{\frac{2}{3}} - 2y^{\frac{1}{4}}z^{\frac{2}{3}}$.

14. $x^4 - x^3 + \frac{x^{\frac{5}{2}}}{2} - \frac{x^{\frac{3}{2}}}{4} + \frac{x^2}{4} + \frac{x}{16}$.

15. $a^{\frac{4}{3}} + \frac{4}{3}b + \frac{1}{4}a^{-2}b^{-\frac{2}{3}} - \frac{4}{3}a^{\frac{2}{3}}b^{\frac{1}{2}} - a^{-\frac{1}{3}}b^{-\frac{1}{3}} + \frac{2}{3}a^{-1}b^{\frac{1}{6}}$.

16. $256x^{\frac{4}{3}} - 512x + 640x^{\frac{2}{3}} - 512x^{\frac{1}{3}} + 304 - 128x^{-\frac{1}{3}} + 40x^{-\frac{2}{3}} - 8x^{-1} + x^{-\frac{4}{3}}$.

17. $\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2 + \frac{4a}{b} - \frac{4b}{a}$.

18. $x^4 - 2x^2\,(y^2 + u^2) + y^4 - 2y^2\,(z^2 - u^2) + z^4 + 2z^2\,(x^2 - u^2) + u^4$.

19. $x^4 - 6x^3 + x^2\,(9 + 2a - 2b) - 6x\,(a - b) + (a - b)^2$.

20. $4(a^2 - ab - ac) - 6\,(2a - b - c) + (b + c)^2 + 9$.

21. $(x^2 + 4y^2 - 4xy)\,m^4 - (2x^2 - 4xy)\,m^3 + (x^2 + 4xy - 6x - 8y^2 + 12y)\,m^2 - (4mxy - 6mx) + 4y^2 - 12y + 9$.

(2) Cube Root.—191.

Find the cube root of the following:

22. $27x^3 - 54x^2y + 36xy^2 - 8y^3$.

23. $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$.

24. $\frac{x^6}{y^3} - 6x^4 + 12x^2y^3 - 8y^6$.

25. $8x^6 + 6x^4 + \frac{3x^2}{2} + \frac{1}{8}$.

26. $x^3 - 3x^2y^2 + 3xy^4 - y^6 + 3x^2z + 3y^4z - 6xy^2z + 3xz^2 - 3y^2z^2 + z^3$.

27. $x^3 + 6x^2y + 12xy^2 + 8y^3 - 3(x+2y)^2 + 3x + 6y - 1$.

28. $a^3 + 8a^{-3} - 12a^2 - 48a^{-2} + 54(a + 2a^{-1}) - 112$.

29. $(a+c-b)^3 + (a+b-c)^3 + (b+c-a)^3 + 24abc$.

30. $\frac{x^6}{y^6} - \frac{6x^4}{y^4} + \frac{15x^2}{y^2} - 20 + \frac{15y^2}{x^2} - \frac{6y^4}{x^4} + \frac{y^6}{x^6}$.

(3) Higher Roots.

Find the 4th root of:

31. $\frac{x^4}{y^4} - \frac{8x^3}{y^2} + 20x^2 - 8xy^2 - 26y^4 + \frac{8y^6}{x} + \frac{20y^8}{x^2} + \frac{8y^{10}}{x^3} + \frac{y^{12}}{x^4}$.

Find the 5th root of:

32. $1 + 5ax + 10a^2x^2 + 10a^3x^3 + 5a^4x^4 + a^5x^5$.

33. $x^{10} - \frac{5x^8y}{2} + \frac{5x^6y^2}{2} - \frac{5x^4y^3}{4} + \frac{5x^2y^4}{16} - \frac{y^5}{32}$.

Find the 6th root of:

34. $1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$.

XVI. RADICALS.

1. SIMPLIFICATION.

(1) Integral Radicals.—199.

Simplify the following quantities:

1. $\sqrt{8}$.
2. $\sqrt{27}$.
3. $\sqrt{50}$.
4. $\sqrt{32}$.
5. $\sqrt{24}$.
6. $\sqrt{48}$.
7. $\sqrt{125}$.
8. $\sqrt{192}$.
9. $\sqrt{338}$.
10. $\sqrt{512}$.
11. $\sqrt{18a^2}$.
12. $\sqrt{80a^3}$.
13. $\sqrt{72a^2b}$.
14. $\sqrt{112x^4y^{-2}}$.
15. $\sqrt{240x^3y^{-3}}$.
16. $\sqrt{336(x+y)^2}$.
17. $\sqrt{54ab^3}$.
18. $\sqrt{45(a^2-b^2)}$.
19. $\sqrt{126(a+b)^3}$.
20. $\sqrt{96a^2b^{-2}}$.
21. $\sqrt[3]{16}$.
22. $\sqrt[3]{24a^2}$.
23. $\sqrt[3]{250ab^3}$.
24. $\sqrt[3]{54x^3y^6}$.
25. $\sqrt[3]{40a^4}$.
26. $\sqrt[3]{32x^3y^6z}$.
27. $\sqrt[3]{108a^3bc^4}$.
28. $\sqrt[3]{(a+b)^3x^2y^6}$
29. $\sqrt[3]{686}$.
30. $\sqrt[3]{448x^4y^5z^6}$.
31. $\frac{1}{2}\sqrt[3]{352}$.
32. $\frac{1}{7}\sqrt{98}$.
33. $a\sqrt[4]{32a^5}$.
34. $3\sqrt[4]{243}$.
35. $\frac{3}{xy^2}\sqrt[3]{54x^3y^6}$.
36. $4\sqrt[4]{64x^4y}$.
37. $\sqrt[5]{224a^5b^6}$.
38. $\sqrt[6]{128x^2y^7}$.
39. $2\sqrt[5]{486}$.
40. $2a\sqrt[4]{1250a^5b^6}$.
41. $\sqrt[4]{96(a+b)^5}$.
42. $\frac{1}{2}\sqrt[4]{162a^2b^3}$.
43. $\sqrt{a^3+a^2b}$.
44. $\sqrt[3]{8a^3+a^6}$.
45. $\sqrt[4]{a^5-a^7b}$.
46. $5\sqrt{722a^5b^3}$.
47. $\frac{1}{3}\sqrt[4]{176x^5y^3}$.
48. $\sqrt[3]{x^3-x^3y}$.
49. $\sqrt{a^{2x}y^3}$.
50. $\sqrt[3]{a^{4n}b^n}$.

51. $5\sqrt{8a^2+32b\,(b-a)}$.

52. $\sqrt[3]{x^4+xy^3+3x^2y(x+y)}$.

53. $(x+y)\sqrt{3ax^2-6axy+3ay^2}$.

54. $\sqrt[3]{3a^2\,(a+3b)+3b^2\,(b+3a)}$.

55. $\sqrt{3a^2(a+b)+3ab(a+b)}$.

56. $(a-b)\sqrt{a^3+a^2b-ab^2-b^3}$.

57. $\sqrt{a^3(a+2b)-b^3(2a+b)}$.

58. $\sqrt[3]{x^4-y^4+2xy(x^2-y^2)}$.

59. $\sqrt{(b^2-a^2)(a-b)}$.

60. $[a^4+b^4-4ab(a^2+b^2)+6a^2b^2]^{\frac{1}{2}}$.

61. $\sqrt[m]{a^{4m}b^{4m+1}}$.

62. $\sqrt[(x+y)]{a^{2(x+y)}b^{x+y+z}}$.

63. $\sqrt[2n]{a^{2n+1}b^{4n}-a^{2n}b^{4n+1}}$.

(2) Fractional Radicals.—199.

Simplify the following:

64. $\sqrt{\frac{1}{2}}$.
65. $\sqrt{\frac{1}{8}}$.
66. $\sqrt[3]{\frac{1}{3}}$.
67. $\sqrt[3]{\frac{1}{9}}$.
68. $\sqrt{\frac{3}{8}}$.
69. $\sqrt{\frac{4}{3}}$.
70. $\sqrt{\frac{5}{16}}$.
71. $\sqrt[3]{\frac{27}{16}}$.
72. $\sqrt[4]{\frac{25}{8}}$.
73. $\frac{1}{21}\sqrt{\frac{28}{9}}$.
74. $2\sqrt{\frac{5x}{2}}$.
75. $3\sqrt[3]{\frac{22}{9}}$.
76. $8\sqrt{\frac{7}{18}}$.
77. $\frac{2}{3}\sqrt{\frac{3}{2}}$.
78. $ab\sqrt[3]{\frac{3}{a^2b}}$.
79. $\sqrt{2\frac{2}{3}}$.
80. $\frac{2}{3}\sqrt{\frac{189}{32}}$.
81. $\sqrt{\frac{x^2(x-y)}{x+y}}$.
82. $\sqrt{90\frac{5}{8}}$.
83. $\sqrt{(x-y)^{-1}}$.
84. $\sqrt[3]{2\frac{1}{40}}$.
85. $\sqrt{\frac{3xy^3}{98z^4}}$.
86. $\frac{2a}{3b}\sqrt[3]{\frac{15b^4c}{4a^3}}$.
87. $\frac{2x}{3}\sqrt[3]{\frac{16}{27x^5}}$.
88. $\frac{x-y}{x+y}\sqrt{\frac{x+y}{x-y}}$.
89. $(x-y)\sqrt{\frac{x+y}{x-y}}$.
90. $(a+b)\sqrt{(a^2-b^2)^{-1}}$.
91. $(a+b)\sqrt{\frac{1}{a-b}}$.
92. $\sqrt[3]{(x+y)(x-y)^{-2}}$.

(3) The Radical being a Complete Power.—201.

Simplify the following:

93. $\sqrt[4]{9}$.
94. $\sqrt[6]{27}$.
95. $\sqrt[8]{16}$.
96. $\sqrt[4]{25a^2}$.
97. $\sqrt[6]{a^2-2ab+b^2}$.
98. $\sqrt[4]{(x+y)^2z^4}$.
99. $\sqrt[9]{(a+b)^3z^6}$.
100. $\sqrt[8]{81a^4b^{12}}$.
101. $\sqrt[6]{125\,(x-y)^3}$.
102. $2\sqrt[10]{32a^5b^{10}}$.
103. $\sqrt[3m]{27a^3b^6}$.
104. $\sqrt[9]{512\,(x-y)^3}$.
105. $\sqrt[6]{\frac{4}{25}}$.
106. $\sqrt[8]{4a^2+4a+1}$.
107. $\sqrt[6]{-8(a+b)^3}$.
108. $\frac{1}{3}\sqrt[4]{36a^3b^2}$.
109. $\sqrt[10]{64\,(a-b)^2}$.
110. $\sqrt[10]{-32\,(a-b)^5}$.
111. $\sqrt[12]{27a^6\,(x-y)^9}$.
112. $\sqrt[9]{-8x^3y^6}$.
113. $\sqrt[6]{x^2+y^2+2xy}$.
114. $\sqrt[9]{a^3-b^3-3ab(a-b)}$.

2. ADDITION AND SUBTRACTION.—204.

1. $2\sqrt{24}+\sqrt{54}-\sqrt{96}$.
2. $3\sqrt{50}-2\sqrt{98}+5\sqrt{72}$.
3. $\sqrt[3]{128}+\sqrt[3]{54}-\sqrt[3]{250}$.
4. $2\sqrt[4]{32}+5\sqrt[4]{162}-3\sqrt[4]{2}$.
5. $\sqrt{\frac{1}{8}}+\sqrt{\frac{75}{9}}+\sqrt{3}$.
6. $\frac{1}{2}\sqrt{\frac{1}{2}}+\frac{1}{4}\sqrt{18}-\sqrt[4]{4}$.
7. $\frac{2}{3}\sqrt[3]{2}+\sqrt[3]{\frac{250}{27}}-2\sqrt[6]{4}$.
8. $2\sqrt{\frac{7}{4}}+3\sqrt{\frac{7}{8}}$.
9. $6\sqrt[3]{16}-4\sqrt[3]{\frac{1}{4}}$.
10. $3\sqrt[4]{64}+8\sqrt[6]{8}$.
11. $\frac{3}{2}\sqrt{\frac{2}{3}}-\frac{2}{3}\sqrt{\frac{3}{2}}$.
12. $\sqrt[5]{2}-3\sqrt[5]{-64}+6\sqrt[5]{\frac{1}{16}}$.
13. $\sqrt[3]{64}+6\sqrt[3]{\frac{1}{2}}+\frac{1}{3}\sqrt[3]{-4}$.
14. $\sqrt{\frac{50}{147}}+\sqrt{\frac{100}{294}}$.
15. $7\sqrt{a^3+a^2b}-\sqrt{4a^2b+4a^3}$.
16. $(x+y)\sqrt{(x-y)^3}+2y^2\sqrt{x-y}$.
17. $\sqrt[6]{25}+\frac{5}{3}\sqrt[3]{\frac{27}{25}}-\sqrt[3]{-625}$.

