Problems in

# Differential Geometry and Topology

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Mir Publishers Moscow

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# СБОРНИК ЗАДАЧ ПО ДИФФЕРЕНЦИАЛЬНОЙ ГЕОМЕТРИИ И ТОПОЛОГИИ

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**TO THE READER**

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make. Our address is: Mir Publishers, 2 Pervy Pizhsky Pereulok, I-110, GSP, Moscow, 129820, USSR.

*На английском языке*

# Preface

This book of problems is the result of a course in differential geometry and topology, given at the mechanics-and-mathematics department of Moscow State University. It contains problems practically for all sections of the seminar course. Although certain textbooks and books of problems indicated in the bibliography list were used in preparation of this volume, a considerable number of the problems were prepared for this book expressly.

The material is distributed over the sections as in textbook [3]. Some problems, however, touch upon topics outside the lectures. In these cases, the corresponding sections are supplied with additional definitions and explanations.

In conclusion, the authors express their sincere gratitude to all those who helped to publish this work.

# 1
# Application of Linear Algebra to Geometry

**1.1.** Prove that a vector set $\mathbf{a}_1, \ldots, \mathbf{a}_k$ in a Euclidean space is linearly independent if and only if

$$\det \|(\alpha_i, \alpha_j)\| \neq 0.$$

**1.2.** Find the relation between a complex matrix $A$ and the real matrix $rA$ of the complex linear mapping.

**1.3.** Find the relations between

$$\det A \text{ and } \det rA, \ \operatorname{Tr} A \text{ and } \operatorname{Tr} rA, \ \det (A - \lambda E) \text{ and } \det (rA - \lambda A).$$

**1.4.** Find the relation between the invariants of the matrices $A$, $B$ and $A \oplus B$, $A \otimes B$.

Consider the cases of det and Tr.

**1.5.** Prove the formula

$$\det e^A = e^{\operatorname{Tr} A}.$$

**1.6.** Prove that

$$e^A e^B = e^{(A+B)} + C'[A, B]C''$$

for a convenient choice of the matrices $C'$ and $C''$, where $[A, B] = AB - BA$.

**1.7.** Prove that if $A$ is a skewsymmetric matrix, then $e^A$ is an orthogonal matrix.

**1.8.** Prove that if $A$ is a skewhermitian matrix, then $e^A$ is a unitary matrix.

**1.9.** Prove that if $[A, A^*] = 0$, then the matrix $A$ is similar to a diagonal one.

**1.10.** Prove that a unitary matrix is similar to a diagonal one with eigenvalues whose moduli equal unity.

**1.11.** Prove that a hermitian matrix is similar to a diagonal one with real eigenvalues.

**1.12.** Prove that a skewhermitian matrix is similar to a diagonal one with imaginary eigenvalues.

**1.13.** Let $A = \|a_{ij}\|$ be a matrix of a quadratic form, and $D_k = \det \|a_{ij}\|_{1 \le i,j \le k}$.

Prove that $A$ is positive definite if and only if for all $k$, $1 \leqslant k \leqslant n$, the inequalities $D_k > 0$ are valid.

**1.14.** With the notation of the previous problem, prove that a matrix $A$ is negative definite if and only if for all $k$, $1 \leqslant k \leqslant n$, the inequality $(-1)^k D_k > 0$ holds.

**1.15.** Put $\|A\|^2 = \sum_{i,k} |a_{ik}|^2$. Prove the inequalities

$$\|A + B\| \leqslant \|A\| + \|B\|,$$

$$\|\lambda A\| \leqslant |\lambda| \cdot \|A\|,$$

$$\|AB\| \leqslant \|A\| \cdot \|B\|.$$

**1.16.** Prove that if $A^2 = E_n$, then the matrix $A$ is similar to the matrix $\begin{pmatrix} E_k & 0 \\ 0 & -E_l \end{pmatrix}$, $k + l = n$.

**1.17.** Prove that if $A^2 = -E$, then the order of the matrix $A$ is $(2n \times 2n)$, and it is similar to a matrix of the form $\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$.

**1.18.** Prove that if $A^2 = A$, then the matrix $A$ is similar to a matrix of the form $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$.

**1.19.** Prove that varying continuously a quadratic form from the class of non-singular quadratic forms does not alter the signature of the form.

**1.20.** Prove that varying continuously a quadratic form from the class of quadratic forms with constant rank does not alter its signature.

**1.21.** Prove that any motion of the Euclidean plane $\mathbf{R}^2$ can be resolved into a composition of a translation, reflection in a straight line, and rotation about a point.

**1.22.** Prove that any motion of the Euclidean space $\mathbf{R}^3$ can be resolved into a composition of a translation, reflection in a plane and rotation about a straight line.

**1.23.** Generalize Problems 1.21 and 1.22 for the case of the Euclidean space $\mathbf{R}^n$.

# 2
# Systems of Coordinates

A set of numbers $q^1, q^2, \dots, q^n$ determining the position of a point in the space $\mathbf{R}^n$ is called its curvilinear coordinates. The relation between the Cartesian coordinates $x_1, x_2, \dots, x_n$ of this point and curvilinear coordinates is expressed by the equalities

$$x_s = x_s(q^1, q^2, \dots, q^n), \tag{1}$$

or, in vector form, by

$$\mathbf{r} = \mathbf{r}(q^1, q^2, \dots, q^n),$$

where $\mathbf{r}$ is a radius vector. Functions (1) are assumed to be continuous in their domain and to have continuous partial derivatives up to the third order inclusive. They must be uniquely solvable with respect to $q^1$, $q^2, \dots, q^n$; this condition is equivalent to the requirement that the Jacobian

$$J = \left| \frac{\partial x_s}{\partial q^k} \right| \tag{2}$$

should not be equal to zero. The numeration of the coordinates is assumed to be chosen so that the Jacobian is positive.

Transformation (1) determines $n$ families of the coordinate hypersurfaces $q^r = q^r_0$. The coordinate hypersurfaces of one and the same family do not intersect each other if condition (2) is fulfilled.

Owing to condition (2), any $n - 1$ coordinate hyperplanes which belong to different families meet in a certain curve. They are called coordinate curves or coordinate lines.

The vectors $\mathbf{r}_k = \dfrac{\partial \mathbf{r}}{\partial q^k}$ are directed as the tangents to the coordinate lines. They determine the infinitesimal vector

$$d\mathbf{r} = \sum_{k=1}^{n} \mathbf{r}_k dq^k$$

in a neighbourhood of the point $M(q^1, q^2, \dots, q^n)$. The square of its length, if expressed in terms of curvilinear coordinates, can be found from the equality

$$ds^2 = (d\mathbf{r}, d\mathbf{r}) = \left( \sum_{s=1}^{n} \mathbf{r}_s dq^s, \sum_{k=1}^{n} \mathbf{r}_k dq^k \right) = \sum_{s,\, k=1}^{n} g_{sk} dq^s dq^k,$$

where (,) is the scalar product defined in $\mathbf{R}^n$.

The quantities $g_{sk} = g_{ks} = (\mathbf{r}_s, \mathbf{r}_k)$ define a metric in the adopted coordinate system.

An orthogonal curvilinear coordinate system is one for which

$$g_{sk} = (\mathbf{r}_s, \mathbf{r}_k) = \begin{cases} 0, & s \neq k \\ H_s^2, & s = k \end{cases}.$$

The quantities $H_s^2$ are called the Lamé coefficients. Thev are equal to the moduli of the vectors $\mathbf{r}_s$:

$$H_s = |\mathbf{r}_s| = \sqrt{\left(\frac{\partial x_1}{\partial q^s}\right)^2 + \left(\frac{\partial x_2}{\partial q^s}\right)^2 + \ldots + \left(\frac{\partial x_n}{\partial q^s}\right)^2}.$$

The square of the linear element in orthogonal curvilinear coordinates is given by the expression

$$ds^2 = H_1^2 dq^{1^2} + H_2^2 dq^{2^2} + \ldots + H_n^2 dqn^2.$$

**2.1.** Calculate the Jacobian $J = \left|\frac{\partial x_s}{\partial q^k}\right|$ of transition from Cartesian coordinates $(x_1, \ldots, x_n)$ to orthogonal curvilinear coordinates $(q^1, q^2, \ldots, q^n)$ in the space $\mathbf{R}^n$.

**2.2.** Calculate the gradient grad $f$ of the function $f: \mathbf{R}^3 \to \mathbf{R}$ in an orthogonal curvilinear coordinate system.

**2.3.** Calculate the divergence div $\mathbf{a}$ of a vector $\mathbf{a} \in \mathbf{R}^3$ in an orthogonal curvilinear coordinate system.

**2.4.** Find the expression for the Laplace operator $\Delta f$ of the function $f: \mathbf{R}^3 \to \mathbf{R}$ in an orthogonal curvilinear coordinate system.

**2.5.** *Cylindrical coordinates in* $\mathbf{R}^3$

$$q^1 = r, \quad q^2 = \varphi, \quad q^3 = z$$

are related to Cartesian coordinates by the formulae

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z.$$

(a) Find the coordinate surfaces of cylindrical coordinates.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator in cylindrical coordinates.

**2.6.** *Spherical coordinates in* $\mathbf{R}^3$

$$q^1 = r, \quad q^2 = \theta, \quad q^3 = \varphi$$

are related to rectangular coordinates by the formulae

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

(a) Find the coordinate surfaces of spherical coordinates.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator in spherical coordinates.

are defined by the formulae

$$x = c\lambda\mu, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}, \quad z = z,$$

where $c$ is a scale factor.

(a) Find the coordinate surfaces of elliptic coordinates.

(b) Compute the Lamé coefficients.

**2.8.** *Parabolic coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = z$$

are related to Cartesian by the formulae

$$x = \frac{1}{2}(\mu^2 - \lambda^2), \quad y = \lambda\mu, \quad z = z.$$

(a) Express parabolic coordinates in terms of cylindrical.

(b) Find the coordinate surfaces of parabolic coordinates.

(c) Compute the Lamé coefficients.

**2.9.** *Ellipsoidal coordinates in* $\mathbf{R}^3$ are introduced by the equations $(a > b > c)$:

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1 \ (\lambda > -c^2) \ \text{(ellipsoid)},$$

$$\frac{x^2}{a^2 + \mu} + \frac{y^2}{b^2 + \mu} + \frac{z^2}{c^2 + \mu} = 1 \ (-c^2 > \mu > -b^2) \ \text{(hyperboloid of}$$

one sheet),

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} + \frac{z^2}{c^2 + \nu} = 1 \ (-b^2 > \nu > -a^2) \ \text{(hyperboloid of}$$

two sheets).

Only one set of values $\lambda$, $\mu$, $\nu$ corresponds to each point $(x, y, z) \in \mathbf{R}^3$.

The parameters

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = \nu$$

are called ellipsoidal coordinates.

(a) Express Cartesian coordinates $x$, $y$, $z$ in terms of ellipsoidal coordinates $\lambda$, $\mu$, $\nu$.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator in terms of ellipsoidal coordinates.

**2.10.** *Degenerate ellipsoidal coordinates* $(\alpha, \beta, \varphi)$ *in* $\mathbf{R}^3$ *for a prolate ellipsoid of revolution are defined by the formulae*

$$x = c \sin\beta \cos\varphi, \quad y = c \sinh\alpha \sin\beta \sin\varphi, \quad z = c \cosh\alpha \cos\beta,$$

where $c$ is a scale factor, $0 \leqslant \alpha < \infty$, $0 \leqslant \beta < \pi$, $-\pi < \varphi \leqslant \pi$.

(a) Find the coordinate surfaces in this coordinate system.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator.

**2.11.** *Degenerate ellipsoidal coordinate system* $(\alpha, \beta, \varphi)$ *in* $\mathbf{R}^3$ *for an oblate ellipsoid of revolution* is defined by the formulae

$$x = c \cosh\alpha \sin\beta \cos\varphi, \quad y = c \cosh\alpha \sin\beta \sin\varphi,$$

$$z = c \cosh\alpha \cos\varphi,$$

$$0 \leqslant \alpha < \infty, \quad 0 \leqslant \beta \leqslant \pi, \quad -\pi < \varphi \leqslant \pi.$$

(a) Find the coordinate surfaces for this coordinate system.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator.

**2.12.** *Toroidal coordinate system* $(\alpha, \beta, \varphi)$ *in* $\mathbf{R}^3$ *is defined by the formulae*

$$x = \frac{c \sinh\alpha \cos\varphi}{\cosh\alpha - \cos\beta}, \; y = \frac{c \sinh\alpha \sin\varphi}{\cosh\alpha - \cos\beta}, \; z = \frac{c \sin\beta}{\cosh\alpha - \cos\beta},$$

where $c$ is a scale factor, $0 \leqslant \alpha < \infty$, $-\pi < \beta \leqslant \pi$, $-\pi < \varphi \leqslant \pi$.

(a) Find the coordinate surfaces in a toroidal coordinate system.

(b) Compute the Lamé coefficients.

(c) Find expression for the Laplace operator.

**2.13.** *Bipolar coordinates in* $\mathbf{R}^3$

$$q^1 = \alpha, \quad q^2 = \beta, \quad q^3 = z$$

are related to Cartesian coordinates $x, y, z$ by the formulae

$$x = \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta}, \quad y = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}, \quad z = z,$$

where $a$ is a scale factor.

Compute the Lamé coefficients for a bipolar coordinate system.

**2.14.** *Bispherical coordinates in* $\mathbf{R}^3$

$$q^1 = \alpha, \quad q^2 = \beta, \quad q^3 = \varphi$$

are defined by the formulae

$$x = \frac{c \sin\alpha \cos\varphi}{\cosh\beta - \cos\alpha}, \quad y = \frac{c \sin\alpha \sin\varphi}{\cosh\beta - \cos\alpha}, \quad z = \frac{c \sinh\beta}{\cosh\beta - \cos\alpha},$$

where $c$ is a constant factor, $0 \leqslant \alpha < \beta$, $-\infty < \beta < \infty$, $-\pi < \varphi \leqslant \pi$.

These formulae can be written shorter:

$$z + i\varrho = ci \cot \frac{\alpha + i\beta}{2} \quad (\varrho = \sqrt{x^2 + y^2}).$$

(a) Find the coordinate surfaces in a bispherical coordinate system.
(b) Compute the Lamé coefficients.
(c) Find expression for the Laplace operator.

**2.15.** *Prolate spheroidal coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = \varphi$$

are defined by the formulae

$$x = c\lambda\mu, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}\cos\varphi,$$

$$z = \sqrt{c(\lambda^2 - 1)(1 - \mu^2)}\sin\varphi,$$

where $\lambda \geqslant 1$, $-1 \leqslant \mu \leqslant 1$, $0 \leqslant \varphi \leqslant 2\pi$, and $c$ is a constant factor.
Compute the Lamé coefficients for this coordinate system.

**2.16.** *Oblate spheroidal coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu, \quad q^3 = \varphi$$

are defined by the formulae

$$x = c\lambda\mu \sin\varphi, \quad y = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}, \quad z = c\lambda\mu\cos\varphi,$$

$$\lambda \geqslant 1, \quad -1 \leqslant \mu \leqslant 1, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Compute the Lamé coefficients for an oblate spheroidal coordinate system.

**2.17.** *Paraboloidal coordinates in* $\mathbf{R}^3$

$$q^1 = \lambda, \quad q^2 = \mu; \quad q^3 = \varphi$$

are defined by the relations

$$x = \lambda\mu\cos\varphi, \quad y = \lambda\mu\sin\varphi, \quad z = \frac{1}{2}(\lambda^2 - \mu^2).$$

(a) Compute the Lamé coefficients for a paraboloidal coordinate system.
(b) Find the coordinate surfaces.

**2.18.** Let $H_1$, $H_2$, $H_3$ be the Lamé coefficients for a certain curvilinear coordinate system in $\mathbf{R}^3$.
Prove the relations

$$(1) \quad \frac{\partial}{\partial q^1}\frac{1}{H_1}\frac{\partial H_2}{\partial q^1} + \frac{\partial}{\partial q^2}\frac{1}{H_2}\frac{\partial H_1}{\partial q^2} + \frac{1}{H_3^2}\frac{\partial H_1}{\partial q^3}\frac{\partial H_2}{\partial q^3} = 0;$$

(2) $\dfrac{\partial}{\partial q^2}\dfrac{1}{H_2}\dfrac{\partial H_3}{\partial q^2}+\dfrac{\partial}{\partial q^3}\dfrac{1}{H_3}\dfrac{\partial H_2}{\partial q^3}+\dfrac{1}{H_1^2}\dfrac{\partial H_2}{\partial q^1}\dfrac{\partial H_3}{\partial q^1}=0;$

(3) $\dfrac{\partial}{\partial q^3}\dfrac{1}{H_3}\dfrac{\partial H_1}{\partial q^3}+\dfrac{\partial}{\partial q^1}\dfrac{1}{H_1}\dfrac{\partial H_3}{\partial q^1}+\dfrac{1}{H_2^2}\dfrac{\partial H_3}{\partial q^2}\dfrac{\partial H_1}{\partial q^2}=0;$

(4) $\dfrac{\partial^2 H_1}{\partial q^2\partial q^3}=\dfrac{1}{H_3}\dfrac{\partial H_3}{\partial q^2}\dfrac{\partial H_1}{\partial q^3}+\dfrac{1}{H_2}\dfrac{\partial H_1}{\partial q^2}\dfrac{\partial H_2}{\partial q^3};$

(5) $\dfrac{\partial^2 H_2}{\partial q^3\partial q^1}=\dfrac{1}{H_1}\dfrac{\partial H_1}{\partial q^3}\dfrac{\partial H_2}{\partial q^1}+\dfrac{1}{H_3}\dfrac{\partial H_2}{\partial q^3}\dfrac{\partial H_3}{\partial q^1};$

(6) $\dfrac{\partial^2 H_3}{\partial q^1\partial q^2}=\dfrac{1}{H_2}\dfrac{\partial H_2}{\partial q^1}\dfrac{\partial H_3}{\partial q^2}+\dfrac{1}{H_1}\dfrac{\partial H_3}{\partial q^1}\dfrac{\partial H_1}{\partial q^2}.$

**2.19.** Prove that if functions $H_1(q^1, q^2, q^3)$, $H_2(q^1, q^2, q^3)$, $H_3(q^1, q^2, q^3)$ of class $C^3$ satisfy the relations of the previous problem, then they are the Lamé coefficients for a certain transformation

$x_s = x_s(q^1, q^2, q^3)$, $s = 1, 2, 3$.

# 3
# Riemannian Metric

**3.1.** Prove that the metric $ds^2 = dx^2 + f(x)dy^2$, $0 < f(x) < \infty$ can be transformed to the form $ds^2 = g(u, v)(du^2 + dv^2)$ (isothermal coordinates).

**3.2.** Prove that local isothermal coordinates can be defined on any real analytic surface $M^2$. Find the conformal representation of the metric $ds^2$.

**3.3.** Mercator's projection is defined as follows: rectangular coordinates $(x, y)$ are defined on a map so that a constant bearing line (where the compass needle remains undeflected) on the earth's surface is put into correspondence with a straight line on the map.

(a) Prove that to a point on the surface of the globe with spherical coordinates $(\theta, \varphi)$ on the map, there corresponds, in Mercator's projection, the point with coordinates $x = \varphi$, $y = \ln \cot\theta/2$.

(b) How can the metric on the terrestrial globe be written in terms of the coordinates $(x, y)$?

**3.4.** Prove that the metric $ds^2$ on the standard hyperboloid of two sheets which is embedded in the pseudo-Euclidean space $\mathbf{R}^3$ coincides with the metric on the Lobachevski plane.

**3.5.** Write the metric on the sphere $S^2$ in complex form.

**3.6.** Find a metric on the two-dimensional space of velocities in relativity theory.

**3.7.** Change the coordinates in the previous problem so that $v \to \tanh\chi$ (where $v$ is the velocity of the moving point).

**3.8.** Write the metric of the previous problem in polar coordinates for the unit circle.

**3.9.** Calculate the length of a circumference and the area of a circle on (a) the Euclidean plane, (b) a sphere, (c) the Lobachevski plane.

**3.10.** Let the Lobachevski plane be realized as the upper half-plane of the Euclidean plane. We call Euclidean semicircumferences with centres on the axis $Ox$ and Euclidean half-lines resting upon the axis $Ox$ and orthogonal to it "straight lines" of the Lobachevski plane. We call a figure formed by three points and the segments of "the straight lines" joining them a triangle in the Lobachevski plane.

Prove that the sum of the angles of a triangle in the Lobachevski plane is less than $\pi$.

**3.11.** (Continuation of Problem 3.10.) Let $ABC$ be an arbitrary triangle in the Lobachevski plane, $a$, $b$, $c$ the non-Euclidean lengths of the sides $BC$, $AC$, $AB$, and $\alpha$, $\beta$, $\gamma$ the values of its angles at the vertices $A$, $B$, $C$. Prove the following relations:

$$(1)\ \cosh a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma};$$

$$(2)\ \cosh b = \frac{\cos\beta + \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha};$$

$$(3)\ \cosh c = \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

**3.12.** (Continuation of Problem 3.11.) Prove the analogue of the law of sines for the Lobachevski plane:

$$\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma} = \frac{\sqrt{Q}}{\sin\alpha\sin\beta\sin\gamma},$$

where $Q = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1$.

**3.13.** (Continuation of Problem 3.12.) Prove the following formulae expressing the angles of a triangle in the Lobachevski plane in terms of its sides:

$$(1)\ \cos\alpha = \frac{\cosh b\cosh c - \cosh a}{\sinh b\sinh c},$$

(2) $\cos\beta = \dfrac{\cosh c \cosh a - \cosh b}{\sinh c \sinh a}$,

(3) $\cos\gamma = \dfrac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$.

**3.14.** (Continuation of Problem 3.13.)
Assume that $\gamma = \pi/2$, i.e., the triangle $ABC$ is right. Prove the following relations:

(1) $\sinh a = \sinh c \sin\alpha$;

(2) $\tanh a = \tanh c \cos\beta$;

(3) $\tanh a = \sinh b \tan\alpha$;

(4) $\cosh c = \cosh a \cosh b$;

(5) $\cosh c = \cot\alpha \cot\beta$;

(6) $\cosh a = \cos\alpha/\sin\beta$.

**3.15.** Let $ABC$ be a spherical triangle on a sphere of radius $R$, $\alpha$, $\beta$, $\gamma$ the values of the angles at the vertices $A$, $B$, $C$ and $a$, $b$, $c$ the lengths of the sides $BC$, $AC$, $AB$. Prove the following relationship

$$\cos\frac{a}{R} = \cos\frac{b}{R}\cos\frac{c}{R} + \sin\frac{b}{R}\sin\frac{c}{R}\cos\alpha.$$

# 4 Theory of Curves

**4.1.** Let $C$ be a plane curve, $M_0$ a point of the curve $C$, and $XOY$ a rectangular system of coordinates given in the plane of the curve. Denote the points of intersection of the tangent and the normal to this curve with the axis $OX$ by $T$ and $N$, respectively. Let $P$ be the projection of the point $M_0$ onto the axis $OX$.

(a) Find the equation of the curve $C$ if its subnormal $PN$ is constant and equal to $a$.

(b) Find the equation of the curve $C$ if its subtangent $PT$ is constant and equal to $a$.

(c) Find the equation of the curve $C$ if the length of its normal $M_0N$ is constant and equal to $a$ (for any point $M_0$ on the curve).

**4.2.** Find the equation of the curve $C$ whose tangent $MT$ is constant in length and equal to $a$.

**4.3.** An arbitrary ray $OE$ intersects the circumference

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

and a tangent to it passing through the point $C$ which is diametrically opposite to $O$ at points $D$ and $E$. Straight lines are drawn through the points $D$ and $E$ parallel to the axes $Ox$ and $Oy$, respectively, to meet each other at a point $M$. Set up the equation of the curve formed by such points $M$ (witch of Agnesi).

**4.4.** A point $M$ moves uniformly along a straight line $ON$ which rotates uniformly around a point $O$. Form the equation of the path of the point $M$ (Archimedes' spiral).

**4.5.** A straight line $OL$ rotates around a point $O$ with constant angular velocity $\omega$. A point $M$ moves along the straight line $OL$ with a velocity which is proportional to the distance $|OM|$. Form the equation of the path described by the point $M$ (logarithmic spiral).

**4.6.** A circle of radius $a$ rolls along a straight line without slipping. Set up the equation of the path of a point $M$ connected to the circle rigidly and placed at a distance $d$ from its centre (when $d = a$, this is a cycloid; when $d < a$, a curtate cycloid; and when $d > a$, a prolate cycloid).

**4.7.** A circumference of radius $r$ rolls without slipping along a circumference of radius $R$ and remains outside it. Form the equation of the path of a point $M$ of the rolling circumference (epicycloid).

**4.8.** A circumference of radius $r$ rolls without slipping along a circumference of radius $R$ and remains inside it. Construct the equation of the path of a point $M$ of the rolling circumference (hypocycloid).

**4.9.** Find a curve given by the equation $\mathbf{r} = \mathbf{r}(t)$, $c < t < d$, if it is known that $\mathbf{r}'(t) = \lambda(t)\mathbf{a}$, where $\lambda(t) > 0$ is a continuous function, and $\mathbf{a}$ is a constant nonzero vector.

**4.10.** Find a curve given by the equation $\mathbf{r} = \mathbf{r}(t)$, $-\infty < t < \infty$, if $\mathbf{r}''(t) = \mathbf{a}$ is a constant nonzero vector.

**4.11.** A vector function $\mathbf{r}(t)$ satisfies the differential equation $\mathbf{r}'' = [\mathbf{r}' \times \mathbf{a}]$, where $\mathbf{a}$ is a constant vector. Express (a) $[\mathbf{r}' \times \mathbf{r}'']^2$; (b) $(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')$ in terms of $\mathbf{a}$ and $\mathbf{r}'$.

**4.12.** Let $\gamma$ be a closed curve of class $C^1$. Prove that, for any vector $\mathbf{a}$, there is a point $x \in \gamma$ at which the tangent to $\gamma$ is orthogonal to $\mathbf{a}$.

**4.13.** Two points move in space so that the distance between them remains constant. Prove that the projections of their velocities onto the direction of the straight line joining these points are equal.

**4.14.** Prove that if a vector function $\mathbf{r}(t)$ is continuous on a segment $[a, b]$ together with its derivative $\mathbf{r}'$, and $\mathbf{r} \parallel \mathbf{r}'$, but $\mathbf{r}' \neq 0$ and $\mathbf{r} \neq 0$, then the hodograph of the vector function $\mathbf{r} = \mathbf{r}(t)$ is a straight line segment.

**4.15.** Prove that if a vector function $\mathbf{r} = \mathbf{r}(t)$ is continuous on a certain segment $[a, b]$ together with its two first derivatives $\mathbf{r}'$ and $\mathbf{r}''$, these derivatives are different from zero for all $t \in [a, b]$, and collinear, i.e., $\mathbf{r}' \parallel \mathbf{r}''$ for all $t \in [a, b]$, then the hodograph of the vector function $\mathbf{r} = \mathbf{r}(t)$ is a straight line segment.

**4.16.** A plane curve is given by the equation $\mathbf{r} = \{\varphi(t), t\varphi(t)\}$. Under what condition does this equation determine a straight line?

**4.17.** Find the function $r = r(\varphi)$, given that this equation describes a straight line in polar coordinates on the plane.

**4.18.** Prove that a point describes a plane path under the action of a central force $\mathbf{F} = F\mathbf{r}$.

**4.19.** A plane translational motion is given on the plane by the laws $\mathbf{r} = \mathbf{r}_1(t)$ and $\mathbf{r} = \mathbf{r}_2(t)$ of motion of the ends of a solid rod. Find the equation of the centre surface (a centre surface is the set of all points of intersection of straight lines passing through the ends of the rod and perpendicular to the directions of the velocities of its ends).

**4.20.** The set of instantaneous centres of rotation with respect to a moving rod is called a centrode in a plane translational motion (see the previous problem). Set up the equation of a centrode.

**4.21.** Prove that the linear velocity $\mathbf{v}$ of a point in any plane translational motion is determined by the relation $\mathbf{v} = \omega[\mathbf{r}]$, where $\mathbf{r}$ is the radius vector of the point $M(\mathbf{R})$ under consideration with respect to the instantaneous centre of rotation (see Problems 4.19, 4.20), and $[\mathbf{r}]$ is the vector obtained from $\mathbf{r}$ by rotation through $+\pi/2$. Express $\omega$ in terms of $\mathbf{r}_1$ and $\mathbf{r}_2$ and find the velocity $\mathbf{v}$ of the point $M(\mathbf{R})$.

**4.22.** The differential equation of the motion of a material particle $M$ is as follows:

$$\mathbf{r}'' = -\frac{\lambda \mathbf{r}}{r^3} \quad (\lambda > 0).$$

Prove, on the basis of this relation, that the point moves along a curve of the second order.

**4.23.** A material particle moves under the action of a central force $\mathbf{F} = F\mathbf{r}^0$. It follows from the result of Problem 4.18 that the motion takes place in a certain plane. Form the equation of the motion and the differential equation of the path in polar coordinates.

Consider the case

$$\mathbf{F} = -k\frac{\mathbf{r}m}{r^3} = -\frac{km}{r^2}\mathbf{r}^0.$$

**4.24.** The motion of an electron in a constant magnetic field is determined by the following differential equation

$$\mathbf{r}'' = [\mathbf{r}' \times \mathbf{H}], \quad \mathbf{H} = \text{const}.$$

Prove that the path is a helix.

4.25. Find the curves determined by the differential equation

$$\mathbf{r}' = [\omega \mathbf{r}].$$

4.26. Find the curves determined by the differential equation

$$\mathbf{r}' = [\mathbf{e} \times [\mathbf{r} \times \mathbf{e}]],$$

where $\mathbf{e}$ is a constant unit vector.

4.27. Find the curves determined by the differential equation

$$\mathbf{r}' = a\mathbf{e} + [\mathbf{e} \times \mathbf{r}],$$

where $a$ = const and $\mathbf{e}$ = const.

4.28. Find the curves determined by the differential equation

$$\bar{\mathbf{r}}' = \frac{1}{2} r^2 \bar{\mathbf{e}} - r(\bar{\mathbf{r}}, \ddot{\mathbf{e}}),$$

where $\mathbf{e}$ = const and $|\mathbf{e}| = 1$.

4.29. Form the equations of the tangent and normal to the following curves:

(1) $\mathbf{r} = \{a \cos t, b \sin t\}$ (ellipse);

(2) $\mathbf{r} = \left\{\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)\right\}$ (hyperbola);

(3) $\mathbf{r} = \{a \cos^3 t, a \sin^3 t\}$ (astroid);

(4) $\mathbf{r} = \{a(t - \sin t), a(1 - \cos t)\}$ (cycloid);

(5) $\mathbf{r} = \left\{\frac{1}{2} t^2 - \frac{1}{4} t^4, \frac{1}{2} t^2 + \frac{1}{3} t^3\right\}$

at the point $t = 0$;

(6) $\mathbf{r} = \{a\varphi \cos\varphi, a\varphi \sin\varphi\}$ (Archimedes' spiral).

4.30. At what angle do the curves $x^2 + y^2 = 8$ and $y^2 = 2x$ intersect?

4.31. At what angle do the curves

$$x^2 + y^2 = 8x, \quad y^2 = x^3/(2 - x)$$

intersect?

4.32. At what angle do the curves

$$x^2 = 4y, \quad y = 8/(x^2 + 4)$$

intersect?

4.33. Prove that the length of the segment of the tangent to the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

between the coordinate axes equals $a$.

$$2 \qquad a - \sqrt{a^2 - x^2}$$

equals $a$.

**4.35.** Prove that the cardioids

$$r = a(1 + \cos\varphi), \qquad r = a(1 - \cos\varphi)$$

are orthogonal.

**4.36.** Find the envelope of the family of straight lines joining the ends of pairs of conjugate diameters of an ellipse.

**4.37.** Find the envelope of the family of straight lines cutting a triangle of constant area off the sides of a right angle.

**4.38.** Find the envelope of the family of straight lines cutting segments of given area off a given parabola.

**4.39.** Find the envelope of the family of straight lines cutting a triangle of given perimeter off the sides of a given angle.

**4.40.** Find the envelope of the family of circumferences constructed on parallel chords of a circumference as on diameters.

**4.41.** Find the envelope of the family of ellipses that have common principal axes and a given semi-axis sum.

**4.42.** A beam of parallel rays falls on a spherical mirror. Find the envelope of the reflected rays (caustic).

**4.43.** Find the envelope of the family of ellipses that have a given area and common principal axes.

**4.44.** Find the envelope of the family of circumferences with centres on an ellipse and passing through one of its foci.

**4.45.** Find the envelope of the family of circumferences of radius $a$ and centres on a curve $\mathbf{r} = \mathbf{r}(v)$.

**4.46.** Find the envelope of the normals of a curve $\mathbf{r} = \mathbf{r}(v)$. The vector function $\mathbf{r}(v)$ is defined, continuous and twice differentiable on a segment $[a, b]$. The vectors $\mathbf{r}'$ and $\mathbf{r}''$ are noncollinear at each point of this segment.

**4.47.** Find the envelope of the rays reflected from a circumference if the luminous point is on the circumference.

**4.48.** Calculate the curvature of the following curves:

(1) $y = \sin x$ at the vertex (sine curve);

(2) $x = a(1 + m)\cos mt - am\cos(1 + m)t$

$y = a(1 + m)\sin mt - am\sin(1 + m)t$ (epicycloid);

(3) $y = a \cosh(x/a)$ (catenary curve);

(4) $x^2y^2 = (a^2 - y^2)(b + y)^2$ (conchoidal curve);

(5) $r^2 = a^2\cos 2\varphi$ (lemniscate);

(6) $r = a(1 + \cos\varphi)$ (cardioid);

(7) $r = a\varphi$ (Archimedes' spiral);

(8) $r = \{a\cos^3 t,\ a\sin^3 t\}$ (astroid).

**4.49.** Calculate the curvature of the following curves:

(1) $y = -\ln\cos x$;

(2) $x = 3t^2,\ y = 3t - t^3$ for $t = 1$;

(3) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$ for $t = \pi/2$;

(4) $x = a(2\cos t - \cos 2t),\ y = a(2\sin t - \sin 2t)$.

**4.50.** Find the curvature of the following curves given in polar coordinates:

(1) $r = a\varphi$; (2) $r = a\varphi^k$; (3) $r = a^\varphi$ at the point $\varphi = 0$.

**4.51.** Find the curvature of the curve given by the equation

$F(x, y) = 0$.

**4.52.** Curves are given by their differential equation $P(x, y)dx + Q(x, y)dy = 0$. Find their curvature.

**4.53.** Calculate the length of the following curves:

(1) $y = a\cosh(x/a)$;

(2) $y = x^{3/2}$;

(3) $y = x^2$;

(4) $y = \ln x$;

(5) $r = a(1 + \cos\varphi)$;

(6) $\mathbf{r} = \{a(t - \sin t),\ a(1 - \cos t)\}$;

(7) $\mathbf{r} = \{a(\cos t + t\sin t),\ a(\sin t - t\cos t)\}$;

(8) $\mathbf{r} = \left\{\frac{a}{3}(2\cos t + \cos 2t),\ \frac{a}{3}(2\sin t + \sin 2t)\right\}$;

(9) $\mathbf{r} = \{a\cos^3 t,\ a\sin^3 t\}$;

(10) $y = e^x$;

(11) $\mathbf{r} = \left\{a\left(\ln\cot\frac{t}{2} - \cos t\right),\ a\sin t\right\}$.

**4.54.** Find the arc length of the curve

$x = -f'(\alpha)\sin\alpha - f''(\alpha)\cos\alpha$,

$y = f'(\alpha)\cos\alpha - f''(\alpha)\sin\alpha$.

The natural equations of a plane curve are equations of the form:

(1) $k = k(s)$, (2) $F(k, s) = 0$ or (3) $k = k(t)$, $s = s(t)$.

If the natural equations of a curve are given, then the parametrization of the curve can be given in the form

$x = \int \cos\alpha(s)ds$, $y = \int \sin\alpha(s)ds$.

**4.55.** Form the natural equations of the curves:

(1) $x = a\cos^3 t$, $y = a\sin^3 t$;

(2) $y = x^{3/2}$;

(3) $y = x^2$;

(4) $y = \ln x$;

(5) $y = a\cosh(x/a)$;

(6) $y = e^x$;

(7) $x = a\left(\ln\tan\dfrac{t}{2} + \cos t\right)$, $y = a\sin t$;

(8) $r = a(1 + \cos\varphi)$;

(9) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

**4.56.** Find the parametric equations of the curves if their natural equations are given (here $R = 1/k$):

(1) $R = as$;

(2) $\dfrac{s^2}{a^2} + \dfrac{R^2}{b^2} = 1$;

(3) $Rs = a^2$;

(4) $R = a + s^2/a$;

(5) $s^2 + 9R^2 = 16a^2$;

(6) $s^2 + R^2 = 16a^2$;

(7) $R^2 = 2as$;

(8) $R^2 + a^2 = a^2 e^{-2s-a}$

**4.57.** Let $p$ be the distance from the origin of radii vectors to the tangent to a curve $\gamma$ at a point $M$, and $r$ the distance from the point $O$ to the point $M$. Prove that

$$k = \left|\frac{dp}{rdr}\right|.$$

**4.58.** At a certain point of a curve $\mathbf{r} = \mathbf{r}(s)$, we have: $k \neq 0$, $\dot{k} \neq 0$. Having taken the equation of the osculating circumference in the form

$(\varrho - \mathbf{r}_0 - R_0\mathbf{n}_0)^2 = R_0^2$, prove that the osculating circumference intersects the given curve in a neighbourhood of the indicated point.

**4.59.** Given that the following conditions are fulfilled at a certain point of a curve: $k_0 \neq 0$, $\dot{k}_0 = 0$, $\ddot{k}_0 \neq 0$, prove that the osculating circumference at this point of the curve does not intersect the curve in a sufficiently small neighbourhood of this point.

**4.60.** Given an equation $R = f(\alpha)$, where $R$ is the curvature radius of a curve, and $\alpha$ the angle from a constant vector $\mathbf{a}$ to the tangent vector $\tau$ to the curve, form the parametric equations of the curve.

**4.61.** Given an equation $\alpha = f(R)$ (see the previous problem), form the parametric equations of the curve.

**4.62.** Given an equation $s = f(\alpha)$, where $s$ is an arc and $\alpha$ the angle from a constant vector $\mathbf{a}$ to the tangent vector $\tau$ to the curve, form the parametric equations of the curve.

**4.63.** Given an equation $\alpha = f(s)$ (see the previous problem), form the parametric equations of the curve.

**4.64.** Given that a beam of luminous rays falls on a plane curve $\mathbf{r} = \mathbf{r}(s)$ from the origin of radii vectors, form the equation of the envelope of the reflected rays (caustic).

**4.65.** What form will the equation of the caustic of a plane curve with respect to the origin of the radii vectors have if the equation of the curve is given in the form $\mathbf{r} = \mathbf{r}(t)$?

**4.66.** A beam of parallel rays with the direction of a vector $\mathbf{e}$ ($|\mathbf{e}| = 1$) falls on a plane curve given by an equation $\mathbf{r} = \mathbf{r}(s)$. Form the equation of the envelope of the rays reflected from the given curve (caustic). Consider the cases where the curve is given by an equation $\mathbf{r} = \mathbf{r}(t)$ and where it is given by an equation $y = f(x)$.

**4.67.** Write the equation of the tangent line and the normal plane of the curve

$$\mathbf{r} = \{u^3 - u^2 - 5,\ 3u^2 + 1,\ 2u^3 - 16\}$$

at the point where $u = 2$.

**4.68.** Find the tangent line and the normal plane at the point $A(3, -7, 2)$ of the curve

$$\mathbf{r} = \{u^4 + u^2 + 1,\ 4u^3 + 5u + 2,\ u^4 - u^3\}.$$

**4.69.** Find the tangent line and the normal plane at the point $A(2, 0, -2)$ of the curve

$$\mathbf{r} = \{u^2 - 2u + 3,\ u^3 - 2u^2 + u,\ 2u^3 - 6u + 2\}.$$

**4.70.** Write the equation of the osculating plane of the curve

$$\mathbf{r} = \{u^2, u, u^3 - 20\}$$

at the point $A(9, 3, 7)$.

**4.71.** Show that the curve

$$\mathbf{r} = \{au + b, cu + d, u^2\}$$

has the same osculating plane at all points.

**4.72.** Form the equations of the osculating plane, principal normal, and binormal of the curve

$$y^2 = x, \quad x^2 = z$$

at the point (1, 1, 1).

**4.73.** Given a helix

$$\mathbf{r} = \{a \cos t, a \sin t, bt\},$$

form the equations of the tangent, normal plane, binormal, osculating plane, and principal normal.

**4.74.** Given a curve

$$\mathbf{r} = \{t^2, 1 - t, t^3\},$$

form the equations of the tangent, normal plane, binormal, osculating plane and principal normal at the point $t = 1$.

**4.75.** Form the equations of the tangent line and the normal plane of the curve given by the intersection of two surfaces

$$F_1(x, y, z) = 0 \quad \text{and} \quad F_2(x, y, z) = 0.$$

**4.76.** The curve in which a sphere meets a circular cylinder, whose base radius is twice less and which passes through the centre of the sphere, is called a Viviani curve. Make up the equation of a Viviani curve in implicit and parametric forms. Find the equations of the tangent, normal plane, binormal, principal normal and osculating plane.

**4.77.** Find the length of the arc of the helix

$$x = 3a \cos t, \quad y = 3a \sin t, \quad z = 4at$$

from the point of intersection with the plane $xOy$ to an arbitrary point $\mathbf{M}(t)$.

**4.78.** Find the length of one turn between the two points of intersection with the plane $xOz$ of the curve

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad z = 4a \cos t/2.$$

**4.79.** Find the length of the arc of the curve

$$x^3 = 3a^2 y, \quad 2xz = a^2$$

between the planes $y = a/3$ and $y = 9a$.

**4.80.** Find the length of the closed curve

$$x = \cos^3 t, \; y = \sin^3 t, \quad z = \cos 2t.$$

**4.81.** Reparametrize the helix

$$\mathbf{r} = \{a\cos t,\ a\sin t,\ bt\}, \qquad b > 0,$$

by the natural parameter.

**4.82.** Reparametrize the curve

$$\mathbf{r} = \{e^t\cos t,\ e^t\sin t,\ e^t\}$$

by the natural parameter.

**4.83.** Reparametrize the curve

$$\mathbf{r} = \{\cosh t,\ \sinh t,\ t\}$$

by the natural parameter.

**4.84.** Find the vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the Frenet frame for the helix

$$\mathbf{r} = \{a\cos t,\ a\sin t,\ bt\}.$$

Calculate the curvature and torsion of the helix.

**4.85.** Given the curve

$$\mathbf{r} = \{t^2,\ 1 - t,\ t^3\},$$

find the vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the Frenet frame. Calculate the curvature and torsion of this curve.

**4.86.** Find the vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the Frenet frame, curvature, and torsion of a Viviani curve (see Problem 4.76).

**4.87.** Find the curvature and torsion of the following curves:

(1) $\mathbf{r} = \{t - \sin t,\ 1 - \cos t,\ 4\sin t/2\}$;

(2) $\mathbf{r} = \{e^t,\ e^{-t},\ t\sqrt{2}\}$;

(3) $\mathbf{r} = \{e^t\sin t,\ e^t\cos t,\ e^t\}$;

(4) $\mathbf{r} = \{2t,\ \ln t,\ t^2\}$;

(5) $\mathbf{r} = \{3t - t^3,\ 3t^2,\ 3t + t^3\}$;

(6) $\mathbf{r} = \{\cos^3 t,\ \sin^3 t,\ \cos 2t\}$.

**4.88.** At each point of the curve

$$x = t - \sin t, \qquad y = 1 - \cos t, \qquad z = 4\sin t/2,$$

a segment equal to four times the curvature at this point is laid off in the positive direction of the principal normal.

Find the equation of the osculating plane of the curve described by the end of the segment.

**4.89.** Calculate the curvature and torsion radii for the curve

$$x^3 = 3a^2y, \qquad 2xz = a^2.$$

**4.90.** Deduce the formulae for the calculation of the curvature and torsion of the curve given by equations $y = y(x)$ and $z = z(x)$ and find the Frenet frame for this curve.

**4.91.** Find the curves intersecting the rectilinear generators of the hyperbolic paraboloid $xy = az$ at right angles.

**4.92.** A curve on a sphere that intersects all the meridians of the sphere at a given angle is called a loxodrome. Find the equation of a loxodrome and the vectors $\tau$, $\nu$, $\beta$ of the Frenet frame for this curve at an arbitrary point. Calculate its curvature and torsion.

**4.93.** Given a curve

$$\mathbf{r} = \{v \cos u,\ v \sin u,\ kv\},$$

where $v = v(u)$, prove that this curve is placed on a cone. Define the function $v(u)$ so that this curve intersects the generators of the cone at a constant angle $\theta$.

**4.94.** The tangent vector $\mathbf{T} = \mathbf{T}(t) \neq 0$ is given at each point of a curve $\mathbf{r} = \mathbf{r}(t)$. The function $\mathbf{r}(t)$ is defined, continuous, and has a continuous derivative $\mathbf{r}'(t)$ on a segment $[a, b]$. The function $\mathbf{T}(t)$ is continuous on the segment $[a, b]$. Prove that this curve can be parametrized so that

$$\frac{d\mathbf{r}}{dt} = \mathbf{T}.$$

**4.95.** A curve $C$ is given by an equation $\mathbf{r} = \mathbf{r}(t)$, the function $\mathbf{r}(t)$ is defined on a segment $[a, b]$ and possesses noncoplanar derivatives $\mathbf{r}'$, $\mathbf{r}''$, $\mathbf{r}'''$ at a point $M$. Prove that the osculating plane of the curve $C$ at the point $M$ intersects the curve $C$.

**4.96.** Prove that if all osculating planes of a curve are concurrent, then the curve is plane.

**4.97.** A curve $C$ is given by an equation $\mathbf{r} = \mathbf{r}(t)$; the function $\mathbf{r}(t)$ is defined on a segment $[a, b]$ and possesses derivatives $\mathbf{r}'$, $\mathbf{r}''$, $\mathbf{r}'''$ at some point $M(t)$ with $\mathbf{r}' \neq \mathbf{r}''$. Calculate the limit

$$\lim_{\Delta t \to 0} \frac{d}{|\Delta t|^3},$$

where $d$ is the distance from the point $M(t + \Delta t)$ to the osculating plane of the curve $C$ at the point $M$. Consider the special case where the curve is given by an equation $\mathbf{r} = \mathbf{r}(s)$ ($s$ being the natural parameter).

**4.98.** Find a necessary and sufficient condition for the given family of curves

$$\mathbf{r} = \varrho(u) + \lambda \mathbf{e}(u) \quad (|\mathbf{e}| = 1)$$

to have the envelope. Find this envelope.

**4.99.** For what value of $b$ is the torsion of the helix

$$\mathbf{r} = \{a\cos t,\ a\sin t,\ bt\} \qquad (a = \text{const})$$

at its maximum?

**4.100.** Prove that if the torsion of a curve $C$ at some of its points $M$ is other than zero, then the osculating plane of the curve $C$ at the point $M$ intersects the curve.

**4.101.** Express $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$, $\dddot{\mathbf{r}}$ in terms of $\tau$, $\nu$, $\beta$, $k$ and $\varkappa$.

**4.102.** Prove that $\tau\dot{\beta}\ddot{\beta} = \varkappa$.

**4.103.** Prove that if the principal normals of a curve form a constant angle with the direction of a vector $\mathbf{e}$, then

$$\frac{d}{ds}\frac{k^2 + \varkappa^2}{k\dfrac{d}{ds}\dfrac{\varkappa}{k}} + \varkappa = 0,$$

and conversely, if this relation is fulfilled, then the principal normals of the curve form a constant angle with the direction of some vector. Find this vector.

**4.104.** Prove that if all normal planes of a line contain a vector $\mathbf{e}$, then this line is either straight or plane.

**4.105.** Prove that if all the osculating planes of a curve which is not a straight line contain the same vector, then this curve is plane.

**4.106.** Prove that if $\beta$ = const, then the curve is plane.

**4.107.** Prove that if the osculating planes of a curve have the same inclination, then the curve is plane.

**4.108.** A space line is called a generalized helix if all its tangents form a constant angle with a fixed direction.

Prove that a line is a generalized helix if and only if one of the following conditions is fulfilled:

(a) the principal normals are perpendicular to a fixed direction;

(b) the binormals make a constant angle with a fixed direction;

(c) the ratio of the curvature to the torsion is constant.

**4.109.** Prove that the condition $\ddot{\mathbf{r}}\dddot{\mathbf{r}}\mathbf{r}^{(4)} = 0$ is necessary and sufficient for a line to be a generalized helix.

**4.110.** Prove that the line $x^2 = 3y$, $2xy = 9z$ is a generalized helix.

Let $\mathbf{r} = \mathbf{r}(s)$ be a curve parametrized by the natural parameter. Then the mapping $\tau$: $(a, b) \to \mathbf{R}^3$ determines a curve $s \to \tau(s)$. This curve may be non-regular. Since $|\tau(s)| = 1$, the image $\tau(s)$ lies on the sphere with radius 1 and the origin at its centre. This curve is called the tangent spherical image of the curve $\mathbf{r} = \mathbf{r}(s)$. The normal spherical image $s \to \nu(s)$ and the binormal spherical image $s \to \beta(s)$ may be defined similarly.

**4.111.** *Find the tangent, normal and binormal spherical images of the* helix

$\mathbf{r} = \{a\cos t,\ a\sin t,\ bt\}.$

**4.112.** Let $\mathbf{r} = \mathbf{r}(s)$ be a curve parametrized by the natural parameter.

(a) Prove that the tangent spherical image of the curve $\mathbf{r} = \mathbf{r}(s)$ degenerates into a point if and only if $\mathbf{r} = \mathbf{r}(s)$ is a straight line.

(b) Prove that the binormal spherical image of the curve $\mathbf{r} = \mathbf{r}(s)$ degenerates into a point if and only if $\mathbf{r} = \mathbf{r}(s)$ is a plane curve.

(c) Prove that the normal spherical image of the curve $\mathbf{r} = \mathbf{r}(s)$ cannot be a point.

**4.113.** Let $\bar{s}$ be the length of the tangent spherical image of a curve $\mathbf{r} = \mathbf{r}(s)$:

$$\bar{s} = \int_0^s |\tau'(\sigma)| d\sigma.$$

(a) Prove that $\dfrac{d\bar{s}}{ds} = k.$

(b) Find necessary and sufficient conditions for the tangent spherical image to be a regular curve.

**4.114.** Les $s^*$ be the length along the normal (resp. binormal) spherical image of a curve $\mathbf{r} = \mathbf{r}(s)$. Prove that

$$\frac{ds^*}{ds} = \sqrt{k^2 + \varkappa^2} \text{ (resp. } |\varkappa|).$$

**4.115.** Let $\mathbf{r} = \mathbf{r}(s)$ be a curve parametrized by the natural parameter, $k\varkappa \neq 0$. Prove that the tangent to the tangent spherical image is parallel to the tangent to the binormal spherical image at the corresponding points.

**4.116.** Let $\mathbf{r} = \mathbf{r}(s)$ be a curve parametrized by the natural parameter. Prove that if the tangent spherical image of this curve lies in a plane passing through the origin, then the curve $\mathbf{r} = \mathbf{r}(s)$ is plane.

**4.117.** Prove that the curve $\mathbf{r} = \mathbf{r}(s)$ is a helix if and only if the tangent spherical image is an arc of a circumference.

By definition, a spherical curve is a curve $\mathbf{r} = \mathbf{r}(t)$ for which there exists a constant vector $\mathbf{m}$ such that

$$< \mathbf{r}(t) - \mathbf{m}, \mathbf{r}(t) - \mathbf{m} > = r^2.$$

**4.118.** Let $\mathbf{r} = \mathbf{r}(t)$ be a regular curve, and $\mathbf{a}$ a point which lies in each normal plane to $\mathbf{r} = \mathbf{r}(t)$. Prove that $\mathbf{r} = \mathbf{r}(t)$ is a spherical curve.

**4.119.** Prove that

$\mathbf{r} = \{-\cos 2t,\ -2\cos t,\ \sin 2t\}$

is a spherical curve by showing that the point $(-1, 0, 0)$ lies in every normal plane.

**4.120.** Let $\mathbf{r} = \mathbf{r}(s)$ be a curve which is parametrized by the natural parameter, $k \neq 0$, $\varkappa \neq 0$, and $\varrho = 1/k$, $\sigma = 1/\varkappa$. Assume that $\varrho^2 + (\varrho'\sigma)^2 = a^2 = \text{const}$, $a > 0$. Prove that the image of the curve $\mathbf{r} = \mathbf{r}(s)$ lies on a sphere of radius $a$.

**4.121.** Prove that if $\mathbf{r} = \mathbf{r}(s)$ is a curve which is parametrized by the natural parameter, $k \neq 0$, $\varkappa \neq 0$, then $\mathbf{r}(s)$ lies on a sphere if and only if

$$\frac{\varkappa}{k} = \left(\frac{k'}{\varkappa k^2}\right)' \quad \left(\text{or } \varkappa\varrho = -\left(\frac{\varrho'}{\varkappa}\right)'\right).$$

**4.122.** Using the results of the previous problems, prove that a curve $\mathbf{r} = \mathbf{r}(s)$ lies on a sphere if and only if there exist constants $A$ and $B$ such that

$$k\left(A \cos\int_0^s \varkappa\, ds + B \sin\int_0^s \varkappa\, ds\right) \equiv 1.$$

**4.123.** Two curves $\mathbf{r} = \mathbf{r}_1(t)$ and $\mathbf{r} = \mathbf{r}_2(t)$ are said to form a pair of Bertrand curves if for any value of the parameter $t_0$, the normal to $\mathbf{r}_1(t)$ coincides with the normal to $\mathbf{r}_2(t)$.

(a) Prove that two arbitrary concentric circumferences which lie in the same plane form a pair of Bertrand curves.

(b) Let

$$\mathbf{r}_1(t) = \frac{1}{2}\left\{\frac{1}{\cos t} - t\sqrt{1 - t^2},\ 1 - t^2,\ 0\right\},$$

$$\mathbf{r}_2(t) = \frac{1}{2}\left\{\frac{1}{\cos t} - t\sqrt{1 - t^2} - t,\ 1 - t^2 + t\sqrt{1 - t^2},\ 0\right\}.$$

Prove that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ form a pair of Bertrand curves.

**4.124.** Prove that the distance between the corresponding points of a pair of Bertrand curves is constant.

**4.125.** Prove that the angle between the tangents to the two curves of a Bertrand pair at corresponding points is constant.

**4.126.** Let $\mathbf{r} = \mathbf{r}_1(s)$ be a curve parametrized by the natural parameter, and $k\varkappa \neq 0$. Prove that the curve $\mathbf{r} = \mathbf{r}_2(s)$ ($s$ is not the natural parameter of $\mathbf{r}_2(s)$) which forms a pair of the Bertrand curves with $\mathbf{r}_1(s)$ exists if and only if there are constants $\lambda$ and $\mu$ such that

$$1/\lambda = k + \mu\varkappa.$$

**4.127.** Let $\mathbf{r} = \mathbf{r}(t)$ be a regular curve of class $C^3$, $\varkappa \neq 0$. Prove that $\mathbf{r}(t)$ is a circular helix if and only if $\mathbf{r}(t)$ possesses at least two different curves which are related in the sense of Bertrand.

Let $\mathbf{m}$ be a constant vector, $\mathbf{r} = \mathbf{r}(s)$ a curve, $c(s) = |\mathbf{r}(s) - \mathbf{m}|^2$, and $a$ a positive number. The curve $\mathbf{r}(s)$ is said to possess at a point $s = s_0$ a spherical contact of order $j$ with the sphere of radius $a$ and centre at the endpoint of $\mathbf{m}$ if

$$c(s_0) = a^2, \; c'(s_0) = c''(s_0) = \ldots = c^{(j)}(s_0) = 0,$$

$$c^{(j+1)}(s_0) \neq 0.$$

**4.128.** Given that $k \neq 0$, calculate the first three derivatives of the function $c(s)$ in terms of $\tau$, $\nu$, $\beta$, $k$ and $\varkappa$.

**4.129.** Prove that a curve $\mathbf{r} = \mathbf{r}(s)$ possesses a spherical contact of order 2 at a point $s = s_0$ if and only if $\mathbf{m} = \mathbf{r}(s_0) + \nu(s_0)/k(s_0) + \lambda\beta(s_0)$, where $\lambda$ is an arbitrary number.

**4.130.** Given that $\varkappa(s_0) \neq 0$, prove that a curve $\mathbf{r} = \mathbf{r}(s)$ possesses a spherical contact of order 3 if and only if

$$\mathbf{m} = \mathbf{r}(s_0) + \frac{1}{k(s_0)}\,\nu(s_0) - \frac{k'(s_0)}{k^2(s_0)\,\varkappa(s_0)}\,\beta(s_0).$$

**4.131.** Let a curve $\mathbf{r} = \mathbf{r}(s)$ be of constant curvature. Prove that the osculating sphere and the circumference have the same radius.

Let $\mathbf{r} = \mathbf{r}(s)$, $s \in [0, a]$, be a plane piecewise regular curve of class $C^2$ parametrized by the natural parameter. The number

$$i_{\mathbf{r}(s)} = \frac{\int_0^a k\,ds + \sum_{i=0}^{n-1} \Delta\theta_i}{2\pi},$$

where $k$ is the curvature of the curve, $s_i (0 \leqslant i \leqslant n - 1)$ are the singular points, $\tau^-(s_i) = \lim_{s \to s_i^-} \tau(s)$, $\tau^+(s_i) = \lim_{s \to s_i^+} \tau(s)$, and $\Delta\theta_i$ is the angle between the vectors $\tau^-(s_i)$ and $\tau^+(s_i)$, is called the rotation number $i_{\mathbf{r}(s)}$ of the curve.

**4.132.** Compute the rotation number of the curve $\gamma$ represented in Fig. 1.

**4.133.** Compute the rotation numbers of the curves given by the following equations (the parametrization is not natural):

(1) $\mathbf{r} = \{a + \varrho\cos t, \; \varrho\sin t\}$, $0 \leqslant t \leqslant 2\pi$, $|a| < \varrho$;

(2) $\mathbf{r} = \{a + \varrho\cos t, \; \varrho\sin t\}$, $0 \leqslant t \leqslant 2\pi$, $0 < \varrho < |a|$;

(3) $\mathbf{r} = \{\varrho\cos 2t, \; -\varrho\sin 2t\}$, $0 \leqslant t \leqslant 2\pi$, $\varrho > 0$;

(4) $\mathbf{r} = \left\{\frac{1}{2}\cos t,\ \sin t\right\},\ 0 \leqslant t \leqslant 2\pi;$

(5) $\mathbf{r} = \{2\cos t,\ -\sin t\},\ 0 \leqslant t \leqslant 6\pi;$

(6) $\mathbf{r} = \{1,\ \sin^2 t\},\ 0 \leqslant t \leqslant 2\pi.$

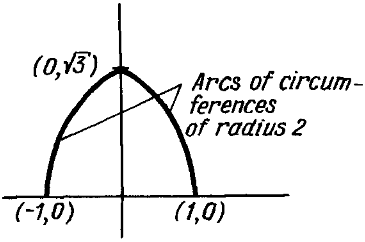


Fig. 1

**4.134.** Prove that if $\mathbf{r}(s)$ is a simple, closed, regular, and plane curve, then the tangent circular image $\tau : [0, L] \to S^1$ of this curve is a mapping "onto".

An oval is a regular, simple, closed and plane curve for which $k > 0$. The vertex of a regular plane curve is a point at which the curvature $k$ has a relative maximum or minimum.

Let $\mathbf{r}(s)$ be an oval and $P$ a point on $\mathbf{r}(s)$. Then there exists a point $P'$ such that the tangent $\tau$ to the oval at this point is opposite to the tangent at the point $P$, i.e., $\tau(P') = -\tau(P)$. The tangents at the points $P$ and $P'$ are parallel. Thus, for a given point $P$, there exists a unique point $P'$ (said to be opposite of $P$) on the oval, so that the tangents at $P$ and $P'$ are parallel and distinct.

The width $w(s)$ of an oval at the point $P = \mathbf{r}(s)$ is the distance between the tangent lines to the oval at the points $P$ and $P'$.

An oval is said to be of constant width if its width at a point $P$ is independent of the choice of $P$.

**4.135*.** Prove that any oval possesses at least four vertices. (This statement is known as the four-vertex theorem.)

**4.136*.** Prove that if $\mathbf{r}(s)$ is an oval of constant width $w$, then its length equals $\pi w$.

**4.137*.** Let $\mathbf{r} = \mathbf{r}(s)$ be an oval of constant width. Prove that the straight line joining a pair of opposite points $P$ and $P'$ of the oval is orthogonal to the tangents at the points $P$ and $P'$.

**4.138*.** Given that $\mathbf{r} = \mathbf{r}(s)$ is an oval, prove that $\tau''$ is parallel to $\tau$ at least at four points.

**4.139.** Prove that the notion of vertex does not depend on the choice of parametrization.

**4.140.** Show that the four-vertex theorem (see Problem 4.135) is not valid if the requirement of closedness is omitted.

**4.141.** Let $\mathbf{r}_1 : [0, a] \to \mathbf{R}^2$ be a segment of a curve parametrized by the natural parameter, and $\mathbf{r}_2(s)$ a curve

$$\mathbf{r}_2(s) = \mathbf{r}_1(s) + (a_0 - s)\tau(s),$$

where $\tau(s)$ is a tangent vector to $\mathbf{r}_1(s)$ and $a_0 > a$ a constant. Show that the unit tangent to $\mathbf{r}_2(s)$ is orthogonal to $\tau(s)$ at every point.

**4.142.** Let $\mathbf{r}(s)$ be a plane curve of constant width. Show that the sum of the curvature radii $1/k$ is constant at opposite points and does not depend on the choice of the points.

**4.143.** (a) Let $\mathbf{r}(s)$ be an oval of length $L$ and with natural parametrization. Denote the angle between the horizontal and tangent vector $\tau(s)$ by $\theta$. Prove that the mapping $\theta : [0, L] \to [0, 2\pi]$ is a parametrization of the oval $\mathbf{r}(s)$.

(b) Let $\varrho(\theta)$ be an oval parametrized by a parameter $\theta$ so that $\mathbf{r}(s) = \varrho(\theta(s))$. Prove that the point which is opposite to $\mathbf{r}(s)$ is $\mathbf{R}(s) = \varrho(\theta(s) + \pi)$.

(c) Prove that the curve $\mathbf{R}(s)$ is regular.

**4.144.** Let $\varrho(\theta)$ be an oval parametrized by an angle $\theta$ in a manner similar to that of the previous problem. Let $w(\theta)$ be the width of the oval at a point $\varrho(\theta)$. Prove that

$$\int_0^{2\pi} w d\theta = 2L,$$

where $L$ is the length of the oval.

**4.145.** Let $\varrho(\theta)$ be an oval parametrized by an angle $\theta$, $k(\theta)$ and $w(\theta)$ its curvature and width, respectively. Prove that

$$\frac{d^2 w}{d\theta^2} + w = \frac{1}{k(\theta)} + \frac{1}{k(\theta + \pi)}.$$

The total curvature of a regular space curve $\mathbf{r} = \mathbf{r}(s)$ parametrized by the natural parameter is the number $\int_0^L k ds$. Since $k = |\tau'(s)|$, the total curvature is the length of the tangent image

$$\tau : [0, L] \to S^2.$$

**4.146*** Prove that if $\mathbf{r} = \mathbf{r}(s)$ is a regular closed curve, then its tangent spherical image cannot lie in any open hemisphere.

**4.147*** Prove that the tangent spherical image of a regular closed curve cannot lie in any closed hemisphere except for the case when it is a great circumference bounding the hemisphere.

**4.148*** Let $\gamma$ be a closed $C^1$-curve on the unit sphere $S^2$. Prove that the image $C$ of the curve $\gamma$ is contained in an open hemisphere if

(a) the length $l$ of the curve $\gamma$ is less than $2\pi$;

(b) $l = 2\pi$, but the image $C$ is not the union of two great semi-circumferences.

**4.149*** Using the results of Problems 4.146-4.148, prove the following statement: the total curvature of a closed space curve $\gamma$ is not less than $2\pi$ and equal to $2\pi$ if and only if $\gamma$ is a plane convex curve (Fenchel theorem).

**4.150*** Let $\gamma$ be a space closed curve. Assume that $0 \leqslant k \leqslant 1/R$ for a certain real number $R > 0$. Prove that the length $l$ of the curve $\gamma$ satisfies the inequality $l \geqslant 2\pi R$.

**4.151.** Calculate the tangent spherical image for the ellipse

$$\mathbf{r} = \{2\cos t,\ \sin t,\ 0\}, \quad 0 \leqslant t \leqslant 2\pi.$$

What can be said about the image taking the Fenchel theorem into account?

Let $\omega$ be an oriented great circumference on the sphere $S^2$. Then there exists on $S^2$ a unique point $w$ associated with $\omega$, viz., the pole of the hemisphere which is on the left when moving along $\omega$ in the positive direction (Fig. 2).

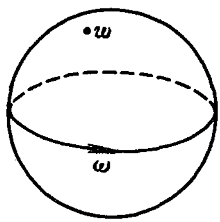


**Fig. 2**

Conversely, every point of $S^2$ is related to a certain orientable great circumference. Thus, the set of oriented great circumferences is in one-to-one correspondence with the points of $S^2$.

The measure of the set of oriented great circumferences is the measure of the corresponding set of points in $S^2$.

If $w \in S^2$, then $w^{\perp}$ denotes the great oriented circumference associated with $w$.

For a regular curve $\gamma$ with the spherical image $C$, we denote the number of points in $C \cap w^{\perp}$ (which may be infinite) by $n_{\gamma}(w)$. Note that the number $n_{\gamma}(w)$ does not depend on a parametrization of the curve $\gamma$.

**4.152*** Let $C$ be the image on $S^2$ of a regular curve $\gamma$ of length $l$. Prove that the measure of the set of oriented great circumferences which intersect $C$ (taking the multiplicities into account) equals $4l$. In other words,

$$\iint_{S^2} n_{\gamma}(w)d\sigma = 4l \text{ (the Crofton formula).}$$

**4.153*** A closed simple curve $\gamma$ is said to be unknotted if there exists a one-to-one continuous function $g : D^2 \to \mathbf{R}^3$ ($D^2$ being the unit disk) which maps the boundary $S^1$ of the disk $D^2$ onto the image of the curve $\gamma$. Otherwise, the curve is said to be knotted.

Prove that if $\gamma$ is a simple, knotted, and regular curve, then its total curvature is greater than or equal to $4\pi$.

**4.154*** Using the Crofton formula, prove that for any closed, regular curve, $\int kds \geqslant 2\pi$.

We call the number $\int_0^L \varkappa ds$ the total torsion of a regular space curve $\mathbf{r} = \mathbf{r}(s)$ parametrized by the natural parameter.

**4.155*** Prove that for any real number $r$, there exists a closed curve $\gamma$ such that its total torsion $\int_0^L \varkappa ds = r$.

**4.156*** Prove that the total torsion $\int \varkappa ds$ of a closed curve $\mathbf{r} = \mathbf{r}(s)$ ($s$ being the natural parameter) placed on the sphere $S^2$ equals zero.

**4.157*** Let $M$ be a surface in $\mathbf{R}^3$ such that $\int \varkappa ds = 0$ for all closed curves placed on $M$. Prove that $M$ is a part of a plane or sphere.

**4.158***. Prove that $\int \frac{\varkappa}{k} ds = 0$ for any closed. spherical curve parametrized by the natural parameter.

# 5
# Surfaces

**5.1.** Make up a parametric equation of the cylinder for which the curve $\varrho = \varrho(u)$ is directing and whose generators are parallel to a vector $\mathbf{e}$.

**5.2.** Make up a parametric equation of the cone with vertex at the origin of the radius vector for which the curve $\varrho = \varrho(u)$ is directing.

**5.3.** Make up a parametric equation of the surface formed by the tangents to a given curve $\varrho = \varrho(u)$. Such a surface is called a developable surface.

**5.4.** A circumference of radius $a$ moves so that its centre is on a given curve $\varrho = \varrho(s)$ and the plane in which the circumference is placed is, at each particular moment, a normal plane to the curve. Make up a parametric equation of the surface described by the circumference.

**5.5.** A plane curve $x = \varphi(v)$, $z = \psi(v)$ revolves about the axis $Oz$. Make up parametric equations of the surface of revolution. Consider the special case where the meridian is given by an equation $x = f(z)$.

**5.6.** The circumference $x = a + b\cos v$, $z = b\sin v$ $(0 < b < a)$ revolves about the axis $Oz$. Make up the equation of the surface of revolution.

**5.7.** A straight line moves translationally with a constant velocity while intersecting another straight line at right angles and uniformly rotating about it. Make up the equation of the surface which is described by the moving straight line (right helicoid).

**5.8.** Make up the equation of the surface formed by the principal normals of a helix.

**5.9.** Make up the equation of the surface formed by the family of normals to a given curve $\varrho = \varrho(s)$.

**5.10.** A straight line moves so that the point $M$ where it meets a given circumference moves along it, the straight line remaining in the plane normal to the circumference at the corresponding point and rotating through an angle equal to the angle $\widehat{MOM_0}$ through which the point was turned while moving along the circumference. Make up the equation of the surface described by the moving straight line assuming that the original position of the moving straight line was the axis $Ox$ and the circumference is given by two equations $x^2 + y^2 = a^2$, $z = 0$.

**5.11.** Given two curves $\mathbf{r} = \mathbf{r}(u)$ and $\varrho = \varrho(v)$. Make up the equation of the surface described by the middle point of the line segment whose extremities lie on the given curves (translation surface).

**5.12.** Make up the equation of the surface formed by the rotation of the catenary line $y = a\cosh x/a$ about the axis $Ox$. This surface is called a catenoid.

**5.13.** Make up the equation of the surface formed by the rotation of the tractrix

$$\varrho = \{a \ln \tan(\pi/4 + t/2) - a\sin t,\ a\cos t\}$$

about its asymptote (pseudosphere).

**5.14.** The surface formed by a straight line moving parallel to a given plane (director plane) so that its generator intersects a given curve (directing curve) is called a conoid. A conoid is determined by a directing line, director plane, and curve which the moving straight line intersects (i.e., the directing curve). Make up the equation of a conoid if the director plane $yOz$, directing line $y = 0$, $z = h$, and the directing curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$ (i.e., ellipse) are given.

**5.15.** Make up the equation of the conoid for which the directing line, director plane and directing curve are given by the following equations, respectively:

(a) $x = a$, $y = 0$;

(b) $z = 0$;

(c) $y^2 = 2pz$, $x = 0$.

**5.16.** We call a cylindroid the surface formed by straight lines which are parallel to a plane. A cylindroid can be determined by two directing curves (lying on it) and a director plane (the generators of the cylindroid being parallel to it). Make up the equation of a cylindroid if its generators are two circumferences $x^2 + z^2 - 2ax = 0, y = 0$ and $y^2 + z^2 - 2ay = 0$, $x = 0$, and the director plane is the plane $xOy$.

**5.17.** A surface given by the parametric equation

$$\mathbf{r} = \mathbf{r}(u, v) = \boldsymbol{\varrho}(u) + v\mathbf{a}(u),$$

where $\boldsymbol{\varrho} = \boldsymbol{\varrho}(u)$ is a vector function determining a certain curve, and $\mathbf{a} = \mathbf{a}(u)$ a vector function determining the distribution of the rectilinear generators of the surface, is said to be ruled. Make up the equation of a ruled surface whose generators are parallel to the plane $y - z = 0$ and intersect two parabolas $y^2 = 2px$, $z = 0$ and $z^2 = -2px$, $y = 0$.

**5.18.** Make up the equation of the ruled surface whose generators intersect the axis $Oz$, are parallel to the plane $xOy$, and intersect the line $xyz = a^3$, $x^2 + y^2 = b^2$.

**5.19.** Make up the equation of the ruled surface whose generators intersect the straight line $\mathbf{r} = \mathbf{a} + u\mathbf{b}$, curve $\boldsymbol{\varrho} = \boldsymbol{\varrho}(v)$, and are perpendicular to a vector $\mathbf{n}$.

**5.20.** Make up the equation of a ruled surface whose generators are parallel to the plane $xOy$ and intersect two ellipses

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad x = a; \quad \frac{y^2}{c^2} + \frac{z^2}{b^2} = 1, \quad x = -a.$$

**5.21.** Make up the equation of a ruled surface formed by the straight lines intersecting the curve $\boldsymbol{\varrho} = \{u, u^2, u^3\}$, parallel to the plane $xOy$, and intersecting the axis $Oz$.

**5.22.** Make up the equation of the surface formed by the straight lines parallel to the plane $x + y + z = 0$, intersecting the axis $Oz$, and circumference $\varrho = \{b, a\cos u, a\sin u\}$.

**5.23.** Make up parametric equations of the surface formed by the straight lines intersecting the circumference $x^2 + z^2 = 1$, $y = 0$ and straight lines $y = 1$, $z = 1$ and $x = 1$, $z = 0$.

**5.24.** Make up the equation of the surface formed by the tangents to the helix $\varrho = \{a\cos v, a\sin v, bv\}$ (developable helicoid).

**5.25.** Make up the equation of the conic surface with the vertex at the point $(0, 0, -c)$ and the directing line $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

**5.26.** Given a straight line $AB$ and a curve $\varrho = \varrho(u)$ in a plane $\pi$. The curve $\varrho$ moves uniformly in the plane $\pi$ so that each of its points travels parallel to $AB$. The plane $\pi$ is, at the same time, in uniform rotation about $AB$. Make up the equation of the surface described by the curve $\varrho$. This surface is called a helical surface. A special case of a helical surface is a right helicoid (see Problem 5.7); in this case, $\varrho = \varrho(u)$ is a straight line orthogonal to $AB$.

**5.27.** Let $\mathbf{r} = \mathbf{r}(u)$ be a curve whose curvature $k$ is other than zero. Normal planes are drawn through each of its points, and a circumference with centre on the curve $\mathbf{r} = \mathbf{r}(u)$ and given radius $a$, $a > 0$, $ak < 1$, is constructed in every such plane. The locus of these circumferences is a tubular surface $S$.

(a) Make up the equation of the surface $S$.

(b) Prove that any normal to the surface $S$ intersects the curve $\mathbf{r} = \mathbf{r}(u)$ and is a normal to this curve.

**5.28.** Find the surface $S$, given that all its normals meet at one point.

**5.29.** Show that the volume of the tetrahedron formed by the intersection of the coordinate planes and the tangent plane to the surface

$$x = u, \quad y = v, \quad z = a^3/uv$$

does not depend on the choice of the point of tangency on the surface.

**5.30.** Show that the sum of the squares of the coordinate axis intercepts of a tangent plane to the surface

$$x = u^3\sin^3 v, \quad y = u^3\cos^3 v, \quad z = (a^2 - u^2)^{3/2}$$

is constant.

**5.31.** Show that the tangent plane meets the conoid

$$x = u\cos v, \quad y = u\sin v, \quad z = a\sin 2v$$

in an ellipse.

**5.32.** Prove that the planes which are tangent to the surface $z = xf(y/x)$ are concurrent.

**5.33.** Make up the equation of the tangent plane and normal to the helicoid

$$\mathbf{r} = \{v\cos u, v\sin u, ku\}.$$

**5.34.** Make up the equation of the tangent plane to the surface

$$xyz = a^3.$$

**5.35.** Given that a surface is formed by the tangents to a curve $C$, prove that this surface possesses the same tangent plane at all points of a tangent to the curve $C$.

**5.36.** Given that a surface is formed by the principal normals of a curve $C$, make up the equation of the tangent plane and the normal at an arbitrary point of the surface.

**5.37.** Make up the equation of the tangent plane and the normal to the surface formed by the binormals of a curve $C$.

**5.38.** Prove that the normal of a surface of revolution coincides with the principal normal to the meridian and intersects the axis of rotation.

**5.39.** Prove that if all normals of a surface intersect one and the same straight line, then the surface is a surface of revolution.

**5.40.** A ruled surface (see the definition in Problem 5.17) is said to be developable if the tangent plane to the surface is the same at all points of an arbitrary generator.

Prove that the ruled surface

$$\mathbf{R} = \mathbf{r}(u) + v\mathbf{a}(u)$$

is developable if and only if

$$\mathbf{r}'\mathbf{a}\mathbf{a}' = 0.$$

**5.41.** Prove that any developable surface may be partitioned into the following parts:

(i) a part of the plane;

(ii) a part of a cylinder;

(iii) a part of a cone;

(iv) a part of a figure consisting of the tangents to a certain non-plane line. In the last case, the indicated line is called an edge of regression.

**5.42.** Find the envelope and the edge of regression of the family of ellipsoids

$$\alpha^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{z^2}{c^2} = 1,$$

where $\alpha$ is the parameter of the family.

**5.43.** Find the envelope of the family of spheres constructed on the chords parallel to the major axis of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

as on diameters.

**5.44.** Find the envelope and the edge of regression of the family of spheres whose diameters are the chords of the circumference

$$x^2 + y^2 - 2x = 0, \quad z = 0,$$

that pass through the origin.

**5.45.** Two parabolas are placed in perpendicular planes, possessing the common vertex and the common tangent to the vertex. Find the envelope of the family of planes which are tangent to both parabolas.

**5.46.** Find the envelope of the family of spheres with constant radius, whose centres are placed on a given curve $\varrho = \varrho(s)$ (canal surface).

**5.47.** Find the edge of regression of the family of spheres with constant radius $a$, whose centres are placed on a curve $\varrho = \varrho(s)$.

**5.48.** Find the envelope and the edge of regression of the family of spheres with radius $a$, whose centres are placed on the circumference

$$x^2 + y^2 = b^2, \quad z = 0.$$

**5.49.** Find the envelope and the edge of regression of the family of spheres passing through the origin and whose centres are placed on the curve

$$\mathbf{r} = \{u^3, u^2, u\}.$$

**5.50.** Find the envelope of the family of ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

whose semi-axis sum

$$a + b + c = l$$

is given.

**5.51.** Find the surface whose tangent planes cut off on the coordinate axes line segments such that the sum of their squares equals $a^2$.

**5.52.** Find the surface whose tangent planes cut a tetrahedron of constant volume $a^3$ off the coordinate angle.

**5.53.** Find the envelope and the edge of regression of the family of planes

$$x\alpha^2 + y\alpha + z = 0,$$

where $\alpha$ is the parameter of the family.

**5.54.** Find the envelope and edge of regression of the family of planes

$$x\sin\alpha - y\cos\alpha + z = a\alpha,$$

where $\alpha$ is the parameter of the family.

**5.55.** Find the envelope, the characteristics, and the edge of regression of the family of osculating planes of a given curve.

**5.56.** Find the envelope, the characteristics, and the edge of regression of the family of normal planes to a given curve.

**5.57.** Find the characteristics, the envelope, and the edge of regression of the family of planes

$$\mathbf{rn} + D = 0,\ \mathbf{n} = \mathbf{n}(u),\ D = D(u),\ |\mathbf{n}| = 1,$$

where $u$ is the parameter of the family.