18. $8\sqrt{\frac{5}{4}}+\frac{1}{2}\sqrt{12}-\frac{4}{3}\sqrt{27}-2\sqrt{\frac{5}{16}}$.

19. $\sqrt[3]{-81}-2\sqrt[3]{-192}+6\sqrt[3]{\frac{1}{3}}$.

20. $\sqrt{5a(a+2b)^2}+2\sqrt{5a(a-b)^2}-\sqrt{45a^3}$.

21. $(x-y)\sqrt{3(x-y)^{-1}}-6\sqrt{\frac{x-y}{3}}$.

22. $\sqrt{a^2(a+3b)+b(3ab+b^2)}-\sqrt{(a-b)(a^2-b^2)}$.

3. MULTIPLICATION AND DIVISION.–205.

1. $5\sqrt{12}\times\frac{1}{2}\sqrt{3}$.
2. $3\sqrt{\frac{1}{8}}\times\frac{1}{2}\sqrt{2}$.
3. $4\sqrt{12}\times3\sqrt{5}$.
4. $\sqrt{2}\times\sqrt{3}\times\sqrt{5}\times\sqrt{15}$.
5. $\sqrt[3]{5}\times\sqrt[3]{25}$.
6. $12\sqrt{15}\div3\sqrt{5}$.
7. $5\sqrt{20}\div10\sqrt{50}$.
8. $\frac{1}{3}\sqrt[3]{16}\times6\sqrt[3]{2}$.
9. $5\sqrt[4]{2}\times2\sqrt[4]{4}\times\frac{1}{10}\sqrt[4]{8}$.
10. $\frac{2}{3}\sqrt{3}\times3\sqrt[4]{225}$.
11. $24\div2\sqrt{8}$.
12. $\frac{5}{14}\sqrt[3]{\frac{2}{8}}\div\frac{5}{21}\sqrt[3]{\frac{1}{4}}$.
13. $\sqrt{2}\times\sqrt[3]{2}$.
14. $\frac{1}{3}\sqrt[3]{2}\times6\sqrt{\frac{1}{3}}$.
15. $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{\frac{1}{2}}\times\sqrt[4]{\frac{1}{3}}$.
16. $(\sqrt{3}-\sqrt{2})\sqrt{6}$.
17. $\sqrt{8}\times\sqrt[3]{5}\times\sqrt[6]{4}$.
18. $(\sqrt{8}+\sqrt[3]{5})\sqrt[6]{4}$.
19. $(\sqrt{x}+\sqrt[3]{y})(\sqrt{x}-\sqrt[3]{y})$.
20. $(\sqrt{5}-\sqrt{1\frac{4}{5}})\sqrt{15}$.
21. $(a^2-b)\div(a-\sqrt{b})$.
22. $(\sqrt{8}+\sqrt{24\frac{1}{2}})\sqrt{10}$.
23. $(\sqrt{54}\div\sqrt{3})+\sqrt[6]{2}$.
24. $(\sqrt{2}-\sqrt{3})^3$.
25. $\sqrt[4]{6-\sqrt{11}}\times\sqrt[4]{6+\sqrt{11}}$.
26. $\sqrt[3]{7-\sqrt{22}}\times\sqrt[3]{7+\sqrt{22}}$.
27. $(x+y)\sqrt{x-y}\times(x-y)\sqrt{x^2-y^2}$.
28. $\sqrt{6a^4+6a^2b^2}\div2a\sqrt{a^2+b^2}$.
29. $(\sqrt{a}-\sqrt[4]{ab}+\sqrt{b})(\sqrt{a}+\sqrt{b}+\sqrt[4]{ab})$.
30. $(\sqrt[3]{a}+1)(\sqrt[3]{a^2}-\sqrt[3]{a}+1)$.

31. $(x\sqrt{x}-x\sqrt{y}+y\sqrt{x}-y\sqrt{y})\,(\sqrt{x}+\sqrt{y})$.

32. $(\sqrt{3}-\sqrt{2})\,(\sqrt{2}+\sqrt{3})-1$.

33. $(a-b+2\sqrt{bc}-c)\div(\sqrt{a}-\sqrt{b}+\sqrt{c})$.

34. $\dfrac{(\sqrt{x}+\sqrt{y})\,(\sqrt{x}-\sqrt{y})\,(x+y)}{x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y}$.

35. $(y\sqrt[4]{x}+\sqrt[3]{y^2})\left(\sqrt[4]{x}-\dfrac{1}{\sqrt[3]{y}}\right)$.

36. $\sqrt[a]{x+y}\times\sqrt[b]{x+y}\times\sqrt[a]{x-y}\times\sqrt[b]{x-y}$.

4. INVOLUTION.—207.

Expand the following:

1. $(\sqrt{3})^2$.
2. $(\sqrt[3]{2})^3$.
3. $(2\sqrt{5})^2$.
4. $2(\sqrt{5})^2$.
5. $\frac{1}{3}(\sqrt[3]{3})^3$.
6. $(\sqrt[3]{2})^2$.
7. $(3\sqrt[4]{5})^2$.
8. $(2\sqrt{2})^3$.
9. $6^2\times(\sqrt{\frac{1}{3}})^3$.
10. $(2\sqrt[10]{4})^5$.
11. $(2\sqrt{5a})^4$.
12. $(a^{\frac{2}{3}}b^{\frac{1}{2}})^6$.
13. $(\sqrt[3]{a^2b^2})^2$.
14. $(2a^{\frac{1}{2}}b^2)^3$.
15. $(\frac{1}{2}a^{-2}b^{\frac{1}{3}})^3$.
16. $(-2a^{\frac{1}{2}}b^3)^2$.
17. $[2(a-b)^{\frac{3}{4}}]^4$.
18. $(3^{\frac{1}{2}}a^{\frac{1}{3}}b^{\frac{1}{6}})^3$.
19. $(\sqrt[3]{\sqrt{2a^3}})^3$.
20. $(\sqrt{2\sqrt{3a}})^4$.
21. $[(a+b)^{\frac{1}{3}}]^6$.
22. $(mn\sqrt[3]{mn})^4$.
23. $(a+b)\sqrt{(a-b)^3}$.
24. $(-2a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{2}{3}})^6$.
25. $\left[\dfrac{2}{3}\left(\dfrac{3x^2}{4y}\right)^{\frac{1}{3}}\right]^5$.
26. $\{[x^2b(a^3xz)^{\frac{1}{3}}]^{\frac{1}{6}}\}^7$.
27. $\left[\left(\dfrac{a^{-2x}}{b^{-2y}}\right)^{\frac{p}{m}}\right]^{\frac{y}{2x}}$

5. EVOLUTION.—208.

Extract the roots indicated in the following:

1. $\sqrt[3]{\sqrt{27a^3}}$.
2. $\sqrt{\sqrt[3]{4x^2}}$.
3. $\sqrt[5]{\sqrt[3]{32a^{10}b^5}}$.
4. $\sqrt[3]{\sqrt{64a^9b^3}}$.
5. $\sqrt{\sqrt[5]{9x^2y^4}}$.
6. $\sqrt{4\sqrt[3]{25x^4}}$.
7. $\sqrt[4]{81\sqrt[3]{16a^3}}$.
8. $\sqrt[3]{27a^3b^6\sqrt{8a^3b^6}}$.
9. $\sqrt[5]{32a^{-5}\sqrt[3]{243a^{10}}}$.
10. $2\sqrt{\frac{a^2}{4}\sqrt[3]{49a^4b^2}}$.
11. $\sqrt{(x+y)^2\sqrt[3]{x-y}}$.
12. $\sqrt[m]{a^m\sqrt{b^mc^{2m}}}$.
13. $\sqrt{2\sqrt{3}}$.
14. $\sqrt[3]{3\sqrt{3}}$.
15. $\sqrt{3\sqrt[3]{3}}$.
16. $\sqrt[3]{\frac{1}{2}\sqrt{32a^6}}$.
17. $\sqrt[4]{\frac{2}{3}\sqrt[3]{\frac{32x^4y^3}{3}}}$.
18. $\sqrt[3]{(a+b)\sqrt{a+b}}$.
19. $\sqrt[4]{2a\sqrt[3]{2a}}$.
20. $\sqrt{c\sqrt{2a}}$.
21. $\sqrt[5]{486a\sqrt[3]{4a^2}}$.
22. $\sqrt[3]{c\sqrt{3}}$.
23. $\sqrt[4]{162x\sqrt[3]{2x}}$.
24. $\sqrt[3]{81\sqrt{3a^3}}$.
25. $\sqrt{a\sqrt{a\sqrt{a}}}$.
26. $\sqrt{3\sqrt{2\sqrt{3\sqrt{2}}}}$.
27. $\sqrt[3]{\frac{a+b}{a-b}\sqrt{\frac{a+b}{a-b}}} \times \sqrt[3]{\frac{a-b}{a+b}\sqrt{\frac{a-b}{a+b}}}$.

6. IMAGINARY QUANTITIES.—209, 210.

Perform the operations indicated in the following:

1. $\sqrt{-72}-\sqrt{-8}$.
2. $\sqrt{-\frac{4}{9}}-\sqrt{-\frac{3}{4}}$.
3. $\frac{1}{2}\sqrt{-\frac{1}{2}}+\frac{3}{2}\sqrt{-8}$.
4. $\sqrt{2}\times\sqrt{-3}$.

5. $\sqrt{3} \times 2\sqrt{-2}$.

6. $\sqrt{\frac{1}{2}} \times \sqrt{-9}$.

7. $2\sqrt{\frac{1}{2}} \times \sqrt{-6}$.

8. $\sqrt{12} \div \sqrt{-2}$.

9. $3\sqrt{-5} \div \sqrt{4}$.

10. $6\sqrt{-10} \div 2\sqrt{-5}$.

11. $\frac{1}{2}\sqrt{-3} \div 2\sqrt{-\frac{1}{3}}$.

12. $2\sqrt{-5} \times 3\sqrt{-2}$.

13. $3\sqrt{-2} \times 2\sqrt{-4}$.

14. $\sqrt{-3} \times \sqrt{-4}$.

15. $\sqrt{-2} \times \sqrt{-8}$.

16. $(-3\sqrt{-3})^2$.

17. $(3+\sqrt{-2})\,(3-\sqrt{-2})$.

18. $(2-\sqrt{-5})^2$.

19. $(\sqrt{-2}+\sqrt{-3})^2$.

20. $(\sqrt{-3}+\sqrt{2})^3$.

21. $(\sqrt{-3}-\sqrt{-2})^4$.

22. $(2\sqrt{-3}-\sqrt{-1})^4$.

23. $(\sqrt{x-y}+\sqrt{y-x})^2$.

24. $(\sqrt{-28}-\sqrt{-7})^4$.

25. $(-1-\sqrt{-3})^6$.

26. $\sqrt{-243}+\sqrt{-27}+\sqrt{-18}$.

27. $\sqrt{-128}-2\sqrt{50}+\sqrt{-18}+10\sqrt{2}$.

28. $4\sqrt{-2}-7\sqrt{-72}+5\sqrt{-8}-\sqrt{-50}$.

29. $2\sqrt{-25}-\sqrt{-49}-3\sqrt{-1}$.