**5.58.** Find the developable surface through the two parabolas

(1) $y^2 = 4ax,\ z = 0;$

(2) $x^2 = 4ay,\ z = b.$

**5.59.** Show that the surface $x = \cos v - (u + v)\sin v,\ y = \sin v + (u + v)\cos v,\ z = u + 2v$ is developable.

**5.60.** Show that the surface $x = u^2 + 1/3\ v,\ y = 2u^3 + uv,\ z = u^4 + 2/3\ u^2 v$ is developable.

**5.61.** Given a paraboloid

$$x = 2au\cos v, \quad y = 2bu\sin v, \quad z = 2u^2(a\cos^2 v + b\sin^2 v),$$

where $a$ and $b$ are constants, make up the equation of a curve on the surface so that the tangent planes to the surface may form a constant angle with the plane $xOy$ along the curve.

Show that the characteristics of this family of tangent planes form a constant angle with the axis $z$. Find the edge of regression of the envelope.

**5.62.** Find the edge of regression of the developable surface which touches the surface $az = xy$ at the points where it meets the cylinder $x^2 = by$.

**5.63.** Show that the developable surface passing through two circumferences $x^2 + y^2 = a^2,\ z = 0$ and $x^2 + z^2 = b^2,\ y = 0$ intersects the plane $x = 0$ in an equilateral hyperbola.

**5.64.** Calculate the first fundamental form of the following surfaces:

(1) $\mathbf{r} = \{a\cos u\cos v,\ a\sin u\cos v,\ a\sin v\}$ (sphere);

(2) $\mathbf{r} = \{a\cos u\cos v,\ b\sin u\cos v,\ c\sin v\}$ (ellipsoid);

(3) $\mathbf{r} = \left\{\frac{a}{2}\left(v + \frac{1}{v}\right)\cos u,\ \frac{b}{2}\left(v + \frac{1}{v}\right)\sin u,\ \frac{c}{2}\left(v + \frac{1}{v}\right)\right\}$

(hyperboloid of one sheet);

(4) $\mathbf{r} = \left\{\frac{a}{2}\frac{uv + 1}{v + u},\ b\,\frac{v - u}{v + u},\ c\,\frac{uv - 1}{v + u}\right\}$ (hyperboloid of one sheet);

(5) $\mathbf{r} = \left\{\frac{a}{2}\left(v - \frac{1}{v}\right)\cos u,\ \frac{b}{2}\left(v - \frac{1}{v}\right)\sin u,\ \frac{c}{2}\left(v - \frac{1}{v}\right)\right\}$

(hyperboloid of two sheets);

(6) $\mathbf{r} = \left\{v\sqrt{p}\cos u,\ v\sqrt{q}\sin u,\ \frac{v^2}{2}\right\}$ (elliptic paraboloid);

(7) $\mathbf{r} = \{(u + v)\sqrt{p},\ (u - v)\sqrt{q},\ 2uv\}$ (hyperbolic paraboloid);

(8) $\mathbf{r} = \{av\cos u,\ bv\sin u,\ cv\}$ (cone);

(9) $\mathbf{r} = \{a\cos u,\ b\sin u,\ v\}$ (elliptic cylinder);

(10) $\mathbf{r} = \left\{\frac{a}{2}\left(u + \frac{1}{u}\right),\ \frac{b}{2}\left(u - \frac{1}{u}\right),\ v\right\}$

(hyperbolic cylinder).

**5.65.** Calculate the first fundamental form of the following surfaces:

(1) $\mathbf{r} = \varrho(s) + \lambda\mathbf{e}$, $\mathbf{e} = \text{const}$ (cylindrical surface);

(2) $\mathbf{r} = v\varrho(s)$ (conical surface);

(3) $\mathbf{r} = \varrho(s) + \lambda\mathbf{e}(s)$ ($|\mathbf{e}(s)| = 1$) (ruled surface);

(4) $\mathbf{r} = \varrho(s) + \nu(s)\cos\varphi + \beta(s)\sin\varphi$ (canal surface);

(5) $\mathbf{r} = \{\varphi(v)\cos u,\ \varphi(v)\sin u,\ \psi(v)\}$ (surface of revolution);

(6) $\mathbf{r} = \{(a + b\cos v)\cos u,\ (a + b\cos v)\sin u,\ b\sin v\}$ (torus);

(7) $\mathbf{r} = \{v\cos u,\ v\sin u,\ ku\}$ (minimal helicoid);

(8) $\mathbf{r} = \varrho(s) + \lambda\nu(s)$ (surface of principal normals);

(9) $\mathbf{r} = \varrho(s) + \lambda\beta(s)$ (surface of binormals).

**5.66.** The first fundamental form of a surface is the following:

$ds^2 = du^2 + (u^2 + a^2)dv^2.$

(i) Find the perimeter of the curvilinear triangle formed by the intersecting curves

$u = \pm 1/2\, av^2, \quad v = 1.$

(ii) Find the angles of this curvilinear triangle.

(iii) Find the area of the triangle formed by the intersecting curves

$u = \pm av, \quad v = 1.$

**5.67.** The first fundamental form of a surface is the following:

$ds^2 = du^2 + (u^2 + a^2)dv^2.$

Calculate the angle at which the curves

$$u + v = 0, \quad u - v = 0$$

intersect each other.

**5.68.** Find the equations of the curves which bisect the angles between the coordinate lines of the paraboloid of revolution

$$x = u\cos v, \quad y = u\sin v, \quad z = 1/2\, u^2.$$

**5.69.** Find the curves intersecting the curve $v =$ const at a constant angle $\theta$ (loxodromes) on the surface

$$x = u\cos v, \quad y = u\sin v, \quad z = a\ln(u + \sqrt{u^2 - a^2}).$$

**5.70.** Given a surface

$$\mathbf{r} = \{u\sin v,\ u\cos v,\ v\}.$$

Find:

(i) the area of the curvilinear triangle $0 \leqslant u \leqslant \sinh v$, $0 \leqslant v \leqslant v_0$;

(ii) the lengths of the sides of this triangle;

(iii) the angles of this triangle.

**5.71.** Let $\mathcal{L}$ be a curve whose equation is $\mathbf{r} = \mathbf{r}(u)$, the curvature $k(u)$, and the torsion $\varkappa(u)$, where $u$ is natural parameter of the curve $\mathcal{L}$. Let $S$ be the surface

$$\mathbf{r}(u, \varphi) = \mathbf{r}(u) + a\nu(u)\cos\varphi + a\beta(u)\sin\varphi,$$

where $\nu$, $\beta$ are the unit vectors of the principal normal and the binormal of the curve $\mathcal{L}$, respectively, $a =$ const $> 0$, $ak(u) < 1$. One and the same point on $S$ is assumed to correspond to the coordinates $(u, \varphi)$ and $(u, \varphi + 2\pi)$.

(i) Find the first fundamental form for the surface $S$.

(ii) Find the curves on $S$ which are orthogonal to the circumferences $u =$ const.

(iii) Calculate the area of the region on the surface $S$ bounded by the circumferences $u = u_1$, $u = u_2$.

(iv) Using the result of (iii), find the area of the torus obtained by rotating the circumference $(x - b)^2 + z^2 = a^2$, $b > a > 0$, about the axis $z$.

(v) Find the area of the surface $S$ in that special case where $\mathcal{L}$ is an arc of the helix $x = r\cos t$, $y = r\sin t$, $z = bt$, $0 \leqslant t \leqslant \pi$, $r > a$, $b \neq 0$.

**5.72.** Given a surface

$$\mathbf{r} = \{\varrho(u)\cos v,\ \varrho(u)\sin v,\ z(u)\},$$

where $\varrho'(u)^2 + z'(u)^2 = 1$, $u_1 < u < u_2$, $0 < v < v_0$, $v_0 < 2\pi$, prove that it can be placed inside a cylinder $x^2 + y^2 = \epsilon^2$ with arbitrarily small positive radius $e$ by bending itself. In bending, a self-covering of the surface is possible.

**5.73.** Find the surface of revolution which is locally isometric to the helicoid

$$\mathbf{r} = \{u\sin v,\ u\cos v,\ v\}.$$

**5.74.** Show that the following helical surface (conoid)

$$x = \varrho\cos v, \quad y = \varrho\sin v, \quad z = \varrho + v$$

covers (i.e., is locally isometric to) the surface of revolution (hyperboloid of revolution)

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = \sqrt{r^2 - 1},$$

the correspondence of the covering points being given by the equations

$$\varphi = v + \tan^{-1}\varrho, \quad r^2 = \varrho^2 + 1.$$

**5.75.** Show that the helical surface

$$x = \varrho\cos v, \quad y = \varrho\sin v, \quad z = a\left(\ln\frac{\varrho}{a} + v\right)$$

covers (is locally isometric to) the surface of revolution

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = a\sqrt{2}\ \ln(r + \sqrt{r^2 - a^2}).$$

**5.76.** Show that any helical surface

$$x = u\cos v, \quad y = u\sin v, \quad z = F(u) + av$$

covers (i.e., is locally isometric to) a surface of revolution so that the helical lines are transformed into parallels.

**5.77.** Prove that with a convenient choice of curvilinear coordinates on a surface of revolution, its first fundamental form can be transformed to the following:

$$ds^2 = du^2 + G(u)dv^2.$$

**5.78.** Transform the first fundamental form of the sphere, torus, catenoid, and pseudosphere to the following:

$$ds^2 = du^2 + G(u)dv^2.$$

**5.79.** A curvilinear coordinate system on a surface is said to be isometric if the first fundamental form of the surface is, with respect to these coordinates, as follows:

$$ds^2 = A(u, v)(du^2 + dv^2).$$

Find isometric coordinates on the pseudosphere.

**5.80.** A right-angled triangle whose sides are arcs of great circumferences of a sphere is given on the sphere. Find (a) the relations between the sides of the triangle, (b) its area.

**5.81.** A spherical lune is a figure formed by two great semicircumferences with common ends. Calculate the area $S$ of a lune with an angle $\alpha$ at the vertex.

**5.82.** Prove that any cylindrical surface is locally isometric to the plane.

**5.83.** Prove that any conical surface is locally isometric to the plane.

**5.84.** A Liouville surface is one whose first fundamental form can be transformed to the following:

$$ds^2 = (f(u) + g(v))(du^2 + dv^2).$$

Prove that a surface locally isometric to a surface of revolution is a Liouville surface.

**5.85.** Prove that any surface of revolution can be locally conformally mapped onto the plane.

**5.86.** Calculate the second fundamental form of the following surfaces of revolution:

(1) $\mathbf{r} = \{R\cos u\cos v,\ R\cos u\sin v,\ R\sin u\}$ (sphere);

(2) $\mathbf{r} = \{a\cos u\cos v,\ a\cos u\sin v,\ c\sin u\}$ (ellipsoid of revolution);

(3) $\mathbf{r} = \{a\cosh u\cos v,\ a\cosh u\sin v,\ c\sinh u\}$ (hyperboloid of revolution of one sheet);

(4) $\mathbf{r} = \{a\sinh u\cos v,\ a\sinh u\sin v,\ c\cosh u\}$ (hyperboloid of revolution of two sheets);

(5) $\mathbf{r} = \{u\cos v,\ u\sin v,\ u^2\}$ (paraboloid of revolution);

(6) $\mathbf{r} = \{R\cos v,\ R\sin v,\ u\}$ (circular cylinder);

(7) $\mathbf{r} = \{u\cos v,\ u\sin v,\ ku\}$ (circular cone);

(8) $\mathbf{r} = \{(a + b\cos u)\cos v,\ (a + b\cos u)\sin v,\ b\sin u\}$ (torus);

(9) $\mathbf{r} = \left\{a\cosh\dfrac{u}{a}\cos v,\ a\cosh\dfrac{u}{a}\sin v,\ u\right\}$ (catenoid);

(10) $\mathbf{r} = \left\{a\sin u\cos v,\ a\sin u\sin v,\ a\left(\ln\tan\dfrac{u}{2} + \cos u\right)\right\}$

(pseudosphere).

**5.87.** Calculate the second fundamental form of the right helicoid

$x = u\cos v, \quad y = u\sin v, \quad z = av.$

**5.88.** Calculate the second fundamental form of the catenoid

$$x = \sqrt{u^2 + a^2}\cos v,$$

$$y = \sqrt{u^2 + a^2}\sin v,$$

$$z = a\ln(u + \sqrt{u^2 + a^2}).$$

**5.89.** Calculate the second fundamental form of the surface $xyz = a^3$.

**5.90.** Given the surface of revolution

$$\mathbf{r}(u, \varphi) = \{x(u), \varrho(u)\cos\varphi, \varrho(u)\sin\varphi\}, \ \varrho(u) > 0,$$

(i) find the second fundamental form;

(ii) find the total curvature $K$ at an arbitrary point of the surface and the dependence of the sign of $K$ on the sense of convexity of the meridian;

(iii) calculate $K$ for the special case $\varrho(u) = u$,

$$x(u) = \pm\left(a\ln\frac{a + \sqrt{a^2 - u^2}}{u} - \sqrt{a^2 - u^2}\right), \quad a > 0$$

(pseudosphere);

(iv) find the mean curvature $H$ at an arbitrary point of the surface of revolution;

(v) select the function $\varrho = \varrho(x)$ for the special case $x = u$ so that $H = 0$ on the whole surface.

**5.91.** Given a curve $\varrho = \varrho(u)$ with the natural parameter $u$, curvature $k = k(u) \neq 0$, and torsion $\varkappa = \varkappa(u) \neq 0$. Let $\tau = \tau(u)$ be the unit tangent vector of this curve. Find (a) $K$, (b) $H$ for the surface of tangents

$$\mathbf{r}(u, v) = \varrho(u) + v\tau(u), \ v > 0.$$

**5.92.** Find the expression for the total curvature of the surface whose first fundamental form with respect to these coordinates is

$$ds^2 = du^2 + G(u, v)dv^2.$$

**5.93.** Find the total curvature of a surface whose first fundamental form is

$$ds^2 = du^2 + e^{2u}dv^2.$$

**5.94.** Find the total curvature of the surface given by the equation $F(x, y, z) = 0$.

**5.95.** Find the total and mean curvatures of a surface $z = f(x, y)$.

**5.96.** Find the principal curvature radii of the surface

$$y = x\tan\frac{z}{a}.$$

**5.97.** Find the principal curvature radii of the surface

$$x = \cos v - (u + v)\sin v,$$
$$y = \sin v + (u + v)\cos v,$$
$$z = u + 2v.$$

**5.98.** Calculate the total and mean curvatures of the helical surface

$$x = u\cos v, \quad y = u\sin v, \quad z = u + v.$$

**5.99.** Calculate the total and mean curvatures of the surface

$$x = 3u + 3uv^2 - u^3,$$
$$y = v^3 - 3v - 3u^2v,$$
$$z = 3(u^2 - v^2).$$

**5.100.** Show that the mean curvature of the helicoid (see Problem 5.7) equals zero.

**5.101.** Show that the principal curvature radii of the right helicoid

$$x = u\cos v, \quad y = u\sin v, \quad z = f(v),$$

where $f(v)$ is an arbitrary analytic function of variable $v$, have unlike signs.

**5.102.** Find the total and mean curvatures of the surface formed by the binormals of a given curve.

**5.103.** Find the total and mean curvatures of the surface formed by the principal normals of a given curve.

**5.104.** Let $S$ be a certain given surface. Mark off segments of the same length and direction on the normals to the surface $S$. The ends of these segments describe a surface $S^*$ "parallel" to the surface $S$. If the equation of the surface $S$ is $\mathbf{r} = \mathbf{r}(u, v)$, then the equation of $S^*$ is

$$\varrho = \mathbf{r}(u, v) + a\mathbf{n}(u, v),$$

where $\mathbf{n}(u, v)$ is a unit normal vector of $S$.

Express the coefficients of the first and second fundamental forms of the surface $S^*$ in terms of the coefficients of the first and second fundamental forms of the surface $S$.

**5.105.** Express the total curvature $K^*$ of the surface $S^*$ "parallel" to a surface $S$ in terms of the total and mean curvatures of the surface $S$.

**5.106.** Express the mean curvature $H^*$ of the surface $S^*$ "parallel" to a surface $S$ in terms of the total and mean curvatures of the surface $S$.

**5.107.** Make up the equation of the minimal surface $S^*$ "parallel" to a surface $S$ if for the surface $S$ the ratio $H/K = \text{const}$.

**5.108.** Given a surface of constant mean curvature $H$. Segments of length $1/2\,H$ are marked off on all its normals. Prove that the total curvature of the surface so formed and "parallel" to the given one is constant.

**5.109.** Segments of length $1/\sqrt{K}$ are marked off on all the normals of a surface with constant positive total curvature $K$. Prove that the mean curvature of the surface so formed is constant. Calculate it.

**5.110.** Prove that the total and mean curvatures at the corresponding points of two parallel surfaces are related by the formula

$$\frac{H^2 - 4K}{K^2} = \frac{H^{*2} - 4K^*}{K^{*2}}.$$

A line on a surface is called a *line of curvature* if it has the principal direction at each of its points. Lines of curvature are determined by the differential equation

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

**5.111.** Find the lines of curvature on the surface

$$x = \frac{a}{2}(u - v), \quad y = \frac{b}{2}(u + v), \quad z = \frac{uv}{2}.$$

**5.112.** Find the lines of curvature of the helicoid

$$x = u\cos v, \quad y = u\sin v, \quad z = av.$$

**5.113.** Prove that, in covering (local isometry) the catenoid

$$x = \sqrt{u^2 + a^2}\cos v, \quad y = \sqrt{u^2 + a^2}\sin v,$$

$$z = a\ln(u + \sqrt{u^2 + a^2})$$

with the helicoid

$$x = u\cos v, \quad y = u\sin v, \quad z = av,$$

the lines of curvature are transformed into asymptotic lines.

**5.114.** Find the lines of curvature of the surface

$$\mathbf{r}(u, v) = \varrho(u) + f(v)\mathbf{a} + g(v)\,[\tau(u) \times \mathbf{a}],$$

where $\tau(u) = \mathbf{r}'(u)$, $|\tau(u)| = 1$, $(\tau(u), \mathbf{a}) = 0$, $|\mathbf{a}| = 1$, $\mathbf{a}$ is a constant vector.

**5.115.** A plane curve $\gamma$ is given by an equation $\varrho = \varrho(u)$, where $u$ is the natural parameter, $k = k(u)$ its curvature $(0 < k < 1/a)$, $\nu$ the principal normal unit vector of $\gamma$, $\mathbf{e}$ the unit normal vector to the plane in which the curve $\gamma$ lies. A surface $S$ is given by the equation

$$\mathbf{r}(u, \varphi) = \varrho(u) + a\nu(u)\cos\varphi + a\mathbf{e}\sin\varphi.$$

(i) Find the Gaussian curvature of the surface $S$.
(ii) Find the mean curvature of the surface $S$.
(iii) Find the lines of curvature of the surface $S$.

**5.116.** Find the lines of curvature of the surface

$$\mathbf{r}(u, \varphi) = \varrho(u) + a\nu(u)\cos\varphi + a\beta(u)\sin\varphi,$$

where $\nu$ and $\beta$ are the unit vectors of the principal normal and the binormal to the curve $\varrho = \varrho(u)$ having the natural parameter $u$, curvature $k(u) < 1/a$, and torsion $\varkappa(u)$.

**5.117.** Find the lines of curvature of the surface

$$\mathbf{r}(u, v) = \left\{u\left(3v^2 - u^2 - \frac{1}{3}\right), v\left(3u^2 - v^2 - \frac{1}{3}\right), 2uv\right\}.$$

Find its total and mean curvatures at each point.

**5.118.** Prove that $H^2 \geqslant K$. When does the equality hold?

**5.119.** Let $\mathbf{X}$ and $\mathbf{Y}$ be orthogonal tangent vectors at some point of a surface. Prove that

$$H = \frac{1}{2}\{\mathrm{I\!I}(\mathbf{X}, \mathbf{X}) + \mathrm{I\!I}(\mathbf{Y}, \mathbf{Y})\},$$

where $\mathrm{I\!I}(,)$ is the second fundamental form of the surface.

**5.120.** Assume that the first fundamental form of a surface is as follows:

$$ds^2 = E\,du^2 + G\,dv^2.$$

Prove that

$$K = -\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial v}\left(\frac{\frac{\partial E}{\partial v}}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{\frac{\partial G}{\partial u}}{\sqrt{EG}}\right)\right\}.$$

**5.121.** Assume that two surfaces $M_1$ and $M_2$ meet in a curve $C$. Let $k$ be the curvature of $C$, $\lambda_i$ normal curvatures of $C$ in $M_i$, and $\theta$ the angle between the normals of $M_1$ and $M_2$. Prove that

$$k^2\sin^2\theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta.$$

Two directions in the tangent plane of a surface which are determined by two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *conjugate* if $\varphi_2(\mathbf{a}, \mathbf{b}) = 0$, i.e., if

$$La_1b_1 + M(a_1b_2 + a_2b_1) + Na_2b_2 = 0,$$

where $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$. A *net of lines* on a surface is said to be *conjugate* if the tangent vectors to the lines of different families of this net are conjugate at each point.

A direction determined by a vector **h** is said to be asymptotic if $\varphi_2(\mathbf{h}, \mathbf{h}) = 0$. A *line* on a surface is said to be *asymptotic* if the tangent has the asymptotic direction at each of its points. An asymptotic line is characterized by the equality $k_n = 0$ which is held at all its points. Asymptotic lines are determined by the differential equation

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0.$$

**5.122.** Find the asymptotic lines of the surface

$$z = a\left(\frac{x}{y} + \frac{y}{x}\right).$$

**5.123.** Find the asymptotic lines of the surface $z = xy^2$.
**5.124.** Find the asymptotic lines of the surface

$$x = u^2 + v, \quad y = u^3 + uv, \quad z = u^4 + \frac{2}{3}u^2 v.$$

Construct the projections of the asymptotic lines, passing through the point $u = 1$, $v = 1/2$, onto the plane $xy$.
**5.125.** Find the asymptotic lines of the surface

$$x = a(1 + \cos u)\cot v, \quad y = a(1 + \cos u), \quad z = a\cos u/\sin v.$$

**5.126.** Prove that for an asymptotic line on the surface, $\varkappa^2 = -K$ (where $\varkappa$ is the torsion and $K$ the total curvature).
**5.127.** Find the torsion of the asymptotic lines of the surface formed by the binormals to a given curve.
**5.128.** Find the torsion of the asymptotic lines of the surface formed by the principal normals to a given curve.
**5.129.** Show that the coordinate lines of the surface

$$x = \frac{a}{2}(u + v), \quad y = \frac{b}{2}(u - v), \quad z = \frac{uv}{2}$$

are straight lines. Find the lines of curvature.
**5.130.** Show that the coordinate lines on the surface

$$x = f_1(u), \quad y = \varphi_1(v), \quad z = f_2(u) + \varphi_2(v)$$

are plane and form a conjugate system.
**5.131.** Show that the coordinate lines on the surface

$$\mathbf{r} = \mathbf{M}(u) + \mathbf{Q}(v)$$

are conjugate.
**5.132.** Prove that the sum of the normal curvature radii for each pair of conjugate directions is the same at an arbitrary point of a surface.

**5.133.** Prove that the product of the normal curvature radii for a pair of conjugate directions attains its minimum for the lines of curvature.

**5.134.** Prove that the ratio of the principal curvature radii is constant for the surface of revolution obtained by rotating a parabola about its directrix.

**5.135.** Prove that if one of the lines of curvature of a developable surface lies on a sphere, then all the remaining non-rectilinear lines of curvature lie on concentric spheres.

**5.136.** Prove that the normal curvature of an orthogonal trajectory of the asymptotic lines of a surface equals the mean curvature of the surface.

**5.137.** Prove that a line of curvature is plane if its osculating plane forms a constant angle with the tangent plane to the surface.

We call a line whose principal normal coincides with the normal to a surface at each of its points a *geodesic line* of the surface. There is a unique geodesic line passing through each point of the surface and having a given direction.

The length of the projection of the curvature vector $k\mathbf{n}$ onto the tangent plane of a surface is called the *geodesic curvature* $k_g$ of a line placed on the surface.

The *geodesic torsion* associated with a given direction is the torsion of the geodesic line passing in this direction.

**5.138.** Prove that a geodesic line on a surface can be fully determined by one of the following properties:

(i) The normal to a surface at each point of the line, where its curvature is other than zero, is a principal normal.

(ii) The normal to a surface lies in the osculating plane of the line at each of its points where its curvature is other than zero.

(iii) The geodesic curvature equals zero at each point of the line.

(iv) The curvature equals the absolute value of the normal curvature at each point of the line.

(v) The rectifying plane coincides with the tangent plane to the surface at each point of the line where its curvature is other than zero.

**5.139.** Prove that any straight line on a surface is a geodesic line.

**5.140.** Given that two surfaces touch each other along a line $l$, prove that if $l$ is a geodesic line on one surface, then it is geodesic on the other.

**5.141.** Prove that the differential equation of the geodesic lines of a surface $\mathbf{r} = \mathbf{r}(u, v)$ can be represented in the form $\mathbf{N} d\mathbf{r} d^2\mathbf{r} = 0$, where $\mathbf{N}$ is the normal vector of the surface.

**5.142.** Prove that geodesic lines of the plane are straight lines and only they.

**5.143.** Find the geodesic lines of a cylindrical surface.

**5.144.** Find the geodesic lines of a developable surface.

**5.145.** Find the geodesic lines of the circular cone $x^2 + y^2 = z^2$.

**5.146.** Find the geodesic lines of the helicoid

$$\mathbf{r} = \{u\cos v,\ u\sin v,\ hv\}.$$

**5.147.** Find the geodesic lines of an arbitrary conical surface.

**5.148.** Prove that the meridians of a surface of revolution are geodesic lines.

**5.149.** Prove that a parallel of a surface of revolution is geodesic if and only if the tangent to a meridian at the points where the meridian meets the parallel is parallel to the axis of rotation.

**5.150.** Find the geodesic lines on the sphere.

**5.151.** Show that the geodesic lines of a surface whose first fundamental form is

$$ds^2 = v(du^2 + dv^2)$$

are parabolas on the plane $u$, $v$.

**5.152.** Prove that a geodesic line is a line of curvature if and only if it is plane.

**5.153.** Prove that a geodesic line is asymptotic if and only if it is straight.

**5.154.** Prove that the geodesic curvature of a line $u = u(s)$, $v = v(s)$ on a surface $\mathbf{r} = \mathbf{r}(u, v)$ can be calculated by the formula

$$k_g = |\mathbf{m}\dot{\mathbf{r}}\ddot{\mathbf{r}}|,$$

where $\mathbf{m}$ is the unit normal vector of the surface.

**5.155.** Find the geodesic curvature of the helical lines of the helicoid

$$\mathbf{r} = \{u\cos v,\ u\sin v,\ av\}.$$

**5.156.** Prove that the geodesic torsion of a line $u = u(s)$, $v = v(s)$ on the surface $\mathbf{r} = \mathbf{r}(u, v)$ can be calculated by the formula

$$\varkappa_g = \dot{\mathbf{r}}\dot{\mathbf{m}}\mathbf{m},$$

where $\mathbf{m}$ is the unit normal vector of the surface.

**5.157.** Prove the following statement: for a line on a surface to be a line of curvature, it is necessary and sufficient that the geodesic torsion should equal zero at each of its points.

**5.158.** Show that on a surface with the first fundamental form

$$ds^2 = [\varphi(u) + \psi(v)]\,(du^2 + dv^2)$$

(a Liouville surface) the geodesic lines are determined by the equation

$$\frac{du}{\sqrt{\varphi(u) + a}} \pm \frac{dv}{\sqrt{\psi(v) - a}} = 0,$$

where $a$ is an arbitrary constant.

**5.159.** Given a triangle $T$ whose area is $\sigma$ and the sides are arcs of great circumferences on a sphere of radius $R_0$, find the sum of the interior angles of the triangle $T$.

**5.160.** Let $T$ be a triangle whose sides are geodesic lines constructed on a surface with constant Gaussian curvature $K = -a^2 < 0$. Assuming that the area $\sigma$ of $T$ is given, find the sum of its interior angles.

**5.161.** Given that a surface $S$ is obtained by a certain bending of a portion of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ determined by the inequalities $x > 0$, $y > 0$, $z > 0$, find the area $\sigma^*$ of the spherical image of the surface $S$.

**5.162.** Given that a surface $\mathbf{R} = \mathbf{R}(u, v)$, $u_1 < u < u_2$, $v_1 < v < < v_2$, has the first fundamental form $ds^2 = du^2 + B^2(u, v)dv^2$, find the area $\sigma^*$ of the spherical image of this surface.

**5.163.** Let $\gamma$ be a closed geodesic line without self-intersections on a closed convex surface $S$. Prove that the spherical image of the curve $\gamma$ divides the Gaussian sphere into two parts equal in area.

**5.164.** Given that $ds^2 = d\varrho^2 + \sinh^2\varrho\, d\varphi^2$ in the geodesic polar coordinates $(\varrho, \varphi)$ on the non-Euclidean plane, find the length $s(\varrho)$, geodesic curvature $k_g(\varrho)$, and rotation $\Pi(\varrho)$ of the geodesic circumference $\varrho = \text{const}$. Calculate $\lim\limits_{\varrho \to \infty} k_g(\varrho)$, $\lim\limits_{\varrho \to \infty} \Pi(\varrho)$. Compare the results obtained with the similar quantities for the Euclidean plane.

**5.165.** Given a plane $P_1$ with the metric $ds^2 = du^2 + \cosh 2u dv^2$, $-\infty < u < \infty$, $-\infty < v < \infty$, and a plane $P_2$ on which $ds^2 = d\varrho^2 + + \sinh^2\varrho d\varphi^2$ with respect to the geodesic polar coordinates $(\varrho, \varphi)$, prove that the planes $P_1$ and $P_2$ (with the first fundamental forms given on them) are isometric.

**5.166.** Given that a surface $S$ is defined by a vector function of the form $\mathbf{R} = \mathbf{R}(u, v)$ of class $C^2$, verify that the quantity $d\mathbf{n}^2 = (d\mathbf{n}, d\mathbf{n})$, where $\mathbf{n}$ is the normal unit vector of the surface $S$, is a quadratic form with respect to the differentials $du$, $dv$ (the so-called *third fundamental form of the surface* $S$). Express $d\mathbf{n}^2$ in terms of the first and second fundamental forms of the surface $S$.

**5.167.** Prove that the sum of the squares of the curvature and torsion of a geodesic line is equal to $-K$ on a minimal surface.

**5.168.** Prove that the plane and catenoid are the unique minimal surfaces of revolution.

**5.169.** Prove that among ruled surfaces, the minimal are the plane and the right helicoid.

**5.170.** Prove that for the mean curvature of a surface $S$, the following formula is valid:

$$H = \lim_{a \to 0} \frac{d\sigma - d\sigma^*}{2a\, d\sigma},$$

where $d\sigma$ and $d\sigma^*$ are the corresponding elements of the area of the parallel surfaces $S$ and $S^*$.

**5.171.** Prove that the area of any portion of a minimal surface cannot be less than the area of the corresponding portion of a parallel surface.

**5.172.** Prove that the limit of the ratio of the area of spherical representation of a surface $S$ to the area of the corresponding region of the surface $S$ equals the total curvature of the surface in magnitude and sign.

**5.173.** Prove that if one of the principal curvature radii of a surface is constant, then the surface is the envelope of a family of spheres with constant radius whose centres lie on a certain curve.

**5.174.** Given that a circular cylinder is intersected by a plane not parallel to the axis of the cylinder, what line will the line of intersection be transformed into in covering (locally isometric mapping) the plane with the cylinder?

**5.175.** Prove that if a material point moving across some surface is not acted upon by external forces, then it is moving along a geodesic line.

**5.176.** Prove that in a locally isometric mapping of surfaces, geodesic lines are transformed into geodesic.

**5.177.** Prove that two surfaces of the same constant Gaussian curvature are locally isometric.

**5.178.** Prove that any surface of constant positive Gaussian curvature is locally isometric to the sphere.

**5.179.** Prove that any surface of constant negative Gaussian curvature is locally isometric to the pseudosphere.

**5.180*.** Prove that all geodesic lines which are different from meridians are closed on the surface $S$ given by the equations

$$x = \frac{1}{2}\cos u \cos\varphi, \quad y = \frac{1}{2}\cos u \sin\varphi,$$

$$z = \int\sqrt{1 - \frac{1}{4}\sin^2 u}\, du,$$

$$-\frac{\pi}{2} \leqslant u \leqslant \frac{\pi}{2}, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

**5.181.** Given the differential equation of motion of a point electrical charge in the field of a magnetic pole, viz.,

$$\mathbf{r}''(t) = c\,|\mathbf{r}(t)|^{-3}\,[\mathbf{r}(t), \mathbf{r}'(t)], \quad c = \text{const},$$

prove that the path of the charge is a geodesic line of a circular cone.

**5.182.** Prove that the Gaussian curvature of the metric $ds^2 = \Phi(u, v)(du^2 + dv^2)$ can be represented in the form

$$K = -\frac{1}{2\Phi}\Delta\ln\Phi,$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplace operator.

**5.183*.** Prove that there are no closed geodesic lines on 1-connected surfaces such that the Gaussian curvature is non-positive at all of their points.

# 6
# Manifolds

**6.1.** Prove that an $n$-dimensional sphere $S^n$ determined in $\mathbf{R}^{n+1}$ by the equation $x_0^2 + x_1^2 + \ldots + x_n^2 = 1$ is a smooth manifold. Construct the atlas of charts for $S^n$.

**6.2.** Prove that the two-dimensional torus $T^2$ obtained by rotating about the axis $Oz$ of a circumference lying in the plane $Oxz$ and not intersecting the axis $Oz$ is a smooth manifold. Construct the atlas of charts.

**6.3.** Prove that the union of two coordinate axes in $\mathbf{R}^{n+1}$ is not a manifold.

**6.4.** Show that an atlas consisting of only one chart cannot be introduced on the sphere $S^n \subset \mathbf{R}^{n+1}$.

**6.5.** Determine whether the following plane curves are smooth manifolds: (a) a triangle, (b) two triangles with only one common point, viz., a vertex.

**6.6.** Prove that the $n$-dimensional projective space $\mathbf{R}P^n$ is a smooth (and real-analytic) manifold.

**6.7.** Prove that the $n$-dimensional complex projective space $\mathbf{C}P^n$ is a smooth (and complex-analytic) manifold.

**6.8.** Prove that the graph of the smooth function $x_{n+1} = f(x_1, \ldots, x_n)$ is a smooth manifold.

**6.9.** Prove that the group $SO(2)$ is homeomorphic to the circumference. What manifold is the group $O(2)$ homeomorphic to?

**6.10.** Prove that the group $SO(3)$ is homeomorphic to the projective space $\mathbf{R}P^3$.

**6.11.** Prove that the groups $GL(n, \mathbf{R})$, $GL(n, \mathbf{C})$ are smooth manifolds.

**6.12.** What manifold is the set of all straight lines on the plane $\mathbf{R}^2$ homeomorphic to?

Form the equations of the following manifolds in $\mathbf{R}^3$:

**6.13.** The cylinder with a directing curve $\varrho = \varrho(u)$ and a generator parallel to a vector $\mathbf{e}$.

**6.14.** The cone with the vertex at the origin and directing curve $\varrho = \varrho(u)$.

**6.15.** The surface made up of the tangents to a curve $\varrho = \varrho(u)$.

**6.16.** The surface formed by a circumference moving so that its centre is on a curve $\varrho = \varrho(u)$ and its plane normal to the curve at each of its points.

**6.17.** Prove that the Jacobian matrix of the composite of smooth mappings is the product of the Jacobian matrices of the factors.

**6.18.** Prove that the rank of a Jacobian matrix does not depend on the choice of a local coordinate system.

**6.19.** Calculate the rank of the Jacobian matrix of the mapping

$$f(x, y) = (x, 0) : \mathbf{R}^2 \to \mathbf{R}^2.$$

**6.20.** Let $f: U \to \mathbf{R}^n$ be a smooth manifold of an open domain $U \subset \mathbf{R}^n$, and the Jacobian $|\partial f| \neq 0$ at a point $p \in U$.

Prove that there exists an open domain $V \subset U$, $p \in V$ such that $f(V) = W$ is an open set, $f|_V$ a homeomorphism, and the inverse mapping $(f|_V)^{-1}$ smooth.

**6.21.** Let $f: U \to V$ be a smooth mapping of open domains in $\mathbf{R}^n$ which has a smooth inverse mapping.

Prove that the Jacobian $|\partial f| \neq 0$ at each point $p \in U$.

**6.22.** Set up explicit formulae for a smooth homeomorphism of the open disk $D^n = \{x \in \mathbf{R}^n: \sum_{i=1}^{n} x_i^2 < R^2\}$ onto the Euclidean space $\mathbf{R}^n$.

**6.23.** Prove that any smooth manifold has an atlas such that each chart is homeomorphic to a Euclidean space.

**6.24.** Give an example of a smooth one-to-one mapping which is not a diffeomorphism.

**6.25.** Construct a smooth function $f(x_1, \ldots, x_n)$ (of class $C^\infty$) equal to unity on a ball of unit radius, vanishing outside a ball of radius 2, and such that $0 \leqslant f \leqslant 1$.

**6.26.** Let $M$ be a manifold, $p \in U \subset M$ a neighbourhood of a point $p$. Prove that there exists a smooth function $f$ such that $0 \leqslant f \leqslant 1$, $f(p) = 1$, $f(x) = 0$ on $M \setminus \overline{U}$.

**6.27.** Let $M$ be a manifold, $A = \overline{A}$ a closed set, and $U \supset A$ an open domain. Prove that there exists a smooth function $f$ such that $0 \leqslant f \leqslant 1$, $f|_A = 1$, $f|_{M/U} = 0$.

**6.28.** Prove that any continuous function in $\mathbf{R}^n$ can be uniformly approximated, as close as we please, by a smooth function.

**6.29.** Prove that any continuous mapping of smooth manifolds can be approximated, as close as we please, by a smooth mapping.

**6.30.** Let a torus $T^2 \subset \mathbf{R}^3$ be formed by rotating a circumference about some axis (standard embedding). Prove that coordinates $x$, $y$, $z$ are smooth functions on the torus $T^2$.

**6.31.** Let a torus $T^2 \subset \mathbf{R}^3$ be standardly embedded in $R^3$, and the function $f: T^2 \to S^2$ associate each point $p \in T^2$ with a vector of unit length normal to the torus $T^2$ at the point $p$. Prove that $f$ is a smooth mapping.

**6.32.** Prove that a mapping $f: S^2 \to \mathbf{R}P^2$ associating a point $p$ on the sphere $S^2$ with the straight line which passes through the origin and the point $p$ is a smooth mapping.

**6.33.** Prove that two smooth structures on a manifold coincide if and only if the spaces of smooth functions (with respect to these structures) coincide.

**6.34.** Let $M^n$ be the solution set of the equations

$$g_i(x_1, \ldots, x_N) = 0, \qquad (i = 1, \ldots, N - n),$$

and the equality for the rank $\|\partial g_i/\partial x_j\| = N - n$ be held.

Prove that $M^n$ is a submanifold.

**6.35.** Let $M^n \subset \mathbf{R}^N$ be a submanifold. Prove that for any point $p \in M^n$, there exists a coordinate set $x_{i_1}, \ldots, x_{i_n}$ such that the projection of $\mathbf{R}^N$ onto the subspace $\mathbf{R}^n = \{x_{i_1}, \ldots, x_{i_n}\}$ is a local diffeomorphism of a neighbourhood of the point $p$ of the manifold $M^n$ onto an open domain in $\mathbf{R}^n$.

**6.36.** Let $M^n \subset \mathbf{R}^N$ be a submanifold. Prove that the manifold $M^n$ is specified locally by a system of equations $g_i(x_1, \ldots, x_N) = 0$ $(i = 1, \ldots, N - n)$, the equality for the rank $\|\partial g_i/\partial x_j\| = N - n$ being fulfilled.

**6.37.** Let $M^n \subset \mathbf{R}^N$ be a compact submanifold. Prove that there exists a set of smooth functions $f_1, \ldots, f_k$ on $\mathbf{R}^n$ such that the solution set of the system of equations $f_1 = f_2 = \ldots = f_k = 0$ coincides with $M^n$ and the rank of the Jacobian matrix $\|\partial f_i/\partial x_j\|$ equals $N - n$ $(k \geqslant N - n)$.

**6.38.** Show that the stereographic projection of a sphere onto a tangent plane from the pole placed opposite the point of contact is a diffeomorphism everywhere except the projection pole.

**6.39.** Prove that the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ are not diffeomorphic when $n \neq m$.

**6.40.** Prove that the groups $SL(n, \mathbf{R})$, $SL(n, \mathbf{C})$ are smooth submanifolds in spaces of real (or complex) square matrices of order $n$.

**6.41.** Prove that the group $SO(n)$ is a smooth submanifold of the space $\mathbf{R}^{n^2}$ of all square matrices of order $n$.

**6.42.** Prove that the groups $U(n)$, $SU(n)$ are smooth submanifolds in the space $\mathbf{C}^{n^2}$ of complex square matrices of order $n$.

**6.43.** Show that the matrix mapping $A \to \exp(A)$ is a smooth homeomorphism in a neighbourhood of the null matrix from the inverse image, and a neighbourhood of the unit matrix from the image. Show that the inverse mapping can be specified by the correspondence $B \to \ln(B)$.

**6.44.** Prove that some of the Cartesian coordinates of the matrix $\ln(A^{-1}X)$ can be taken as local coordinate systems in a neighbourhood $U_A$ of a matrix $A$ on each of the groups listed in Problems 6.40-6.42. Show that coordinate changes are smooth functions of class $C^\infty$ respective to the coordinate systems indicated.

**6.45.** The Riemann surface of the algebraic function $w = \sqrt[n]{P(z)}$, where $P(z)$ is a polynomial, is given by the equation $w^n - P(z) = 0$. Find a condition for the roots of the polynomial under which the Riemann surface is a two-dimensional submanifold in $\mathbf{C}^2$.

**6.46.** Show that the projection of the direct product $X \times Y$ of two manifolds $X$ and $Y$ onto the factor $X$ is a smooth mapping.

**6.47.** Prove that a compact smooth manifold $M^n$ can be embedded in the Euclidean space $\mathbf{R}^N$ for a convenient dimension $N < \infty$.

**6.48.** Prove that a smooth function on a compact smooth manifold $M$ can be represented as a coordinate under a certain embedding $M \subset \mathbf{R}^N$.

**6.49.** Prove that the product of spheres can be embedded in $\mathbf{R}^N$ of codimension 1.

**6.50.** Prove that if $\dim X < \dim Y$ and $f: X \to Y$ is a smooth mapping, then the image of the mapping $f$ does not coincide with $Y$.

**6.51.** Prove that a two-dimensional, compact, smooth and closed manifold can be immersed into $\mathbf{R}^3$.

**6.52.** (Whitney lemma.) Prove that a compact, smooth and closed manifold $M^n$ can be embedded in the Euclidean space $\mathbf{R}^{2n+1}$ and immersed into $\mathbf{R}^{2n}$.

**6.53.** Let $f: X \to Y$ be a smooth mapping of a compact and closed manifold $X^n$ into a manifold $Y^n$. Let $y_0 \in Y$ be a regular point of the mapping $f$. Prove that the inverse image $f^{-1}(y_0)$ consists of a finite number of points.

Let $f: X \to Y$ be a smooth mapping of smooth manifolds, and $M \subset Y$ a smooth submanifold. *The mapping $f$ is said to be transverse along the submanifold $M$* if for every point $x \in f^{-1}(M)$, the tangent space $T_{f(x)}(Y)$ to the manifold $Y$ is the sum (generally speaking, not direct) of the tangent space $T_{f(x)}(M)$ to the manifold $M$ and the image $\partial f(T_x(X))$ of the tangent space to the manifold $X$. Two submanifolds $M_1$ and $M_2$ of the manifold $X$ are said to intersect transversally if an embedding of one of them is transverse along the other.

**6.54.** Prove that if $y \in Y$ is a regular point of a mapping $f: X \to Y$, then $f$ is a mapping transverse along $y$.

**6.55.** Prove that the definition of a transversal intersection does not depend on the choice of order in the pair $M_1$, $M_2$.

**6.56.** Prove that if $f: X \to Y$ is a mapping transverse along a submanifold $M \subset Y$, then the inverse image $f^{-1}(M)$ is a submanifold of the manifold $X$. Calculate the dimension of $f^{-1}(M)$.

**6.57.** Investigate whether the following submanifolds intersect transversally: (a) the plane $xy$ and the axis $z$ in $\mathbf{R}^3$; (b) the plane $xy$ and plane spanned by the vectors $\{(3, 2, 0), (0, 4, -1)\}$ in $\mathbf{R}^3$; (c) the subspace $V \times \{0\}$ and the diagonal of the product $V \times V$; (d) the spaces of symmetric and skewsymmetric matrices in the space of all matrices.

**6.58.** For what values of $a$ will the surface $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally?

**6.59.** Let all the points of a mapping $f: X \to Y$ be regular, and $X$, $Y$ compact manifolds. Prove that $f$ is a locally trivial fibre map (or fibration), i.e., the inverse image $f^{-1}(U)$ of a sufficiently small neighbourhood of each point $y \in Y$ is homeomorphic to the direct product $U \times f^{-1}(y)$. In particular, if $Y$ is a connected manifold, then all submanifolds $f^{-1}(y)$, $y \in Y$ are pairwise homeomorphic.

**6.60.** Let $f: S^n \to \mathbf{R}P^n$ be a mapping associating a point $x \in S^n$ with the straight line passing through the point $x$ and the origin in $\mathbf{R}^{n+1}$. Prove that all points of the mapping $f$ are regular.

**6.61.** Let $f: SO(n) \to S^{n-1}$ associate every orthogonal matrix with its first column. Prove that all points of the mapping $f$ are regular. Find the inverse image $f^{-1}(y)$.

**6.62.** Let $f: U(n) \to S^{2n-1}$ associate every unitary matrix with its first column. Prove that all points of the mapping $f$ are regular. Find the inverse image $f^{-1}(y)$.

**6.63.** Show that the set $V_{n,k}$ of all orthonormal systems consisting of $k$ vectors from the Euclidean space $\mathbf{R}^n$ admits a smooth manifold structure. Find its dimension. Show that $V_{n,1} = S^{n-1}$, $V_{n,n} = O(n)$.

**6.64.** Show that the set $G_{n,k}$ of all $k$-dimensional subspaces in the Euclidean space $\mathbf{R}^n$ admits a smooth manifold structure. Find its dimension. Show that $G_{n,1} = \mathbf{R}P^{n-1}$.

**6.65.** Let $f: V_{n,k} \to V_{n,s}$, $s \leqslant k$ be a mapping associating an orthonormal system consisting of $k$ vectors with its first $s$ vectors. Prove that every point for the mapping $f$ is regular. Show that the inverse image $f^{-1}(y)$ is homeomorphic to the manifold $V_{n-s,\, k-s}$.

**6.66.** Let $f: O(n) \to G_{n,k}$ be a mapping associating every orthogonal matrix with the subspace generated by the first $k$ columns. Show that all points for the mapping $f$ are regular. Prove that the inverse image $f^{-1}(y)$ is homeomorphic to the manifold $O(n-k) \times O(k)$.

**6.67.** Let $f: X \times Y \to M$ be a smooth mapping, and $m_0 \in M$ a regular point. Consider the family of mappings $f_y: X \to M$, $f_y(x) = f(x, y)$.

Prove that the point $m_0$ is regular for the mappings $f_y$ almost for all values of the parameter $y$, i.e., when $y$ ranges over an open, everywhere dense subset of $Y$.

**6.68.** Solve Problem 6.67 if the point $m_0$ is replaced by a submanifold $N \subset M$, and its regularity by transversality of the mappings along the submanifold $N$.

**6.69.** Verify whether the following manifolds are orientable: (a) a sphere $S^n$; (b) a torus $T^n$; (c) a projective space $\mathbf{R}P^n$; (d) a complex projective space $\mathbf{C}P^n$; (e) groups $GL(n, R)$, $U(n)$, $SO(n)$.

**6.70.** Prove that the Klein bottle is a non-orientable, two-dimensional manifold.

**6.71.** Prove that an arbitrary complex analytic manifold is orientable.

**6.72.** Let $M$ be a manifold with boundary $\partial M$. Prove that the manifold $M$ can be embedded in the half-space $(x_{N+1} \geqslant 0)$ of the Euclidean space $\mathbf{R}^{N+1}$ so that $\partial M$ lies in the subspace $(x_{N+1} = 0)$.

**6.73.** Let a boundary $\partial M$ consist of two connected components $\partial M = M_1 \cup M_2$, $M_1 \cap M_2 = \varnothing$. Prove that the manifold $M$ can be embedded in $\mathbf{R}^N \times [0, 1]$ so that $M_1$ lies in $\mathbf{R}^N \times \{0\}$, and $M_2$ in $\mathbf{R}^N \times \{1\}$.

**6.74.** Prove that an orientable two-dimensional surface possesses a complex structure.

**6.75.** Prove that the manifolds $S^1 \times S^{2n-1}$, $S^{2n-1} \times S^{2n-1}$ possess a complex structure.

**6.76.** Prove that a compact closed odd-dimensional Riemannian manifold of positive curvature is orientable.

A function $w = f(z^1, \dots, z^n)$, $z^k = x^k + iy^k$ is said to be holomorphic if it is continuously differentiable and its differential is a complex linear form at each point $(z^1, \dots, z^n)$.

**6.77.** Show that if $f$ is a holomorphic function, then

$$\frac{\partial \operatorname{Re} f}{\partial x^k} = \frac{\partial \operatorname{Im} f}{\partial y^k},$$

$$\frac{\partial \operatorname{Im} f}{\partial x^k} = -\frac{\partial \operatorname{Re} f}{\partial y^k}.$$

**6.78.** Let $w^j = f^j(z^1, \dots, z^n)$ be a holomorphic vector function mapping $\mathbf{C}^n$ into $\mathbf{C}^m$. Find the relation between the real Jacobian matrix of this mapping and its complex Jacobian matrix.

**6.79.** Prove that a holomorphic vector function $f: \mathbf{C}^n \to \mathbf{C}^n$ produces a local coordinate system if and only if its complex Jacobian is other than zero.

**6.80.** Show that $S^2$ admits a complex analytic structure. Describe the simplest atlas of charts explicitly.

**6.81.** Show that complex projective spaces $\mathbf{C}P^2$ admit a complex analytic structure. Describe the simplest atlas of charts explicitly.

**6.82.** Identify $S^2$ with $\mathbf{C}P^1$.

# 7
# Transformation Groups

**7.1** Prove that all one-parameter smooth homeomorphism groups on a compact manifold are in one-to-one correspondence with smooth vector fields of point trajectory velocities.

**7.2.** Let $X$ be a smooth connected manifold, and $x_0$, $x_1$ two arbitrary points. Find a one-parameter group of smooth transformations $\varphi_t$ such that $\varphi_1(x_0) = x_1$. Show that we can assume, without loss of generality, all the transformations $\varphi_t$ to be identity outside a certain compactum.

**7.3.** Give an example of a vector field on a non-compact manifold whose trajectories are not generated by the action of any one-parameter transformation group.

**7.4.** Let $\xi$ be a constant vector field respective to angular coordinates on the two-dimensional torus $T^2$. Investigate under which conditions for the coordinates of the field $\xi$, the integral curves are closed.

**7.5.** Generalize the previous problem to the case of the torus $T^n$, viz., let $\xi = (\xi^1, \ldots, \xi^n)$ be a constant vector field respective to angular coordinates on the torus $T^n$. Prove that the closure of any trajectory is homeomorphic to the torus $T^k$, where $k$ is the number of linearly independent numbers $\xi^1, \ldots, \xi^n$ over the field of rational numbers.

**7.6.** Let a finite group $G$ act smoothly on a smooth manifold $X$. Prove that if the action of the group $G$ is free (i.e., each point $x \in X$ is transformed into itself only under the action of the unit element of the group $G$), then the factor space $X/G$ is a manifold.

**7.7.** Show that the projective space $\mathbf{R}P^n$ is a factor space $S^n/\mathbf{Z}_2$ under a certain action of the group $\mathbf{Z}_2$ on the sphere $S^n$.

**7.8.** Show that the complex projective space $\mathbf{C}P^n$ is the factor space $S^{2n+1}/S^1$ under the action of the group $S^1$ on the sphere $S^{2n+1}$.

**7.9.** Let a finite group $G$ act smoothly on a manifold $X$, and $x_0 \in X$ be a fixed point under the action of any element of the group $G$. Prove that in a neighbourhood of the point $x_0$, there is a local coordinate system with respect to which the action of the group $G$ is linear.

**7.10.** Generalize the previous problem to the case of an arbitrary compact Lie group.

**7.11.** Prove that the set of all fixed points under the action of a finite group $G$ on a smooth manifold is the union of smooth submanifolds (generally speaking, of different dimensions).

**7.12.** Let $G$ be a Lie group. Show that the action of the group $G$ on itself via left (or right) translations is smooth.

**7.13.** Let a Lie group $G$ act on itself via inner automorphisms. Prove that the set of fixed points coincides with the centre of the group $G$.

**7.14.** Prove that the group of isometry of a Riemannian space is a smooth manifold.

**7.15.** List all finite-dimensional Lie groups of transformations of the straight line $\mathbf{R}^1$.

**7.16.** Find the group of all linear fractional transformations preserving the disk $|z| \leqslant 1$ in the complex plane. Prove that this group is isomorphic to the group $SL(2, \mathbf{R})/\mathbf{Z}_2$ and also to the group of all transformations preserving the form $dx^2 + dy^2 - dt^2$ in $\mathbf{R}^3(x, y, t)$. Establish a relation to Lobachevskian geometry.

**7.17.** Prove that the connected component of the unit element of the isometry group on the Lobachevski plane (under the standard metric of constant curvature) is isomorphic to $SL(2, \mathbf{R})/\mathbf{Z}_2$. Find the total number of components in the group of motions of the Lobachevski plane.

**7.18.** A solid ball is pressed in between two parallel planes (which are tangent to it). With the planes moving (so that they remain parallel and at the same distance from each other), the ball rotates without slipping at the points of contact. Consider all motions of the ball induced by the motion of the upper plane such that the lower point of contact of the ball describes a closed trajectory on the lower plane, i.e., the point of contact returns to the original position. What part of the group $SO(3)$ can be obtained by such ball rotations (rotations of the ball are considered after its centre returns to the original point)?

**7.19.** Prove that the isometry group of Euclidean space is generated by orthogonal transformations and parallel displacements.

**7.20.** Prove that the isometry group of the standard $n$-dimensional sphere is isomorphic to the group of orthogonal transformations of the $(n + 1)$-dimensional Euclidean space.

**7.21.** Prove that the groups $Sp(1)$ and $SU(2)$ are isomorphic (as Lie groups). Prove that they are diffeomorphic to the sphere $S^3$. Establish the relation to quaternions.

**7.22.** Prove that in the algebra of quaternions, multiplication by a quaternion $A : x \to Ax$ generates the transformation group $SU(2)$. Prove that the transformations of the form $x \to AxB$, where $A, B$ are quaternions, generate the group $SO(4)$. Prove that *SO(4) is isomorphic to the factor* *group* $S^3 \times S^3/\mathbf{Z}_2$, where $S^3$ is supplied with the structure of the group $SU(2) \cong Sp(1)$. Find the fundamental group $SO(4)$, and also $SO(n)$ for any $n$.

**7.23.** Prove that the Lie groups $SO(n)$, $SU(n)$, $U(n)$, $Sp(n)$ are connected. Prove that there are two connected components in the group $O(n)$. Find the number of connected components in the group of motions of the pseudo-Euclidean plane of index 1. Prove that the group $SL(2, \mathbf{R})/\mathbf{Z}_2$ is connected.

**7.24.** Let us realize the group $U(n)$ and its Lie algebra $u(n)$ as submanifolds in the Euclidean space of all square complex matrices of order $n \times n$ and consider the natural embedding of unitary and skewhermitian matrices in this space.

(a) Prove that $U(n) \subset S^{2n^2-1}$, where the sphere $S^{2n^2-1}$ is standardly embedded in $\mathbf{R}^{2n^2} = \mathbf{C}^{n^2}$ and has radius $\sqrt{n}$.

(b) Prove that the Riemannian metric induced on the group $SU(n)$, which is considered as a submanifold of $S^{2n^2-1}$, coincides with the Cartan-Killing metric invariant on the group $SU(n)$.

(c) Find the intersection $U(n) \cap u(n)$ by considering these sets as submanifolds in the space $\mathbf{C}^{n^2}$.

Solve similar problems for the groups $O(n)$ and $Sp(n)$.

**7.25.** Find the factor group $\mathfrak{G}/\mathfrak{G}_0$, where $\mathfrak{G}$ is the group of motions of the Lobachevski plane (under the standard metric), $\mathfrak{G}_0$ the connected component of the unit element. Indicate all conformal transformations of the standard metric.

**7.26.** Find all discrete subgroups of the group $\mathfrak{G}$ of affine transformations of the straight line $\mathbf{R}^1$.

**7.27.** Describe all discrete normal subgroups of the following compact Lie groups: $O(n)$, $SO(n)$, $SU(n)$, $U(n)$, $Sp(n)$.

**7.28.** Find all symmetry groups of all regular polygons. Find all symmetry groups (groups of motions) of all regular convex polyhedra in $\mathbf{R}^3$. Indicate the non-commutative groups among them.

**7.29.** Prove that left-invariant vector fields on a Lie group $G$ are in one-to-one correspondence with the vectors of the tangent space $T_e(G)$ to the group $G$ at the unit element.

**7.30.** Prove that the Poisson bracket of two left-invariant vector fields is also a left-invariant vector field, i.e., the commutator operation transforms the space $T_e(G)$ into a Lie algebra.

**7.31.** Let $\xi$ be a left-invariant vector field, and $\varphi_t$ a one-parameter transformation group associated with it. Prove that $\varphi_t$ is a right translation for any $t$, i.e., $\varphi_t(g) = gh_t$, $t_t \in G$.

Let $G$ be a Lie group, and $x^1, \ldots, x^n$ a local coordinate system in a neighbourhood of the unit element (we will assume its coordinates to be zeroes). Then the operation of multiplication induces the vector-valued function $q = q(x, y) = xyx^{-1}y^{-1}, x = (x^1, \dots, x^n), y = (y^1, \dots, y^n)$. If the function $q = q(x, y)$ is expanded into Taylor's series, then it will assume the following form:

$$q^i = \sum_{j,k} c^i_{jk} x^j y^k + \epsilon^i_2,$$

where $\epsilon^i_2$ is an infinitesimal of the third order with respect to the coordinates $x^i$, $y^i$.

The bilinear expression

$$\zeta^i = \sum_{j,k} c^i_{j,k} \xi^j \eta^k \quad (\zeta = [\xi, \eta])$$

determines a certain operation (called the Poisson bracket of the vectors $\xi$ and $\eta$) over the tangent vectors in the unit element of the group $G$. Thus, the tangent space $T_e(G)$ has been transformed into an algebra called the Lie algebra of the Lie group $G$. Usually, it is denoted by the small letter $g$.

**7.32.** Show that the following properties are fulfilled in a Lie algebra:

(a) $[\xi, \eta] = -[\eta, \xi]$;

(b) $[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$.

**7.33.** Verify that an operation in a Lie algebra $g$ is transformed into the Poisson bracket of vector fields if a vector $\xi$ is associated with a (right-) left-invariant vector field.

**7.34.** Let $x(t)$, $y(t)$ be two curves passing through the unit element of the group $G$,

$$\xi = \frac{dx}{dt}(0), \qquad \eta = \frac{dy}{dt}(0).$$

Show that

$$[\xi\ \eta] = \frac{d}{dt}(x(\sqrt{t})\, y(\sqrt{t})\, x^{-1}(\sqrt{t})\, y^{-1}(\sqrt{t}))|_t = 0.$$

**7.35.** Let $\gamma(t)$ be a one-parameter subgroup of a Lie group. Assume that $\gamma$ intersects itself. Show that there exists a number $L > 0$ such that $\gamma(t + L) = \gamma(t)$ for all $t \in \mathbf{R}$.

**7.36.** Let $G$ be a compact, connected Lie group. Show that each point $x \in G$ belongs to a certain one-parameter subgroup.

**7.37.** Let $G$ be a compact group acting smoothly on a manifold $M$. Show that there is a Riemannian metric on $M$ such that $G$ is the isometry group.

**7.38.** Show that a commutative, connected Lie group is locally isomorphic to a finite-dimensional vector space.

**7.39.** Show that a compact, commutative, connected Lie group is isomorphic to the torus.

**7.40.** Show that a commutative, connected Lie group is isomorphic to the product of the torus and a vector space.

**7.41.** Let a Lie group $G$ be a subgroup of the matrix group $GL(n, \mathbf{C}) \subset \mathbf{C}^{n^2} = \mathrm{End}(n, \mathbf{C})$. Show that the commutator operation in the Lie algebra $g$ of the group $G$, which is understood to be a subspace of $\mathrm{End}(n, \mathbf{C})$, coincides with the usual commutator of matrices, i.e., $[\xi, \eta] = \xi\eta - \eta\xi$, $\xi, \eta \in g$.

**7.42.** Describe the Lie algebras of the following matrix Lie groups:

$SL(n, \mathbf{C})$, $SL(n, \mathbf{R})$, $U(n)$, $O(n)$, $O(n, m)$, $Sp(n)$.

**7.43.** Prove that the operator $Y : \vec{x} \to [\vec{y}, \vec{x}]$ is determined by a skew-symmetric matrix. Find the relation between the coefficients of this matrix and the coordinates of the vector $\vec{y}$.

**7.44.** Let $Y$, $Z$ be two matrices of vector multiplication operators by two vectors $y$, $z$. Prove that the matrix of the vector multiplication operator by $[y, z]$ equals $[Y, Z] = YZ - ZY$.

**7.45.** Prove that a finite group cannot operate effectively on $\mathbf{R}^n$.

# 8
# Vector Fields

**8.1.** Prove the equivalence of the three definitions of a tangent vector to a manifold at a point $P$:

(a) tensor of rank (1, 0);

(b) differentiation operator of smooth functions at the point $P$;

(c) a class of osculating curves at the point $P$.

**8.2.** Find the derived function $f$ at a point $P$ in the direction of the vector $\xi$:

(a) $f = \sqrt{x^2 + y^2 + z^2}$; $P = (1, 1, 1)$, $\xi = (2, 1, 0)$;

(b) $f = x^2y + xz^2 - 2$; $P = (1, 1, -1)$, $\xi = (1, -2, 4)$;

(c) $f = xe^y + ye^x - z^2$; $P = (3, 0, 2)$, $\xi = (1, 1, 1)$;

(d) $f = \dfrac{x}{y} - \dfrac{y}{x}$; $P = (1, 1)$, $\xi = (4, 5)$.

**8.3.** Find the derivative of the function $f = \ln(x^2 + y^2)$ at the point $P = (1, 2)$ along the curve $y^2 = 4x$.

**8.4.** Find the derivative of the function $f = \tan^{-1}(y/x)$ at the point $P = (2, -2)$ along the curve $x^2 + y^2 - 4x = 0$.

**8.5.** Find the derivative of the function $f$ at the point $P$ along the curve $\gamma$:

(a) $f = x^2 + y^2$, $P = (1, 2)$, $\gamma : x^2 + y^2 = 5$;

(b) $f = 2xy + y^2$, $P = (\sqrt{2}, 1)$, $\gamma : \dfrac{x^2}{4} + \dfrac{y^2}{2} = 1$;

(c) $f = x^2 - y^2$, $P = (5, 4)$, $\gamma : x^2 - y^2 = 9$;

(d) $f = \ln(xy + yz + xz)$, $P = (0, 1, 1)$, $\gamma : x = \cos t$, $y = \sin t$, $z = 1$;

(e) $f = x^2 + y^2 + z^2$, $P = (0, R, \pi a/2)$, $\gamma : x = R\cos t$, $y = R\sin t$, $z = at$.

**8.6.** Find the derivative of the function $f = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$ at an arbitrary point $P = (x, y, z)$ in the direction of the radius vector of this point.

**8.7.** Find the derivative of the function $f = 1/r$, $r = \sqrt{x^2 + y^2 + z^2}$ in the direction of its gradient.

**8.8.** Find the derivative of the function $f = yze^x$ in the direction of its gradient.

**8.9.** Find the derivative of the function $f = f(x, y, z)$ in the direction of the gradient of the function.

**8.10.** Let $\nabla$ be a vector differential operator in $\mathbf{R}^3$ whose components are as follows: $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$. Show that

(a) $\mathrm{grad} F = \nabla F$;

(b) $\mathrm{div} X = (\nabla, X)$;

(c) $\mathrm{rot} X = [\nabla \times X]$.

**8.11.** Prove the formula

$$\mathrm{div}(uX) = u \,\mathrm{div}\, X + (X, \mathrm{grad} u),$$

where $X$ is a vector field, and $u$ a function in $\mathbf{R}^3$.

**8.12.** Prove the formula

$$\mathrm{rot}(uX) = u \,\mathrm{rot}\, X - [X, \mathrm{grad} u].$$

**8.13.** Calculate $\mathrm{div} X[X \times X]$.

**8.14.** Prove that the vector $x = u \,\mathrm{grad}\, v$ is orthogonal to rot $X$.

**8.15.** Show that

(a) $\mathrm{div}\,(\mathrm{rot}\, X) = 0$;

(b) $\mathrm{rot}\,\mathrm{rot}\, X = \mathrm{grad}\,\mathrm{div} X - \Delta X$,

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where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

**8.16.** Let $X = (x, y, z)$. Show that

(a) $\operatorname{div} X = 3$;

(b) $\operatorname{rot} X = 0$;

(c) $\operatorname{div}\left(\frac{X}{|X|^3}\right) = 0$;

(d) $\operatorname{rot}\left(\frac{X}{|X|^3}\right) = 0$;

(e) $\operatorname{grad} \frac{1}{|X|} = -\frac{X}{|X|^3}$.

Find a function $\varphi$ such that $X = \operatorname{grad}\varphi$.

**8.17.** Let $\mathbf{v}(x, y, z)$ be the field of velocities of a solid rotating about some axis. Show that

(a) $\operatorname{div}(\mathbf{v}) = 0$;

(b) $\operatorname{rot}(\mathbf{v}) = 2\mathbf{w}$,

where $\mathbf{w}$ is an angular velocity vector.

**8.18.** Let $X = (x, y, z)$, and $Y$ a constant vector field. Show that $\operatorname{rot}[Y \times X] = 2Y$.

**8.19.** Show that $\operatorname{rot} \operatorname{grad} F = 0$.

**8.20.** Prove the formula

$$\Delta(FG) = F\Delta G + G\Delta F + 2(\operatorname{grad} F, \operatorname{grad} G).$$

**8.21.** Solve the equation $\operatorname{rot} X = Y$ if

(a) $Y = (1, 1, 1)$;

(b) $Y = (2y, 2z, 0)$;

(c) $Y = (0, 0, e^x - e^y)$;

(d) $Y = (6y^2, 6z, 6x)$;

(e) $Y = (3y^2, -3x^2, -(y^2 + 2x))$;

(f) $Y = (0, 2\cos xz, 0)$;

(g) $Y = (-y/(x^2 + y^2), x/(x^2 + y^2), 0)$;

(h) $Y = (ye^{x^2}, 2yz, -(2xyze^{x^2} + z^2))$.

**8.22.** Prove that to each smooth vector field on a manifold, there corresponds a one-parameter group of diffeomorphisms $\varphi_t$ whose trajectories are tangent to the given vector field.

**8.23.** Show that the Poisson bracket of vector fields (as differentiation operators) is a vector field.

**8.24.** Let $\varphi_t$ be a one-parameter group of diffeomorphisms associated with a vector field $\xi$. Show that

$$[\eta, \xi] = \frac{d}{dt}(\varphi_t^*(\eta) - \eta).$$

**8.25.** Let $\xi$, $\eta$ be two vector fields, and $f$, $g$ two smooth functions. Prove the formula

$$[f\xi, g\eta] = fg[\xi, \eta] + g\,\eta(f)\,\xi - f\xi(g)\,\eta.$$

**8.26.** Let $\xi$, $\eta$ be two vector fields, and $\varphi_t$, $\psi_t$ the one-parameter transformation groups associated with them. Show that if $[\xi, \eta] = 0$, then the transformations $\varphi_t$ and $\psi_t$ commute.

**8.27.** Let $V$ be a linear finite-dimensional space of vector fields which is closed under the Poisson bracket operation, i.e., $[\xi, \eta] \in V$ when $\xi, \eta \in V$. Show that $V$ is a Lie algebra.

**8.28.** (See the previous problem.) Show that the Lie group $G$ corresponding to the algebra $V$ acts on a manifold, each field $\xi \in V$ specifying a one-dimensional subgroup of the group $G$ whose orbits under this action are tangent to the vector field $\xi$.

**8.29.** Let $P$, $Q$ be two arbitrary points of the disk $D_n \subset \mathbf{R}^n$. Find a diffeomorphism $\varphi$ on $\mathbf{R}^n$ such that $\varphi(P) = Q$, $\varphi(x) = x$, when $x \in D_n$.

**8.30.** Let $\xi$ be a vector field on a manifold $X$, $P \in X$. Show that if $\xi|_P \neq 0$, then there exists a local coordinate system $(x^1, \ldots, x^n)$ in a neighbourhood of the point $P$ such that $\xi = \partial/\partial x^1$.

**8.31.** Construct three linearly independent smooth vector fields at each point of the standard sphere $S^3$. Find the explicit forms of the integral curves on these fields.

**8.32.** Construct the integral curves of the following vector fields on the plane:

(a) $\xi = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$;

(b) $\xi = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$;

(c) $\xi = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$;

(d) $\xi = (x + y)\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$;

(e) $\xi = (x - y)\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$;

(f) $\xi = x^2\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y}$.

**8.33.** Prove that the singular points (zeroes) of the vector fields $\operatorname{grad}\operatorname{Re}(f(z))$, $\operatorname{grad}\operatorname{Im} f(z)$ coincide with the zeroes of the derivative $f'_z(z)$.

**8.34.** Find the integral curves of a flow $\mathbf{v}_1(x)$ orthogonal to a flow $\mathbf{v}_2(x)$, where $\mathbf{v}_2(x) = \operatorname{grad} f(x)$, $x \in \mathbf{R}^2$, $f(x)$ is the magnitude of the angle $AxB$ ($A$, $B$ being certain two points of the plane $\mathbf{R}^2$ and $x$ a variable point).

**8.35.** Specify the quality characteristics for the integral curve distribution of the flows $\mathbf{v}_1 = \operatorname{grad}\operatorname{Re}(f(z))$, $\mathbf{v}_2 = \operatorname{grad}\operatorname{Im}(f(z))$ of the complex-analytic functions $f(z)$ listed below. Find the singular points of the flows $\mathbf{v}_1$, $\mathbf{v}_2$. Investigate the stability of the singular points. Specify the quality characteristics of the behaviour of the trajectories of the flows $\mathbf{v}_1$, $\mathbf{v}_2$ on the sphere $S^2$ (extended plane $\mathbf{R}^2 : S^2 = \mathbf{R}^2 \cup \infty$). Specify the resolution process of the singularity $z = 0$ of these vector fields for a small perturbation of the original function $f(z)$ leading to a function $g(z)$ with all the singular points of the flows $\mathbf{v}_1$, $\mathbf{v}_2$ non-degenerate:

(a) $f(z) = z^n$ (where $n$ is an integer);

(b) $f(z) = z + 1/z$ (Zhukovski function);

(c) $f(z) = z + 1/z^2 b$ (where $b$ is an integer);

(d) $f(z) = z + 1/(z - 2)$;

(e) $f(z) = z^4(2(z - 5)^2 + 12z^6)$ (investigate in a neighbourhood of the point $z = 0$);

(f) $f(z) = z^3(z - 1)^{100}(z - 2)^{900}$;

(g) $f(z) = 2z - \ln z$;

(h) $f(z) = 1 + z^4(z^4 - 4)^{44}(z^{44} - 44)^{444}$ (investigate in a neighbourhood of the point $z = 0$);

(i) $f(z) = \dfrac{1}{100}\ln[(z - 2i)/(z - 4)]^3$;

(j) $f(z) = 1/(z^2 + 2z - 1)$;

(k) $f(z) = \dfrac{2}{z} + 21\ln z^2$;

(l) $f(z) = z^5 + 2\ln z$;

(m) $f(z) = 2\ln(z - 1)^2 - 4/3\ln(z + 10i)^3$;

(n) $f(z) = 1/z^3 - 1/(z - i)^3$;

(o) $f(z) = (2 + 5i/2)\ln[(4z - 2)/(64z + i)]$;

(p) $f(z) = (1 - i/2)^4 \ln \left( \dfrac{18z - i}{10z + 1} \right)^{\dot{2}}$.

**8.36.** Prove that the irrotational flow $\mathbf{v} = (\mathbf{P}, Q)$, where $\mathbf{P}$, $Q$ are the components of the flow on the plane $\mathbf{R}^2(x, y)$, is potential and $\mathbf{v} = \operatorname{grad} f(x, y)$ for a certain smooth function $f$. What can be said about the potential of $f$, given additionally that the flow is incompressible, i.e., $\operatorname{div}(\mathbf{v}) = 0$?

**8.37.** Let a vector field $\xi$ satisfy the condition $\operatorname{div}(\xi) = 0$. Show that the displacement operator along the integral curves is unitary.

**8.38.** Find all homotopy classes of the vector fields on the torus $T^2$.

**8.39.** Prove that if a vector field $X$ on the two-dimensional torus is homotopic to $d\varphi_1$, then it possesses a periodic trajectory.

**8.40.** Find the greatest number of linearly independent tangent vector fields on a smooth closed surface $M^2$.

**8.41.** Prove that the indices of two vector fields on an arbitrary two-dimensional and closed surface are equal. Does the statement hold for a manifold of any dimension?

**8.42.** Let $m$, $n$ be the rotation numbers of the vector field on the torus $T^2$, $\lambda = (m, n)$. Prove that this field has $\lambda$ periodic solutions (closed trajectories).

**8.43.** (The Poincaré-Bendixson theorem.) Prove that if an arbitrary integral curve of some vector field on the plane is compact and contains no singular points, then it is periodic.

**8.44.** Prove that if $P$ on the plane is a limit point for some trajectory of a vector field, then the trajectory passing through $P$ is limiting for the original trajectory.

**8.45.** Prove that the set of vector fields possessing only isolated singularities is connected.

**8.46.** Prove that the sum of the indices at singularities of a vector field on a compact and closed manifold is unaltered in smooth deformations.

**8.47.** Prove that the set of all integral curves of the vector field $\mathbf{v}(x) = (x^1, -x^0, x^3, -x^2)$, where $x = (x^0, x^1, x^2, x^3) \in S^3 : (|x| = 1) \subset \mathbf{R}^4$, is homeomorphic to the sphere $S^2$. Find the relation to the Hopf map $S^3 \xrightarrow{S^1} S^2$. How is this vector field related to quaternions?

**8.48.** Let $\mathbf{v}(x)$ be a smooth vector field on the plane $\mathbf{R}^2$, $L$ a smooth self-intersecting contour on the plane $\mathbf{R}^2$, $j_L$ the index of the contour $L$ in the vector field $\mathbf{v}(x)$, $J$ the number of points where the field $\mathbf{v}$ and contour $L$ touch internally, and $E$ the number of points touching externally.

Prove that if the number of all points of contact of the field with the contour is finite, then $j_L \leqslant \frac{1}{2}(2 + J - E)$. Prove that the index of any isolated singular point of a smooth irrotational vector field $\mathbf{v}(x) = \operatorname{grad} f(x)$ is always not greater than unity (note that this singular point may, certainly, be degenerate).

**8.49.** How many solutions can the equation $\sin z = z$ have over the field of complex numbers?

**8.50.** Put $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$; $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. Show that a function $f$ is holomorphic if and only if $\frac{\partial}{\partial \bar{z}^k}(f) \equiv 0$ for all $k$.

**8.51.** Show that a vector field $\xi$ is holomorphic if and only if it has the form $\xi = \sum a^i \frac{\partial}{\partial z^i}$, where $a^i$ are holomorphic functions, with respect to a local coordinate system $(z^1, \ldots, z^n)$.

# 9
# Tensor Analysis

**9.1.** Determine the type of the following tensors:

(a) $T_i \frac{\partial f}{\partial x^i}$;

(b) $T_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ at those points where the gradient of the function $f$ vanishes;

(c) $T^i_j$, i.e., the components of the matrix of a linear operator on a vector space;

(d) $T_{ij}$, i.e., the components of the matrix of a bilinear form on a vector space.

**9.2.** Let

$$\delta^i_j = \begin{cases} 0, & \text{when} \quad i \neq j \\ 1, & \text{when} \quad i = j \end{cases}.$$

Show that $\{\delta^i_j\}$ yields a tensor of type (1, 1).

**9.3.** Let $\{\xi^{ij}\}$ be a tensor of type (2, 0). Show that the numbers $\eta_{ij}$ satisfying the condition $\xi^{ij}\eta_{jk} = \delta^i_k$ yield a tensor of type (0, 2).

**9.4.** Show that if $f: V^m_n \to V^{m^1}_{n^1}$ is a linear mapping of tensor spaces, then the mapping components yield a tensor of type $(m, n)$.

**9.5.** Determine the dimension of the tensor space $V^m_n$.

**9.6.** Show that any tensor of type (2, 0) can be decomposed uniquely into the sum of a symmetric and a skewsymmetric addend.

**9.7.** Determine the dimension of the space $\Lambda^k$ of skew-symmetric tensors.

**9.8.** Determine the dimension of the space $S^k$ of symmetric tensors.

**9.9.** Prove the formula

$$\Lambda_k(V_1 \oplus V_2) = \bigoplus_{\alpha+\beta=R} \Lambda_\alpha(V_1) \otimes \Lambda_\beta(V_2).$$

**9.10.** Calculate the components of the fundamental tensor of the plane with respect to a system of polar coordinates.

**9.11.** Calculate the components of the fundamental tensor of $\mathbf{R}^3$:

(a) with respect to a system of cylindrical coordinates;

(b) with respect to a system of spherical coordinates.

**9.12.** Calculate the components of the fundamental tensor of the sphere $S^2$:

(a) with respect to a system of spherical coordinates;

(b) with respect to Cartesian coordinates on the stereographic projection.

**9.13.** Assuming that the gradient of a function $f$ is the composite of two operations, viz., that of partial differentiation and that of raising the indices, write the gradient of the function with respect to:

(a) a system of polar coordinates;

(b) a system of cylindrical coordinates;

(c) a system of spherical coordinates.

**9.14.** Find the gradient of the function $f = \ln\sqrt{x^2 + y^2 + z^2}$.

**9.15.** Derive the following formulae for the functions $f$ and $g$ with respect to an arbitrary system of coordinates:

(a) $\operatorname{grad}(\lambda f) = \lambda \operatorname{grad} f$; $\lambda = \text{const}$;

(b) $\operatorname{grad}(f \pm g) = \operatorname{grad} f \pm \operatorname{grad} g$;

(c) $\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$;

(d) $\operatorname{grad}(f/g) = \dfrac{g \operatorname{grad} f - f \operatorname{grad} g}{g^2}$, $g \neq 0$;

(e) $\operatorname{grad}(f(g)) = \dfrac{df}{dg} \operatorname{grad} g$.