30. $\sqrt{-9(a+1)}+\sqrt{-16(a+1)^3}$.

31. $\sqrt{(a^2-1)\,(1-a)}+\sqrt{-a-1}$

32. $\sqrt[4]{-16}-\sqrt[4]{-81}+\sqrt{-25}$.

33. $(\sqrt{2}-\sqrt{-2}) \times (\sqrt{2}+\sqrt{-2})$.

34. $\sqrt[3]{3+3\sqrt{-2}} \times \sqrt[3]{3-3\sqrt{-2}}$.

35. $\sqrt[4]{2+2\sqrt{-3}} \times \sqrt[4]{2-2\sqrt{-3}}$.

36. $\sqrt[3]{\sqrt{5}-\sqrt{-3}} \times \sqrt[3]{\sqrt{5}+\sqrt{-3}}$.

37. $\sqrt{6+\sqrt{-13}} \times \sqrt{6-\sqrt{-13}}$.

38. $\dfrac{1}{1+\sqrt{-3}}+\dfrac{-1}{\sqrt{-3}-1}$.

39. $\dfrac{5+\sqrt{-3}}{5-\sqrt{-3}}+\dfrac{5-\sqrt{-3}}{5+\sqrt{-3}}$.

7. RATIONALIZATION OF DENOMINATORS.—206.

1. $\dfrac{3}{\sqrt{2}}$.

2. $\dfrac{\sqrt{b}}{\sqrt{a}}$.

3. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$.

4. $\dfrac{3}{2\sqrt{2}}$.

5. $\dfrac{1}{\sqrt[3]{2}}$.

6. $\dfrac{2}{\sqrt[3]{4}}$.

7. $\dfrac{a}{\sqrt[3]{2a}}$.

8. $\dfrac{3}{\sqrt{-3}}$.

9. $\dfrac{2}{\sqrt[4]{8}}$.

10. $\dfrac{8}{\sqrt[4]{2}}$.

11. $\dfrac{5}{\sqrt[3]{-5}}$.

12. $\dfrac{3}{\sqrt[5]{9}}$.

13. $\dfrac{3}{\sqrt{3}-\sqrt{2}}$.

14. $\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$.

15. $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.

16. $\dfrac{5-\sqrt{3}}{5+\sqrt{3}}$.

17. $\dfrac{4+\sqrt{3}}{2-\sqrt{3}}$.

18. $\dfrac{\sqrt{8}+\sqrt{12}}{\sqrt{3}-\sqrt{2}}$.

19. $\dfrac{\sqrt{27}-\sqrt{18}}{\sqrt{2}+\sqrt{3}}$.

20. $\dfrac{1}{\sqrt[3]{3}-\sqrt[3]{2}}$.

21. $\dfrac{\sqrt{3}-8}{\sqrt[4]{3}-2\sqrt{2}}$.

22. $\dfrac{\sqrt{-2}+\sqrt{-3}}{\sqrt{-8}}$.

23. $\dfrac{\sqrt{-3}-\sqrt{-5}}{\sqrt{-15}}$.

24. $\dfrac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{2}+3\sqrt{3}}$.

25. $\dfrac{6}{\sqrt{3}+\sqrt{2}-\sqrt{5}}$.

26. $\dfrac{\sqrt{10}}{\sqrt{5}+\sqrt{2}-\sqrt{7}}$.

27. $\dfrac{4+2\sqrt{10}}{\sqrt{2}+\sqrt{5}-\sqrt{3}}$.

28. $\dfrac{8-2\sqrt{35}-2\sqrt{3}}{\sqrt{7}-\sqrt{5}-\sqrt{3}-1}$.

29. $\dfrac{2}{\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}}$.

30. $\dfrac{1}{\sqrt[4]{a^3}-\sqrt{a}+\sqrt[4]{a}-1}$.

31. $\dfrac{5}{\sqrt{3}+\sqrt{2}-\sqrt[4]{24}}$.

32. $\dfrac{15}{6+\sqrt{3}+\sqrt{6}+3\sqrt{2}}$.

33. $\dfrac{\sqrt{a+b}}{\sqrt{a+b}-\sqrt{a-b}}$.

34. $\dfrac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}$.

35. $\dfrac{\sqrt{-6}}{\sqrt{-3}-\sqrt{-2}}$.

36. $\dfrac{\sqrt{-3}-\sqrt{-2}}{\sqrt{-3}+\sqrt{-2}}$.

37. $\dfrac{2+\sqrt{3}}{\sqrt{-2}-\sqrt{-3}}$.

38. $\dfrac{1}{\sqrt{-2}-\sqrt{-3}+\sqrt{-5}}$.

39. $\dfrac{40}{\sqrt{-7}-\sqrt{14}+\sqrt{-2}-7}$.

8. EQUATIONS CONTAINING RADICALS.—216.

Solve the following equations:

1. $\sqrt{x-9}=-2$.
2. $3\sqrt{-x}=\sqrt{54}$.
3. $\sqrt{x^2-6x+24}=x+2$.
4. $\sqrt{x-16}+2=\sqrt{x}$.
5. $\sqrt{5x-9}=9-\sqrt{5x}$.
6. $\sqrt{\sqrt{x-16}+\sqrt{x}}=\sqrt[4]{x}$.
7. $\sqrt{\sqrt{x+15}-\sqrt{x}}=\sqrt[4]{x}$.
8. $\sqrt{x+8}-\sqrt{x-8}=\sqrt{2}$.
9. $\sqrt{5x+1}+\sqrt{5x-6}=7$.
10. $\dfrac{1}{\sqrt{x}-1}=\dfrac{2}{\sqrt{x}+2}$.
11. $\dfrac{\sqrt{x}+1}{\sqrt{x}-1}+\dfrac{\sqrt{x}-1}{\sqrt{x}+1}=a$.
12. $\dfrac{1}{\sqrt{x}-2}+\dfrac{1}{\sqrt{x}+2}=\dfrac{3}{\sqrt{x}}$.
13. $\dfrac{\sqrt{ax}-\sqrt{a}}{\sqrt{b}(\sqrt{x}-1)}=\dfrac{\sqrt{x}+a}{\sqrt{x}+b}$.
14. $\sqrt{\sqrt{\sqrt{12x}-2}-1}=1$.

15. $\sqrt{6+\sqrt{5+\sqrt{2x}}}=3.$

16. $\sqrt{x}+\sqrt{x+\sqrt{x+1}}=1.$

17. $\sqrt[3]{x}+\sqrt[3]{a}=\sqrt[3]{a+x}.$

18. $\sqrt{ax}-\sqrt{bx}=a-b.$

19. $\sqrt{x+12}+\sqrt{x-12}=12.$

20. $\dfrac{1}{\sqrt{x}-1}+\dfrac{2}{\sqrt{x}-2}=\dfrac{3}{\sqrt{x}-3}.$

21. $\sqrt{x}-\sqrt{2-x}=\dfrac{\sqrt{x}+\sqrt{2-x}}{2}.$

22. $\sqrt[6]{x+2a\sqrt{a+x}}=\sqrt[3]{a-\sqrt{a+x}}.$

23. $\sqrt{x+a}+\sqrt{x-b}=\sqrt{a-b}.$

24. $\sqrt{\dfrac{x}{3}-3}+\sqrt{\dfrac{x}{3}+3}=\sqrt{\dfrac{4x-18}{3}}.$

25. $\sqrt{2x+3+\sqrt{4x^2-12}}=3.$

26. $\sqrt{5x-7a}+\sqrt{5x}=7\sqrt{a}.$

27. $-\sqrt{x}+\sqrt{6+x}=\dfrac{2}{\sqrt{6+x}}.$

28. $\sqrt{x}+\sqrt{a+x}=\dfrac{b}{\sqrt{a+x}}.$

29. $\sqrt{x}+\sqrt{x+3a}=\dfrac{6a}{\sqrt{x+3a}}.$

30. $\sqrt[3]{8-12x+x^2\sqrt{6-x}}=2-x.$

31. $a\sqrt{x}-b\sqrt{x}=a^2-b^2.$

32. $\dfrac{2}{x}+\dfrac{3}{a}=\sqrt{\dfrac{9}{a^2}+\dfrac{2}{x}\sqrt{\dfrac{3}{a}+\dfrac{4}{x^2}}}.$

33. $\dfrac{\sqrt[6]{10+5\sqrt{-1}}}{\sqrt{2-\sqrt{x}}}=\dfrac{\sqrt{2+\sqrt{x}}}{\sqrt[6]{10-5\sqrt{-1}}}.$

34. $\sqrt{x}-\sqrt{a+x}=\left(\frac{a}{x}\right)^{\frac{1}{2}}$.

35. $x=m-n\pm\sqrt{m^2+x^2}$.

36. $\frac{\sqrt[4]{x-20}}{\sqrt{\sqrt{x}-4}}=\frac{\sqrt{\sqrt{x}+4}}{\sqrt[4]{x-10}}$.

37. $\sqrt[3]{x^2-3x-1}=\frac{2-x}{\sqrt[3]{2-x}}$.

38. $(m+x)^{\frac{1}{c}}=\sqrt[2c]{x^2+8mx+r^2}$.

39. $\sqrt[3]{1+x}=\sqrt[3]{2-\sqrt[3]{1-x}}$.

40. $\frac{x-9}{\sqrt{x}+3}=\frac{1}{5}(2\sqrt{x}-3)$.

41. $\frac{9x-1}{\sqrt{9x}+1}=4+\frac{\sqrt{9x}-1}{2}$.

42. $\frac{ax-b^2}{\sqrt{ax}+b}-\frac{\sqrt{ax}-b}{c}=c$.

43. $\sqrt{\frac{4}{3+x}}+\sqrt{\frac{2}{3-x}}=\sqrt[4]{\frac{32}{9-x^2}}$.

44. $\sqrt{\sqrt{x}+3}-\sqrt{\sqrt{x}-3}=\sqrt{2\sqrt{x}}$.

45. $\sqrt[4]{(1-a)^2\times\frac{1+x}{1-x}}+\sqrt[4]{(1+a)^2\times\frac{1-x}{1+x}}=2\sqrt[4]{1-a^2}$.

46. $\frac{2\sqrt{-x}-5}{4\sqrt{-x}-11}=\frac{2\sqrt{-x}-3}{4\sqrt{-x}-9}$.

47. $\frac{1}{a}\sqrt{a+x}+\frac{1}{x}\sqrt{a+x}=\frac{1}{b^{\frac{3}{2}}}\sqrt{ax}$.

48. $\frac{1}{x+\sqrt{x^2-1}}+\frac{1}{x-\sqrt{x^2-1}}=10$.

49. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\dfrac{5}{3}+\dfrac{\sqrt{x^2-a^2}}{a}$.

50. $\dfrac{\sqrt{36x+1}+\sqrt{36x}}{\sqrt{36x+1}-\sqrt{36x}}=9$.

51. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=4x$.

52. $\sqrt{\sqrt{x}+\sqrt{1-\sqrt{1+4x}}}+\sqrt{\sqrt{x}-\sqrt{1-\sqrt{1+4x}}}=\sqrt{2}$.

XVII. INEQUALITIES.—217.

Show that the following propositions are true for all positive unequal values of the various letters considered. State which are true for all possible negative values also:

1. $a^2+b^2>2ab$.
2. $\dfrac{a}{b}+\dfrac{b}{a}>2$.
3. $a+b>2\sqrt{ab}$.
4. $a^n+b^n>2a^{\frac{n}{2}}b^{\frac{n}{2}}$.
5. $a^2c+b^2c>2abc$.
6. $a^3b+ab^3>2a^2b^2$.
7. $(a+b)^2>4ab$.
8. $\dfrac{a+b}{2}>\dfrac{2ab}{a+b}$.
9. $a^3+b^3>ab(a+b)$.
10. $\dfrac{a}{b^2}+\dfrac{b}{a^2}>\dfrac{1}{b}+\dfrac{1}{a}$.
11. $\dfrac{a^2}{b}+\dfrac{b^2}{a}>a+b$.
12. $a^3+b^3+c^3>3abc$.
13. $(a+b+c)^3>27abc$.
14. $a^2+b^2+c^2>ab+ac+bc$.
15. $a^3+1>a^2+a$ unless $a=1$.
16. $a^3+8>2a^2+4a$ unless $a=2$.
17. $a^3+b^3+c^3>\frac{1}{2}[ab(a+b)+ac(a+c)+bc(b+c)]$.