**9.16.** Let $f = f(u, v)$, where $u$, $v$ are two functions. Show that $\operatorname{grad} f =$

$$= \frac{\partial f}{\partial u} \operatorname{grad}(u) - \frac{\partial f}{\partial v} \operatorname{grad}(v).$$

**9.17.** Write the formula for the derivative of a function $f$ with respect to an arbitrary system of coordinates:

(a) in the direction of its gradient;
(b) in the direction of the gradient of the function.

**9.18.** Derive a formula enabling us to determine the greatest change of a function $f$ at a given point with respect to an arbitrary system of coordinates.

**9.19.** Write a solid medium deformation tensor

$$u^{ik} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} + \frac{\partial u^l}{\partial x^i}\frac{\partial u^l}{\partial x^k}\right)$$

with respect to an arbitrary system of coordinates using the fundamental tensor. Write out separately similar formulae for the terms which are linear respective to $u^i$.

**9.20.** Prove that the Christoffel symbols of two connections differ by addends which are the components of a tensor.

**9.21.** Show that the covariant derivative along a curve depends on the value of the Christoffel symbols of this curve.

**9.22.** Show that if two submanifolds osculate to some curve $\gamma$, then the parallel displacement operation does not depend on the choice of a submanifold.

**9.23.** Show that the parallel displacement operation can be obtained on a submanifold by passage to the limit of the composite of a parallel displacement in the ambient manifold and the orthogonal projection onto the tangent space to the submanifold.

**9.24.** Calculate the angle through which the tangent vector to a right circular cone turns after parallel displacement along a closed curve. Establish the dependence on the kind of the curve.

**9.25.** Calculate the angle through which the tangent vector of a sphere turns after parallel displacement along a curve $\gamma$ if:

(a) $\gamma$ is a parallel;
(b) $\gamma$ is made up of two meridians and a part of the equator which is included in between them;

(c) $\gamma$ is made up of two meridians and a part of the parallel which is included in between them.

**9.26.** Establish a dependence between the angle of rotation of the tangent vector to a sphere after parallel displacement along a closed curve $\gamma$ and the area of the region bounded by the curve $\gamma$.

**9.27.** Generalize Problem 9.26 to the case of a surface with constant Gaussian curvature.

**9.28.** Prove that if the curvature tensor of a Riemannian manifold is identically equal to zero, then the operation of parallel displacement along a curve $\gamma$ does not depend on a homotopy of the path $\gamma$.

**9.29.** Calculate the scalar curvature of the following Riemannian manifolds:

(a) $S^2$;
(b) the torus $T^2$ embedded in $\mathbf{R}^3$;
(c) the Lobachevski plane;
(d) the right circular cone;
(e) the cylinder;
(f) the group $SO(n)$ with a bi-invariant metric.

**9.30.** Show that any two sufficiently near points on any compact, Riemannian manifold can be joined by a geodesic line, the geodesic of the least length being unique.

**9.31.** Describe the geodesics in the following Riemannian manifolds:

(a) $\mathbf{R}^2$;
(b) the torus $T^2$ under the flat metric;
(c) $S^2$;
(d) the Lobachevski plane.

**9.32.** Let $\sum_k^7 = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^{6k-1} = 0\} \cap \cap \left\{ \sum_{\alpha=1}^{5} |z_\alpha|^2 = 1 \right\}$ be the Brieskorn spheres ($k = 1, \ldots, 28$). Prove that the Riemannian metric induced on these spheres (in embedding $\sum_k^7 \to S^y \subset \mathbf{C}^5 = \mathbf{R}^{10}$) is not a metric with positive curvature.

**9.33.** Prove that there always exist a pair of conjugate points on a compact, 1-connected manifold $M^n$. What happens if $M^n$ is not 1-connected? Is there a conjugate point on every geodesic $\gamma(t)$ (the geodesics emanating from one point)?

**9.34.** The following kinds of manifolds of strictly positive curvature are known, being the classical symmetric spaces of rank 1, viz., $S^n$, $RP^n$,

$CP^n$, $QP^n$, $K^{16}$. Calculate the curvature (find the limits within which the curvature varies) of $R(\sigma)$ on $CP^n$, $QP^n$, $K^{16}$.

**9.35.** Let $\mathfrak{G}$ be a Lie group, and $\langle,\rangle_g$ a bi-invariant metric on $\mathfrak{G}$, where $g \in \mathfrak{G}$ is the variable point on $\mathfrak{G}$.

Recall that a metric is said to be bi-invariant if it is preserved under left and right translations, viz.,

$$L_{g_0} : g \to g_0 g; \quad R_{g_0} : g \to g g_0.$$

Prove that it follows from the bi-invariance of the metric $\langle,\rangle_g$ on $\mathfrak{G}$ that the form $\langle,\rangle_e$ is invariant under all transformations of the form $\mathrm{Ad}_g : X \to gXg^{-1}$.

**9.36.** Give examples of matrix Lie algebras $G$ such that the quadratic form $\langle XY \rangle_e = \mathrm{Tr}(X\bar{Y}^T)$ is non-singular, where $X, Y \in G$, the bar denotes complex conjugacy, and Tr transposing.

**9.37.** Let $\langle,\rangle_g$ be a bi-invariant Riemannian metric on a Lie group $\mathfrak{G}$. Let $\nabla$ be a symmetric Riemannian connection on $\mathfrak{G}$, compatible with the metric $\langle,\rangle_g$. Prove that the geodesics of the connection $\nabla$ are the following trajectories only: one-parameter subgroups of the group $\mathfrak{G}$ and their translations (left and right).

**9.38.** Let $X$ be a left-invariant vector field on a group $\mathfrak{G}$. Prove that the integral curves $\gamma(t)$ of this field are left translations of a one-parameter (i.e., one-dimensional) subgroup which passes through the unit element of the group in the direction of the vector $X(e)$, where $X(e)$ is the value of the field $X$ at a point $e \in \mathfrak{G}$.

**9.39.** Let $X$, $Y$ be two invariant vector fields on a group $\mathfrak{G}$, and $\nabla$ a symmetric connection compatible with a bi-invariant Riemannian metric on $\mathfrak{G}$. Prove that $\nabla_X(Y) = 1/2\,[X, Y]$, where $[X, Y] = XY - YX$ (the Lie group being a matrix).

**9.40.** Let $\langle,\rangle$ be a bi-invariant metric on the Lie algebra $G$ of a group $\mathfrak{G}$, and $[X, Y] = XY - YX$ the commutator in $G$. Prove that $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$.

**Note.** The operation $X \to [X, Y]$ is sometimes denoted as $\mathrm{ad}_Y : X \to [X, Y]$; then the required relation is written as

$$\langle \mathrm{ad}_Y X, Z\rangle = -\langle X, \mathrm{ad}_Y Z\rangle.$$

**9.41.** Let $\mathfrak{G}$ be a compact Lie group with a bi-invariant metric, and $X$, $Y$, $Z$ left-invariant vector fields on $\mathfrak{G}$. Prove that $R(X, Y)Z = 1/4\,[[X, Y], Z]$.

**9.42.** Let $\mathfrak{G}$ be a compact Lie group with the bi-invariant metric $\langle , \rangle$, and $X, Y, Z, W$ left-invariant vector fields on $\mathfrak{G}$.

Prove that

$$\langle R(X, Y)Z, W\rangle = 1/4 \langle [X, Y], [Z, W]\rangle.$$

Let $\mathfrak{G}$ be a compact Lie group, and $X, Y$ two orthogonal unit vectors (bi-invariant metric $\langle , \rangle$ is given on $\mathfrak{G}$). We call the number $\sigma(X, Y) = \langle R(X, Y)X; Y\rangle$ the sectional curvature determined by the vectors $X, Y$.

Prove that $\sigma(X, Y) \geqslant 0$ and $\sigma(X, Y) = 0$ if and only if $[X, Y] = 0$.

**Hint:** $\sigma(X, Y) = 1/4 \langle [X, Y], [X, Y]\rangle = 1/4 \|[X, Y]\|^2 \geqslant 0$.

# 10
# Differential Forms, Integral Formulae, De Rham Cohomology

**10.1.** Prove that if vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ are linearly dependent, then $T(\mathbf{v}_1, \ldots, \mathbf{v}_p) = 0$ for any form $T \in \Lambda^p(V^*)$.

**10.2.** Prove that if forms $\varphi_1, \ldots, \varphi_p \in V^*$ are linearly dependent, then $\varphi_1 \wedge \ldots \wedge \varphi_p = 0$.

**10.3.** Let $\varphi_1, \ldots, \varphi_n \in V^*$, and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Prove that

$$(\varphi_1 \wedge \ldots \wedge \varphi_n)(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \frac{1}{n!} \det\|\varphi_i(\mathbf{v}_j)\|^n_{i,j}.$$

**10.4.** Prove that the element of volume equals $\sqrt{\det(g_{ij})}dx^1 \wedge \ldots \wedge dx^n$, where $g_{ij}$ is the Riemannian metric with respect to the coordinates $(x^1, \ldots, x^n)$.

**10.5.** Show that the exterior differentiation operation of a differential form can be represented as the composite of the gradient covariant component operation and that of alteration for an arbitrary symmetric connection on a manifold.

**10.6.** Calculate the exterior differential of the following differential forms:

(a) $z^2 dx \wedge dy + (z^2 + 2y)dx \wedge dz$;

(b) $13x dx + y^2 dy + xyz dz$;

(c) $(x + 2y^3)(dz \wedge dx + 1/2\, dy \wedge dx)$;
(d) $(xdx + ydy)/(x^2 + y^2)$;
(e) $(ydx - xdy)/(x^2 + y^2)$;
(f) $f(x^2 + y^2)(xdx + ydy)$;
(g) $fdg$ ($f$, $g$ being two smooth functions);
(h) $f(g(x^1, \ldots, x^n))dg(x^1, \ldots, x^n)$.

**10.7.** Prove the validity of the formula

$$2(dw)(X, Y) = X(w(Y)) - Y(w(X)) - w([X, Y]),$$

where $w$ is a differential form of degree 1 and $X$, $Y$ two vector fields.

**10.8.** Generalize the formula in 10.7 to the case of differential forms of an arbitrary degree.

Given a scalar product on the vector space $\mathbf{R}^n$, there are two isomorphic operations. One of them associates each vector $X$ with a linear form $w = V(X)$ such that $(X, Y) = V(X)(Y)$. The other associates each multilinear skewsymmetric form $w$ of degree $p$ with a form $*(w)$ of degree $n - p$ as follows: let $w_1, \ldots, w_n$ be the orthonormal basis consisting of linear forms, and $w = f$, $w_{i_1} \wedge \ldots \wedge w_{i_p}$. Then $*(w) = (-1)^\sigma f w_{j_1} \wedge \ldots \wedge w_{j_{n-p}}$, where $\sigma$ is parity of the permutation $(i_1 \ldots i_p j_1 \ldots j_{n-p})$.

**10.9.** Show that the following formulae hold for the space $\mathbf{R}^3$:

(a) $\text{grad} F = V^{-1}(dF)$;
(b) $\text{div} X = *d*V^{-1}(X)$;
(c) $\text{rot} X = -V*dV^{-1}(X)$.

**10.10.** Show that Green's, Stokes' and Ostrogradsky's formulae are special cases of general Stokes' theorem for differential forms.

**10.11.** Derive the formula of integration with respect to the volume $V$ bounded by a closed surface $\Sigma$:

(a) $$\iiint_V (\varphi \Delta \psi + (\text{grad}\varphi, \text{grad}\psi))dv = \oint\oint_\Sigma \varphi \frac{\partial \psi}{\partial n}\, d\sigma;$$

(b) $$\iiint_V (\varphi \Delta \psi - \psi \Delta \varphi)dv = \oint\oint_\Sigma \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right) d\sigma,$$

where $\partial/\partial n$ denotes the derivative in the direction of the normal to the surface $\Sigma$.

**10.12.** Calculate the surface integral $\oint\!\!\oint \varphi \frac{\partial \psi}{\partial n} d\sigma$ with respect to a closed surface $\Sigma$:

(a) for $\varphi = z^2$, $\psi = x^2 + y^2 - z^2$ if $\Sigma$ bounds the region $x^2 + y^2 + z^2 \leqslant 1$ and $y \geqslant 0$;

(b) for $\varphi = 2x^2$, $\psi = x^2 + z^2$ if $\Sigma$ bounds the region $x^2 + y^2 \leqslant 1$ and $0 \leqslant z \leqslant 1$;

(c) for $\varphi = \psi = (x + y + z)/\sqrt{3}$ if $\Sigma$ is the sphere $x^2 + y^2 + z^2 = \mathbf{r}^2$;

(d) for $\varphi = 1$, $\psi = e^x \sin y + e^y \sin x + z$ if $\Sigma$ is the tri-axial ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

**10.13.** Find the gradients of the functions with respect to cylindrical coordinates:

(a) $u = \varrho^2 + 2\varrho\cos\varphi - e^z \sin\varphi$;

(b) $u = \varrho\cos\varphi + z\sin^2\varphi - e^{\varrho}$.

**10.14.** Find the gradients of the functions with respect to spherical coordinates:

(a) $u = r^2\cos\theta$;

(b) $u = 3r^2\sin\theta + e^r\cos\varphi - r$;

(c) $u = \cos\theta/r^2$.

**10.15.** Find div$X$ with respect to cylindrical coordinates:

(a) $X = (\varrho, z\sin\varphi, e\varphi\cos z)$;

(b) $X = (\varphi\tan^{-1}\varrho, 2, -z^2e^z)$.

**10.16.** Find the divergence of the vector field $X = (r^2, -2\cos^2\varphi, \varphi/(r^2 + 1))$ with respect to spherical coordinates.

**10.17.** Find the rotors of the vector fields with respect to spherical coordinates:

(a) $X = (2r + \alpha\cos\varphi, -\alpha\sin\theta, r\cos\theta)$, $\alpha = \text{const}$;

(b) $X = (r^2, 2\cos\theta, -\varphi)$.

**10.18.** Verify that the following vector fields are potential with respect to spherical coordinates $(r, \theta, \varphi)$:

(a) $X = (2\cos\theta/r^3, \sin\theta/r^3, 0)$;

(b) $X = (f(r), 0, 0)$.

**10.19.** Find the potentials on the following vector fields with respect to cylindrical coordinates $(\varrho, \varphi, z)$:

(a) $X = (1, 1/\varrho, 1)$;

(b) $X = (\varrho, \varphi/\varrho, z)$;

(c) $X = (\varphi z, z, \varrho\varphi)$;

(d) $X = (-e^{\varrho}\sin\varphi, e^{\varrho}\cos\varphi/\varrho, 2z)$;

(e) $X = (\varphi\cos z, \cos z, -\varrho\varphi\sin z)$.

**10.20.** Find the potentials on the following vector fields with respect to spherical coordinates:

(a) $X = (\theta, 1, 0)$;

(b) $X = (2r, 1/r\sin\theta, 1/r)$;

(c) $X = (\varphi^2/2, \varphi/\sin\theta, \theta/r)$;

(d) $X = (\cos\varphi\sin\theta, \cos\varphi\cos\theta, -\sin\varphi)$;

(e) $X = (e^r\sin\theta, e^r\cos\theta/r, 2\varphi/(1 + \varphi^2)r\sin\theta)$.

**10.21.** Calculate the circulation of the vector field $X = (r, 0, (R + r)\sin\theta)$ with respect to spherical coordinates along the circumference $\{r = R, \theta = \pi/2\}$.

**10.22.** Calculate the line integral along the line $L$ of the vector field $X$, both given with respect to cylindrical coordinates:

(a) $X = (z, \varrho\varphi, \cos\varphi)$, $L$ is the line-segment $\{\varrho = a, \varphi = 0, 0 \leqslant z \leqslant 1\}$;

(b) $X = (\varrho, 2\varrho\varphi, z)$, $L$ is the semi-circumference $\{\varrho = 1, z = 0, 0 \leqslant \varphi \leqslant \pi\}$;

(c) $X = (e^{\varrho}\cos\varphi, \varrho\sin\varphi, \varrho)$, $L$ is the helix $\{\varrho = R, z = \varphi, 0 \leqslant \varphi \leqslant 2\pi\}$;

(d) $X = (z, \varrho z, \varrho)$, $L$ is the circumference $\{\varrho = 1, z = 0\}$;

(e) $X = (\varrho\sin\varphi, -\varrho^2 z, \varrho^2)$, $L$ is the circumference $\{\varrho = R, z = R\}$;

(f) $X = (z\cos\varphi, \varrho, \varphi^2)$, $L$ is $\{\varrho = \sin\varphi, z = 1\}$.

**10.23.** Calculate the line integral of the vector field $X$ along the line $L$ given in spherical coordinates:

(a) $X = (e^r\cos\theta, 2\theta\cos\varphi, \varphi)$, $L = \{r = 1, \varphi = 0, 0 \leqslant \varphi \leqslant \pi\}$;

(b) $X = (4r^3\tan\varphi/2, \theta\varphi, \cos^2\varphi)$, $L = \{\varphi = \pi/2, \theta = \pi/4, 0 \leqslant r \leqslant 1\}$;

(c) $X = (\sin^2\theta, \quad \sin\theta, \quad r\varphi\theta)$, $L = \{\varphi = \pi/2, \quad r = 1/\sin\theta,$ $\pi/4 \leqslant \theta \leqslant \pi/2\}$;

(d) $X = (r\theta, 0, r\sin\theta)$, $L = \{r = 1, \theta = \pi/4\}$;

(e) $X = (r\sin\theta, \theta e^{\theta}, 0)$, $L = \{r = \sin\varphi, \theta = \pi/2, 0 \leqslant \varphi \leqslant \pi\}$;

(f) $X = (0, 0, r\varphi\theta)$, $L$ is the contour bounding the half-disk $\{r \leqslant R,$ $\varphi = \pi/4\}$.

**10.24.** Find the flow of the vector field $X$ on the surface $S$ given in cylindrical coordinates:

(a) $X = (\varrho, -\cos\varphi, z)$, $S$ bounds the region $\{\varrho \leqslant 2, 0 \leqslant z \leqslant 2\}$;

(b) $X = (\varrho, \varrho\varphi, -2z)$, $S$ bounds the region $\{\varrho \leqslant 1, 0 \leqslant \varphi \leqslant \pi/2,$ $-1 \leqslant z \leqslant 1\}$.

**10.25.** Find the flow of the vector field $X$ on the surface $S$ given in spherical coordinates:

(a) $X = (1/r^2, 0, 0)$, $S$ encloses the origin;

(b) $X = (r, r\sin\theta, -3r\varphi\sin\theta)$, $S$ bounds the region $\{r \leqslant R, \theta \leqslant \pi/2\}$;

(c) $X = (r^2, 0, R^2\cos\varphi)$, $S = \{r = R\}$;

(d) $X = (r, 0, 0)$, $S$ bounds the region $\{r \leqslant R, \theta \leqslant \pi/2\}$;

(e) $X = (r^2, 0, R^2 r\sin\theta\cos\varphi)$, $S$ bounds the region $\{r \leqslant R,$ $0 \leqslant \varphi \leqslant \pi/2, \theta \leqslant \pi/2\}$.

**10.26.** Let $f_t : X \times [0, 1] \to Y$ be a smooth mapping, and $w$ a differential form on $Y$, $dw = 0$. Prove that $f_0^*(w) - f_1^*(w) = d\Omega$ for a convenient form $\Omega$ on $X$.

**10.27.** Prove that if a manifold $X$ is contractible, then, for any form $w(dw = 0)$, the equation $d\Omega = w$ is solvable.

**10.28.** Let $F$ be a vector field in a three-dimensional region $W$ with a smooth boundary $\partial W$, and $n$ a vector normal to $\partial W$.

Prove that

$$\int_W (dwF)dxdydz = \int_{\partial W} (nF)dA,$$

where $dA$ is the element of area on $\partial W$.

**10.29.** (See the previous problem). Prove that

$$\int_S (\mathrm{rot} F, n)dA = \int_{\partial S} (f_1 dx_1 + f_2 dx_2 + f_3 dx_3),$$

where $S$ is a smooth surface with a smooth boundary $\partial S$.

**10.30.** Calculate the de Rham cohomology groups of the following manifolds: (a)$S^1$, (b)$S^2$, (c)$\mathbf{R}P^2$, (d)$T^2$, (e)$T^n$, (f) the plane with exclusion of a finite number of points.

**10.31.** Describe the differentials of left-invariant forms on a Lie group in terms of the commutator in the Lie algebra.

**10.32.** Prove that the bi-invariant forms are closed on a compact Lie group.

Prove that bi-invariant forms are not homologous to zero on a compact Lie group.

**10.33.** Prove that there is a bi-invariant metric on a compact Lie group.

**10.34.** Show that the de Rham cohomology on a compact Lie group is isomorphic to the space of bi-invariant forms.

**10.35.** Prove that each differential form on a complex manifold with respect to complex coordinates $z^1, \ldots, z^n$; $z^k = x^k + iy^k$ is as follows:

$$w = \Sigma w_{k_l} dz^k \wedge d\bar{z}^l,$$

where $dz_k = dz_{i_1} \wedge \ldots \wedge dz_{i_r}$, $d\bar{z}^l = d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_s}$.

**10.36.** Let $X$ be a complex-analytic manifold of dimension $n$, and $w$ a holomorphic form of degree $n$. Show that the integral of the form $w$ along the boundary of an $(n + 1)$-dimensional real submanifold in $X$ equals zero.

**10.37.** Derive the Cauchy theorem $\oint_\gamma f(z)dz = 0$ from Stokes' formula for a holomorphic function in a region bounded by a curve $\gamma$.

**10.38.** Derive the Cauchy residue theorem from Stokes' formula.

**10.39.** Let $f: M \to N$ be a smooth mapping of orientable, closed, and compact manifolds of dimension $n$, and $w$ an $n$-dimensional differential form on the manifold $N$.

Prove that $\int_M f^*(w) = \deg f \int_N w.$

**10.40.** Let $p$ and $q$ be two arbitrary polynomials in variables $(z^1, \ldots, z^n)$, and $a$, $k$ real numbers. Let there exist a differential form $w$ such that

$dp \wedge w = pdz$, $dw = adz$, $dq \wedge w = kdz$. Prove that $d(p^{-k-a}qw) = 0$ (where $dz = dz^1 \wedge \ldots \wedge dz^n$).

**10.41.** Let $G = S^1$, $\varphi$ a curvature form, and $w$ a connection form. Prove that

(a) $d\varphi = f^*(w)$, $dw = 0$;

(b) $\oint_\gamma w$ are integers for any closed cycle $\gamma$.

**10.42.** Prove that if $\oint_\gamma w$ are integers for any closed cycle, then there exists a connection such that $w$ is its curvature form.

**10.43.** Construct a connection in the fibration $G \to G/H$ (where $G$ is a Lie group and $H$ its subgroup) so that the form is invariant with respect to all the motions.

**10.44.** Let $M^2$ be a smooth, closed and compact manifold, $g_{ij}$ the components of its fundamental tensor, and $K(x)$ the Gaussian curvature. Let

$$I(g;\, M^2) = \int_{M^2} K(x)d\sigma(g).$$

Given that $\delta_{(g)}I(g,\, M^2) = 0$, derive the classical Gauss-Bonnet formula

$$\frac{1}{2\pi}\int_{M^2} K(x)d\sigma(g) = \frac{1}{2\pi}\int_{M^2} K(x)d\sigma = \chi(M^2).$$

**10.45.** Prove that one-dimensional de Rham cohomology groups are isomorphic to the group $\mathrm{Hom}(\pi_1(X),\, \mathbf{R}^1)$.

**10.46.** Let a smooth triangulation of a manifold $M$ be given. Prove that the simplicial cohomology groups with real coefficients are isomorphic to the de Rham cohomology groups.

# 11
# General Topology

**11.1.** Prove that any finite $CW$-complex can be embedded in a finite-dimensional Euclidean space $\mathbf{R}^N$ (of sufficiently large dimension).

**11.2.** If a compact, smooth and closed manifold is taken as a $CW$-complex, then the result formulated in the previous problem can be made more precise, viz.,

(a) prove that $M^n$ can be embedded in the Euclidean space $\mathbf{R}^{2nk}$, where $k$ is the number of open balls $D^n$ forming a covering of $M^n$;

(b) prove that $M^n$ can be embedded in the Euclidean space $\mathbf{R}^{nk}$, where the number $k$ has been defined in item (a).

**11.3.** Prove that any compact, smooth and closed manifold $M^n$

(a) can be embedded in the Euclidean space $\mathbf{R}^{2n+1}$;

(b) can be immersed into the Euclidean space $\mathbf{R}^{2n}$.

**11.4.** Construct the immersion of the projective plane $\mathbf{R}P^2$ into the Euclidean space $\mathbf{R}^3$.

**11.5.** Describe the set of nodes of the immersion of $\mathbf{R}P^2$ into $\mathbf{R}^3$ constructed in Problem 11.4. Indicate the multiplicities of the nodes, i.e., how many sheets of the surface intersect at each node of the surface.

**11.6.** Consider the immersion of $\mathbf{R}P^2$ into $\mathbf{R}^3$ described in Problem 11.4. Denote the image of $\mathbf{R}P^2$ in $\mathbf{R}^3$ by $i(\mathbf{R}P^2)$. Consider a line-segment of length $2\epsilon$ orthogonal to $i(\mathbf{R}P^2)$ with the centre at each point $x \in i(\mathbf{R}P^2)$ which is not a node of the surface, where $\epsilon$ is sufficiently small. Since $i$ is a smooth mapping, the pencil of orthogonal line-segments obtained can be additionally defined at each node. In doing so, we shall obtain as many line-segments at each node as the multiplicity of the node is. Consider in $\mathbf{R}^3$ the set consisting of the ends of all the orthogonal line-segments. Prove that it is the image of a two-dimensional sphere under a certain smooth immersion into $\mathbf{R}^3$.

**11.7.** Construct an example of a topological space not satisfying the first countability axiom (resp. the second countability axiom).

**11.8.** Given two continuous functions $f(x)$, $g(x)$ on the two-dimensional sphere $S^2$ such that $f(x) = -f(\tau x)$, $g(x) = -g(\tau x)$, where $\tau$ is the reflection through the centre of the sphere. Prove that these functions have a common zero.

**11.9.** Construct an example of a topological space $X$ such that a certain of its subsets $Y \subset X$ (indicate $Y$) is closed and bounded, but not compact.

**11.10.** Prove that a one-dimensional cellular complex is a space of type $K(\pi, 1)$, where $\pi$ is a free group.

**11.11.** Prove that any finite simplicial complex is a subcomplex of a simplex of sufficiently large dimension. In particular, it can be embedded in the Euclidean space so that the embedding is linear on each simplex.

**11.12.** Prove that a contractible space is homotopy equivalent to a point.

**11.13.** Prove that a universal covering space of $X$ is also a covering space for any other covering space.

**11.14.** Prove that any two spaces of type $K(\pi, n)$ are weakly homotopy equivalent.

**11.15.** Prove that

(a) $S^n \wedge S^k = S^{n+k}$;

(b) $S^n/S^k$ is homotopy equivalent to $S^n \vee S^{k+1}$, $S^n \setminus S^k$ homotopy equivalent to $S^{n-k-1}$ and $S^{n-k}$ diffeomorphic to $S^{n-k-1} \times \mathbf{R}^{k+1}$ if $S^k \subset S^n$ is the standard embedding.

**11.16.** Prove that a function is continuous on a *CW*-complex if and only if it is continuous on each finite subcomplex.

**11.17.** Let $M = X \times Y$, where $X$, $Y$ are two topological spaces. We shall consider a set from $M$ to be open if it is the product of open sets from $X$ and $Y$ or the union of any number of such sets.

Prove that such a system satisfies all the axioms determining a topology on the set $M$.

**11.18.** Prove that if a space $X$ is both Hausdorff and locally compact, while a space $Y$ Hausdorff, then, for any space $T$, the spaces $H(X \times Y, T)$ and $H(Y, H(X, T))$ are homeomorphic, where $H(X, Y) = Y^X$.

**11.19.** Prove that the standard fibre map $EX \xrightarrow{\Omega X} X$ (Serre fibre map), where $X$ is a manifold, is a locally trivial fibre map.

**11.20.** Prove that there exists a homeomorphism of the Cantor discontinuum onto itself commuting two given points.

**11.21.** Let a mapping $f: E \to F$ be a continuous mapping "onto", and let $E$ be compact. Prove that $F$ is compact.

**11.22.** Prove that the $n$-dimensional sphere ($n < \infty$) is compact. Is it true for $n = \infty$?

**11.23.** Let $A$, $B$ be two connected spaces and $A \cap B \neq \varnothing$. Prove that $A \cup B$ is connected.

**11.24.** Prove that if $E$, $F$ are two connected spaces, then $E \times F$ is connected.

**11.25.** Let $f: E \to F$ be a continuous mapping "onto", and $E$ connected. Prove that $F$ is a connected space.

**11.26.** Prove that:

(a) the intervals $0 < x < 1$, $0 \leqslant x \leqslant 1$, $0 \leqslant x < 1$ are connected;

(b) if $A \subset \mathbf{R}^1$ is connected, then $A$ is of the form $a < x < b$, $a \leqslant x \leqslant b$, $a < x \leqslant b$, $a \leqslant x < b$, where $a$, $b$ may assume the values $\pm\infty$.

**11.27.** Let $f: E \to F$, $E = A \cup B$, $A = \bar{A}$, $B = \bar{B}$. Then $f$ is continuous if and only if $f|_A$ and $f|_B$ are continuous. If $A \neq \bar{A}$, then, generally speaking, this does not hold. Give an example.

**11.28.** Prove that $f: E \to F$ is continuous if and only if $f^{-1}(U)$ is open for any open subset $U \subset F$.

**11.29.** Let $f: X \to Y$. Prove that $f$ is continuous if and only if the inverse image of every closed set is closed.

**11.30.** Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

**11.31.** Let $\mathrm{Int}(A) = \cup \{G \subset A : G \text{ is open}\}$. Then $p \in \mathrm{Int}(A)$ if and only if there exists a neighbourhood $U$ of the point $p$ such that $U \subset \mathrm{Int}(A)$, $p \in \overline{A} = \cap \{F \supset A : F \text{ is closed}\}$ if and only if we have $U \cap \overline{A} \neq \varnothing$ for any neighbourhood $U \ni p$.

**11.32.** Prove that the open disk ($|x| < 1$) in Euclidean space is an open set.

**11.33.** Let $X$ be a locally path-connected metric space. Prove that if $X$ is connected, then $X$ is path-connected.

**11.34.** Let $X$ be a metric, compact and connected space. Can any two of its points be connected with a continuous path?

**11.35.** Prove that the cube $I^n$ and sphere $S^n$ are connected.

**11.36.** Let $G \subset I^1$ be an open set on a closed line-segment. Prove that $G$ is the union of disjoint open intervals.

**11.37.** Let $X$ be a metric space. Prove that each of its one-point subsets is closed.

**11.38.** Prove that if the product of two topological spaces is homeomorphic to the suspension of some topological space, then either both factors or one of them is contractible to a point.

**11.39.** Let $X$ be a compact space, $Y$ a metric space, and $f: X \to Y$ a continuous map. Prove that $f$ is a uniformly continuous map.

**11.40.** Prove that if $f: X \to Y$ is a sequence of continuous mappings and $f_n$ uniformly converges to $f$($Y$ being a metric space), then $f$ is continuous.

**11.41.** Let $X \subset Y$, and $Y$ a compact space. Prove that $X$ is a compact space if and only if $X$ is a closed subspace.

**11.42.** Let $A \cap B \neq \varnothing$, and $X = A \cup B$. Prove that if $A$ and $B$ are connected spaces, then $X$ is a connected space.

**11.43.** Prove that the cube $I^n$ is a compact space.

**11.44.** Prove that the sphere $S^\infty$ and ball $D^\infty$ are homeomorphic to cellular spaces.

**11.45.** Prove that the ellipsoid $\left\{ \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} = 1 \right\}$ is homeomorphic to the sphere $S^n$.

11.46. Prove that the ball $\left\{\sum_{i=1}^{n} x_i^2 \leqslant 1\right\}$ and the upper hemisphere $\left\{\sum_{i=1}^{n+1} x_i^2 = 1,\ x_{n+1} \geqslant 0\right\}$ are homeomorphic.

11.47. Prove that the cube $\{|x_i| \leqslant 1,\ i = 1, 2, \ldots, n\}$ and the ball $\left\{\sum_{i=1}^{n} x_i^2 \leqslant 1\right\}$ are homeomorphic. Prove that an open cube and an open ball are diffeomorphic.

11.48. Are the line-segment $0 \leqslant x \leqslant 1$ and the letter $T$ homeomorphic?

11.49. Prove that the interval $(-1, 1)$ is homeomorphic to the straight line $(-\infty, \infty)$. Prove that any two intervals are homeomorphic.

11.50. Are a ball and a sphere homeomorphic?

11.51. Prove that the Hamming metric on the $n$-dimensional cube cannot be embedded in any $\mathbf{R}^n$, i.e., there exists no embedding such that the Hamming metric is induced by the standard Euclidean metric (the cube being considered only as the set of its vertices, i.e., as a discrete set, and then the distance $\varrho(a, b)$, where $a$ and $b$ are the vertices of the cube, equals the number of different coordinates of these two vertices).

11.52. Let $f: M^2 \to S^2$ be a mapping of class $C^2$ of a closed, smooth, and compact manifold $M^2$ onto $S^2$, $f$ being open (i.e., the image of any open set is open) and finitely multiple (i.e., the inverse image of each point $x \in S^2$ is a finite number of points).

Prove that $M^2$ is diffeomorphic to the sphere $S^2$. What can be said about a similar mapping $f: M^n \to S^n$?

11.53. Prove that a metric topological space satisfies Hausdorff separation axiom.

11.54. Is it true that the distance between two disjoint, closed sets on the plane (straight line) is always greater than zero?

Recall that the distance between two subsets $X$ and $Y$ of a metric space $Z$ is the number $\varrho(X, Y) = \sup_{x \in X} \inf_{y \in Y} r(x, y) + \sup_{y \in Y} \inf_{x \in X} r(x, y)$, where $r(x, y)$ is the distance between two points $x$ and $y$ in the space $Z$. This is not the only way of defining the distance between two subsets of a metric space. Give other examples of the metric $\varrho(X, Y)$.

11.55. Prove that a set whose elements are closed subsets of a metric space can itself be transformed into a metric space in a natural manner by introducing a metric described, e.g., in Problem 11.32.

**11.56.** Prove that any contracting mapping of a metric space is continuous.

A mapping $f : X \to X$ of a metric space $X$ into itself is said to be contracting if there exists a real constant $\lambda < 1$ such that $\varrho(f(x), f(y)) \leqslant \lambda\varrho(x, y)$ for any two points $x, y \in X$.

**11.57.** Prove that any contracting mapping of a complete metric space into itself always has a fixed point (which is unique).

**11.58.** Give an example showing that a condition for a metric space to be complete (see Problem 11.57) cannot be discarded.

**11.59.** Show that a group of orthogonal matrices of order $3 \times 3$ is a compact topological space.

**11.60.** Prove that the group of orthogonal transformations of the $n$-dimensional Euclidean space is a compact topological space.

**11.61.** Prove that the group $SO(3)$ considered as a topological space (in embedding $SO(3)$ in the linear space of all real matrices of order $3 \times 3$) is homeomorphic to $\mathbf{R}P^3$.

**11.62.** Prove that $SO(n)$ is a connected topological space, and that $O(n)$ consists of two connected components. Prove that $U(n)$ and $SU(n)$ are connected topological spaces.

**11.63.** Prove that the open disk $x^2 + y^2 < 1$ and the plane $\mathbf{R}^2(x, y)$ are homeomorphic. Prove that the open square $\{|x| < 1, |y| < 1\}$ and plane $\mathbf{R}^2(x, y)$ are homeomorphic. Prove that the interval $0 < x < 1$ and the open square $\{|x| < 1, |y| < 1\}$ are not homeomorphic.

**11.64.** Prove that the group of unitary matrices $U(n)$ considered as a topological space is homeomorphic to the direct product of $S^1$ and $SU(n)$ (as topological spaces).

**11.65.** Prove that the group $GL(n, \mathbf{G})$ considered as a subset in the space of all complex matrices of order $n \times n$ is an open and connected subset.

**11.66.** Prove that the group $GL^+(n, \mathbf{R})$ consisting of real matrices of order $n \times n$ with positive determinants is a connected topological space.

**11.67.** Prove that the group $GL(n, \mathbf{R})$ of real non-singular matrices of order $n \times n$ is a topological space consisting of two connected components.

**11.68.** Prove that a Möblius strip is not homeomorphic to the direct product of a line-segment by a circumference.

**11.69.** Construct an immersion of a Möbius strip into Euclidean three-dimensional space so that the boundary circumference of the former may be standardly embedded in Euclidean two-dimensional plane.

**11.70.** Prove that the set of all straight lines on the Euclidean plane $\mathbf{R}^2$ is homeomorphic to a Möbius strip.

**11.71.** Prove that for any compact set $K \subset \mathbf{R}^n$, there exists a smooth real-valued function $f$ such that $K = f^{-1}(0)$.

**11.72.** The group $SO(3)$ is, naturally, embeddable, in the Euclidean space $\mathbf{R}^9$ (each element of $SO(3)$ is a real matrix of order $3 \times 3$). Prove that, virtually, $SO(3) \subset S^8 \subset \mathbf{R}^9$, where $S^8$ is the sphere of radius $\sqrt{3}$ standardly embedded in $\mathbf{R}^9$.

# 12
# Homotopy Theory

**12.1.** Represent (a) the torus, (b) Klein bottle and (c) suspension of a complex $K$ as cellular complexes.

**12.2.** Prove that the topology of a $CW$-complex is the weakest of all topologies respective to which all the characteristic mappings are continuous.

**12.3.** Prove that a torus with a disk generated by a meridian are homotopy equivalent to the wedge $S^1 \vee S^2$.

**12.4.** Prove that a torus with a disk generated by a meridian and a parallel are homotopy equivalent to the sphere $S^2$.

**12.5.** Generalize Problems 12.3 and 12.4 to the case of the product $S^k \times S^{n-k}$.

**12.6.** Prove the homotopy equivalence of the spaces $(X \times S^n)/(X \vee S^n)$ and $\Sigma^n X$.

**12.7.** Prove the homotopy equivalence

(a) $\Sigma(X \vee Y) \sim \Sigma X \vee \Sigma Y$;

(b) $\Sigma(X \wedge Y) \sim \Sigma(X \times Y)/(\Sigma X \vee Y)$.

**12.8.** Let $\{X; A, B\}$ be the space of paths starting at $A$ and ending at $B$, and $A \subset B$. Prove that $\{X; A, B\}$ contains a subspace homeomorphic to $A$.

**12.9.** Let $f: X \to \Sigma$ be a continuous mapping of simplicial complexes, and $Y \subset X$ a subcomplex such that the mapping $f$ is simplicial on it. Prove that there exists a subdivision of the complex $X$ such that it is identity on $Y$ and the mapping $f$ is homotopic to a certain simplicial mapping $q$, the homotopy being constant on $Y$.

**12.10.** Let $X$ be a simplicial complex, and $S_x$ the star of a vertex $x \in X$. Prove that any two simplexes of the star $S_x$ meet in a certain face.

**12.11.** Prove that a simplicial mapping of simplicial complexes is continuous.

**12.12.** Let $X$ be a simplicial complex, and $\epsilon > 0$. Prove that there exists a subdivision of the complex $X$ such that the diameter of each new simplex is less than $\epsilon$.

**12.13.** Let $f$ be a mapping of the unit line-segment $[0, 1]$ into itself, and $f(0) = 0, f(1) = 1$. Prove that there exists a homotopy which leaves the endpoints of the line-segment fixed and deforms the mapping $f$ into the identity.

**12.14.** Is the vector space $\mathbf{R}^n$ contractible to a point on itself?

**12.15.** Let a space $X$ be contractible to a point on itself. Prove that any two paths with the same endpoints are homotopic to each other (a fixed endpoint homotopy).

**12.16.** Let a space $X$ be contractible to a subspace $Y$, with the homotopy leaving $Y$ fixed (constant). Prove that any path in $X$ with the endpoints in $Y$ is homotopic to a path wholly lying in $Y$ (a fixed endpoint homotopy).

**12.17.** Prove that any two paths are homotopic on the sphere $S^n$, $n > 1$ (the endpoints are the same, and the homotopy is fixed endpoint).

**12.18.** Prove that any connected, cellular complex is homotopy equivalent to a cellular complex with one vertex.

**12.19.** Prove that the sphere $S^{n-1}$ can be represented as the union $(S^r \times D^{n-r}) \cup (D^{r+l} \times S^{n-r-1})$ with the common boundary $S^r \times S^{n-r-1}$.

**12.20.** Consider the standard sphere $S^{n-1}$ in the Euclidean space $\mathbf{R}^n$ and two spheres embedded in it:

$$S^{r-1} = \{x_{r+1} = \ldots = x_n = 0\},\ S^{n-r-1} = \{x_1 = \ldots = x_r = 0\}.$$

Prove that any pair of points $y \in S^{r-1}$, $x \in S^{n-r-1}$ can be joined by a unique arc on a great circle having no other points of intersection with these spheres.

**12.21.** Find the topological type of the closed hyperboloid of one sheet $\Gamma = \{x^2 + y^2 - z^2 = 1\}$ in the projective space $\mathbf{R}P^3$.

**12.22.** Cut a Möbius strip (embedded in $\mathbf{R}^3$) along its midline.

Is the manifold obtained orientable?

Repeat the cutting process several times. Describe the manifold obtained (it is disconnected) and find the linking number of any two connected components.

**12.23.** Prove that the space of polynomials of the third degree without multiple roots is homotopy equivalent to the complement of a trifolium in the sphere $S^3$. Construct an explicit deformation.

**12.24.** Consider the set of points $\mathbf{C}^n$ with pairwise different coordinates. Show that the space obtained has the same type as the Eilenberg-MacLane complex $K(\pi, 1)$.

**12.25.** Construct an example of two spaces $X_1$, $X_2$, which are not homotopy equivalent, and also of two continuous one-to-one mappings $f: X_1 \to \mathbf{X}_2$, $g: X_2 \to X_1$ such that the spaces themselves may not be homotopy equivalent.

**12.26.** Let $h : X \to X^1$ be a continuous mapping, and the correspondence $\Phi : \{X', \ Y\} \to \{X, \ Y\}$ be defined by the formula $\Phi(\alpha) = \alpha \cdot h$.

Prove that the correspondence $\Phi$ transforms homotopic mappings into homotopic.

**12.27.** Prove that the following homotopy equivalence holds: $\Sigma(S^n \times S^m) \sim S^{n+1} \vee S^{m+1} \vee S^{n+m+1}$.

**12.28.** Prove that the finite-dimensional sphere $S^\infty$ is contractible to a point on itself.

**12.29.** Prove that a connected finite graph is homotopy equivalent to the wedge of circumferences $\vee S^1$.

**12.30.** Let a mapping $p : X \to Y$ satisfy the covering homotopy axiom. Prove that the inverse images of the points are homotopy equivalent.

**12.31.** Let a space $X$ be contractible to its path-connected subspace $A$. Prove that the space $X$ is path-connected.

**12.32.** Fix two points $x_0$ and $x_1$ in a space $X$. Let $Y$ be the space of paths starting at $x_0$ and passing through $x_1$. Prove that the space $Y$ is contractible.

**12.33.** Prove that the space of all paths $\{X; X; X\}$ is contractible to $X \subset \{X; X; X\}$ with $X$ held fixed.

**12.34.** Let a sequence of spaces $X_n \subset X_{n+1}$ be given, and let $X_{n+1}$ be contractible to $X_n$ with $X_n$ held fixed.

Prove that the space $X = \bigcup_n X_n$ is contractible to $X_0$ with $X_0$ held fixed.

**12.35.** Prove that any open $n$-dimensional manifold is homotopy equivalent to an $(n - 1)$-dimensional complex.

**12.36.** Prove that if a space $X$ is contractible to a subspace $A$ on itself with $A$ held fixed, then $A$ is homotopy equivalent to $X$.

**12.37.** Calculate the sets $\pi(S^1 \times S^1, S^2)$ and $\pi(S^k \times S^{n-k}, S^n)$.

**12.38.** Find $\mathrm{Cat}_1(\mathbf{R}P^n)$ and $\mathrm{Cat}_2(\mathbf{R}P^n)$, where $\mathrm{Cat}_1(\mathbf{R}P^n)$ and $\mathrm{Cat}_2(\mathbf{R}P^n)$ are the minimal numbers of closed subsets $X_i$ such that $X = \bigcup_i X_i$ and the embeddings $X_i \subset X$ are homotopic to constant mappings.

**12.39.** Calculate $\mathrm{Cat}_1(K)$ and $\mathrm{Cat}_2(K)$ for the case of a sphere with three identified points.

**12.40.** Let $M^2$ be a compact, closed, oriented, and 2-dimensional manifold of genus $h$, i.e., $M^2$ is the sphere $S^2$ with $h$ handles. Find $\Sigma^2 M^2$ (i.e., double suspension) up to homotopy type.

**12.41.** Consider some standard chart with non-homogeneous coordinates $x_1, \ldots, x_n$ in $\mathbf{R}P^n$. Find the homotopy type of

(a) $\mathbf{R}P^n \setminus \check{S}^k$, where $\check{S}^k = \{x_1^2 + \ldots + x_{k+1}^2 = 1, \ x_{k+2} = \ldots = x_n = 0\}$;

(b) $\mathbf{R}P^n \setminus \check{M}_q^k$, where $\check{M}_q^k = \{x_1^2 + \ldots + x_k^2 - x_{q+1}^2 - \ldots - x_{n+1}^2 - 1 = 0, x_{k+2} = \ldots = x_n = 0\}$;

(c) $\check{S}^k$ and $\check{M}_q^k$.

**12.42.** Consider a small ball $D^n$ in the open manifold $\mathbf{R}^n \times S^{n-k}$ and glue the projective space $\mathbf{R}P^n$ in its place, i.e., identify points $x$ and $-x$ on the boundary of the ball $\partial D^n = S^{n-1}$.

Prove that the space obtained is homotopy equivalent to $\mathbf{R}P^{n-1} \vee S^{n-k}$.

**12.43.** Given a topological manifold $M^n$ whose boundary is a topological manifold $P^{n-1}$, the boundary of $P^{n-1}$ being contractible to a point in the manifold $M^n$.

(a) Prove that the manifold is contractible to a point.

(b) Prove that if the manifold $P^{n-1}$ is 1-connected, then the manifold $M^n$ is homeomorphic to the disk $D^n$ (assuming that $P^{n-1}$ is contractible to a point in $M^n$).

(c) Give an example of a pair $(M^n, P^{n-1})$ such that the manifold $P^{n-1}$ is contractible to a point in the manifold $M^n$, but $M^n$ is not homeomorphic to the disk $D^n$. As a corollary, prove that $\pi_1(P^{n-1}) \neq 0$.

**12.44.** Find the homotopy type of the space $\mathbf{C}^n \setminus \Delta$, where $\Delta = \bigcup_{ij} \Delta_{ij}$, $\Delta_{ij} = \{x \in \mathbf{G}^n | x_i = x_j\}$.

**12.45.** Calculate the number of mappings (up to homotopy):

(a) $\mathbf{R}P^n \to \mathbf{R}P^n$; (a′) $\mathbf{C}P^n \to \mathbf{C}P^n$;

(b) $\mathbf{R}P^{n+1} \to \mathbf{R}P^n$; (b′) $\mathbf{C}P^{n+1} \to \mathbf{C}P^n$;

(c) $\Sigma\mathbf{R}P^n \to \mathbf{R}P^n$; (c′) $\Sigma\mathbf{C}P^n \to \mathbf{C}P^n$;

(d) $\Sigma\mathbf{R}P^n \to \mathbf{R}P^{n+1}$; (d′) $\Sigma\mathbf{C}P^n \to \mathbf{C}P^{n+1}$.

**12.46.** Prove that

(a) Cat [join$(X, Y)$] = min[Cat$(X)$, Cat$(Y)$], where Cat is the Lusternik-Schnirelmann category (the spaces $X$ and $Y$ being connected).

(b) Find Cat$(S^1 \times S^2)$.

**12.47.** Let spaces $X_i$, $1 \leqslant i \leqslant N$, be path-connected, and $X = X_1 \times X_2 \times \ldots \times X_N$.

Prove that $[\mathrm{Cat}(X_i)] \leqslant \mathrm{Cat}(X) \leqslant 1 + \sum_{i=1}^{N} [\mathrm{Cat}(X_i) - 1]$.

**12.48.** (a) Calculate Cat$(\mathbf{R}P^n)$; Cat$(T^n)$; Cat$(S^m \times S^n)$.

(b) Prove that if the sphere $S^n$ is covered by $q$ closed sets (not necessarily connected) $V_1, V_2, \ldots, V_q$, where $q \leqslant n$, then there always exists at least one set $V_i$ such that it contains two diametrically opposite points of the sphere $S^n$, viz., $-x$ and $x$.

**12.49.** Let $M \subset \mathbf{R}^n$ be an arbitrary subset of Euclidean space (e.g., smooth submanifold), and let $\mathbf{R}^n \subset \mathbf{R}^{n+1}$ be the standard embedding. Prove that the following homotopy equivalence holds $\mathbf{R}^{n+1} \setminus M \simeq \Sigma(\mathbf{R}^n \setminus M)$.

**Reminder.** Let $X$ be a topological space, and $X \times I$ its direct product by a line-segment. After identifying the "upper base" $X \times \{1\} \subset X \times I$ with a point and the "lower base" $X \times \{0\} \subset X \times I$ with another point, we obtain a factor space called the suspension $\Sigma X$ of $X$.

**12.50.** *The relation between the Lusternik-Schnirelmann category and "cuplong" of a compex (or manifold).* Let $M^n$ be a smooth, compact, connected, and closed manifold. Consider the ring $H^*(M^n; G)$, where $G = \mathbf{Z}$ if $M^n$ is orientable, and $G = \mathbf{Z}_2$ if $M^n$ is non-orientable. Denote by $l(M^n; G)$ the greatest integer for which there exists a sequence of elements $x_1, x_2, \ldots, x_l$ of the ring $H^*(M^n; G)$ ($\deg x_\alpha > 0$, $1 \leqslant \alpha \leqslant l$) such that $x_1 \wedge x_2 \wedge \ldots \wedge x_{l(M^n;G)} \neq 0$ in the ring $H^*(M^n; G)$. The number $l(M^n; G)$ is denoted by cuplong $(M^n)$. Prove that

$$\mathrm{Cat}(M^n) \geqslant l(M^n; G).$$

**12.51.** Prove that for any path-connected topological space $X$ and any of its points $x_0$, the group $\pi_1(\Omega X, x_0)$ is Abelian.

**12.52.** Prove that any contractible space is 1-connected.

**12.53.** Prove that the group $\pi_1(\vee_A S^1)$ is a free group with $A$ generators.

**12.54.** Prove that if $X$ and $Y$ are homotopy equivalent, then the isomorphisms hold: $\pi_1(X) \cong \pi_1(Y)$ and $\pi_k(X) \cong \pi_k(Y)$, $k \geqslant 2$.

**12.55.** Prove that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$, where $\pi_1(X) * \pi_1(Y)$ is the free product of the groups $\pi_1(X)$ and $\pi_1(Y)$.

**12.56.** Find the knot group of the trefoil in $\mathbf{R}^3$ (and also in the sphere $S^3$) and prove that one cannot "untie" the trefoil, i.e., there exists no homeomorphism of Euclidean space (or sphere) onto itself which would transform the trefoil knot into the standardly embedded, unknotted circumference, i.e., trivial knot.

**12.57.** Find the knot group of a knot $\Gamma$ in $\mathbf{R}^3$ given thus: the circumference which represents the knot is placed on the two-dimensional standardly embedded torus $T^2 \subset \mathbf{R}^3$, on which it traverses its parallel $p$ times, and its meridian $q$ times, $p$ and $q$ being prime to one another. (The trefoil knot from Problem 12.56 can be represented as such a knot $\Gamma$, where $p = 2$, $q = 3$.) Make out the role of the condition for the numbers $p$ and $q$ to be prime to one another.

**12.58.** Let $X = Y \cup_W Z$, where $Y$, $Z$, $W$ are finite $CW$-complexes, $W = Y \cap Z$, $W$ is path-connected, and $X = Y \cup_W Z$ is the complex obtained by gluing $Y$ and $Z$ relative to the common subset $W$. Calculate the group $\pi_1(X)$, given the groups $\pi_1(Y)$, $\pi_1(Z)$ and $\pi_1(W)$. Consider the case where $W$ is disconnected separately.

**12.59.** Given an arbitrary group $G$ with a finite number of generators and relations, prove that there exists a finite complex $X$ whose fundamental group is isomorphic to $G$. Can a finite-dimensional manifold, e.g., four-dimensional. be selected as such a complex $X$?

**12.60.** Calculate the group $\pi_1(X)$, where $X$ is the wedge of three circumferences.

**12.61.** Construct a two-dimensional complex $X$ whose fundamental group equals $\mathbf{Z}/p\mathbf{Z}$. For which values of $p$ can a two-dimensional, smooth, closed, and compact manifold be selected as such a complex?

**12.62.** Calculate the fundamental group of the two-dimensional sphere with three handles. Check this group on commutativity and find its commutator subgroup. Calculate the fundamental group of the two-dimensional torus.

**12.63.** Let a simplicial complex $X$ have $N$ one-dimensional simplexes. Prove that its fundamental group has no more than $N$ generators.

**12.64.** Prove that $\pi_1(X) = \pi_1(X_2)$, where $X$ is a $CW$-complex and $X_2$ its two-dimensional skeleton, i.e., the union of all cells of dimensions 1 and 2.

**12.65.** Find $\pi_2(X)$, where $X = S^1 \vee S^2$. Is this group finitely generated?

**12.66.** Find the knot group of the figure-of-eight (i.e., wedge of two circumferences).

**12.67.** Let $f$ be a path in $X$, $\alpha \in \pi_1(X, x_0)$, and $f(0) = x_0$. Prove that there exists a path $g$ such that $g(0) = x_0$, $g(1) = f(1)$ and $fg^{-1} \in \alpha$.

**12.68.** Let $X$ be a path-connected space. Prove that the group $\pi_1(X, x_0)$ is isomorphic to the group $\pi_1(X, y)$ for any two points $x, y \in X$.

**12.69.** Calculate $\pi_1(X)$ and $\pi_n(X)$, where $X$ is the wedge $S^1 \vee S^n$.

**12.70.** Prove that if $X$ is a one-dimensional $CW$-complex, then $\pi_1(X)$ is a free group.

**12.71.** Prove that the group $\mathbf{G} = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ cannot be the fundamental group of any three-dimensional manifold.

**12.72.** Calculate $\pi_1(P_g)$, where $P_g$ is a two-dimensional, compact, closed, and orientable surface of genus $g$.

**12.73.** Calculate $\pi_1(TP_g)$, where $TP_g$ is the manifold of linear elements of a surface of genus $g$.

**12.74.** Calculate the fundamental group of the Klein bottle by constructing the covering space with the action of a discrete group.

**12.75.** Let $P$ be a two-dimensional surface with non-empty boundary (i.e., open surface). Prove that $\pi_1(P)$ is a free group.

**12.76.** Prove that if $X$ is a $CW$-complex, then $\pi_1(X)$ is a group whose generators are one-dimensional cells, and the whole set of relations is determined by the boundaries of two-dimensional cells.

**12.77.** Let $G$ be a topological groupoid with identity. Prove that $G$ is homotopically simple in all dimensions and, as a corollary, that $\pi_1(G)$ is an Abelian group.

**12.78.** Let $X$ be a topological groupoid with identity, and $G \subset \pi_1(X)$ a subgroup. Prove that

(a) it is possible to introduce the operation of multiplication in $\hat{X}_G$ so that $p_G$: $X_G \to X$ (where $p_G$ is the projection of the covering space $\hat{X}_G$ onto $X$) becomes a homomorphism;

(b) if $X$ is a group, then $X_G$ (i.e., covering space relative to the subgroup $G$) is also a group. Consider the example $\mathbf{Z}_2 \to \mathrm{Spin}(n) \to SO(n)$, $n > 2$.

**12.79.** Prove that the following isomorphism holds

$$\pi_n(\underbrace{S^n \vee \ldots \vee S^n}_{k \text{ times}}) \cong \underbrace{\pi_n(S^n) \oplus \ldots \oplus \pi_n(S^n)}_{k \text{ times}}$$

**12.80.** Prove that the groups $\pi_i(X)$ are commutative when $i > 1$ for any $CW$-complex $X$.

**12.81.** Demonstrate by way of example that the excision axiom does not hold for the group $\pi_i(X, Y)$ (the axiom being held for the usual (co) homology theories), i.e., there exist pairs $(X, Y)$ such that

$$\pi_i(X, Y) \neq \pi_i(X/Y).$$

**12.82.** Prove that for any path-connected space $Y$ and any point $x_0 \in Y$, the isomorphism holds $\pi_g(Y, x_0) \cong \pi_{g-1}(\Omega_{x_0}, Y, w_{x_0})$, where $w_{x_0}$ is the constant loop at the point $x_0$.

**12.83.** Prove that $\pi_1(\mathbf{R}P^n) = \mathbf{Z}_2$, $n > 1$, and $\pi_k(\mathbf{R}P^n) = \pi_k(S^n)$, $n \geqslant 1$, $k > 1$, where $\mathbf{R}P^n$ is the real projective space.

**12.84.** Prove that if

(a) $A$ is a contractible subspace in a space $X$ ($X$ and $A$ being $CW$-complexes) to a point $x_0 \in A$, then the homomorphism $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is trivial when $n \geqslant 1$, and when $n \geqslant 3$, the decomposition

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) \oplus \pi_{n-1}(A, x_0)$$

holds;

(b) $i : X \vee Y \to X \times Y$ is an embedding, then the exact sequence is given rise:

$$\pi_g(X \vee Y) \xrightarrow{i_*} \pi_g(X \times Y) \to 0.$$

**12.85.** Prove that $\pi_1(\mathbf{C}P^n) = 0$; $\pi_2 = (\mathbf{C}P^n) = \mathbf{Z}$, $n > 0$; $\pi_k = (\mathbf{C}P^n) = \pi_k(S^{2n+1})$, $k \geqslant 2$.

**12.86.** Prove that if a $CW$-complex $X$ has no cells of dimensions from 1 to $k$ inclusive, then $\pi_1(X) = 0$ when $i \leqslant k$.

**12.87.** Let $X$, $Y$ be two $CW$-complexes. Prove that $\pi_i(X \times Y) = \pi_i(X) \oplus \pi_i(Y)$. Calculate the action of $\pi_1(X \times Y)$ on $\pi_1(X \times Y)$. Construct a universal covering of $X \times Y$.

**12.88.** Find the homotopy groups $\pi_q(S^n)$ $(0 \leqslant q \leqslant n)$ and prove that $\pi_n(S^n) = \mathbf{Z}$, where $S^n$ is a sphere.

**12.89.** Prove that $\pi_i(S^3) = \pi_i(S^2)$ when $i \geqslant 3$ and, as a corollary, that $\pi_3(S^2) = \mathbf{Z}$.

**12.90.** Prove that

(a) $\pi_1(SO(3)) = \mathbf{Z}_2$, $\pi_2(SO(3)) = \pi_2(SO) = 0$, where $SO = \varinjlim SO(n)$;

(b) $\pi_3(SO(4)) = \mathbf{Z}$, $\pi_1(U) = \mathbf{Z}$, $\pi_2(U) = 0$, where $U = \varinjlim U(n)$ (embeddings $U(n) \subset U(n+1)$ and $SO(n) \subset SO(n+1)$ being standard);

(c) $\pi_3(SO(5)) = \mathbf{Z}$.

**12.91.** Find the groups $\pi_g(S^1 \vee S^1)$, $g \geqslant 0$.

**12.92.** Calculate the groups $\pi_1(X)$, $\pi_n(X)$ and action of the group $\pi_1(X)$ on the group $\pi_n(X)$ for the following cases: (a) $X = \mathbf{R}P^n$; (b) $X = S^1 \vee S^n$; (c) $X = \partial B(\xi^{n+1})$, where $B$ $(\xi^{n+1})$ is the division ring of a non-trivial $O(n+1)$-fibration of disks on $S^1$.

**12.93.** If a mapping $f(X, A) \to (Y, B)$ sets up the isomorphisms $\pi_g(X) \approx \pi_g(Y)$ and $\pi_g(A) \approx \pi_g(B)$ for all $g$, then it sets up the isomorphisms $\pi_g(X, A) \approx \pi_g(X, B)$ for all $g$.

**12.94.** Calculate the groups $\pi_{n-k}(V^{\mathbf{R}}_{n,k})$, where $V^{\mathbf{R}}_{n,k}$ is the real Stiefel manifold.

**12.95.** Prove that the groups $\pi_k(S^n)$ cannot become trivial, as $k$ increases, beginning with a certain number $k$.

**12.96.** Prove that $\pi_3(S^2) = \mathbf{Z}$, and $\pi_{n+1}(S^n) = \mathbf{Z}_2$, when $n \geqslant 3$.

**12.97.** Find $\pi_3(S^2 \vee S^2)$, $\pi_3(S^1 \vee S^2)$, and $\pi_3(S^2 \vee S^2 \vee S^2)$.

**12.98.** Calculate the one-dimensional relative homotopy group of the pair $(\mathbf{C}P^2, S^2)$, where $S^2 \cong \mathbf{C}P^1 \subset \mathbf{C}P^2$ standardly.

**12.99.** Prove that if:

(a) a three-dimensional, compact, and closed manifold $M^3$ is 1-connected, then $M$ is homotopy equivalent to the sphere (i.e., $M^3$ is a homotopy sphere);

(b) $M^n$ is a smooth compact and closed manifold such that $\pi_i(M^n) = 0$ when $i \leqslant \left[\frac{n}{2}\right]$, then $M^n$ is homotopy equivalent to the sphere $S^n$.

**12.100.** Construct an example of a three-dimensional, closed, and compact manifold $M^3$ such that $M^3$ is a homology sphere (i.e., it has the same integral homology as $S^3$), but $\pi_1(M^3) \neq 0$. Construct an example of a finitely generated group $G$ which coincides with its first commutator subgroup.

**12.101.** Prove that the set of homotopy classes of mappings $[S^n, X]$ is isomorphic to the set of classes of conjugate elements of the group $\pi_n(X, x_0)$ under the action of $\pi_1(X, x_0)$ ($X$ being a connected complex).

**12.102.** Calculate $\pi_2(\mathbf{R}^2, X)$, where $\mathbf{R}^2$ is a plane and $X$ a figure-of-eight embedded into the two-dimensional plane.

**12.103.** Calculate $\pi_i(\mathbf{C}P^n)$ when $i \leqslant 2n + 1$.

**12.104.** Let $\pi_n(X) = 0$, and a finite group $G$ act on $X$ and $Y$ without fixed points. Prove that there exists, and is unique up to homotopy, a mapping $f: Y \to X$ which commutes with the action of the group $G$.

**12.105.** Prove that $[\mathbf{C}P^2, S^2] = \pi_4(S^2)$, where $[X, Y]$ is the set of homotopy classes of mappings of $X$ into $Y$.

**12.106.** Let $(X, A)$, $X \supset A$, be a pair of topological spaces, and $X$ path-connected. Let $\Lambda$ be the set of paths in the space $X$ starting at a certain point $x_0$ and ending at points of the subspace $A$. Prove that $\pi_g(X, A, a) = \pi_{g-1}(\Lambda, \lambda_a)$, where $\lambda_a$ is an arbitrary path from $x_0$ to $a \in A$.

**12.107.** Prove that the following conditions are equivalent to $n$-connectedness:

(a) $\pi_0(S^q, X)$ consists of one element when $q \leqslant n$ (base-point preserving maps);

(b) any continuous mapping $S^q \to X$ can be extended to any continuous mapping of the disk $D^{q+1} \to X$, $q \leqslant n$.

**12.108.** Prove that $\pi_0(X, \Omega\Omega Z)$ is an Abelian group, where $X$, $Z$ are two topological spaces, and $\Omega X$ is the loop space. Prove that $\Omega X$ is an $H$-space.

**12.109.** Let $A$ be a retract of $X$. Prove that when $n \geqslant 1$, for any point $x_0 \in A$, the homomorphism induced by the embedding

$$i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$$

is a monomorphism, and when $n \geqslant 2$, determines the following decomposition into the direct sum

$$\pi_n(X, x_0) \cong \pi_n(A, x_0) \oplus \pi_n(X, A, x_0).$$

**12.110.** Prove that $\pi_0(\Sigma\Sigma Z, X)$ is an Abelian group. Establish a relation with $\pi_0(Z, \Omega\Omega X)$.

**12.111.** Let $TS^n \to S^n$ be the standard tangent bundle over the sphere $S^n$. Calculate the homomorphism $\partial_* : \pi_n(S^n) \to \pi_{n-1}(S^{n-1})$ from the exact homotopy sequence of this bundle.

**12.112.** Let $f : X \to Y$ be a continuous mapping ($f(x_0) = y_0$). Prove that the induced mapping $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a group homomorphism.

**12.113.** Let

$$\begin{array}{l} Y \supset F_0 \ni y_0 \\ \downarrow \\ X \ni x_0 \end{array}$$

be a fibration with fixed points $x_0$, $y_0$ and fibre $F$. Prove that $\pi_n(Y, F_0; y_0) \cong \pi_n(X, x_0)$.

**12.114.** Let $E$, $X$ be two topological spaces, $X$ path-connected, and $p : E \to X$ a continuous mapping such that for any points $x \in X$, $y \in p^{-1}(x)$, the isomorphism holds true

$$p_* : \pi_i(E, p^{-1}(x), y) \xrightarrow{\approx} \pi_i(X, x), \ i \geqslant 0$$

(for $i = 0, 1$, a set isomorphism is valid, the sets being stripped of the group structure). Prove that for any points $x_1$ and $x_2$, the topological spaces $p^{-1}(x_1)$ and $p^{-1}(x_2)$ are weakly homotopy equivalent.

**12.115.** Prove that for the homotopy groups of a pair $(X, A)$ the exact sequence is valid

$$\ldots \to \pi_i(A) \to \pi_i(X) \to \pi_i(X, A) \to \pi_{i-1}(A) \to \ldots$$

**12.116.** Prove that if $X$ is a smooth, compact and closed submanifold of codimension one in Euclidean space, then $X$ is orientable.

**12.117.** Prove that if the fundamental group of a compact closed manifold is trivial, then the manifold is orientable. Prove that if a manifold $X$ is non-orientable, then there is a subgroup of index two in $\pi_1(X)$.

**12.118.** Prove that if $X$ is a non-orientable space, then the suspension $\Sigma X$ is not a manifold.

**12.119.** Prove that the Euler characteristic $\chi(X)$ of any compact, closed manifold is trivial.

**12.120.** Give examples of

(a) a non-orientable manifold doubly embedded in another manifold (whose dimension is one greater);

(b) an orientable manifold singly embedded in another manifold.

**12.121.** Let $X_1$ and $X_2$ be two tori (meaning the solids), $f: \partial X_1 \to \partial X_2$ a diffeomorphism, and $M_f^3 = X_1 \cup_f X_2$. Give examples of diffeomorphisms $f$ such that the manifold $M_f^3$ is diffeomorphic to: (a) $S^3$, (b) $S^2 \times S^1$, (c) $\mathbf{R}P^3$.

**12.122.** *With the notation of the previous problem, consider the* mapping

$$f_* : \pi_1(\partial X_1) \to \pi_1(\partial X_2), \quad \text{i.e.,} \quad f_*\mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z},$$

which is induced by the diffeomorphism between the solid tori $X_1$ and $X_2$. It is obvious that the homomorphism $f_*$ is given by the integral matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove that this matrix is unimodular and calculate the fundamental group of the manifold $M_f^3$ in terms of the matrix $f_*$.

**12.123.** Let $X_n$ be the space of polynomials $f_n(z)$ (of one complex variable) without multiple roots. Find the groups $\pi_k(X_n)$.

**12.124.** Prove that a finite $CW$-complex is homotopy equivalent to a manifold with boundary.

# 13
# Covering Maps, Fibre Spaces, Riemann Surfaces

**13.1.** Let $p: X \to Y$ be a covering map such that $f_*[\pi_1(X, x_0)]$ is a normal subgroup of the group $\pi_1(Y, y_0)$, $p(x_0) = y_0$. Prove that each element $\alpha \in \pi_1(Y, y_0)$ generates a homeomorphism $\varphi$ of the covering, i.e., $p\varphi(x) = p(x)$.

**13.2.** Let $p: X \to Y$ be a covering map, $p(x_0) = y_0$. Prove that $p_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism.

**13.3.** Let $p: X \to Y$ be a covering map, and $p(x_0) = y_0$. Prove that the induced mapping $p_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism.

**13.4.** Let $p: X \to Y$ be a covering map, and $\pi_1(Y) = 0$. Prove that each element $\alpha \in \pi_1(X)$ is determined by a homeomorphism of the space $Y$ onto itself, $\alpha: Y \to Y$, and the diagram

$$\begin{tikzcd} Y \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is commutative.

**13.5.** Let $p : X \to Y$ be a covering map where the space $X$ is connected, and let $F = p^{-1}(y_0)$ be the inverse image of a point $y_0 \in Y$, $x_0 \in F$.

Prove that $F$ and $\pi_1(Y, y_0)$ are in one-to-one correspondence if $\pi_1(X, x_0) = 0$.

**13.6.** Let $p : X \to Y$ be a covering map, $F : I^2 \to Y$ a continuous function, where $I^2$ is a square, and $f : I^1 \to X$ also continuous, with $pf(t) = F(t, 0)$.

Prove that $f$ can be extended to the mapping $G : I^2 \to X$, with $pG = F$, $G(t, 0) = f(t)$.

**13.7.** Let $p : X \to Y$ be a covering map, $f$, $g$ two paths on $X$, and $f(0) = g(0)$. Let $pf(1) = pg(1)$, and the paths $pf$ and $pg$ homotopic. Prove that $f(1) = g(1)$.

**13.8.** Let $p : X \to Y$ be a covering map, $f$, $g$ two paths on $X$, and $f(0) = g(0)$. Does it follow from $pf(1) = pg(1)$ that $f(1) = g(1)$?

**13.9.** Let $p : X \to Y$ be a covering map, $f$, $g$ two paths on $Y$, and $\bar{f}$, $\bar{g}$ two paths on $X$ such that $p\bar{f} = f$, $p\bar{g} = g$, $\bar{f}(0) = \bar{g}(0)$.

Prove that if $f$ and $g$ are homotopic, then $\bar{f}$ and $\bar{g}$ are also homotopic.

**13.10.** Let $p : X \to Y$ be a covering map, $f$ a path in $Y$, and $x_0$ a point in $X$ such that $p(x_0) = f(0)$. Prove that there exists, and is unique, a path $g$ in $X$ such that $pg = f$.

**13.11.** Prove that a covering map is a Serre fibre map.

**13.12.** Prove that any two-sheeted covering is regular. What purely algebraic fact corresponds to this statement?

**13.13.** Prove that a three-sheeted covering of a pretzel (i.e., sphere with two handles) is non-regular.

**13.14.** Let $M^2$ be a non-orientable, compact, smooth and closed manifold. Prove the existence of a two-sheeted covering map $p : M^2_+ \to M^2$, where $M^2_+$ is an orientable manifold, and find $M^2_+$ in explicit form. What is the property of the fundamental group of a non-orientable manifold?

**13.15.** Construct the covering map $S^n \xrightarrow{\mathbf{Z}_2} \mathbf{R}P^n$ and prove that:

(a) $\mathbf{R}P^n$ is orientable when $n = 2k-1$, and non-orientable when $n = 2k$;

(b) $\pi_1(\mathbf{R}P^n) = \mathbf{Z}_2$, $\pi_i(\mathbf{R}P^n) = \pi_i(S^n)$ when $n > 1$, $i > 1$.

**13.16.** Prove that a covering space is regular if and only if its paths lying over one path in a basis are either all closed or all non-closed.

**13.17.** Let $p : \hat{X} \to X$ be a covering map. Prove that any path in $X$ can be covered in $\hat{X}$ in a unique way up to the choice of the origin of the path in the inverse image, and the multiplicity of the projection $p$ is the same at all points of the base space.

**13.18.** Construct all coverings over the circumference and prove that $\pi_1(S^1) = \mathbf{Z}$, $\pi_i(S^1) = 0$ when $i \geqslant 2$.

**13.19.** Construct the regular covering map $p : P_k \xrightarrow{Z_{k-1}} P_2$, where $k > 2$ and $P_k$ is a sphere with $k$ handles.

**13.20.** Construct a universal covering of $\vee_A S^1$ and prove that $\pi_i(\vee_A S^1) = 0$ when $i > 1$. Find $\pi_1(\vee_A S^1)$.

**13.21.** Construct a covering map $\varphi : \hat{X} \to P_2$, where $P_2$ is a pretzel, such that $\hat{X}$ is contractible to the graph. Prove, as a corollary, that

(a) a universal covering space $P_2$ is contractible, $P_2 \sim K(\pi, 1)$;

(b) if $M^2$ is a two-dimensional closed manifold and $\pi_1(M^2)$ an infinite group, then $M^2 \sim K(\pi, 1)$ (i.e., homotopy equivalent).

**13.22.** Establish the relation between universal coverings over $P_k$ (i.e., sphere with $k$ handles) and the Lobachevski plane.

**13.23.** Prove that all covering maps of the torus $T^2$ are regular and find them. Construct an example of two non-equivalent, but homeomorphic covering maps of the torus $T^2$.

**13.24.** Let $X$ be a finite complex. Find the relation between $G \subset \pi_1(X)$ (arbitrary subgroup), $\chi(X)$ (the Euler characteristic) and $\chi(X_G) \to (X_G)$ (covering map constructed relative to the subgroup $G \subset \pi_1(X)$).

**13.25.** Construct a universal covering of the torus $P_1$ (i.e., sphere with one handle) and Klein bottle (i.e., sphere with two cross-caps) and calculate the homotopy groups of $P_1$ and $N_2$. Can the torus $P_1$ be a two-sheeted and regular covering of the Klein bottle? If so, find the covering and calculate the image of the group $\pi_1(P_1)$ in $\pi_1(N_2)$ under the covering monomorphism.

**13.26.** Prove that if $\pi_1(M^n) = 0$ or $\pi_1(M^n)$ is a simple or finite group of order $p \neq 2$ (where $p$ is prime), then the manifold $M^n$ is orientable.

**13.27.** Construct the explicit form of seven smooth linearly independent vector fields on the sphere $S^7$. Use the algebra of octaves (Cayley numbers). Construct the integral curves of these vector fields.

**13.28.** Prove that if $k$ linear operators $A_1, \ldots, A_k$ are given in $\mathbf{R}^n$ such that $A_i^2 = -E$ and $A_iA_j + A_jA_i = 0$ (for all $i, j$), then $k$ linearly independent vector fields can be specified on the sphere $S^{n-1} \subset \mathbf{R}^n$.

**13.29.** If the homotopy groups of the base space and fibre of a fibre space have finite rank, then the homotopy groups of the total space also have finite rank, the rank of the $q$-dimensional group of the total space not exceeding the sum of the ranks of the $q$-dimensional homotopy groups of the base space and fibre.

**13.30.** Let the fibre map $p : E \to B$ admit an image set of a section $\chi : B \to E$, and $e_0 = \chi(b_0)$. Prove that when $n \geqslant 1$, the mapping $p_*$ is an epimorphism, and when $n \geqslant 2$, it determines a decomposition into the direct sum $\pi_n(E, e_0) = \pi_n(B, b_0) \oplus \pi_n(F, e_0)$.

**13.31.** Prove that if all the homotopy groups of the base space and fibre are finite, then the homotopy groups of the total space are also finite and their orders do not exceed the product of the orders of the homotopy groups of the base space and fibre of the same dimension.

**13.32.** Prove that the mapping $p : EX \xrightarrow{\Omega X} X$ satisfies the covering homotopy axiom (Serre fibre map).

**13.33.** Prove that if $p_{1,2} : \hat{X}_{1,2} \to X$ are covering maps and $\mathrm{Im}(p_1)_* = \mathrm{Im}(p_2)_*$, then $(\hat{X}_1, p_1, X)$ and $(\hat{X}_2, p_2, X)$ are fibre homeomorphic, where $\mathrm{Im}(p_i)_* \subset \pi_1(X)$.

**13.34.** Prove that there exists a covering map $p : \hat{X} \to X$ over any connected complex $X$ such that $\pi_1(\hat{X}) = 0$ (i.e., existence of a universal covering).

**13.35.** Prove that the set of vector bundles with a structural group $G$ over the sphere $S^n$ is isomorphic to $\pi_{n-1}(G)$, and that the group $G$ is path-connected.

**13.36.** Show by way of example that there exists no exact homology sequence of a fibration.

**13.37.** Let $p : E \xrightarrow{F} B$ be a locally trivial fibre map, and $B$, $F$ finite complexes. Then $\chi(E) \leqslant \chi(B)\ \chi(F)$.

**13.38.** Given that a material particle moves with constant (in modulus) velocity (a) on the torus $T^n$, (b) sphere $S^n$, find phase space for this system.

**13.39.** Let $p : E \xrightarrow{F} B$ be a fibre map with path-connected $B$ and $F$. Let $\widetilde{\mathrm{Cat}} = \mathrm{Cat} - 1$ be a reduced Lusternik-Schnirelmann category, i.e., $\widetilde{\mathrm{Cat}}$ (of a point) $= 0$. Prove that $\widetilde{\mathrm{Cat}}(E) \leqslant \widetilde{\mathrm{Cat}}_E(F)\ \widetilde{\mathrm{Cat}}(B) + \widetilde{\mathrm{Cat}}(B) + \mathrm{Cat}_E(F)$, where $\mathrm{Cat}_E(F)$ is a relative category of the fibre $F$ respective to $E$.

**13.40.** Prove that if $p : X \to Y$ is a Serre fibre map, then $p$ is a mapping "onto".

**13.41.** Prove that if $p : X \to Y$ is a Serre fibre map, then $p^{-1}(y_1)$ and $p^{-1}(y_2)$ are homotopy equivalent for any $y_1, y_2 \in Y$.

**13.42.** Prove that the manifold of linear elements of a manifold $M$ is a fibration with the base space $M$.

**13.43.** Prove that a locally trivial fibration (twisted product) is a Serre fibration.

**13.44.** Prove that the space of paths $EX$ whose starting point is fixed in the space $X$ is a Serre fibre space with the base space $X$.

**13.45.** Prove that if $M^n$ is a smooth manifold, then the spaces of its total and unimodular tangent bundles (fibrations) are orientable.

**13.46.** Prove that the direct product of topological spaces $X \times Y$ with the projection onto one of the factors is a Serre fibration.

**13.47.** Let $p : X \to Y$ be a covering map, $p(x_0) = y_0$, and $f$, $g$ two paths such that $f(0) = g(0) = y_0$, $f(1) = g(1)$. Let $fg^{-1} \in p_*(\pi_1(X, x_0))$ and let $\hat{f}$, $\hat{g}$ be two coverings of these paths. Prove that $\hat{f}(1) = \hat{g}(1)$.