18. $a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 > 6abc.$

19. $a^4 + b^4 + c^4 + d^4 > 4abcd.$

20. $(a + b + c + d)^4 > 256abcd.$

21. The truth of what general theorem may be inferred from the following?

$$a^2 + b^2 > 2ab;\ a^3 + b^3 + c^3 > 3abc;$$
$$a^4 + b^4 + c^4 + d^4 > 4abcd.$$

22. From the following?

$$(a + b)^2 > 2^2ab;\ (a + b + c)^3 > 3^3abc;$$
$$(a + b + c + d)^4 > 4^4abcd.$$

23. If the fractions $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, be unequal, then will $\frac{a + c + e}{b + d + f}$ be intermediate in value between the greatest and the least of the three fractions.

24. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, show that $xy > ad + bc$, and $xy > ac + bd$.

25. If $a + b > c$, $a + c > b$, and $b + c > a$, prove $2(ab + ac + bc) > a^2 + b^2 + c^2$.

26. $(a + b - c)^2 + (a - b + c)^2 + (b + c - a)^2 > ab + ac + bc.$

27. $(a + b)(a + c)(b + c) > 8abc.$

28. $3(a^3 + b^3 + c^3) > (a + b + c)(ab + ac + bc).$

29. $(a + b + c)(a^2 + b^2 + c^2) > 9abc.$

30. $a^4 + b^4 + c^4 > abc(a + b + c).$

31. $8(a^3 + b^3 + c^3) > 3(a + b)(b + c)(a + c).$

32. $\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} > \frac{3}{2}.$

33. $\left(\frac{a+b+c}{3}\right)^3 < \frac{a^3+b^3+c^3}{3}$.

34. $abc > (a+b-c)\,(a-b+c)\,(-a+b+c)$.

XVIII. QUADRATIC EQUATIONS.

1. PURE QUADRATICS.

Abstract Problems.—227, 228.

Solve the following:

1. $\frac{1}{10}(2x^2-28)=\frac{1}{6}(x^2-4.)$
2. $(x+3)^2-6x=25$.
3. $(2x+3)^2=12x+18$.
4. $16x^2=4\,(3-x)^2$.
5. $\frac{2}{x+5}-\frac{2}{x-5}=\frac{5}{4}$.
6. $\frac{x}{5}+\frac{5}{x}=\frac{x}{6}+\frac{6}{x}+\frac{1}{5x}$.
7. $\sqrt{x^2-17}=2\sqrt{\frac{65-x^2}{2}}$.
8. $\frac{9x}{16-x}=\frac{25\,(16-x)}{x}$.
9. $\frac{x}{\frac{120}{x}}=\frac{5}{6}(x-9)^2$.
10. $(3x+\frac{1}{3})^2=2x+1$.
11. $\frac{x}{\frac{x+18}{28}}=\frac{x+18}{\frac{4x}{63}}$.
12. $\frac{75(x-7)}{x-4}=\frac{48\,(x-4)}{x-7}$.
13. $\frac{b}{a-x}+\frac{b}{a+x}=2ab$.
14. $\sqrt{c+x}=\sqrt{a}-\sqrt{c-x}$.
15. $ax^2-bx^2=a^2-b^2$.
16. $\frac{x+b}{2}+\frac{2}{x+b}=b$.
17. $\frac{x+\sqrt{2-x^2}}{x-\sqrt{2-x^2}}=\frac{4}{3}$.
18. $\frac{4}{2+x}+\frac{4}{2-x}=\frac{16}{3}$.
19. $\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{x+2}{x-2}}=4$.
20. $(x-3)\,(x-2)-(x-4)\,(x+5)=3\,(x-1)^2+11$.

21. $\frac{1}{3}(3x^2+5)-\frac{1}{4}(x^2-1)=\frac{1}{3}(4x^2+9)-13.$

22. $(x^2-1)(x^2+2)=(x^2-1)^2+1.$

23. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{80}{x^2-4}.$

24. $\frac{x-5}{x+5}+\frac{x+5}{x-5}=\frac{148}{x^2-25}.$

25. $\frac{8x^2-9}{16}+\frac{3x^2-7}{2(x^2-14)}=\frac{x^2+27}{2}.$

26. $\frac{4x^2+9}{14}-\frac{3(x^2+4)}{x^2-6}=\frac{2x^2-23\frac{1}{2}}{7}.$

27. $\frac{8x^2+9}{10}-\frac{5x^2+9}{3x^2-9}=\frac{4x^2-10\frac{1}{2}}{5}.$

28. $(x-c)(x-2c)+(x-b)(x+3b)=a^2-x(3c-2b).$

29. $\frac{(x+c)(x-c)}{a+b}+\frac{c^3-b^3}{(a+b)(a+c)}=\frac{(x-b)(x+b)}{a+c}.$

30. $\frac{bc(x-a)^2}{(a-c)(a-b)}+\frac{ac(x-b)^2}{(b-a)(b-c)}$
$+\frac{ab(x-c)^2}{(c-a)(c-b)}=4.$

2. AFFECTED QUADRATICS.

Abstract Problems.—230, 231.

Solve the following:

1. $x^2-8x=-12.$
2. $x^2+2x=35.$
3. $x^2-3x=10.$
4. $x^2+5x-14=0.$
5. $x^2-7x=-12.$
6. $x^2-7x=3x-16.$

7. $x^2 - 2x - 99 = 0$.

8. $x^2 + 10x + 21 = 0$.

9. $x^2 + x = 30$.

10. $\frac{x^2}{6} - \frac{2}{3} = \frac{x}{2}$.

11. $x^2 = 6(x+2) + 4$.

12. $\frac{x^2}{15} = 10 + \frac{x}{3}$

13. $(x-2)(x+6) = 20$.

14. $(x-8)(x-6) = x-2$.

15. $2x^2 + x = 3$.

16. $4x^2 - 3(x-1) = 10x$.

17. $5x^2 - 9x = -2\frac{1}{4}$.

18. $3(x^2+2) = 11(x+2) + 4$.

19. $(3x-5)^2 = 20x$.

20. $\frac{x^2}{3} + \frac{3x}{2} = 7(x-3)$.

21. $\frac{3}{x} - \frac{x}{3} = 5\frac{5}{6}$.

22. $x^2 - \frac{14x}{15} = -\frac{1}{5}$.

23. $x^2 - \frac{7x}{12} = -\frac{1}{12}$.

24. $2x^2 - \frac{2x^3 - 7}{x+3} = 6\frac{1}{3}$.

25. $2x^2 - 3 = \frac{x+5}{2} + 11$.

26. $\frac{3}{x+2} + \frac{5}{x+4} = \frac{14}{x+6}$.

27. $\frac{(x+2)(x+3)}{(x-3)(x-4)} = 28$.

28. $\frac{x+3}{x-2} + \frac{x-2}{x+3} = 2\frac{1}{2}$.

29. $\frac{x-4}{x+2} - \frac{x-2}{x+4} = -\frac{1}{4}$.

30. $\frac{2 - \frac{1}{x}}{1 - \frac{1}{x}} - \frac{2 - \frac{3}{x}}{1 - \frac{2}{x}} = -\frac{1}{6}$.

31. $\frac{x+1}{2} - \frac{2}{x+1} = \frac{2x+3}{6}$.

32. $\frac{1 + \frac{2}{x}}{x-1} - \frac{\frac{4}{x} - 1}{2x} = \frac{7}{3x}$.

33. $x^2 + 3a^2 = 4ax$.

34. $x^2 - b^2 = 2ax - a^2$.

35. $x^2 + x(a-b) = ab$.

36. $x^2 - (a+b)x = -ab$.

37. $x^2 + a(c-b)x = a^2bc$.

38. $x^2 - x(2a-b) = 2ab$.

39. $bx^2 + (b^2 - a)x = ab$.

40. $a^2(x^2 + a^2) = 2a^3x + 1$.

41. $(a^2 - b^2)x^2 - 2abx = a^2$.

42. $x^4 - 29x^2 = -100$.

43. $x^6 - 9x^3 = -8$.

44. $x^8 - 17x^4 + 16 = 0$.

45. $x^{10} - 31x^5 = 32$.

46. $x^2 + x + 1 = \frac{21}{x^2 - x + 1}$.

47. $x^8 + x^4 = 30$.

48. $x + 2x^{\frac{1}{2}} = 15.$

49. $x + 2\sqrt{x} = 3.$

50. $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} = 36.$

51. $x^{-2} - 2x^{-1} = 8.$

52. $x^{\frac{2}{3}} - 9x^{\frac{1}{3}} = 400.$

53. $x^{\frac{1}{2}} + x^{\frac{1}{4}} = 56.$

54. $6x^{-1} + x^{-\frac{1}{2}} = 1.$

55. $x^{-2} + x^{-1} = 12.$

56. $x^{-2} + ax^{-1} = 2a^2.$

57. $x^{-2n} + x^{-n} = 6.$

58. $x^{\frac{1}{n}} + x^{\frac{1}{2n}} = 2.$

59. $3x^{\frac{1}{n}} + 2x^{\frac{2}{n}} = 5.$

60. $x^{\frac{1}{2}} + \dfrac{3}{2x^{\frac{1}{2}}} = \dfrac{7}{2}.$

61. $x + 3\sqrt{5x} = 20.$

62. $x^{\frac{1}{3}} + 3 = \dfrac{x + 42}{x^{\frac{2}{3}} + 6}.$

63. $3\sqrt{4x} - 5x = -8.$

64. $\frac{1}{3}\sqrt{x} + \dfrac{2}{\sqrt{9x}} = \dfrac{11}{9}.$

65. $x^{-\frac{1}{4}} + 2x^{-\frac{1}{2}} = 10.$

66. $2(x^n + x^{-n}) = 5.$

67. $x + 3 - \sqrt{x+3} = 12.$

68. $x + \sqrt{x+9} = 11.$

69. $3x + 7 - 2\sqrt{3x+7} = 15.$

70. $2x - 5 + \sqrt{2x+3} = 4.$

71. $x - 3 + \sqrt{x+15} = 12.$

72. $\sqrt{x+16} - 2\sqrt[4]{x+16} = 3.$

73. $(x-5) - \sqrt{2x-11} = 8.$

74. $\sqrt{5x+a} + \sqrt{3x} = 7\sqrt{a}.$

75. $(2x+3)(3x+2) = (x+4)(2x+1) + 5.$

76. $(x-4)^2 + 17 = (2x-1)(x-3).$

77. $2x + \dfrac{x^2+1}{x^2-3} = \dfrac{2(x^3-4)}{x^2-3}.$

78. $\dfrac{x-5}{2x+3} + \dfrac{2x+3}{x-5} = -\dfrac{85}{18}.$

79. $\dfrac{2}{x-1} + \dfrac{x+1}{2} = \dfrac{x+2}{2} + \dfrac{x^2}{2(x-1)}.$

80. $x^2 - (a-b+c)x = -c(a-b).$

81. $x^2 - x(a+b) = (a+1)(1-b).$

82. $2x^2 - 3ax + bx = b^2 - a^2$.

83. $bdx^2 + x(ad + bc) = (b - a)(c + d)$.

84. $4x + \frac{a^2}{x} - 4a - \frac{b^2}{x} = 0$.

85. $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$.

86. $x^2 - \frac{x}{ac}(bc + a^2) = -\frac{b}{c}$.

87. $(a^2 - b^2)x^2 - x(2a^2 + 2b^2) = b^2 - a^2$.

88. $\frac{x^2}{2c} - x\left(\frac{b+a}{2abc}\right) = -\frac{1}{2abc}$.

89. $(x - c)\sqrt{ab} = (a - b)\sqrt{cx}$.

90. $3x^2(b + c)^2 = 2a^2 + ax(b + c)$.