**13.48.** Represent the torus $T^2$ as $T^2 = \{g\}$, where $g = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$.

Consider the following equivalence relations $R$:

(a) $(e^{i\varphi_1}, e^{i\varphi_2})R(-e^{i\varphi_1}), e^{-i\varphi_2})$;

(b) $(e^{i\varphi_1}, e^{i\varphi_2})R(-e^{-i\varphi_1}, -e^{-i\varphi_2})$;

(c) $(e^{i\varphi_1}, e^{i\varphi_2})R(e^{-i\varphi_1}, e^{-i\varphi_2})$.

Find the space $X = T^2/R$ and calculate the image $f_*(\pi_1(T^2)) \subset \pi_1(X)$, where $f : T^2 \to X = T^2/X$ is the projection induced by the relation $R$. Is $f$ a covering map?

**13.49.** How many fibrations are there of the following form:

(a) $T^3 \to S^1_i$ where $T^3$ is a three-dimensional torus;

(b) $T^n \to S^1$, where $T^n$ is an $n$-dimensional torus (fibrations are considered up to homotopy equivalence)?

**13.50.** Let $C = A * B$ be the free product of arbitrary groups $A$ and $B$. Prove that for any subgroup $M \subset C$, the equality $M = A_1 * B_1 * F$ holds, where $A_1 \subset A$, $B_1 \subset B$ and $F$ is a free group. Give a topological proof using covering spaces.

**13.51.** Let $\mathfrak{G}$ be a 1-connected compact Lie group, and $\sigma : \mathfrak{G} \to \mathfrak{G}$ an arbitrary involutive automorphism (i.e., $\sigma^2 = 1$). Put $\mathfrak{H} = \{g \in \mathfrak{G} | \sigma(g) = g\}$; $V = \{g \in \mathfrak{G} | \sigma(g) = g^{-1}\}$. Prove that $\mathfrak{G} = V\mathfrak{H}V$, i.e., any element $g \in \mathfrak{G}$ admits a representation in the form

$$g = vhv, \quad v \in V,\ h \in \mathfrak{H}.$$

Prove that $V \equiv \mathfrak{G}/\mathfrak{H}$ (homogeneous space).

**13.52.** The following construction (by Cartan) is known. Let $\sigma : \mathfrak{G} \to \mathfrak{G}$ be an arbitrary involutive automorphism of a compact connected Lie group. Put $\mathfrak{H} = \{g \in \mathfrak{G} \mid \sigma(g) = g\}$; $V = \{g \in \mathfrak{G} | \sigma(g) = g^{-1}\}$. Then $V \cong \mathfrak{G}/\mathfrak{H}$ and $V \subset \mathfrak{G}$ is a totally geodesic submanifold. Therefore, $V$ is a symmetric space. The submanifold $V$ is called Cartan's model of the symmetric space $\mathfrak{G}/\mathfrak{H}$. Any symmetric space admits Cartan's model (which is almost always uniquely determined).

(a) Prove that the projection $p : \mathfrak{G} \to V$, where $p(g) = g\sigma(g^{-1})$ determines the principal fibration $0 \to \mathfrak{H} \to \mathfrak{G} \to \mathfrak{G}/\mathfrak{H} \to 0$.

(b) Let $V$ be Cartan's model, $\pi_1(V) = 0$, $e \in V$, and $e$ the identity element in $\mathfrak{G}$. Prove that if a point $x \in V$ is conjugate of $e$ along a geodesic $\gamma(t) \subset V$ in the group $\mathfrak{G}$, then the point $x$ is conjugate of $e$ along $\gamma$ in the manifold $V \subset \mathfrak{G}$ itself.

**13.53.** Prove that a compact, closed manifold $M^2$ with Euler characteristic $N$ can be represented as a $(2N + 4)$-gon such that some of its sides are glued together to yield the word

$$a_1 a_2 \ldots a_{N+2} a_1^{-1} a_2^{-1} \ldots a_{N+2}^{-1}$$

(where $a_1, a_2, \ldots, a_{N+2}$ are the designations of the sides) in traversing the sides one after another. Prove that the last factor is $a_{N+2}^{-1}$ if and only if $M^2$ is orientable.

**13.54.** Classify compact, closed smooth and connected manifolds $M$ and calculate their fundamental groups in terms of the generators and relations.

**13.55.** Prove that any orientable, two-dimensional, and compact manifold is determined by a unique invariant, viz., the genus of the manifold.

**13.56.** Prove that any non-orientable, two-dimensional and compact manifold can be represented as the connected sum of an orientable manifold and several replicas of projective planes.

**13.57.** Describe the semigroup of two-dimensional manifolds under the connected sum operation.

**13.58.** Calculate the homotopy groups $\pi_i(T_g)$ $(i \geqslant 1)$ of a two-dimensional manifold $T_g$ of genus $g$.

**13.59.** Let $M^2$ be a compact, closed, oriented and two-dimensional manifold of genus $g$. Find the homotopy type of $\Sigma^2 M^2$.

**13.60.** (a) Prove that $\mathbf{R}P^2 \setminus D^2$ is diffeomorphic to the Möbius strip.

(b) What spaces is the sphere $S^2$, with a Möbius strip glued into, homeomorphic to? with two Möbius strips?

**13.61.** Let $S^1 \times S^1 \subset \mathbf{R}^3$ be the standard embedding of the torus in Euclidean space. Prove that there exists no homeomorphism of the pair $(\mathbf{R}^3, S^1 \times S^1)$ onto itself whose restriction to the torus is determined by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

**13.62.** Given two odd functions on the sphere $S^2$, prove that they have a common zero.

**13.63.** Let $\pi$ be the fundamental group of a two-dimensional surface, and $f: \pi \to \pi$ an epimorphism. Prove that $f$ is an isomorphism.

**13.64.** Prove by three totally different techniques that there exists no continuous vector field without singular points (i.e., different from zero at each of its points) on the sphere $S^2$.

**13.65.** Let in $\mathbf{C}^2(z, w)$ the Riemann surface of the algebraic function $w = \sqrt{P_n(z)}$ be given, where the polynomial $P_n(z)$ has no multiple roots. Prove that this Riemann surface turns, after completing it with a point at infinity, into a two-dimensional, smooth, compact, and orientable manifold.

**13.66.** Find the genus of a two-dimensional manifold described in the previous problem as a function in the degree $n$ of the polynomial $P_n$.

**13.67.** Can the two-dimensional projective plane $\mathbf{R}P^2$ be the Riemann surface of a certain algebraic function $w = w(z)$ in $\mathbf{C}^2(z, w)$ in the sense of Problem 13.65, i.e., after completing the Riemann surface with a point at infinity?

**13.68.** Prove that the Riemann surface of an algebraic function in $\mathbf{C}^2$ is always an orientable manifold.

**13.69.** Investigate what happens to the Riemann surface of the function $w = \sqrt{P_n(z)}$ when some roots of the polynomial $P_n(z)$ merge to yield a multiple root. For example, what is the structure of the Riemann surface of the function $w = \sqrt{z^2(z - b)}$?

**13.70.** Describe the topological structure of the Riemann surfaces of the following analytic functions:

$$w = \sqrt[3]{1 + z}, \quad z = w + 1/w, \quad z^n + w^n = 1.$$

**13.71.** Prove that the Riemann surface of the function $w = \ln z$ is homeomorphic to a finite part of the complex plane.

**13.72.** Let $p : X \to Y$ be a two-sheeted covering. Prove that any path in $Y$ can then be covered by precisely two paths.

**13.73.** Construct a universal covering space for the orthogonal group $SO(n)$.

**13.74.** Prove that any two-dimensional, closed, oriented and smooth manifold can be locally isometrically covered by the Lobachevski plane (which is supplied with the standard metric of constant negative curvature). In other words, prove that the fundamental group of a surface of the indicated form can be represented as a discrete subgroup (operating effectively) of the Lobachevski plane isometry group.

**Corollary.** A two-dimensional, compact, closed, orientable, and smooth manifold can be supplied with the Riemannian metric of constant negative curvature.

**13.75.** What spaces can cover the Klein bottle?

**13.76.** Let $S_g$ be a sphere with $g$ handles. What $S_h$ can cover $S_g$?

**13.77.** Prove that for any compact, non-orientable, and two-dimensional manifold, there is precisely one compact, two-dimensional, and orientable manifold which serves as its two-sheeted covering.

**13.78.** Prove that the Beltrami surface (i.e., surface of constant negative curvature standardly embedded in $\mathbf{R}^3$) can be infinitely-sheeted and locally isometrically covered by a certain region lying in the Lobachevski plane.

Find this region. Prove that it is homeomorphic to the two-dimensional disk. Find the corresponding group of this covering (it is the group $\mathbf{Z}$).

**13.79.** Can a two-dimensional torus be a two-sheeted covering of the Klein bottle?

**13.80.** Calculate the permutation group of the sheets of the Riemann surface of the algebraic function $w = \sqrt[q]{z}$ arising in traversing around the branch point of this function (point 0).

**13.81.** Let $M^2$ be an ellipsoid, and $p$ one of its vertices. Consider all geodesics emanating from the point $p$. Find the locus of the first conjugate points (i.e., mark the first conjugate point of $p$ on each geodesic and describe this set).

**13.82.** Prove that the fundamental group of a complete Riemannian manifold of non-positive curvature contains no elements of finite order. Prove that $\pi_1(M)$ (where $M$ is a complete Riemannian manifold of strictly negative curvature) possesses the following property: if two elements commute ($ab = ba$, $a, b \in \pi_1(M)$), then $a$ and $b$ belong to the same cyclic subgroup.

**13.83.** Prove that a closed, orientable Riemannian manifold $M^n$ of strictly positive curvature and even dimension is 1-connected.

**13.84.** (a) Prove that any compact, closed Riemannian manifold of constant curvature $\lambda$ is isometric either to the sphere $S^n$ or $\mathbf{R}P^n$ (of radius $1/\sqrt{\lambda}$). Use Problem 13.83.

(b) Let $M^n$ be a compact, closed, 1-connected, complete Riemannian manifold, and $C(l)$ the set of the first conjugate points of a certain point $l \in M^n$.

Prove that if $M^n$ is a symmetric space, then the complement $M^n/C(l)$ is homeomorphic to the open disk.

**13.85.** Prove that a complete, non-compact Riemannian manifold of positive curvature and dimension $m$, where either $m = 2$ or $m \geqslant 5$, is diffeomorphic to $\mathbf{R}^m$.

**13.86.** Let $x$, $y$ be two near points on the standard sphere $S^2$, and a function $f(z)$ the area of the geodesic triangle with vertices at the points $x$, $y$, $z$.

(a) Is the function $f(z)$ harmonic on the sphere $S^2$?

(b) Investigate the case of the $n$-dimensional sphere (where $f(z)$ is the volume of the geodesic simplex whose one face is fixed, and $z$ is a free vertex).

(c) Investigate the same problem for the Lo achevski plane.

**13.87.** Prove that if $M^n$ is a complete, 1-connected Riemannian manifold such that $n$ is odd and there exists a point $p$ on $M^n$, the set of the first conjugate points of $p$ being regular and each point of constant order $k$, then $k = n - 1$, $M^n$ is homeomorphic to the sphere $S^n$ (order of a point is understood to be its multiplicity).

**13.88.** Let $\gamma \subset \mathbf{R}^2$ be a simple closed curve of length $l$ bounding a region $G$ of area $S$ (on the plane).

Prove that $l^2 \geqslant 4\pi S$ and that the equality holds if and only if $\gamma$ is a circumference.

**13.89.** Let $\gamma \subset \mathbf{R}^2$ be a closed curve (not necessarily simple, i.e., in contrast with the previous problem, self-intersecting is possible). Prove that $l^2 \geqslant 4\pi \int\limits_{R^2} \omega(x)dS$, where the function $\omega(x)$ is the number of rotations of the curve about a point $x \in \mathbf{R}^2$.

**13.90.** Is it true that if $n(x, y)$ is the refractive index of a planar, transparent, isotropic, but non-homogeneous substance filling the two-dimensional plane, then the integral curves of the vector field grad $n(x, y)$ $(n = c/v)$ are the trajectories of light rays? (Certainly, not only they.)

# 14
# Degree of Mapping

**14.1.** Calculate the degree of the mapping $f(z) : S^1 \to S^1$, where $f(z) = z^k$, $|z| = 1$.

**14.2.** Calculate the degree of the mapping $f : S^2 \to S^2$, where $f(z) = z^k$, $z \in \mathbf{C} \cup \{\infty\}$.

**14.3.** Let $M^n$ be an orientable, smooth, and compact manifold. Prove that the homotopy class of a mapping $M^n \to S^n$ is fully determined by the degree of the mapping.

**14.4.** Let $f : S^{2n} \to S^{2n}$ be a continuous mapping. Prove that there is a point for which either $f(x) = x$ or $f(x) = -x$.

**14.5.** Let the degree of the mapping $f: S^n \to S^n$ be equal to $2k + 1$. Prove that there exists a point $x$ such that $f(x) = -f(-x)$. Prove that there exists no even tangent vector field $v(x)$ without singularities (i.e., $v(-x) = v(x)$ has no zeroes) on the sphere $S^{2p-1}$.

**14.6.** Let $f, g: S^n \to S^n$ be two simplicial mappings. Prove that:

(a) the inverse image of each interior point consists of the same number of points (meaning the difference between the numbers of positively oriented and negatively oriented points);

(b) if $f$, $g$ are homotopic, then the difference between the number of positively oriented and negatively oriented points of the inverse image is the same for the two mappings;

(c) if the difference between the number of positively oriented and negatively oriented points of the inverse image coincides for the two mappings, then they are homotopic.

**14.7.** Let $f: M \to S^2$ be a mapping of the normals of a closed surface in $\mathbf{R}^3$. Prove that $f^*(\omega) = K(\omega')$ where $\omega$ and $\omega'$ are elements of area and $K$ the Gaussian curvature. Prove that $2 \deg f = \int K d\omega$ and also equals the Euler characteristic of the surface.

**14.8.** Prove that any continuous mapping of the ball $D^n$ into itself always has a fixed point.

**14.9.** Let $f: SU(n) \to SU(n)$ be a smooth mapping, and $f(g) = g^3$. Find $\deg f$.

**14.10.** Let $f: M^n \to \mathbf{R}^p$ be a smooth mapping of a connected, compact, orientable, and closed $n$-dimensional ($n < p$) manifold in $\mathbf{R}^p$. Let $\nu(f)$ be the normal bundle of this immersion, and $S\nu(f)$ the associated fibre bundle of spheres, i.e., $S\nu(f) = \partial\nu(f)$ is the boundary of a certain sufficiently small tubular neighbourhood of the submanifold $f(M^n) \subset \mathbf{R}^p$. Let $T: S\nu(f) \to S^{p-1}$ be a usual Gauss (spherical) mapping.

Find $\deg T$ ($\dim S\nu(f) = p - 1$) if the Euler characteristic of the manifold $M^n$ is known. Does $\deg T$ depend on the method of immersing $M^n$ into $\mathbf{R}^p$? What happens if $M^n$ is non-orientable? Separately consider the case where $p = n + 1$.

**14.11.** Given that a two-dimensional, orientable, closed, and compact manifold $M^2$ of genus $g$ is embeddable in the Euclidean space $\mathbf{R}^3$, find the minimal number of the saddle points (generally speaking, degenerate) of the function $f(p) = z$, $p \in i(M^2)$, where $i$ is an embedding and $f$ the height function.

**14.12.** Prove that non-degenerate critical points of a smooth function on a smooth manifold are isolated.

**14.13.** Let $f(x)$ be a function on a two-dimensional, compact, orientable surface of genus $g$ (i.e., sphere with $g$ handles) having a finite number of critical points, all of them being non-degenerate. Prove that the number of minima minus the number of saddle points plus the number of maxima equals $2g - 2$.

**14.14.** Let $f: M^n \to \mathbf{R}$ be a smooth function on a smooth manifold. Prove that almost every value of the function $f$ is regular.

**14.15.** Prove that the alternating sum of singular (critical) points of a smooth function $f(x)$ (assuming that all its singularities are non-degenerate) given on a smooth compact manifold does not depend on the function (by the alternating sum, we understand

$$\sum_{\lambda=0}^{n} (-1)^{\lambda} m_{\lambda}, \quad \text{where} \quad n = \dim M,$$

$\lambda$ the index of a critical point, and $m_\lambda$ the number of the critical points of index $\lambda$).

**14.16.** Let $f(x)$ be a complex analytic function of one variable $x$. Prove that the set of critical values of the function $f(x): S^2 \to S^2$ has measure zero.

**14.17.** Let $M_c^n = \{x \mid f(x) = c\}$. Prove that if $M_c^n$ contains no critical points of the function $f$, then $M_c^n$ is a submanifold in $M^n$ and $\operatorname{codim} M_c^n = 1$.

**14.18.** Prove that the notion of non-degenerate critical point of a smooth function does not depend on the choice of the local chart containing this point.

**14.19.** Show that for the standard embedding of the torus $T^2 \subset \mathbf{R}^3$ (i.e., surface of revolution about the axis $Oz$), the coordinate $x$ orthogonal to the axis of rotation of $T^2$ has only non-degenerate critical points.

**14.20.** (a) Construct functions with only non-degenerate critical points on $\mathbf{R}P^n$ and $\mathbf{C}P^n$ so that their values at all critical points may be different.

(b) Construct functions on $\mathbf{R}P^n$ and $\mathbf{C}P^n$ such that $f(x_\lambda) = \lambda = \operatorname{ind}(x_\lambda)$, where $x_\lambda$ are non-degenerate critical points of index $\lambda$.

**14.21.** Let $F(x, y)$ be a non-degenerate bilinear form on $\mathbf{R}^n$. Consider a smooth function $f(x) = F(x, x)$, where $|x| = 1$, i.e., $F(x, x)$ is a function on the sphere $S^{n-1} \subset \mathbf{R}^n$. Let $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_{n-1}$ be all the eigenvalues of the form $F$ (recall that all $\lambda_i$, $0 \leqslant i \leqslant n - 1$, are real).

Prove that $\lambda_i$ are the critical values of the function $F(x, x)$ on the sphere $S^{n-1}$. Find all the critical points of the function $F(x, x)$. Prove that $\lambda_i = \inf_{S^i} \{\max_{x \in S^i} f(x)\}$, where $S^i$ are the standard $i$-dimensional equators of the sphere $S^{n-1}$.

**14.22.** Prove that if a point $p$ is a non-degenerate critical point for a smooth function $f(x)$ on a smooth manifold, then there exists a local coordinate system in which the function $f(x)$ in a neighbourhood of the point $p$ can be represented as a non-singular quadratic form.

**14.23.** Prove that if $M_c$ is a non-critical level for a function $f(x)$ on a manifold $M$ (i.e., the level hypersurface $f(x) = c =$ const not containing critical points for $f(x)$), then the neighbourhood $M_c$ is diffeomorphic to $M_c \times I$.

**14.24.** If $M_{c_1}$ and $M_{c_2}$ are consecutive critical levels, then the interval between them is diffeomorphic to $M_c \times I$, where $c_1 < c < c_2$.

**14.25.** If there are no critical levels between $M_{c_1}$ and $M_{c_2}$ (i.e., level hypersurfaces $f(x) =$ const with critical points) and $M_{c_1}$ and $M_{c_2}$ are non-critical either, then they are diffeomorphic.

**14.26.** (a) Construct a smooth function $f(x)$ having one point of maximum, one point of minimum (both being non-degenerate), and another critical point, perhaps, degenerate, on every compact, orientable, two-dimensional, and smooth manifold $M^2$. Find the relation between such a function and the representation of $M^2$ as the Riemann surface of a certain many-valued analytic function. Investigate the case of a non-orientable two-dimensional manifold $M^2$ (e.g., case of the projective plane $\mathbf{R}P^2$).

(b) Construct a smooth function $f(x)$ having only non-degenerate critical points, precisely one point of maximum, precisely one point of minimum and $s$ saddle points (find the number $s$) on every compact manifold $M^2$. Construct the function so that it takes the same value at all the saddle points. Investigate the non-orientable case. Indicate the relation to the problem of point (a) and construct the confluence of all the saddle points into one degenerate critical point.

# 15
# Simplest Variational Problems

**15.1.** Prove that the extremals of the action functional $E[\gamma] = \int_0^1 |\dot{\gamma}|^2 dt$ on a smooth Riemannian manifold $M^n$ (where $\gamma(t)$ are smooth trajectories on $M^n$, $0 \leqslant t \leqslant 1$, and $\dot{\gamma}(t)$ is the velocity vector of the curve $\gamma(t)$) are geodesics.

**15.2.** Establish the relation between the extremals of the length functional $L[\gamma] = \int_0^1 |\dot{\gamma}| dt$ and action functional $E[\gamma] = \int_0^1 |\dot{\gamma}|^2 dt$. Prove that any extremal $\gamma_0(t)$ of $E[\gamma]$ is that of $L[\gamma]$. Prove that if $s_0(t)$ is an extremal of $L[\gamma]$, then by replacing the parameter $t = t(\tau)$ by $s_0(t)$, this trajectory can be transformed into an extremal of $E[\gamma]$.

**15.3.** Let $S(f) = \iint_D \sqrt{EG - F^2}\, dudv$ be a functional associating each smooth function $z = f(x, y)$ which is defined on a bounded region $D = D(x, y) \subset \mathbf{R}^2(x, y)$ (where $x, y, z$ are Cartesian coordinates in $\mathbf{R}^3$) with the area of the graph of the function $z = f(x, y)$. Prove that the extremality of a function $f_0$ relative to a functional $S$ is equivalent to the condition $H = 0$, where $H$ is the mean curvature of the graph of $z = f(x, y)$ considered as a two-dimensional, smooth submanifold in $\mathbf{R}^3$.

**15.4.** Prove the statement formulated in the previous problem for the case of the $(n - 1)$-dimensional graphs of $x^n = f(x^1, \ldots, x^{n-1})$ in $\mathbf{R}^n$.

**15.5.** Prove that the action functional $E[\gamma]$ and length functional $L[\gamma]$ are related by the formula $(L[\gamma])^2 \leqslant E[\gamma]$, the equality being held if and only if $\gamma(t)$ is a geodesic.

**15.6.** Prove that the areal functional

$$S[\vec{r}] = \iint_D \sqrt{EG - F^2}\, dudv$$

(where $\vec{r} = \vec{r}(u, v)$ is a radius vector in $\mathbf{R}^3$ depending smoothly on $(u, v)$) and Dirichlet's functional $D[\vec{r}[ = \iint_D \frac{E + G}{2} dudv$ are related by the formula $S[\vec{r}] \leqslant D[\vec{r}]$.

**15.7.** Remember that the radius vector $\vec{r}(u, v)$ determining a two-dimensional surface $M^2$ in Euclidean three-dimensional space is said to be harmonic if $\vec{r}(u, v)$ is an extremal of Dirichlet's functional $D[\vec{r}] = \frac{1}{2}\iint_D (E + G) dudv$. Prove that if the mean curvature $H$ of a surface $M^2$ given by a radius vector $\vec{r}(u, v)$ equals zero, then local coordinates $(p, q)$ can be introduced in a neighbourhood of each point on the surface so that the radius vector $\vec{r}(p, q)$ in these coordinates becomes harmonic.

**15.8.** Construct an example of a harmonic radius vector $\vec{r}(u, v)$ such that the surface $M^2 \subset \mathbf{R}^3$ described by it may not be minimal (i.e., so that $H \not\equiv 0$).

**15.9.** *The Wirtinger Inequality.* Let $H$ be a hermitian symmetric positive definite form in $\mathbf{C}^n$, and $\alpha : \mathbf{C}^n \to \mathbf{R}^{2n}$ a realification of $\mathbf{C}^n$. Then $H \to H^R = \begin{pmatrix} S & A \\ -A & S \end{pmatrix}$, where $H = S + iA$, $S$ and $A$ are real matrices and $S^T = S$, $A^T = -A$, $\overline{H}^T = H$.

The form $S$ defines the Euclidean scalar product in $\mathbf{R}^{2n}$ and the form $A$ an exterior 2-form $\omega^{(2)}$ in $\mathbf{R}^{2n}$. For simplicity, we may assume that $\omega^{(2)} = \sum_{k=1}^{n} dz^k \wedge d\bar{z}^k$. Consider the form $\Omega^{(2r)} = \frac{1}{r!} \underbrace{\omega \wedge \ldots \wedge \omega}_{r}$, $r \geqslant n$.

(a) If $\omega_1, \ldots, \omega_{2r}$ is an arbitrary orthonormal basis in $\mathbf{R}^{2n} \cong \mathbf{C}^n$ relative to the scalar product $S = \operatorname{Re} H$, then

$$|\Omega^{(2r)}(\omega_1, \ldots, \omega_{2r})| \leqslant 1 \quad \text{and} \quad |\Omega^{(2r)}(\omega_1, \ldots, \omega_{2r})| = 1$$

if and only if the plane $L(\omega_1, \ldots, \omega_{2r})$ generated by the vectors $\omega_1, \ldots, \omega_{2r}$ is a complex subspace in $\mathbf{R}^{2n} \cong \mathbf{C}^n$.

**Hint:** Let $r = 1$, and $\omega_1$, $\omega_2$ be an orthonormal pair of vectors. It is required to prove that $\omega(\omega_1, \omega_2) \leqslant 1$, where $\omega(\omega_1, \omega_2) = A(\omega_1, \omega_2)$. Consider

$$H = (\omega_1, \omega_2) = (S + iA)(\omega_1, \omega_2) =$$

$$= S(\omega_1, \omega_2) + iA(\omega_1, \omega_2) = iA(\omega_1, \omega_2).$$

Hence, $|H(\omega_1, \omega_2)| = |A(\omega_1, \omega_2)| \leqslant |\omega_1| |\omega_2| = 1$. Now, let $A(\omega_1, \omega_2) = 1$. Then $H(\omega_1, \omega_2) = i$, i.e., $S(i\omega_1, \omega_2) = 1$. Since $|\omega_2| = |i\omega_1| = 1$, it follows that $\omega_2 = i\omega_1$, i.e., the two-dimensional plane spanned by $\omega_1$, $\omega_2$ is the complex straight line. For $r > 1$, the relation $\Omega^{(2r)}(\omega_1 - \omega_2) = \sqrt{\det g_{ij}}$ should be used, where $g_{ij}$ is the skewsymmetric scalar product defined by the 2-form $\omega^{(2)}$.

(b) Let $W^r \subset \mathbf{C}^n$, $r < n$ ($r$ being complex dimension) be a complex submanifold in $\mathbf{C}^n$ (if $W^r$ is an algebraic submanifold, then singular points on $W^r$ are possible). Let $V^{2r}$ be a real submanifold in $\mathbf{C}^n$ such that $V \cup W = \partial Z^{2r+1}$, where $Z^{2r+1}$ is a real $(2r+1)$-dimensional submanifold in $\mathbf{C}^n$ whose boundary is $V \cup W$. Let $K = V \cap W$. Then $\mathrm{vol}_{2r}(V \setminus K) \geqslant \mathrm{vol}_{2r}(W/K)$.

**Note.** This statement means that complex submanifolds $W$ in the complex space $\mathbf{C}^n$ are minimal submanifolds, i.e., after any "perturbation" of $V$, the $2r$-dimensional volume ($\mathrm{vol}_{2r}$) does not decrease.

**Hint:** The statement follows from the Wirtinger inequality (see above) and Stokes' formula. In fact, consider the exterior 2-form

$$\omega = \sum_{k=1}^{n} dz^k \wedge d\bar{z}^k,$$

and let

$$\Omega^{(2r)} = \frac{1}{r!}\,\omega\wedge \ldots \wedge\omega \quad (r \text{ times}).$$

Since $d\omega = 0$, $d\Omega^{(2r)} = 0$. It follows from Stokes' formula that

$$\int_W \Omega^{(2r)} = \int_V \Omega^{(2r)}.$$

While integrating the form $\Omega^{(2r)}$ with respect to a $2r$-dimensional submanifold, the expression of the sort $\Omega^{(2r)}(\omega_1, \ldots, \omega_{2r})dx^1 \wedge \ldots \wedge dx^{2r}$, where $\omega_1, \ldots, \omega_{2r}$ is an orthonormal basis in the tangent plane to the submanifold (with respect to the Riemannian metric induced by the underlying Euclidean metric in $\mathbf{C}^n = \mathbf{R}^{2n}$) should be considered (in local coordinates $x^1, \ldots, x^{2r}$). If the submanifold $W$ is complex, then

$$\Omega^{(2r)}(\omega_1, \ldots, \omega_{2r}) = 1 \quad \text{and} \quad \mathrm{vol}_{2r}(W) = \int_W \Omega^{(2r)}.$$

If the submanifold $V$ is of general form (i.e., real), then $\Omega^{(2r)}(\omega_1, \ldots, \omega_{2r}) \leqslant 1$, i.e., $\int_V \Omega^{(2r)} \leqslant \mathrm{vol}(V)$, which proves the statement.

(c) Prove that the statement of problem (b) remains valid if $\mathbf{C}^n$ is replaced by any Kähler manifold, i.e., complex manifold supplied with an exterior 2-form $\omega^{(2)}$ (non-degenerate and closed).

**15.10.** Consider functions of the form $F(x^1, \ldots, x^n)$ on $\mathbf{R}^n(x^1, \ldots, x^n)$ and the functional $J[F] = \int_D |\mathrm{grad} F| d\sigma^n$, where $D$ is the domain of the functions $F$. Let $F_0$ be an extremal of the functional $J$. Prove that the level surfaces $F_0(x^1, \ldots, x^n) = \mathrm{const}$ considered as hypersurfaces in $\mathbf{R}^n(x^1, \ldots, x^n)$ are locally minimal.

# Answers and Hints

## 2
## Systems of Coordinates

**2.1.** $J = H_1 H_2 \ldots H_n$.

**2.2.** $\operatorname{grad} f = \left\{ \dfrac{1}{H_1} \dfrac{\partial f}{\partial q^1}, \dfrac{1}{H_2} \dfrac{\partial f}{\partial q^2}, \dfrac{1}{H_3} \dfrac{\partial f}{\partial q^3} \right\}$.

**2.3.** $\operatorname{div} \mathbf{a} = \dfrac{1}{H_1 H_2 H_3} \left[ \dfrac{\partial}{\partial q^1}(H_2 H_3 a_1) + \dfrac{\partial}{\partial q^2}(H_3 H_1 a_2) + \right.$

$\left. + \dfrac{\partial}{\partial q^3}(H_1 H_2 a_3) \right]$, where $a_1$, $a_2$, $a_3$ are the coordinates of the vector $\mathbf{a}$.

**2.4.** $\Delta f = L \dfrac{1}{H_1 H_2 H_3} \left[ \dfrac{\partial}{\partial q^1} \left( \dfrac{H_2 H_3}{H_1} \dfrac{\partial f}{\partial q^1} \right) + \dfrac{\partial}{\partial q^2} \left( \dfrac{H_3 H_1}{H_2} \dfrac{\partial f}{\partial q^2} \right) + \right.$

$$\left. + \frac{\partial}{\partial q^3} \left( \frac{H_1 H_2}{H_3} \frac{\partial f}{\partial q^3} \right) \right].$$

**2.5.** (a) The coordinate surfaces are: cylinders $r = \text{const}$, planes $\varphi = \text{const}$ and planes $z = \text{const}$.

(b) $H_1 = 1$, $H_2 = r$, $H_3 = 1$.

(c) $\Delta u = \dfrac{1}{r} \dfrac{\partial}{\partial r}\left( r \dfrac{\partial u}{\partial r} \right) + \dfrac{1}{r^2} \dfrac{\partial^2 u}{\partial \varphi^2} + \dfrac{\partial^2 u}{\partial z^2}$.

**2.6.** (a) The coordinate surfaces are: concentric spheres $r = \text{const}$, planes $\varphi = \text{const}$ and cones $\theta = \text{const}$.

(b) $H_1 = 1$, $H_2 = r$, $H_3 = r \sin\theta$.

(c) $\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) +$

$+ \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}.$

**2.7.** (a) The coordinate surfaces are: cylinders of elliptic section and foci at the points $x = \pm c$, $y = 0$ when $\lambda = \text{const}$, the family of confocal hyperbolic cylinders $\mu = \text{const}$ and planes $z = \text{const}$.

(b) $H_1 = c\sqrt{\frac{\lambda^2 - \mu^2}{\lambda^2 - 1}}$, $H_2 = c\sqrt{\frac{\lambda^2 - \mu^2}{1 - \mu^2}}$, $H_3 = 1$.

**2.8.** (a) $q^1 = \sqrt{2r}\sin\frac{\theta}{2}$, $q^2 = \sqrt{2r}\cos\frac{\theta}{2}$, $q^3 = z$.

(b) The coordinate surfaces are: parabolic cylinders with generators parallel to the axis $z$ when $\lambda = \text{const}$, $\mu = \text{const}$.

(c) $H_1 = H_2 = \sqrt{\lambda^2 + \mu^2}$, $H_3 = 1$.

**2.9.**

**(a)** $x = \pm\sqrt{\frac{(\lambda + a^2)(\mu + a^2)(\nu + a^2)}{(b^2 - a^2)(c^2 - a^2)}}$,

$y = \pm\sqrt{\frac{(\lambda + b^2)(\mu + b^2)(\nu + b^2)}{(c^2 - b^2)(a^2 - b^2)}}$,

$z = \pm\sqrt{\frac{(\lambda + c^2)(\mu + c^2)(\nu + c^2)}{(a^2 - c^2)(b^2 - c^2)}}$.

(b) $H_1 = \frac{1}{2}\sqrt{\frac{(\lambda - \mu)(\lambda - \nu)}{R^2(\lambda)}}$, $H_2 = \frac{1}{2}\sqrt{\frac{(\mu - \nu)(\mu - \lambda)}{R^2(\mu)}}$,

$H_3 = \frac{1}{2}\sqrt{\frac{(\nu - \lambda)(\nu - \mu)}{R^2(\nu)}}$,

where $R(s) = \sqrt{(s + a^2)(s + b^2)(s + c^2)}$, $s = \lambda, \mu, \nu$.

$$\text{(c) } \Delta u = \frac{4}{(\lambda - \mu)(\lambda - \nu)(\mu - \nu)} \left[ (\mu - \nu)R(\lambda) \frac{\partial}{\partial \lambda}\left(R(\lambda)\frac{\partial u}{\partial \lambda}\right) + \right.$$

$$\left. + (\nu - \lambda)R(\mu) \frac{\partial}{\partial \mu}\left(R(\mu)\frac{\partial u}{\partial \mu}\right) + (\lambda - \mu)R(\nu) \frac{\partial}{\partial \nu}\left(R(\nu)\frac{\partial u}{\partial \nu}\right) \right].$$

**2.10.** (a) The coordinate surfaces are: prolate ellipsoids of revolution $\alpha = \text{const}$, hyperboloids of revolution of two sheets $\beta = \text{const}$ and planes $\varphi = \text{const}$.

(b) $H_1 = H_2 = c\sqrt{\sinh^2\alpha + \sin^2\beta}$, $H_3 = c \sinh\alpha \sin\beta$.

$$\text{(c) } \Delta u = \frac{1}{c^2(\sinh^2\alpha + \sin^2\beta)} \left[ \frac{1}{\sinh\alpha} \frac{\partial}{\partial \alpha}\left(\sinh\alpha \frac{\partial u}{\partial \alpha}\right) + \right.$$

$$\left. + \frac{1}{\sin\beta} \frac{\partial}{\partial \beta}\left(\sin\beta \frac{\partial u}{\partial \beta}\right) + \left(\frac{1}{\sinh^2\alpha} + \frac{1}{\sin^2\beta}\right) \frac{\partial^2 u}{\partial \varphi^2} \right] = 0.$$

**2.11.** (a) The coordinate surfaces are: oblate ellipsoids of revolution $\alpha = \text{const}$, hyperboloids of revolution of one sheet $\beta = \text{const}$ and planes $\varphi = \text{const}$ passing through the axis $z$.

(b) $H_1 = H_2 = c\sqrt{\cosh^2\alpha - \sin^2\beta}$, $H_3 = c \cosh\alpha \sin\beta$.

$$\text{(c) } \Delta u = \frac{1}{c^2(\cosh^2\alpha - \sin^2\beta)} \left[ \frac{1}{\cosh\alpha} \frac{\partial}{\partial \alpha}\left(\cosh\alpha \frac{\partial u}{\partial \alpha}\right) + \right.$$

$$\left. + \frac{1}{\sin\beta} \frac{\partial}{\partial \beta}\left(\sin\beta \frac{\partial u}{\partial \beta}\right) + \left(\frac{1}{\sin^2\beta} - \frac{1}{\cosh^2\alpha}\right) \frac{\partial^2 u}{\partial \varphi^2} \right].$$

**2.12.** (a) The coordinate surfaces are: the tori $\alpha = \text{const}$,

$$(\varrho - c \coth\alpha)^2 + z^2 = \left(\frac{c}{\sinh\alpha}\right)^2 \quad (\varrho = \sqrt{x^2 + y^2}),$$

the sphere $\beta = \text{const}$,

$$(z - c \cot\beta)^2 + \varrho^2 = \left(\frac{c}{\sin\beta}\right)^2,$$

and the plane $\varphi = \text{const}$;

(c) $\Delta u = \dfrac{\partial}{\partial \alpha}\left(\dfrac{\sinh\alpha}{\cosh\alpha - \cos\beta}\dfrac{\partial u}{\partial \alpha}\right) +$

$$+ \frac{\partial}{\partial \beta}\left(\frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\frac{\partial u}{\partial \beta}\right) + \frac{1}{(\cosh\alpha - \cos\beta)\sinh\alpha}\frac{\partial^2 u}{\partial \varphi^2}.$$

**2.13.** $H_1 = H_2 = \dfrac{a}{\cosh\alpha - \cos\beta}$, $H_3 = 1$.

**2.14.** (a) The coordinate surfaces are: spindle-shaped surfaces of revolution $\alpha = \text{const}$

$$(\varrho - c\cot\alpha)^2 + z^2 = \left(\frac{c}{\sin\alpha}\right)^2,$$

spheres $\beta = \text{const}$,

$$\varrho^2 + (z - c\cot\beta)^2 = \left(\frac{c}{\sinh\beta}\right)^2,$$

(b) $H_\alpha = H_\beta = \dfrac{c}{\cosh\alpha - \cos\beta}$, $H_\varphi = \dfrac{c\sinh\alpha}{\cosh\alpha - \cos\beta}$;

and planes $\varrho = \text{const}$.

(b) $H_1 = H_2 = \dfrac{c}{\cosh\beta - \cos\alpha}$, $H_3 = \dfrac{c\sin\alpha}{\cosh\beta - \cos\alpha}$.

(c) $\Delta u = \dfrac{\partial}{\partial \alpha}\left(\dfrac{\sin\alpha}{\cosh\beta - \cos\alpha}\dfrac{\partial u}{\partial \alpha}\right) +$

$$+ \frac{\partial}{\partial \beta}\left(\frac{\sin\alpha}{\cosh\beta - \cos\alpha}\frac{\partial u}{\partial \beta}\right) + \frac{1}{\sin\alpha(\cosh\beta - \cos\alpha)}\frac{\partial^2 u}{\partial \varphi^2}.$$

**2.15.** $H_1 = c\sqrt{\dfrac{\lambda^2 - \mu^2}{\lambda^2 - 1}}$, $H_2 = c\sqrt{\dfrac{\lambda^2 - \mu^2}{1 - \mu^2}}$,

$H_3 = c\sqrt{(\lambda^2 - 1)(1 - \mu^2)}$.

**2.16.** $H_1 = c\sqrt{\dfrac{\lambda^2 - \mu^2}{\lambda^2 - 1}}$, $H_2 = c\sqrt{\dfrac{\lambda^2 - \mu^2}{1 - \mu^2}}$, $H_3 = c\lambda\mu$.

**2.17.** (a) $H_1 = H_2 = \sqrt{\lambda^2 + \mu^2}$, $H_3 = \lambda\mu$.

(b) The coordinate surfaces are paraboloids $\lambda$ = const, $\mu$ = const of revolution about the axis of symmetry $Oz$.