91. $2x - 5 + \sqrt{3x + 4} = x + 23$.

92. $x^2 - 4x + 2\sqrt{x^2 - 4x + 7} = 1$.

93. $x^2 + x + \sqrt{x^2 + x + 5} = 25$.

94. $x^2 + 2x + 1 + \sqrt{x^2 + 3x - 5} = 62 - x$.

95. $x^2 + \frac{1}{x^2} + 2 + x + \frac{1}{x} = 2$.

96. $x^2 + x + \frac{1}{2x} + \frac{1}{4x^2} = 6\frac{5}{16}$.

97. $x^2 + 3 + \frac{1}{2x} - x + \frac{1}{4x^2} = 5\frac{5}{16}$.

98. $x^2 + \frac{1}{x^2} + 3\left(x - \frac{1}{x}\right) = 8\frac{3}{4}$.

99. $(2a - bx)^2 + (2a - bx) = a^2 + a$.

100. $\sqrt[3]{x + 22} - \sqrt[3]{x + 3} = 1$.

3. FORMING EQUATIONS FROM GIVEN ROOTS.—234.

Form the equation whose roots are:

1. 3 and 2.
2. -5 and -4.
3. ± 6.
4. 1, 2, and 3.
5. ± 2 and 4.
6. ± 5 and ± 1.
7. ± 3 and $\pm\sqrt{-3}$.
8. $1 \pm \sqrt{5}$.
9. $\sqrt{3} \pm \sqrt{-5}$.
10. $\pm\sqrt{3}$ and $\pm\sqrt{-5}$.
11. $\dfrac{1 \pm \sqrt{5}}{2}$.
12. $a \pm \sqrt{b}$ and $b \pm \sqrt{a}$.
13. $\dfrac{a \pm b}{c}$ and $a - b$.
14. $1 \pm \sqrt{-3}$ and $2 \pm \sqrt{-2}$.

Form an equation in each of the following exercises whose two roots shall satisfy the given conditions:

15. Their sum is a, and their product b.
16. They differ by 6, and have a quotient of 4.
17. They differ by $\dfrac{a^2 - b^2}{ab}$, and have a product $= 1$.
18. They differ by 2, and the difference of their cubes equals the sum of their squares.
19. They differ by 3, and their cubes differ by 9.
20. They differ by 5, and the sum of the square and fourth roots of the greater root of the equation is 2.
21. Each is less by unity than the roots of the equation $x^2 - 2ax + a^2 - 5 = 4x - 4a$.
22. They are the factors of $6a^2 + 5a - 6$.
23. They are reciprocals of the roots of $3x^2 - 5x = 2$.

4. QUADRATICS INVOLVING ONE UNKNOWN QUANTITY.

Concrete Problems.

1. Find two numbers having the ratio of 3 to 7, and the sum of whose squares is 232.

2. Find two numbers whose sum is to the greater as 7 is to 5, and whose sum multiplied by the less equals 56.

3. Find a number whose square taken from 40 leaves a remainder equal to that left by taking 32 from the square of the number.

4. Find two numbers whose difference is 3, and the difference of whose reciprocals is $\frac{3}{10}$.

5. Divide 7 into two parts, such that the quotient of one part by the other, added to the reciprocal of this fraction, shall equal $2\frac{1}{12}$.

6. The sum of two numbers added to the sum of their squares is 32, and the product of the numbers is 12. Find the numbers.

7. The difference of two numbers is 2, and the difference of their cubes is 26. Find the numbers.

8. One number exceeds another by 3, and if the sum of the numbers be increased by the square root of their sum the result is 12. Find the numbers.

9. The difference of two numbers is 4, and the sum of their fourth powers is 626. Find the numbers. (*Let* $x-2$, *and* $x+2=$ *the numbers.*)

10. Find two numbers, the sum of whose cubes is 35, and the cube of whose sum is 125.

11. Separate 21 and 31, each into two parts, such that the first part of 21 may be $\frac{1}{2}$ the first part of 31,

and the sum of the squares of the remaining parts may be 130.

12. Separate 50 into two parts, so that the difference of the two quotients formed by dividing each part by the other shall be $\frac{5}{6}$.

13. A number is composed of three factors, which have a consecutive difference of 2; and, if the number be divided by each of its factors in turn, the sum of the quotients will be 104. Find the number.

14. Find two numbers whose sum is $\frac{9}{4}$ of the less, and the difference of whose squares is 9.

15. Find two numbers whose difference is $\frac{3}{5}$ of the greater, and the difference of whose cubes equals 117.

16. Find two numbers whose sum is 7, and twice their product equals $\frac{3}{8}$ of the cube of the greater.

17. Find two fractions whose sum is $\frac{17}{15}$; the denominator of the first is 1 more than its numerator, and the denominator of the second is 4 more than its numerator, and 2 more than the numerator of the first. (*In this problem and the next, use the last two conditions to represent the fractions, but form the equation with the first condition.*)

18. Find two fractions whose product is $\frac{12}{35}$; the denominator of the first exceeds its numerator by 2; the denominator of the second exceeds its own numerator by 3, and exceeds the denominator of the first by 2.

19. Divide $\frac{10}{21}$ into two factors, such that the denominator of the first is 1 more than its numerator; the denominator of the second exceeds its numerator by the numerator of the first, and exceeds the denominator of the first by twice the numerator of the first.

20. The numerator of a fraction is 2 less than the denominator. If 1 be added to the fraction, the result will equal $\frac{7}{12}$ divided by the fraction. Find the fraction.

21. If from the square of my age 56 be taken, and the result be divided by 4, the quotient will be 5 more than the square of my age 11 years ago. How old am I?

22. A and B each spent \$48 in buying sheep. B got 4 more for his money than A, and paid \$2 apiece less. How many did each buy, and at what prices?

23. A party of 180 persons chartered a train; by putting 15 more in each car, one car less would be needed. How many cars were chartered?

24. What are oranges worth a dozen, if, by buying 15 more for a dollar, the price would be lowered 4 cents a dozen?

25. A boy was sent to market to buy eggs to the amount of a dollar. On the way home he broke 20, and thus made the remainder cost 2 cents a dozen more. How many did he buy?

26. A man bought a certain number of bushels of wheat for \$25. If he had bought 5 bushels more for the same money, they would have cost 25 cents less per bushel. How many bushels did he buy?

27. A party hired a boat for \$3.60. After using it, but before paying for it, 3 of the men left, thus making it cost each 20 cents more than it would have cost had all paid. How many used the boat?

28. If a rectangular bar of brass, weighing 45 pounds, be drawn out 6 feet longer, it will weigh 2 pounds less per linear foot. How long is it?

29. A man bought a number of oranges for \$1.20; he ate 10 of them, and sold the remainder at 2 cents apiece more than he paid for them; by so doing, he gained 30 cents on all. How many did he buy?

30. A regiment was drawn up with 3 men in front more than in depth. Coming to a pass where only 3 could march abreast, the line was 243 men longer. How many were in the regiment?

31. A boy bought a certain number of oranges, paying as many cents a dozen as the number of oranges he bought; if he had bought 1 dozen fewer for the same money, they would have cost him $4\frac{1}{2}$ cents apiece. How many oranges did he buy?

32. A man bought a certain number of sheep, paying $\frac{1}{4}$ as many dollars per head as there were sheep; he lost four, and thus the remainder averaged \$7$\frac{1}{2}$ a head. How many sheep did he buy?

33. A farmer bought a square field, and cut off from it a path 2 yards wide surrounding the field. If the area of the path were increased by 216 square yards, the sum would be $\frac{1}{10}$ of the original area of the field. How much was the area of the field?

34. A and B hired a pasture, into which A put 4 horses, and B as many as cost him \$18 a week. Afterwards B put in 2 additional horses, and found he must pay \$20 per week. At what price was the pasture hired?

35. A and B were hired by the day. A received \$125, and B, who had worked 7 days less, received \$80. If A had lost the 7 days and B had worked full time, their wages would have been equal. How many days did each work, and at what rates?

36. Two men were hired to work the same num-

ber of days, but at different prices per day. A worked the whole time, and received \$45; B lost 4 days, and received \$20. Had A and B exchanged daily wages, they would have received equal sums, For how many days were they hired, and at what rates?

37. A sold two flocks of sheep, at different rates per head, receiving the same amount for each flock. Had he sold the first flock at the same price per head as the second, and the second at the same price per head as the first, he would have received \$250 and \$160 respectively. There were 90 sheep in both flocks. How many were there in each, and at what prices per head were they sold?

38. Two men were hired to work by the day. A worked 3 days more than B, but received \$6 less for total wages. Had A lost the 3 days and B worked full time, A would have received \$72 and B \$120. How many days did each work, and at what rates?

39. A banker had two kinds of money. It required 6 pieces more of one kind than of the other to make a dollar. He gave 20 pieces of the more valuable and 30 pieces of the other to pay a bill of \$8. What was the value of each coin?

40. A and B sold 130 yards for \$42, of which A sold 40 and B 90. A sold for \$1, $\frac{1}{3}$ of a yard more than B did. How many yards did each sell for \$1?

41. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards. If the circumference of each wheel be increased by 1 yard, the fore-wheel will make 14 revolutions more than the hind-wheel in going 420 yards. What is the circumference of each wheel?

21. $\frac{1}{8}(3x^2+5)-\frac{1}{4}(x^2-1)=\frac{1}{8}(4x^2+9)-13.$

22. $(x^2-1)(x^2+2)=(x^2-1)^2+1.$

23. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{80}{x^2-4}.$

24. $\frac{x-5}{x+5}+\frac{x+5}{x-5}=\frac{148}{x^2-25}.$

25. $\frac{8x^2-9}{16}+\frac{3x^2-7}{2(x^2-14)}=\frac{x^2+27}{2}.$

26. $\frac{4x^2+9}{14}-\frac{3(x^2+4)}{x^2-6}=\frac{2x^2-23\frac{1}{2}}{7}.$

27. $\frac{8x^2+9}{10}-\frac{5x^2+9}{3x^2-9}=\frac{4x^2-10\frac{1}{2}}{5}.$

28. $(x-c)(x-2c)+(x-b)(x+3b)=a^2-x(3c-2b).$

29. $\frac{(x+c)(x-c)}{a+b}+\frac{c^3-b^3}{(a+b)(a+c)}=\frac{(x-b)(x+b)}{a+c}.$

30. $\frac{bc(x-a)^2}{(a-c)(a-b)}+\frac{ac(x-b)^2}{(b-a)(b-c)}$
$+\frac{ab(x-c)^2}{(c-a)(c-b)}=4.$

2. AFFECTED QUADRATICS.

Abstract Problems.—230, 231.

Solve the following:

1. $x^2-8x=-12.$
2. $x^2+2x=35.$
3. $x^2-3x=10.$
4. $x^2+5x-14=0.$
5. $x^2-7x=-12.$
6. $x^2-7x=3x-16.$

50. A ship springs two leaks, one of which would sink her in 4 hours less time than the other. Both run for 2 hours, when the smaller one is stopped, and, the other still running, she sinks in 4 hours, 40 minutes after. How long would it have taken each separately to sink the ship?

51. A, B, and C together do a piece of work which it would take A 4 days longer, B 3 times as long, and C 20 days longer to do alone. How long would it take each alone, and how long all together, to do a piece of work twice as large?

52. A cistern could be filled by an inlet pipe in 2 hours less time than it could be emptied by an outlet pipe. Both ran together for 4 hours, when the outlet pipe became stopped, and the cistern was full in 7 hours, 12 minutes afterwards. How long would it take the inlet pipe to fill the cistern if the outlet pipe were closed?

53. A and B could do a piece of work in 9 days. After working together 3 days, B left, when it took A 12 days longer to finish than it would have taken B to do all. How long would it have taken each alone to do it?

54. A, B, and C can build a wall in $4\frac{4}{9}$ days. A can build twice as much in a day as B, and B can do the whole in 3 days less than C. How long would it take C alone to build it?

55. Find a number of two digits whose units' digit is twice the tens', and which is equal to 8 more than the square of the units.

56. Find a number of two places, such that the units' digit increased by 1, shall equal the square of the tens' digit, and if 2 be taken from the number,

the remainder shall equal 4 times the square of the tens' digit.

57. Find a number of two digits whose units' digit is 3 more than the tens', and if the square of the number be divided by the units' digit, the quotient will be 125.

58. The square of the time past noon is twice the time to midnight. What time is it?

59. The square root of the time past noon equals the time to midnight. What time is it?

60. The square of the time past noon, increased by the time to midnight, equals the time past midnight. What is the time?

61. The square of the time past noon, increased by the time to midnight, equals the time past noon, increased by the time past midnight. What is the time?

62. The hands of a watch are between 3 and 4, and the square of the distance the minute-hand is beyond 2, is 16 more than the distance it lacks of being at 10. What is the time?

63. The hour and minute-hands of a watch are together between two consecutive numbers, the difference of whose squares is 7 less than the square of the less number. What time is it?

64. The hour and minute-hands of a watch are opposite, and the square of the distance the minute-hand is past 12 is $5\frac{25}{121}$ more than the distance the hour-hand lacks of being at 12. What is the time?

65. The hands of a watch are at right angles, and the square of the number just behind the hour-hand is 1 more than 5 times the number just ahead of it. What is the time?

66. A man loaned two sums of money, which differed by \$100. His rate on each was 1 per cent of its principal, and the interest on both was \$61. What was each principal?

67. A man traveled 72 miles in a certain time. Had he traveled 2 miles per hour faster, he would have been 3 hours less in performing the journey. How many miles did he travel per hour?

68. A and B each traveled 96 miles. B, by going 2 miles per hour more than A, finished the journey 4 hours sooner. What was A's rate of travel?

69. A and B have a distance of 90 miles to travel; B, who travels 3 miles per hour slower than A, starts 2 hours before him and arrives 3 hours after him. What is the rate of each?

70. An oarsman can row a boat $7\frac{1}{2}$ miles down a river and back in 3 hours, 20 minutes. If the current be 3 miles per hour, what is the rate of rowing?

71. When the current is 2 miles per hour an oarsman can row 14 miles and back in 6 hours and 40 minutes. How long would it take him to row the same distance in still water?

72. An excursion steamer went 40 miles down a river and back in 16 hours, the current being 3 miles per hour. Owing to an accident to her machinery, she could steam back at only $\frac{1}{2}$ the rate she steamed down. What was her rate down and back?

73. A and B set out at the same instant to meet each other; upon meeting, it appeared that A had traveled 18 miles more than B, but that A could have gone B's journey in 12 days, and B would have required 27 days to perform A's. How far did each travel?

74. Generalize problem 73 and solve. State what numbers will give integral solutions.

75. A and B, being 72 miles apart, set out at the same time to meet each other. A traveled 2 miles per hour more than B, and the number of hours before they met was equal to $\frac{1}{2}$ the number of miles B traveled per hour. What were their rates?

76. A set out from C and B from D, distant 60 miles, at the same time. A arrived in D 2 hours, and B in C 8 hours, after they met. Find their rates.

77. A rode 64 miles, and walked the rest of his journey. He could ride $\frac{1}{8}$ of his whole journey in the same time that it took him to walk 4 miles. Had he walked at the same rate as he rode, he would have gone 154 miles further than he did. Find the number of miles he walked, and the comparative rates of riding and walking.

78. A rode 56 miles, and walked the rest of his journey. Had he ridden the whole distance, he would have completed the journey at the same moment that he completed the 20th mile of his walk. Had he ridden during the whole time, he would have gone 40 miles further. How far did he walk? If he walked at the rate of 4 miles per hour, what was his rate of riding?

79. A, who traveled daily $\frac{1}{20}$ of the distance between C and D, left C 7 days after B left D. After traveling as many days as he went miles in a day, he met B, whose rate was 4 miles per day. How many miles had B traveled when he met A, and what is the distance C D?

80. A's rate is 24 miles per day, and A's rate exceeds B's by as many per cent as B goes miles per

day. Find their comparative times in going equal distances.

81. A can go 10 miles in 15 minutes less than B, and can go 15 miles further in $7\frac{1}{2}$ hours. How far can each go per hour?

82. A man starts from the foot of a mountain and walks to its summit in 12 hours. During the first half of the distance, his rate is one mile per hour faster than during the second half. He descends in 4 hours by walking 2 miles per hour faster than during the first half of the ascent. Find the distance to the summit.

83. From a vessel containing 1,024 gallons of wine, a certain quantity was drawn, and replaced with water. The same quantity of the mixture was drawn and replaced with water, and so on for 5 draughts, when 243 gallons of pure wine remained in the cask. How much wine was drawn each time? How much wine would remain after 10 draughts? After n draughts?

84. Twenty-seven gallons are drawn from a cask full of wine, and it is then filled up with water; then 27 gallons of the mixture are drawn and the cask is again filled up with water, and so on for four draughts. After the fourth refilling, the quantity of wine in the cask is to the quantity of water as 16 is to 65. How much does the cask hold?

85. From a cask full of wine a gallons are drawn, and it is then filled up with water; then a gallons of the mixture are drawn, and the cask is again filled up with water, and so on for n draughts, when the wine in the cask is b^{-1} times the original quantity. How much does the cask hold?

5. SPECIAL SOLUTIONS IN HIGHER EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

Abstract Problems.

Solve the following equations:

1. $x^3 - 13x = -12$.
2. $x^3 - 3x = 18$.
3. $2x^3 - x^2 = 12$.
4. $x^3 - 6x + 9 = 0$.
5. $x^3 - 6x^2 + 10x - 8 = 0$.
6. $9x^3 - 45x^2 + 75x - 39 = 0$.
7. $4x^3 = 39x - 35$.
8. $x^2 - \frac{4}{9}x^{-1} = 1\frac{4}{9}$.
9. $x + x^{-\frac{1}{2}} = 2$.
10. $\dfrac{(1+x)^3}{1+x^3} = \dfrac{76}{16}$.
11. $\dfrac{1+x^5}{(1+x)^5} = \dfrac{11}{81}$.
12. $\dfrac{1+x^4}{(1-x)^4} = 17$.
13. $4x^2 - 27x + 28x^{\frac{3}{2}} - 140x^{\frac{1}{2}} = -100$.
14. $x^4 - 2x^3 + x = 132$.
15. $x^4 + x^3 - 4x^2 + x + 1 = 0$.
16. $4x^4 - 4x^3 + 5x^2 - 2x = 8$.
17. $x^4 - 8x^3 + 10x^2 + 24x = -5$.
18. $x^4 + \frac{1}{3}x^3 - 39x = 81$.
19. $12x^4 - 104x^3 + 209x^2 - 104x = -12$.
20. $x^4 - 10x^3 + 35x^2 - 50x = -24$.

21. $x^3 - 12x^2 + 41x = 30.$

22. $x^3 - 10x^2 + 17x = -28.$

23. $x^4 + 10x^3 + 27x^2 + 10x = 8.$

24. $\sqrt[3]{x+21} - \sqrt[3]{x+2} = 1.$

25. $\sqrt{x^2+8} + \sqrt{x^2-8} = 1 + \sqrt{17}.$

26. $\frac{1}{2}\sqrt{3a-4x} + \sqrt{x(3a-1)} = \frac{3a}{2}\sqrt{1-4x}.$

27. $2^{x+1} + 4^x = 80.$ Find x.

28. $a^x + \sqrt{a^x} = \frac{3}{4}.$ Find a^x.

29. $5^x + 25^x = \frac{6}{25}.$ Find x.

30. Find three cube roots of 1.

31. Find four fourth roots of 16.

32. Find six sixth roots of 64.

33. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}} = \dfrac{c}{x}.$

34. $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{(a^2-ax)}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}} = c.$

35. $\sqrt{\dfrac{b+x}{x}} + 2\sqrt{\dfrac{b}{b+x}} = c^2\left(\dfrac{x}{b+x}\right)^{\frac{1}{2}}.$

36. $\sqrt{3y^2+8y+14} + \sqrt{3y^2-8y+14} = 8y.$

37. $(x+2)^{\frac{1}{3}} - 6(x+2)^{\frac{1}{6}}(x-1)^{\frac{1}{6}} + 8(x-1)^{\frac{1}{3}} = 0.$

38. $x^5 - 3x^4 - 8x^3 + 24x^2 - 9x = -27.$

39. $\sqrt[3]{x^3+8x^2+15x-8} = (x+3)\sqrt[3]{x+8}.$

40. $12x^4 - 4x^3 + 9x^2 + 3x = 2.$

41. $x^4 - 10x^3 + 120x = 144.$

42. $x^4 - 10x^3 + 120x = -144.$

43. $\dfrac{x^4 + 6x^2 + 1}{4x(x^2+1)} = \dfrac{81x^4 + 16}{81x^4 - 16}.$

44. $(x-1)^3 + (x+4)^3 = 8x^3 + 27.$

45. $(2x-1)^3 + (x+2)^3 = 27x^3 + 1.$

46. $(2x-1)^3 + (x-2)^3 = 9x^3 - 9.$

47. $(2x-5)^3 - (x-2)^3 = x^3 - 27.$

48. $(x-2)^4 - (2x-1)^4 = 9 - 9x^4.$

49. $\sqrt{x^2+1} - \sqrt{1-x^2} = \sqrt{x^4+1}.$

50. $\sqrt{1+x^2} - \sqrt{1-x^2} = \sqrt{1-x^4}.$

51. $(x^2 - 4x - 8)^2 = 9(x^2 + 2x + 9) - 4(x-2).$

52. $(1+x)(1+x^2)(1+x^3) = 30x^3.$

53. $\dfrac{(x+4)^5 + (x+6)^5}{(x+3)^5 + (x+7)^5} = \frac{16}{121}.$ Find one value of x.

54. $\dfrac{(x+1)^4 + (x+3)^4}{(x+4)^4 + (x+2)^4} = \frac{1}{8}.$ Find one value of x.

55. $\sqrt{x - \dfrac{1}{x}} + \sqrt{1 - \dfrac{1}{x}} = x.$

56. $\sqrt{x^2 - \dfrac{a^4}{x^2}} + \sqrt{a^2 - \dfrac{a^4}{x^2}} = \dfrac{x^2}{a}.$

57. $(x^2 - 3x - 2)^3 + (3x^2 + 3x + 1)^3 =$
$(2x-1)(2x+1)(3x^2 + 3x + 1)(x^2 - 3x - 2).$

58. $(x^2 - ax - b)^3 + (x^2 + 2ax + b)^3 =$
$x(2x+a)(x^2 - ax - b)(x^2 + 2ax + b).$

59. $(x^2 - ax + 2b)^3 - (x^2 + 2ax - b)^3 =$
$9(b - ax)(x^4 + ax^3 + bx^2 - 2a^2x^2 + 5abx - 2b^2).$

6. QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.–244, 245.

Abstract Problems.

Solve the following:

1. $x^2 + y^2 = 65$, $xy = 28$

2. $x^2 + y^2 = 13$, $xy = -6$

3. $4x^2 + 9y^2 = 288$, $xy = 24$

4. $x + y = 41$, $\sqrt{xy} = 20$

5. $25x^2 + y^2 = 26$, $3xy = 3$

6. $x + y = 10$, $xy = 24$

7. $x - y = 6$, $xy = 16$

8. $2x + 3y = 23$, $5xy = 100$

9. $2x + \frac{1}{2}y = 18$, $\frac{1}{4}xy = 8$

10. $3x - 2y = 8$, $4xy = 32$

11. $x^2 - y^2 = 20$, $xy = 24$

12. $3x - 5y = -16$, $xy = 15$

13. $x^3 + y^3 = 224$, $xy = 12$

14. $x + y = 20$, $\sqrt[3]{xy} = 4$

15. $x^4 - y^4 = 175$, $xy = 12$

16. $x^4 + y^4 = 17$, $xy = 2$

17. $x^{\frac{3}{2}} + y^{\frac{1}{2}} = 13$, $x^{\frac{3}{4}}y^{\frac{1}{4}} = 6$

18. $x^2 + y^2 = 80$, $xy = 32$

19. $x^2 + y^2 = 106$, $x + y = 14$

20. $x^2 + y^2 = 25$, $x - y = 1$

21. $x^2 + y^2 = 74$, $x + y = 12$

22. $x^4 + y^2 = 82$
$x^2 + y = 10$

23. $4x^2 + 25y^2 = 2$
$2x + 5y = 2$

24. $16x^4 + y^2 = 265$
$4x^2 + y = 19$

25. $4x^{\frac{2}{3}} + y^{\frac{1}{2}} = 68$
$2x^{\frac{1}{3}} + y^{\frac{1}{4}} = 10$

26. $9x^2 + 25y^{\frac{1}{2}} = 306$
$x + \frac{5}{3}y^{\frac{1}{4}} = 8$

27. $x^{\frac{1}{2}} + y^{\frac{1}{3}} = 7$
$x^{\frac{1}{4}} - y^{\frac{1}{6}} = 2 - \sqrt{3}$

28. $x^2 - y^2 = 84$
$x + y = 14$

29. $x^2 - y^2 = 24$
$x - y = 4$

30. $x^4 - y^4 = 65$
$x^2 + y^2 = 13$

31. $x^4 - y^4 = 544$
$x^2 - y^2 = 16$

32. $x^3 - y^3 = 56$
$x - y = 2$

33. $x^3 + y^3 = 9$
$x + y = 3$

34. $9x^2 - 16y^2 = 287$
$3x + 4y = 7$

35. $8x - 9y = -65$
$2\sqrt{2x} - 3\sqrt{y} = -5$

36. $27x^3 + 8y^3 = 432$
$3x + 2y = 12$

37. $x - y = 16$
$\sqrt{x} + y^{\frac{1}{2}} = 8.$

38. $x^4 + x^2y^2 + y^4 = 133$
$x^2 + xy + y^2 = 19$

39. $x^3 + y^3 = 468$
$x^2 - xy + y^2 = 39$

40. $x^2y - xy^2 = 2$
$x - y = 1$

41. $x^2 + xy = 180$
$x^2 - y^2 = 135$

42. $x^2y + xy^2 = 6$
$x^2 + xy = 6$

43. $x^2 + xy + y^2 = 108$
$x + y + \sqrt{xy} = 18$

44. $x^2 + 9xy + 20y^2 = 90$
$x + 4y = 9$

45. $x^2 - xy - 30y^2 = -40$
$x - 6y = -2$

46. $6x^2 - 13xy - 5y^2 = 10$
$2x - 5y = 1$

47. $4x^2 - 27xy - 7y^2 = 492$
$x - 7y = 6$

48. $x^4 + 4y^4 = 580$
$x^2 + 2xy + 2y^2 = 58$

49. $x^4 - 7x^2y^2 + y^4 = 451$
$x^2 + 3xy + y^2 = 41$

50. $9x^4 + 21x^2y^2 + 25y^4 = 11\frac{1}{16}$
$3x^2 - 3xy + 5y^2 = \frac{1}{8}$

51. $x^3 + y^{-3} = 27\frac{1}{8}$
$x + y^{-1} = 3\frac{1}{2}$

52. $3(x^3 - y^3) = 19xy$
$x - y = 2$

53. $x^2 + y^2 = 93 - xy$
$x^3 - y^3 = 279$

54. $x^2y^2 = 504 - 10xy$
$x + 4y = 17$

55. $5xy = 266 - x^2y^2$
$x - y = 5$

56. $x^2y^2 + xy = 72$
$x + y = 6$

57. $x + y + x^2 + y^2 = 32$
$xy = 12$

58. $x^2 + 4x - y = 60 - 2xy$
$y^2 + 4y - x = 10$

59. $x^2 - y^2 - x - y = 14$
$xy + y^2 = 14$

60. $x^2 + y = 23 - 2xy$
$x + y^2 = 7$

61. $a - b^{-1} = 1$
$a^3 - b^{-3} = 19$

62. $\sqrt{\frac{3x}{x+y}} + \sqrt{\frac{x+y}{3x}} = 2$
$xy - (x + y) = 54$

63. $x^2 + 2y = 4y^2 - x$
$x^2 + y^2 = 34$

64. $a + b = 3$
$a^4 + b^4 = 17$

65. $x^2 + y^2 = 97$
$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5$

66. $a - b = 1$
$a^4 + b^4 = 97$

67. $x + y = 3$
$x^5 + y^5 = 33$

68. $x - y = 3$
$x^3 + y^3 = 133$

69. $x + y = 6$
$x^4 - y^4 = 624$

70. $x^2 + xy = 28$
$2xy + y^2 = 33$

71. $x(x+y) = 15$
$y(x-y) = 2$

72. $y(x+y) = x^2$
$x^3 - y^3 = x^2 + y^2$

73. $x^3 - y^3 = 62(x-y)^2$
$xy = 320$

74. $x^3 + y^2 = 43$
$x + y^2 = 19$

75. $x^2 + y = 19$
$x + y^2 = 13$

76. $xy(x^2 + y^2) = 160$
$x^2 - y^2 = 12$

77. $x^3 + y^3 = 35$
$(x+y)(x^2 + y^2) = 65$

78. $a^2 + b^2 = 10$
$a^3 + b^3 = 28$

79. $a^2 + b^2 = 13$
$a^3 - b^3 = 19$

80. $x^2 - y^2 = 8$
$x^3 + y^3 = 28$

81. $x^2 + xy - 6y^2 = 21$
$xy - 2y^2 = 4$

82. $2x^2 + xy - 3y^2 = 7y$
$x^2 + 2xy = 8y$

83. $\dfrac{x}{x-y} - \dfrac{x-y}{x+y} = 1\frac{2}{3}$
$2 + 3xy = 4x$

84. $(x+y)^3 + (x+y) = 30$
$x - y = 1$

85. $2xy = x^2 + y^2 - 9$
$x^2 + y^2 = x^3 - y^3$

86. $\dfrac{(x+y)^4}{x^4 + y^4} = \dfrac{81}{17}$
$x^2 + y^2 = 5$

87. $x\sqrt{x+y} = 48$
$(x+y)\sqrt{y} = 32$

88. $x - y = 1$
$(x^2+y^2)(x^3-y^3) = 247$

89. $x^3 + \dfrac{1}{x^3} = y + \dfrac{1}{y}$
$y^3 + \dfrac{1}{y^3} = 9\left(x + \dfrac{1}{x}\right)$

90. $x^3 - \dfrac{1}{x^3} = y + \dfrac{1}{y}$
$y^3 + \dfrac{1}{y^3} = 9\left(\dfrac{1}{x} - x\right)$

91. $\dfrac{x+y}{x-y} = 5\frac{1}{6} + \dfrac{2y}{x+y}$
$x^2 + y^2 = 74$

92. $\sqrt{x^2+y^2} + \sqrt{x^2-y^2} = 2y$
$x^4 - y^4 = a^4$

93. $\left.\begin{matrix} x^3 - y^3 = 56 \\ x^2 - y^2 = 12 \end{matrix}\right\}$ Find one value of x and y.

94. $(x-y) + xy(x-y) + x^2y^2 = 43$
$xy + (x-y)^2 + xy(x-y) = 13$

95. $4(x+y)^2(x^2 - xy + y^2) = 27x^2y^2$
$x^2 + y^2 = 20$

96. $\frac{1}{2}[(x+y)^3 + 5] = (x^2+y^2)(x^3 + y^3 - 5xy)$
$x + y = 5$

97. $(x-2y)^3 + (2x+y)^3 = 27x^3 - 30xy^2 + 9y^3$
$y - y^{\frac{1}{2}} = 6$

98. $(x+y)(x-y)^2 = 160$
$(x+y)(x^2+y^2) = 580$

99. $(x+y)(xy+1) = 8xy$
$(x^2+y^2)(x^2y^2+1) = 28x^2y^2$

100. $\left.\begin{matrix} (x+y)(xy+1) = 7xy \\ 3(x^2+y^2)(x^2y^2-1) = 70x^2y^2 \end{matrix}\right\}$ Find two values.

101. $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{61}{\sqrt{xy}} + 1$
$\sqrt[4]{x^3y} + \sqrt[4]{xy^3} = 78$

7. ROOTS OF BINOMIAL SURDS.—241.

Reduce each of the following expressions to its simplest form:

1. $\sqrt{8+2\sqrt{15}}+\sqrt{12-2\sqrt{35}}$.
2. $\sqrt{3+2\sqrt{2}}+\sqrt{19-6\sqrt{2}}$.
3. $\sqrt{9+6\sqrt{2}}+\sqrt{27-12\sqrt{2}}$.
4. $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$.
5. $\sqrt{37+12\sqrt{7}}+\sqrt{29-4\sqrt{7}}$.
6. $\sqrt{8+4\sqrt{3}}+\sqrt{14-8\sqrt{3}}$.
7. $\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}$.
8. $\sqrt{12-6\sqrt{3}}-\sqrt{39+12\sqrt{3}}$.
9. $\sqrt{15-4\sqrt{11}}-\sqrt{13-2\sqrt{22}}$.
10. $\sqrt{11+6\sqrt{2}}+\sqrt{3-2\sqrt{2}}$.
11. $(\sqrt{31-10\sqrt{6}}-\sqrt{10-4\sqrt{6}})^{\frac{1}{2}}$.
12. $\sqrt{7\sqrt{3}-12}+\sqrt{4\sqrt{3}-6}$.
13. $\sqrt{12+5\sqrt{6}}-\sqrt{24+11\sqrt{6}}$.
14. $\dfrac{1}{\sqrt{9-2\sqrt{14}}}+\dfrac{1}{\sqrt{4+3\sqrt{2}}}$.
15. $\dfrac{1}{\sqrt{17+2\sqrt{30}}+\sqrt{17-2\sqrt{30}}}$.

16. $\sqrt[4]{17+12\sqrt{2}}+\sqrt[4]{49-20\sqrt{6}}$

17. $\dfrac{\sqrt[8]{3}}{\sqrt[4]{7\sqrt{3}+12}}+\dfrac{\sqrt[8]{6}}{\sqrt[4]{120+49\sqrt{6}}}$.

18. $\sqrt{a+b-c-2\sqrt{ab-bc}}$.

19. $\sqrt{a+b-\sqrt{c(2a+2b-c)}}$.

20. $\sqrt{3a\sqrt{-1}}+\sqrt{5a\sqrt{-1}}$.

21. $\sqrt{x-y-\sqrt{z(2x-2y-z)}}$.

22. $\sqrt{x+y-\sqrt{-z(2x+2y+z)}}$.

23. $\sqrt{(a+b)^2-2(a+b)\sqrt{ab}}$.

24. $\sqrt{xy+z^2+\sqrt{(x+z)(x-z)(y+z)(y-z)}}$.

8. QUADRATIC EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

Concrete Problems.

1. The sum of two numbers is 9, and their product is 20. Find the numbers.

2. The product of the sum and difference of two numbers is 16, and the sum of their squares is 34. Find the numbers.

3. The sum of two numbers multiplied by their product, is 56; and their difference multiplied by their product, is 42. Find the numbers.

4. The greater of two numbers multiplied by the square root of their sum, is 36; the less multiplied

by the square root of their sum, is 28. Find the numbers.

5. The sum of two numbers multiplied by the square root of the greater, is 39; and the sum, multiplied by the square root of the less, is 26. Find the numbers.

6. Find two numbers, the sum of whose square roots is 5; and the square of the difference of whose square roots equals the difference of their square roots increased by 6.

7. Find two numbers, such that their sum being divided by their difference, and the quotient increased by its reciprocal, shall be $\frac{10}{3}$; and the difference of whose squares shall be 48.

8. The product of two numbers is 20 times their difference, and the sum of their squares is 41. Find the numbers.

9. Find two numbers, such that their difference multiplied by the difference of their squares, is 40; and their sum multiplied by the sum of their squares, is 520.

10. Find two numbers, such that their sum plus their product, is 39; and the sum of their squares diminished by their sum, is 54.

11. Find two numbers, such that their product is $\frac{2}{5}$ of their sum, and the sum of their squares is 68.

12. The product of two numbers added to their difference, is 13, and the sum of their squares equals 10 times their difference, less 1. Find the numbers.

13. Divide 12 into three parts, such that the sum of the squares of two of the parts shall equal the square of the other part, and the sum of the squares of all the parts shall equal 50.

14. If a man had worked 5 days less, and had received \$4 less per day, he would have earned \$12; if he had worked 4 days less, and had earned \$5 less per day, his wages would have been \$7. How much did he earn?

15. A party of emigrants had sufficient food to last 6 days. If there had been 3 fewer, and each had eaten 1 pound less per day, the food would have lasted 12 days. If there had been 6 more, and each had eaten 1 pound more per day, it would have lasted only 3 days. How many pounds of food had they?

16. Find two numbers whose difference is 208, and the difference of whose cube roots is 4.

17. Find two numbers whose sum is 35, and the sum of whose cube roots is 5.

18. Find two numbers whose product equals $\frac{8}{10}$ of the sum of their squares, and the difference of whose squares is 96 times the quotient of the less divided by the greater.

19. If the numerator of a fraction be increased by 1, and its denominator decreased by 1, the resulting fraction will be the reciprocal of the first; and, if $\frac{7}{40}$ be subtracted from the fraction, the remainder will equal $\frac{1}{2}$ the reciprocal of the original fraction. Find the fraction.

20. A father's age, diminished by the square of his son's age, leaves a remainder equal to three times the son's age. Four years ago, the square root of the father's age was 6 times the square root of the son's age. How old is each?

21. If A be younger than B, and to the square of A's age the age of B be added, the sum will equal

the square of the difference of their ages; and, if to the square of B's age, the age of A be added, the sum will equal $\frac{1}{2}$ the square of the sum of their ages, less 10. Find the ages.

22. The cube of A's age, divided by the square of B's, equals $\frac{8}{3}$; the cube of B's, divided by the square of A's, equals 405. Find their ages.

23. A father has four children, whose ages differ successively by 2 years. Four years ago, the father's age was $\frac{3}{2}$ of the united ages of his children. Sixteen years hence, his age will be 5 times the square root of their united ages. Find the age of each.

24. A father has four children, whose ages differ successively by 2 years. Six years ago, his age was 3 times their united ages; 10 years hence, the square root of his age will be $\frac{1}{10}$ of their united ages. Find the age of each.

25. A number is equal to three times the product of its two digits, and, if 18 be added to it, the number will be reversed. Find the number.

26. If a number of two digits be increased by 1, and the sum be divided by the sum of its digits, the quotient will be 6; if the digits be in reversed order, and the square of the number thus resulting be subtracted from the square of the original number, the remainder will be 99 times a number which is 1 more than the double of the units' digit. Find the number.

27. Find a number of two digits which, being divided by the reverse of the number, gives a quotient $\frac{8}{5}$; if 3 times the sum of its digits be divided by the number, and the quotient be subtracted from the number, the remainder will be 26.

28. Find a number of two digits, such that if the square of the units' digit be divided by the square of the tens', the quotient will be the same as if the units' digit were divided by the square root of the tens' digit; and, if 36 be added to the number, the sum will equal the number reversed.

29. Find two numbers whose product, added to the sum of their squares, is 63, and the difference of whose squares is 27.

30. A man has a rectangular garden, on the inner edge of whose sides he makes a path 6 feet wide. The contents of the garden being 9,600 square feet, and of the path 2,256 square feet, find the lengths of the sides of the garden.

31. A rectangular stone was cut down $\frac{1}{24}$ of its length and $\frac{1}{8}$ of its breadth, and thus lost $7\frac{3}{4}$ square feet in area, and 2 feet in perimeter. What was its original length and breadth?

32. A certain number of horses, each carrying a load an hour, requires 5 hours to move a quantity of merchandise. If there were one horse more, and if each horse should carry 300 pounds less at a load, it would require $6\frac{2}{3}$ hours. If there were 1 horse less, and if each horse should carry 300 pounds more, it would require $4\frac{4}{15}$ hours. How many horses are there? and how much does each carry at a load?

33. A began a piece of work alone, and worked for $\frac{1}{2}$ the time it would take B to do the entire work. B then finished the job. If both had worked together after B began, the task would have been completed 1 day sooner, and A would have done, in all, $\frac{2}{3}$ of what he left for B. How long would it require each to do the work separately?

34. A and B engage to do a piece of work for $18. B works $\frac{1}{3}$ as many days as it would take A to do the whole, and then leaves. A then works $\frac{1}{4}$ as many days as it would take B to do the whole. B then returns, and together they finish the work in $1\frac{2}{3}$ days. Had A worked $\frac{1}{3}$ as many days as it would take B to do the whole, and then left; and had B continued for $\frac{1}{4}$ as many days as it would take A to do the whole, they could together have finished the remainder in $1\frac{9}{10}$ days. How much should each receive for his work?

35. Two men set out, at the same time, from two places, and travel at a uniform rate towards each other. When they meet it is found that A has traveled 20 miles more than B, and that if they continue at the same rates, they will reach the place from which the other set out in 16 and 25 hours respectively, from the time of meeting. Find the distance between the two places, and the rate of each.

36. A went from M to N, a distance of 24 miles, in 5 hours, riding $\frac{1}{2}$ the way and walking the other half. Returning, he rode $\frac{1}{2}$ the way at a rate 2 miles faster than when he went out, and walked the other half at a rate 2 miles slower than before. He reached M in $7\frac{1}{2}$ hours. Find his rates of riding and walking.

37. A sets out to go from M to N at the same time that B starts from N to M. A's time in performing the whole journey exceeds B's by the quotient of 180 divided by the product of their rates. They meet at the end of 9 hours. If, after meeting, B should turn about and travel with A $4\frac{1}{2}$ hours at A's rate, he could then reach M at the same time that A reaches N. Find the distance MN, and the two rates.

38. A and B set out at the same time from C and D, respectively, to travel towards each other. On meeting, it appears that A has traveled 32 yards more than B. A reaches D 8 seconds before B reaches C; but if, from the point of meeting, B had increased his speed $8\frac{1}{3}$ yards per second, they would have reached their destinations at the same time. What is the distance CD?

39. A sets out from C towards D at the same time that B leaves D for C. On meeting, it appears that A has gone 12 miles more than B, and to reach D takes A 2 hours less time than he has already traveled. If, after meeting, B had increased his speed 3 miles per hour, B would have reached C 1 hour before A reached D. What is the distance CD?

XIX. HIGHER EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

Solve:

1. $x^2+y^2+z^2=14$
   $x+y=6-z$
   $xy=6$

2. $xy=15$
   $x+y+z=9$
   $x^2+y^2=33+z^2$

3. $y^3z=\frac{8}{3}x$
   $\frac{y}{x^3z}=\frac{2}{27}$
   $\frac{yz^2}{x^2}=\frac{2}{9}$

4. $x^2+y^2=16+z^2$
   $x=4+y+z$
   $xy=8$

5. $xyz=-3$
   $yzt=3$
   $xzt=9$
   $xyt=-9$

6. $x+z=4+y$
   $x(z-y)=3$
   $yz=20$

$$x^2+xy+xz=8$$
7. $y^2+yz+xy=4$
$$z^2+xz+yz=-8$$

$$x^2-(y-z)^2=48$$
8. $y^2-(x-z)^2=16$
$$z^2-(x-y)^2=12.$$

$$x^2+y^2+z^2=70$$
9. $x+y+z=14$
$$x(y+z)=33$$

10. Find rational values in
$$xyz=24$$
$$xy+yz+xz=26$$
$$x^2+y^2=1+2xy.$$

11. Find the rational values only in
$$x^2+y^2-z^2=5$$
$$x+y-z=3$$
$$yz=30$$

$$x+y+z=2$$
12. $xy+xz+yz=-23$
$$xyz=-60$$

$$xy+xz+yz=11$$
13. $3xy-xz+2yz=15$
$$4xy+3xz-yz=11$$

$$(x+y)(x+z)=56$$
14. $y^2+y(x+z)+xz=40$
$$z^2+xz+yz+xy=35$$

$$\frac{1}{xy}+\frac{1}{xz}=\frac{1}{8}$$
15. $\frac{1}{xy}+\frac{1}{yz}=\frac{1}{6}$
$$\frac{1}{xz}+\frac{1}{yz}=\frac{5}{24}$$

$$x^3y^2z=486$$
16. $x^{-1}yz^2=\frac{1}{2}$
$$\frac{x^2y}{z}=108$$

$$x+y+z=6$$
17. $x^2+y^2+z^2=14$
$$xyz=6$$

$$x^2+y^2+z^2=26$$
18. $xy+xz+yz=-11$
$$x=y-z$$

$$x+y+z=14$$
19. $x^2+y^2+z^2=84$
$$xz+yz=\frac{3xy}{4}$$

$$x^4+y^4+z^4=18$$
20. $x^2+y^2+z^2=6$
$$xyz=2$$

$$x^2+y^2=7-xy$$
21. $x^2+z^2=19-xz$
$$y^2+z^2=13-yz$$

22.
$$\begin{aligned} &x^3+y^3+z^3=1\\ &x+y+z=1\\ &xyz=-4 \end{aligned}$$

23.
$$\begin{aligned} &x^3+y^3+z^3=8\\ &x^2+y^2+z^2=6\\ &xy+xz+yz=-1 \end{aligned}$$

24.
$$\begin{aligned} &(x+t)(y+z)=25\\ &(x+z)(y+t)=24\\ &(x+y)(z+t)=21\\ &x^2+y^2+z^2+t^2=30 \end{aligned}$$

25.
$$\begin{aligned} &x^3+y^3+z^3=216\\ &3x=y+z\\ &2y=x+z \end{aligned}$$

26.
$$\begin{aligned} &\frac{xyz}{x+y}=4\tfrac{4}{5}\\ &\frac{xyz}{x+z}=6\\ &\frac{xyz}{y+z}=8 \end{aligned}$$

27.
$$\begin{aligned} &(x^2+3)=\frac{16}{\sqrt{y^2+7}}\\ &\sqrt{z^2-9}=\frac{64}{7+y^2}\\ &(z^2-9)\sqrt{x^2+3}=32 \end{aligned}$$

28.
$$\begin{aligned} &(x+a)\sqrt{y+b}=m^2\\ &(y+b)\sqrt{z+c}=n^2\\ &(z+c)\sqrt{x+a}=r^2 \end{aligned}$$

29.
$$\begin{aligned} &x^2+y^2+z^2=6\,(x+y+z)=84\\ &xy=z^2 \end{aligned}$$

30.
$$\begin{aligned} &\left[\frac{1}{x}+\frac{1}{y}+6\right]\sqrt{\frac{1}{x}+\frac{1}{z}+1}=48\\ &\left[\frac{1}{x}+\frac{1}{z}+1\right]\sqrt{\frac{1}{y}+\frac{1}{z}-2}=18\\ &\left[\frac{1}{y}+\frac{1}{z}-2\right]\sqrt{\frac{1}{x}+\frac{1}{y}+6}=16 \end{aligned}$$

31.
$$\begin{aligned} &(x+y+z)+x^2+y^2+z^2=8\\ &xy+xz+yz=-1\\ &x^2y^2z^2=6+xyz \end{aligned}$$

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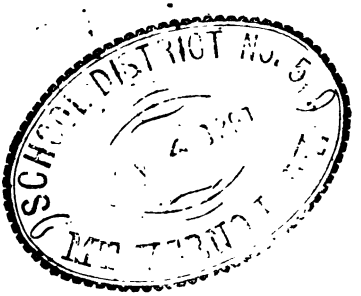
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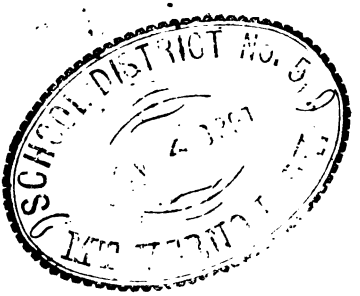
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