# 3
# Riemannian Metric

**3.2.** Let the surface $M^2 \subset \mathbf{R}^3$ be given by equations $x_i = x_i(p, q)$, $i = 1, 2, 3$, and the variables $p$ and $q$ have a plane region as their domain. Let the functions $x_i = x_i(p, q)$ be real-analytic. The pair $(p, q)$ can be regarded as the coordinates of a point on the surface $M^2$. A curve $C$ on $M^2$ is given by the equations

$$p = p(t), \quad q = q(t), \quad a \leqslant t \leqslant b.$$

An element of arc length is expressed in terms of the vector $\mathbf{x} = (x_1, x_2, x_3)$ thus:

$$ds^2 = d\mathbf{x}d\mathbf{x} = (\mathbf{x}_p dp + \mathbf{x}_q dq)(\mathbf{x}_p dp + \mathbf{x}_q dq),$$

or

$$ds^2 = (\mathbf{x}_p, \mathbf{x}_p)dp^2 + 2(\mathbf{x}_p, \mathbf{x}_q)dpdq + (\mathbf{x}_q, \mathbf{x}_q)dq^2$$

$$= Edp^2 + 2Fdpdq + Gdq^2,$$

where $E = (\mathbf{x}_p, \mathbf{x}_p)$, $F = (\mathbf{x}_p, \mathbf{x}_q)$, $G = (\mathbf{x}_q, \mathbf{x}_q)$.

Since the element of length $ds^2$ is always positive, $W^2 = EG - F^2$ is also positive. Let us find the coordinate system $(u, v)$ with the element of arc $ds^2 = \lambda(u, v)(du^2 + dv^2)$. We have

$$ds^2 = \left(\sqrt{E}dp + \frac{(F + iW)}{\sqrt{E}}dq\right)\left(\sqrt{E}dp + \frac{(F - iW)}{\sqrt{E}}dq\right).$$

Assume that we can find an integrating factor $\sigma = \sigma_1 + i\sigma_2$ such that

$$\sigma\left(\sqrt{E}dp + \frac{(F + iW)}{\sqrt{E}}dq\right) = du + idv.$$

Then

$$\bar{\sigma}\left(\sqrt{E}dp + \frac{(F - iW)}{\sqrt{E}}dq\right) = du - idv,$$

and, finally, $|\sigma|^2 ds^2 = du^2 + dv^2$. Assuming $\sigma|^2 = 1/\lambda$, we obtain the required isothermal coordinates $(u, v)$. Thus, we have obtained isothermal coordinates by having found the integrating factor which transforms the expression

$$\sqrt{E}dp + \frac{(F + iW)}{\sqrt{E}}dq$$

into a total differential. The differential $du + idv$ can be written in the following form:

$$du + idv = \left(\frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p}\right)dp + \left(\frac{\partial u}{\partial q} + i\frac{\partial v}{\partial q}\right)dq.$$

Further,

$$\frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p} = \sigma\sqrt{E}, \qquad \frac{\partial u}{\partial q} + i\frac{\partial v}{\partial q} = \sigma\frac{(F + iW)}{\sqrt{E}}.$$

Eliminating $\sigma$, we obtain

$$E\left(\frac{\partial u}{\partial q} + i\frac{\partial v}{\partial q}\right) = (F + iW)\left(\frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p}\right),$$

or

$$E\frac{\partial u}{\partial q} = F\frac{\partial u}{\partial p} - W\frac{\partial v}{\partial p}, \qquad E\frac{\partial v}{\partial q} = W\frac{\partial u}{\partial p} + F\frac{\partial v}{\partial p}.$$

Solving this system for the unknowns $\partial v/\partial p$ and $\partial v/\partial q$, we obtain

$$\frac{\partial v}{\partial p} = \frac{F\frac{\partial u}{\partial p} - F\frac{\partial u}{\partial q}}{\sqrt{EG - F^2}},$$

$$\frac{\partial v}{\partial q} = \frac{G\frac{\partial u}{\partial p} - F\frac{\partial u}{\partial q}}{\sqrt{EG - F^2}}. \tag{1}$$

Similarly,

$$\frac{\partial u}{\partial p} = \frac{E\dfrac{\partial v}{\partial q} - F\dfrac{\partial v}{\partial p}}{\sqrt{EG - F^2}},$$

$$\frac{\partial u}{\partial q} = \frac{F\dfrac{\partial v}{\partial q} - G\dfrac{\partial v}{\partial p}}{\sqrt{EG - F^2}}. \tag{2}$$

Therefore, $u$ satisfies the equation

$$\frac{\partial}{\partial q}\left(\frac{F\dfrac{\partial u}{\partial p} - E\dfrac{\partial u}{\partial q}}{W}\right) + \frac{\partial}{\partial p}\left(\frac{F\dfrac{\partial u}{\partial q} - G\dfrac{\partial u}{\partial p}}{W}\right) = 0$$

which is called the Beltrami-Laplace equation. Given a second family of isothermal coordinates $(x, y)$ in a neighbourhood of a point, we have $ds^2 = \mu(dx^2 + dy^2)$. Using the coordinates $(x, y)$ instead of the coordinates $(p, q)$, we obtain $E = G = \mu$, $F = 0$ and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

Thus, the Cauchy-Riemann equations have been obtained, and hence the functions $u$ and $v$ are conjugate harmonic functions, whereas the function $f = u + iv$ is analytic in $z = x + iy$. The Beltrami equation assumes the form of the well-known Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. A complex-valued function $f(p, q)$ defined on $M^2$ is said to be a complex potential on $M^2$ if its real and imaginary parts satisfy equations (1). Thus, the real and imaginary parts of a complex potential on the manifold $M^2$ determine isothermal coordinates in a neighbourhood of every point on $M^2$ (the coordinates being local and not serving, generally speaking, the whole of the two-dimensional manifold; while transferring from one point to another, the complex potential will vary).

**3.3** (a) Consider some curve $\varphi = \varphi(\theta)$ on the surface of the sphere. In moving along this curve, the compass needle forms an angle $\psi$ with the direction of motion determined by the relations

$$\tan\psi = \sin\theta \frac{d\varphi}{d\theta}. \tag{1}$$

(The angle $\psi$ is measured from the $y$ axis clockwise.) We obtain on the map:

$$\frac{dy}{dx} = \tan\left(\psi + \frac{\pi}{2}\right) = -\frac{1}{\tan\psi}. \tag{2}$$

It follows from relations (1) and (2) that

$$\sin\theta\,\frac{d\varphi}{d\theta} = -\frac{\dfrac{dx}{d\theta}}{\dfrac{dy}{d\theta}} = -\frac{\dfrac{\partial x}{\partial\theta} + \dfrac{\partial x}{\partial\varphi}\dfrac{d\varphi}{d\theta}}{\dfrac{\partial y}{\partial\theta} + \dfrac{\partial y}{\partial\varphi}\dfrac{d\varphi}{d\theta}},$$

$$\left(\frac{\partial y}{\partial\theta} + \frac{\partial y}{\partial\varphi}\frac{d\varphi}{d\theta}\right)\frac{d\varphi}{d\theta}\sin\theta = -\frac{\partial x}{\partial\theta} - \frac{\partial x}{\partial\varphi}\frac{d\varphi}{d\theta}. \tag{3}$$

Since relation (3) must be fulfilled at the point in question for any value of $d\varphi/d\theta$, we obtain, by equalizing the coefficients of the same powers of the derivative $d\varphi/d\theta$ on the right-hand and left-hand sides, that

$$\frac{\partial y}{\partial\varphi} = 0, \quad y = y(\theta), \tag{4}$$

$$\frac{\partial x}{\partial\theta} = 0, \quad x = x(\varphi), \tag{5}$$

$$-\sin\theta\,\frac{\partial y}{\partial\theta} = \frac{\partial x}{\partial\varphi}. \tag{6}$$

It follows from (4) and (5) that the left-hand side of relation (6) depends only on $\theta$, whereas the right-hand side only on $\varphi$; therefore, both sides of this relation should be constant. We put this constant equal to unity. Thus, in Mercator's projection, the mapping is given by the formulae

$$x = \varphi, \quad y = -\int\frac{d\theta}{\sin\theta} = \ln\cot\frac{\theta}{2}.$$

(b) $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 = \sin^2\theta(dx^2 + dy^2) = (dx^2 + dy^2)/\cosh^2 y.$

**3.5.** $ds^2 = \dfrac{dz d\bar{z}}{(1 + z\bar{z})^2}.$

**3.6.** $ds^2 = \dfrac{dv^2}{\left(1 - \dfrac{v^2}{c^2}\right)^2} + \dfrac{v^2 d\varphi^2}{\left(1 - \dfrac{v^2}{c^2}\right)}.$

**3.7.** $ds^2 = d\chi^2 + \sinh^2\chi d\varphi^2.$

**3.8.** $ds^2 = \dfrac{(d\varrho^2 + \varrho^2 d\varphi^2)}{(1 - \varrho^2)^2}.$

**3.9.** (a) In polar coordinates,

$$ds^2 = dr^2 + r^2 d\varphi^2, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant r < \infty.$$

(b) If the sphere has radius $a$, then

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

(c) $ds^2 = a^2(d\chi^2 + \sinh^2\chi d\varphi^2), \ 0 \leqslant \chi < \infty, \ 0 \leqslant \varphi \leqslant 2\pi.$

In case (a), $\mathfrak{G} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, and the equation of the circumference is

$$r(t) = R = \text{const}, \quad \varphi(t) = t, \quad 0 \leqslant t \leqslant 2\pi,$$

whereas the length of the circumference is

$$L = \int_0^{2\pi} \sqrt{R^2}\, dt = 2\pi R.$$

The circle is given by the relations $0 \leqslant r \leqslant R$, $0 \leqslant \varphi \leqslant 2\pi$, and its area equals $S = \int_0^{2\pi} \int_0^R r dr d\varphi = \pi R^2.$

(b) $\mathfrak{G} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix}$, and the equation of the circumference is $a\theta(t) = R = \text{const}$, $a\varphi(t) = t$, $0 \leqslant t \leqslant 2\pi a$, $\theta(t) = R/a$, $\varphi(t) = t/a$, whereas the length of the circumference is

$$L = \int_0^{2\pi a} a \sqrt{\sin^2 \frac{R}{a} \frac{1}{a^2}}\, dt = 2\pi a \sin \frac{R}{a}.$$

The circle is given by the relations $0 \leqslant a\theta \leqslant R$, $0 \leqslant \varphi \leqslant 2\pi$, i.e.,

$$0 \leqslant \theta \leqslant \frac{R}{a}, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad \sqrt{g} = a^2 \sin\theta,$$

and the area of the circle is

$$S = \int_0^{2\pi} \int_0^{R/a} a^2 \sin\theta d\theta d\varphi = 2\pi a^2 \left(1 - \cos \frac{R}{a}\right) .$$

(c) $\mathfrak{G} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sinh\chi \end{pmatrix}$, and the equation of the circumference is

$$a\chi(t) = R = \text{const}, \quad a\varphi(t) = t, \quad 0 \leqslant t \leqslant 2\pi a,$$

and

$$\chi(t) = R/a, \quad \varphi(t) = t/a,$$

whereas the length of the circumference is

$$L = \int_0^{2\pi a} a \sqrt{\sinh^2 \frac{R}{a} \frac{1}{a^2}}\, dt = 2\pi a \sinh \frac{R}{a}.$$

The circle is given by the relations $0 \leqslant a\chi \leqslant R$, $0 \leqslant \varphi \leqslant 2\pi$, i.e.,

$$0 \leqslant \chi \leqslant \frac{R}{a}, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad \sqrt{g} = a^2 \sinh\chi,$$

and the area of the circle is

$$S = \int_0^{2\pi} \int_0^{R/a} a^2 \sinh\chi d\chi d\varphi = 2\pi a^2 \left(\cosh \frac{R}{a} - 1\right).$$

# 4
# Theory of Curves

**4.1.** (a) $y^2 = 2ax - 2C$ is the parabola with the axis $OX$ and parameter $p = |a|$. The curve opens leftward when $a < 0$, and rightward when $a > 0$.

(b) $y = Ce^{-x/a}$.

(c) $(x - C)^2 + y^2 = a^2$ is the circumference with radius $a$ and centre on the axis $OX$.

**4.2.** The condition for the length of the tangent to be constant is written in the form

$$y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = a. \tag{1}$$

We will consider the curve only in the upper half-plane and therefore put $|y| = y > 0$.

Consider the angle $\varphi$, $0 < \varphi < \pi$ determined by the condition

$$\tan\varphi = dy/dx. \tag{2}$$

Replacing $dx/dy$ in (1) by $\cot\varphi$, we obtain $y/\sin\varphi = a$, or

$$y = a\sin\varphi.$$

Hence,

$$dy = a\cos\varphi d\varphi.$$

But it follows from (2) that

$$dx = \cot\varphi dy.$$

Substituting the expression obtained for $dy$, we obtain

$$dx = a\frac{\cos^2\varphi}{\sin\varphi}d\varphi,$$

or

$$dx = a\left(\frac{1}{\sin\varphi} - \sin\varphi\right)d\varphi.$$

Integrating termwise, we find

$$x = a\left(\ln\tan\frac{\varphi}{2} + \cos\varphi\right) + C.$$

This curve is called a *tractrix*.

**4.3.** $y = a^3/(x^2 + a^2)$; $x = a\cot t$, $y = a\sin^2 t$.
**4.4.** $r = a\varphi$.
**4.5.** $r = r_0 e^{k\varphi}$, where $\varphi = \omega t$.
**4.6.** $x = at - d\sin t$, $y = a - d\cos t$.

**4.7.** $$x = (R + r)\cos\frac{r}{R}t - r\cos\frac{R+r}{R}t,$$

$$y = (R + r)\sin\frac{r}{R}t - r\sin\frac{R+r}{R}t.$$

**4.8.** $$x = (R - mR)\cos mt + mR\cos(t - mt),$$
$$y = (R - mR)\sin mt - mR\sin(t - mt),\ m = r/R.$$

**4.9.** The equation of the required curve is

$$\mathbf{r}(t) = \mu(t)\mathbf{a} + \mathbf{b},$$

where $\mathbf{b}$ is a constant vector, and $\mu(t)$ the antiderivative of the function $\lambda(t)$, $c < t < d$. Geometrically, the following cases are possible: a straight line collinear with $\mathbf{a}$ if $\int_c^d \lambda(t)dt$ diverges when $t = c$ and $t = d$; a ray with the direction of the vector $\mathbf{a}$ if $\int_c^d \lambda(t)dt$ converges when $t = c$, but diverges when $t = d$; a ray with the direction of the vector $-\mathbf{a}$ if $\int_c^d \lambda(t)dt$ diverges when $t = c$, but converges when $t = d$; an open line-segment collinear with $\mathbf{a}$ if $\int_c^d \lambda(t)dt$ converges.

**4.10.** The equation of the required curve is

$$r(t) = \frac{1}{2}t^2\mathbf{a} + t\mathbf{b} + \mathbf{c},$$

where $\mathbf{b}$, $\mathbf{c}$ are arbitrary constant vectors.

If $\mathbf{b} \neq 0$, then this equation determines (with $\mathbf{b}$ and $\mathbf{c}$ fixed) a parabola with the axis whose direction coincides with that of the vector $\mathbf{a}$. If $\mathbf{b} = 0$, then we obtain two coincident rays parallel to $\mathbf{a}$.

**4.11.** (a) $(\mathbf{r}')^2\ [\mathbf{r}' \times \mathbf{a}]^2$;

(b) $-(\mathbf{r}', \mathbf{a})\ [\mathbf{r}' \times \mathbf{a}]^2$.

**4.12.** Apply the Rolle theorem to the function $(\mathbf{a}, \mathbf{r}(t) - \mathbf{r}(t_0))$.

**4.13.** Use the equality $(\mathbf{r}_2(t) - \mathbf{r}_1(t))^2 = \text{const}$, where $\mathbf{r}_1$, $\mathbf{r}_2$ are radii vectors of the moving points and $t$ is time.

**4.14.** Put $\frac{\mathbf{r}'}{\mathbf{r}} = \lambda$, $\lambda(t)$ being a function continuous on the segment $[a, b]$ and having the same sign on it. We have $\mathbf{r}' - \lambda\mathbf{r} = 0$, whence $\mathbf{r} = \mathbf{a}e^{\int\lambda dt}$. Since the derivative of the function $e^{\int\lambda dt}$ equals $\lambda e^{\int\lambda dt}$, it does not change sign on the segment $[a, b]$, i.e., $e^{\int\lambda dt}$ is a monotonic and continuous function of $t$.

**4.15.** Applying the method of solution of the previous problem, we have $\mathbf{r}' = \mathbf{a}e^{\int\lambda dt}$, whence

$$\mathbf{r} = \mathbf{a}\int e^{\int\lambda dt}dt + \mathbf{b}.$$

The derivative of $\int e^{\int\lambda dt}dt$ equals $e^{\int\lambda dt} > 0$; therefore, $\int e^{\int\lambda dt}dt$ is a monotonic increasing function of $t \in [a, b]$.

**4.16.** $\mathbf{r}' = \{\varphi', \varphi + t\varphi'\}$, $\mathbf{r}'' = \{\varphi'', 2\varphi' + t\varphi''\}$, $[\mathbf{r}' \times \mathbf{r}''] = 2\varphi'^2 - \varphi\varphi''$. The given equation determines a straight line if and only if $2\varphi'^2 - \varphi\varphi'' = 0$. Solving this equation, we find $\varphi = 1/(at + b)$, where $a$ and $b$ are constants.

**4.17.** $\mathbf{r} = r\mathbf{r}^0$, $\frac{d\mathbf{r}}{d\varphi} = r'\mathbf{r}^0 + r\frac{d\mathbf{r}^0}{d\varphi}$. Since $\mathbf{r}^0 = \{\cos\varphi, \sin\varphi\}$, $\frac{d\mathbf{r}^0}{d\varphi} = \{-\sin\varphi, \cos\varphi\}$, i.e., $\frac{d\mathbf{r}^0}{d\varphi}$ is obtained from $\mathbf{r}^0$ by rotating it through $+\pi/2$. Denote the vector obtained from $\mathbf{r}^0$ by rotating it through $+\pi/2$ by $[\mathbf{r}^0]$. Therefore,

$$\frac{d\mathbf{r}}{d\varphi} = r'\mathbf{r}^0 + r[\mathbf{r}^0].$$

Furthermore,

$$\frac{d^2\mathbf{r}}{d\varphi^2} = r''\mathbf{r}^0 + 2r''[\mathbf{r}^0] - r\mathbf{r}^0 = (r'' - r)\mathbf{r}^0 + 2r'[\mathbf{r}^0],$$

$$\left[\frac{d\mathbf{r}}{d\varphi} \times \frac{d^2\mathbf{r}}{d\varphi^2}\right] = \begin{vmatrix} r' & r \\ r'' - r & 2r' \end{vmatrix} = 2r'^2 - rr'' + r^2 = 0.$$

Putting $r' = \omega$, we find

$$r'' = \frac{d\omega}{d\varphi} = \frac{d\omega}{dr}r' = \omega\frac{d\omega}{dr},$$

$$2\omega^2 - \omega r\frac{d\omega}{dr} + r^2 = 0, \quad \frac{2\omega^2}{r^2} - \frac{\omega d\omega}{rdr} + 1 = 0.$$

Put $\omega^2 = p$, $r^2 = q$, then $dp/dq = 2p/q + 1$. Solving this equation, we find that $p = aq^2 - q$, or $\omega^2 = ar^4 - r^2$, $r' = r\sqrt{ar^2 - 1}$. Substituting $1/r = \xi$, we easily obtain

$$\frac{1}{r} = C_1 \sin(\varphi + C_2),$$

where $C_1$ and $C_2$ are arbitrary numbers.

**4.18.** $\mathbf{F} = F\mathbf{r} = m\mathbf{r}''$. Differentiating, we obtain $\lambda'\mathbf{r} + \lambda\mathbf{r}' = m\mathbf{r}'''$; therefore, the vectors $\mathbf{r}'$, $\mathbf{r}''$, $\mathbf{r}'''$ are coplanar.

**4.19.** The radius vector $\varrho$ of an arbitrary point of the centre surface can be determined by one of the relations:

$$\varrho = \mathbf{r}_1 + \lambda[\mathbf{r}_1'] = \mathbf{r}_2 + \mu[\mathbf{r}_2'],$$

$$\mathbf{r}_1 - \mathbf{r}_2 + \lambda[\mathbf{r}_1'] = \mu[\mathbf{r}_2'],$$

$$(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{r}_2' + \lambda[\mathbf{r}_1']\mathbf{r}_2' = 0,$$

$$\lambda = \frac{(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_2'}{|[\mathbf{r}_1' \times \mathbf{r}_2']|}.$$

Therefore,

$$\varrho = \mathbf{r}_1 + \frac{(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_2'}{|[\mathbf{r}_1' \times \mathbf{r}_2']|}[\mathbf{r}_1'],$$

and in coordinates,

$$\xi = x_1 - \frac{(x_2 - x_1)x_2' + (y_2 - y_1)y_2'}{x_1'y_2' - x_2'y_1'}y_1',$$

$$\eta = y_1 - \frac{(x_2 - x_1)x_2' + (y_2 - y_1)y_2'}{x_1'y_2' - x_2'y_1'}x_1'.$$

**4.20.** Consider the vector $\lambda[\mathbf{r}_1']$, where $\lambda = \dfrac{(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_2'}{|[\mathbf{r}_1' \times \mathbf{r}_2']|}$. If this vector is marked off from the end $M_1$ of the rod, then its end will fall on the instantaneous centre of rotation. The projections of the vector $\lambda[\mathbf{r}_1']$ onto the vectors $\mathbf{r}_2 - \mathbf{r}_1$ and $[\mathbf{r}_2 - \mathbf{r}_1]$ are equal, respectively, to

$$\frac{\lambda[\mathbf{r}_1'](\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|} \quad \text{and} \quad \frac{\lambda[\mathbf{r}_1'][\mathbf{r}_2 - \mathbf{r}_1]}{|\mathbf{r}_2 - \mathbf{r}_1|}.$$

Therefore, the equations of the centrode are:

$$x = \frac{(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_1'}{|[\mathbf{r}_1' \times \mathbf{r}_2']|}\,\frac{[\mathbf{r}_1' \times (\mathbf{r}_2 - \mathbf{r}_1)]}{|\mathbf{r}_2 - \mathbf{r}_1|},$$

$$y = \frac{(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_2' \ \mathbf{r}_1'(\mathbf{r}_2 - \mathbf{r}_1)}{|[\mathbf{r}_1 \times \mathbf{r}_2']| \ |\mathbf{r}_2 - \mathbf{r}_1|},$$

or

$$x = \frac{\{(x_2 - x_1)x_2' + (y_2 - y_1)y_2'\} \begin{vmatrix} x_1' & y_1' \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix}}{\begin{vmatrix} x_1' & y_1' \\ x_2' & y_2' \end{vmatrix} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}},$$

$$y = \frac{\{(x_2 - x_1)x_2' + (y_2 - y_1)y_2'\}\{(x_2 - x_1)x_1' + (y_2 - y_1)y_1'\}}{\begin{vmatrix} x_1' & y_1' \\ x_2' & y_2' \end{vmatrix} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}.$$

**4.21.** $\mathbf{R} = \mathbf{r}_1 + \mathbf{p} = \mathbf{r}_1 + \xi\mathbf{a} + \eta[\mathbf{a}]$, where $\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$, $\xi = \text{const}$, $\eta = \text{const}$ (point $M$ being rigidly connected with the rod) and $\mathbf{R}' = \mathbf{r}_1' + \xi\mathbf{a}' + \eta[\mathbf{a}']$. Since $|\mathbf{a}| = |\mathbf{r}_2 - \mathbf{r}_1| = \text{const}$, $\mathbf{a}' \perp \mathbf{a}$. Therefore,

$$\mathbf{a}' = s[\mathbf{a}], \ \mathbf{r}_2' - \mathbf{r}_1' = s[\mathbf{r}_2 - \mathbf{r}_1],$$

$$(\mathbf{r}_2' - \mathbf{r}_1')[\mathbf{r}_1'] = s[\mathbf{r}_2 - \mathbf{r}_1][\mathbf{r}_1'],$$

$$[\mathbf{r}_1']\mathbf{r}_2' = s(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_1',$$

$$s = \frac{|[\mathbf{r}_1' \times \mathbf{r}_2']|}{(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{r}_1'} = \frac{1}{\lambda}.$$

Thus:

$$\mathbf{a}' = \frac{1}{\lambda}[\mathbf{a}], \ [\mathbf{a}'] = -\frac{1}{\lambda}\mathbf{a},$$

$$\mathbf{R}' = \mathbf{r}_1' + \frac{\xi}{\lambda}[\mathbf{a}] - \frac{\eta}{\lambda}\mathbf{a} = \frac{1}{\lambda}(\lambda\mathbf{r}_1' + \xi[\mathbf{a}] - \eta\mathbf{a}).$$

On the other hand,

$$\mathbf{r} = \mathbf{R} - \varrho = \mathbf{r}_1 + \xi\mathbf{a} + \eta[\mathbf{a}] - \mathbf{r}_1 - \lambda[\mathbf{r}_1'] = \xi\mathbf{a} + \eta[\mathbf{a}] - \lambda[\mathbf{r}_1'],$$

$$[\mathbf{r}] = \lambda\mathbf{r}_1' - \xi[\mathbf{a}] - \eta\mathbf{a};$$

therefore,

$$\mathbf{R}' = \frac{1}{\lambda}[\mathbf{r}], \ \omega = \frac{1}{\lambda}.$$

**4.22.** $\mathbf{r}'' \parallel \mathbf{r}$, $[\mathbf{r}, \mathbf{r}''] = 0$, $[\mathbf{rr}']' = [\mathbf{rr}'']$. Therefore, $[\mathbf{rr}'] = \mathbf{a} = \text{const}$, and

$$[\mathbf{ar}''] = -\frac{1}{r^3}[[\mathbf{rr}']\mathbf{r}] = \frac{\lambda}{r^3}(\mathbf{r}'\,\mathbf{r}^2 - \mathbf{r}(\mathbf{rr}'))$$

$$= -\lambda\frac{\mathbf{r}'\mathbf{r} - \mathbf{rr}'}{r^2} = -\lambda\left(\frac{\mathbf{r}}{r}\right)'.$$

Thus,

$$\frac{d}{dt}\left\{[\mathbf{ar}'] + \lambda\frac{\mathbf{r}}{r}\right\} = 0,$$

$$[\mathbf{ar}'] + \lambda\frac{\mathbf{r}}{r} = \mathbf{b} = \text{const}.$$

Multiplying both sides of this equality by $\mathbf{r}$ and noticing that $[\mathbf{ar}']\mathbf{r} = \mathbf{a}[\mathbf{r}'\mathbf{r}] = -\mathbf{a}^2$, we have: $-\mathbf{a}^2 + \lambda r = \mathbf{br}$. The motion is in the same plane perpendicular to the vector $\mathbf{a}$ (since it follows from the relation $[\mathbf{rr}'] = \mathbf{a}$ that $\mathbf{ar} = 0$). Introducing a polar coordinate system on this plane and making the pole coincident with the origin of the radii vectors, while directing the polar axis along the vector $\mathbf{b}$, we obtain $-a^2 + \lambda r = br\cos\varphi$, whence $r = a^2/(\lambda - b\cos\varphi)$ is a curve of the second order.

**4.23.** $u^2\left(\dfrac{d^2u}{d\varphi^2} + u\right) = -\dfrac{F}{mc^2}$, $u = 1/r$, $c = \text{const}$.

In the case of the Newtonian force, $F = -km/r^2 = -kmu^2$, whence

$$\frac{d^2u}{d\varphi^2} + u = \alpha \quad (\alpha = k/c^2).$$

**4.25.** The circumferences whose centres are placed on the straight line passing through the origin of the radii vectors and collinear with the vector $\omega$, whereas the planes of these circumferences are perpendicular to the indicated straight line.

**4.26.** The straight lines along which the planes perpendicular to the vector $\mathbf{e}$ intersect with those passing through the straight line drawn through the pole 0 and collinear with the vector $\mathbf{e}$.

**4.27.** Introducing Cartesian rectangular coordinates with the axis $Oz$ collinear with the vector $\mathbf{e}$, we have $a\mathbf{e} + [\mathbf{er}] = -y\mathbf{i} + x\mathbf{j} + a\mathbf{e}$, and the given differential equation assumes the following form: $x' = -y$, $y' = x$, $z' = a$. We find from the relations $x' = -y$, $y' = x$ that $x^2 + y^2 = C_1$ is the family of circular cylinders whose axes coincide with the straight line passing through the origin of the radii vectors and collinear with the vector **e**. Furthermore,

$$\frac{dx}{dz} = -\frac{y}{a}, \frac{dy}{dz} = \frac{x}{a},$$

whence

$$\frac{xdy - ydx}{dz} = \frac{x^2 + y^2}{a}, a\frac{xdy - ydx}{x^2} = \left(1 + \frac{y^2}{x^2}\right)dz,$$

$$\frac{ad\frac{y}{x}}{1 + \frac{y^2}{x^2}} = dz, \quad \text{and} \quad z + C_2 = a\tan^{-1}\frac{y}{x}$$

is the family of right helicoids whose axis is the axis of the cylinders mentioned above. The integral curves are helical. Finally, $z = at + C_3$. Now, to express $x$, $y$, $z$ in terms of $t$ is easy from the relations obtained.

**4.28.** Semi-circumferences touching the axis $Oz$ (which is collinear with the vector **e**) at the origin.

**4.31.** $\pi/4$ and $\pi/2$.

**4.32.** $\tan^{-1}3$.

**4.36.** $\dfrac{x^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \dfrac{y^2}{\left(\frac{b}{\sqrt{2}}\right)^2} = 1$, where $a$ and $b$ are the semi-axes of the given ellipse.

**4.37.** $xy = \pm s/2$, where $s$ is the given area.

**4.38.** $y = ax^2 + \sqrt[3]{\dfrac{9as^2}{16}}$, where the parabola is given by the equation $y = ax^2$, and $s$ is the area of a segment.

**4.39.** $\left(x - \dfrac{l}{\cos\frac{\alpha}{2}}\right)^2 + y^2 = \left(l\tan\dfrac{\alpha}{2}\right)^2$, where $\alpha$ is the given angle, and $l$ the semi-perimeter of the triangle.

**4.40.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{2a^2} = 1$, where $a$ is the radius of the given circumference.

**4.41.** $\mathbf{r} = \{l\cos^3 v, l\sin^3 v\}$, where $l$ is the given semi-axis sum.

4.42. $x = \frac{a}{4}(3\cos v - \cos 3v)$, $y = \frac{a}{4}(3\sin v - \sin 3v)$ is a hypocycloid.

4.43. $xy = \pm\frac{1}{2}\sqrt{c}$, where $c$ is the given area.

4.44. $(x - c)^2 + y^2 = 4a^2$, where $a$ is the major semi-axis of the ellipse, $c = \sqrt{a^2 - b^2}$.

4.45. $\boldsymbol{\varrho} \pm \mathbf{r} \pm a\frac{[\mathbf{r}']}{|\mathbf{r}'|}$.

4.46. $\boldsymbol{\varrho} = \mathbf{r} + [\mathbf{r}']\frac{\mathbf{r}'^2}{|[\mathbf{r}' \times \mathbf{r}'']|}$,

and in coordinates,

$$\xi = x - y'\frac{x'^2 + y'^2}{x'y'' - x''y'}, \qquad \eta = y + x'\frac{x'^2 + y'^2}{x'y'' - x''y'}.$$

4.47. A cardioid.

4.48. (1) 1; (2) $\frac{4am(1 + m)}{1 + 2m}\sin\frac{t}{2}$; (3) $\frac{y^2}{a}$;

(4) $\frac{a(y^4 - 2by^3 + a^2b^2)^{3/2}}{y^3|2a^2b - y^3 - 3by^2|}$; (5) $\frac{a^2}{3r}$;

(6) $\frac{4}{3}a\cos\frac{\varphi}{2}$; (7) $a\frac{(1 + \varphi^2)^{3/2}}{2 + \varphi^2}$;

(8) $3a|\sin t\cos t|$.

4.49. (1) $|\cos x|$; (2) 1/6; (3) $\frac{2}{a}\pi$; (4) $\frac{3}{8a|\sin t/2|}$.

4.50. (1) $\frac{2 + \varphi^2}{a(1 + \varphi^2)^{3/2}}$;

(2) $\frac{k(k + 1) + \varphi^2}{a\varphi^{k-1}(k^2 + \varphi^2)^{3/2}}$;

(3) $\frac{1}{\sqrt{1 + \ln^2 a}}$.

**4.51.** $k = \dfrac{\operatorname{mod}\begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix}}{(F_x^2 + F_y^2)^{3/2}}.$

**4.52.** $k = \dfrac{\left[P\left(Q\dfrac{\partial Q}{\partial x} - P\dfrac{\partial Q}{\partial y}\right) + Q\left(P\dfrac{\partial P}{\partial y} - Q\dfrac{\partial P}{\partial x}\right)\right]}{(P^2 + Q^2)^{3/2}}.$

**4.53.** (1) $s = \displaystyle\int_0^x \sqrt{1 + y'^2}\,dx = \frac{a}{2}(e^{x/a} - e^{-x/a});$

(2) $s = \displaystyle\int_0^x \sqrt{1 + y'^2}\,dx = \frac{1}{27}\,[(4 + 9x)^{3/2} - 8];$

(3) $s = \displaystyle\int_0^x \sqrt{1 + y'^2}\,dx = \frac{x}{2}\sqrt{1 + 4x^2} + \frac{1}{4}\ln(2 - x + \sqrt{1 + 4x^2});$

(4) $s = \displaystyle\int_1^x \sqrt{1 + y'^2}\,dx = \sqrt{1 + x^2} + \ln\frac{\sqrt{1 + x^2}}{x} - \sqrt{2} - \ln(\sqrt{2} - 1);$

(5) $s = \displaystyle\int_0^\varphi \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\,d\varphi = 4a\sin\frac{\varphi}{2};$

(6) $s = \displaystyle\int_0^t \sqrt{x'^2 + y'^2}\,dt = 4a\left(1 - \cos\frac{t}{2}\right);$

(7) $s = \displaystyle\int_0^t \sqrt{x'^2 + y'^2}\,dt = \frac{at^2}{2};$

(8) $s = \displaystyle\int_0^t \sqrt{x'^2 + y'^2}\,dt = \frac{8a}{3}\sin\frac{t}{2};$

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(9) $s = \int_0^t \sqrt{x'^2 + y'^2}dt = \frac{3a}{2}\sin^2 t;$

(10) $s = \int_0^x \sqrt{1 + y'^2}dx$

$$= \sqrt{1 + e^{2x}} + \frac{1}{2}\ln\frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1} - \sqrt{2} - \ln(\sqrt{2} - 1);$$

(11) $s = \int_{\pi/2}^t \sqrt{x'^2 + y'^2}dt = a \ln \sin t.$

**4.54.** $f(\alpha) + f''(\alpha)$.

**4.55.** (1) $R^2 + 4s^2 - 6as = 0;$

(2) $(27s + 8)^2 = \left[4 + 9\frac{36R^2}{(27s + 8)^2}\right]^3;$

(3) $s = \frac{1}{4}\sqrt{\sqrt[3]{4R^2} - 1} + \sqrt[3]{2R} + \frac{1}{4}\ln\left[\sqrt{\sqrt[3]{4R^2} - 1} + \sqrt[3]{2R}\right].$

(4) The parametric natural equations are

$s = \sqrt{1 + x^2} + \ln\frac{\sqrt{1 + x^2} - 1}{x}$ and $k = \frac{x}{(1 + x^2)^{3/2}};$

(5) $R = a + s^2/a;$

(6) The parametric natural equations are

$s = \sqrt{1 + e^{2x}} + \frac{1}{2}\ln\frac{\sqrt{1 + e^{2x}} - 1}{\sqrt{1 + e^{2x}} + 1}$ and $k = \frac{e^x}{(1 + e^{2x})^{3/2}};$

(7) $R^2 + a^2 = a^2e^{-2s/a};$

(8) $s^2 + 9R^2 = 16a^2;$

(9) $R^2 = 2as.$

**4.56.** (1) $r = Ce^{\varphi}$, a logarithmic spiral;

(2) $x = \frac{a}{2}\left(\frac{b}{a + b}\sin\frac{(a + b)}{b}t + \frac{b}{a - b}\sin\frac{a - b}{b}t\right),$

$$y = \frac{a}{2}\left(-\frac{b}{a + b}\cos\frac{a + b}{b}t - \frac{b}{a - b}\cos\frac{a - b}{b}t\right);$$

(3) $\mathbf{r} = \left\{\int_0^s \cos\frac{s^2}{2a^2}\,ds, \int_0^s \sin\frac{s^2}{2a^2}\,ds\right\}$, a clothoid;

(4) $x = a\ln\tan\left|\frac{\pi}{4} + \frac{t}{2}\right|$, $y = \frac{a}{\cos t}$, a catenary line;

(5) $\mathbf{r} = \left\{\frac{a}{2}(\sin 4t + 2\sin 2t), -\frac{a}{2}(\cos 4t + 2\cos 2t)\right\}$;

(6) $\mathbf{r} = \{a(2t + \sin 2t), a(2 - \cos 2t)\}$, a cycloid;

(7) $\mathbf{r} = \{a(\cos t + t\sin t), a(\sin t - t\cos t)\}$, the evolute of the circumference;

(8) $\mathbf{r} = \left\{a\cos t, a\ln\tan\left|\frac{\pi}{4} + \frac{t}{2}\right| - a\sin t\right\}$, a tractrix.

**4.57.** $p = |\mathbf{rn}|$. Assume that $\mathbf{rn} > 0$. Then $p = \mathbf{rn}$. Hence

$$\frac{dp}{ds} = \dot{\mathbf{r}}\mathbf{n} + \mathbf{r}\dot{\mathbf{n}} = -\mathbf{r}\tau k = -\mathbf{r}\dot{\mathbf{r}}k = -r\dot{r}k = -r\frac{dr}{ds}k,$$

whence the required relation.

**4.58.** Rewrite the equation $(\varrho - \mathbf{r}_0 - R_0\mathbf{n}_0)^2 = R_0^2$ in the form $(\varrho - \mathbf{r}_0)^2 - 2R_0\mathbf{n}_0(\varrho - \mathbf{r}_0) = 0$ and consider the function $\varphi(s) = (\mathbf{r} - \mathbf{r}_0)^2 - 2R_0\mathbf{n}_0(\mathbf{r} - \mathbf{r}_0)$. We have $\varphi'(s) = 2(\mathbf{r} - \mathbf{r}_0)\tau - 2R_0\mathbf{n}_0\tau$, $\varphi'(s_0) = 0$, $\varphi''(s) = 2 + 2k\mathbf{n}(\mathbf{r} - \mathbf{r}_0) - 2R_0\mathbf{n}_0k\mathbf{n}$, $\varphi''(s_0) = 0$, $\varphi'''(s) = 2\dot{k}\mathbf{n}(\mathbf{r} - \mathbf{r}_0) - 2k^2\tau(\mathbf{n} - \mathbf{n}_0) - 2R_0\mathbf{n}_0\dot{k}\mathbf{n} + 2R_0\mathbf{n}_0k\tau$, $\varphi'''(s_0) = -2R_0k_0 \neq 0$; therefore $\varphi(s)$ changes sign when $s$ crosses through $s_0$, and since $\varphi(s)$ is the index of the point on the circumference, the proposition has been proved.

**4.59.** See the solution to the previous problem. We have

$$\varphi'(s_0) = \varphi''(s_0) = \varphi'''(s_0) = 0,$$

$$\varphi^{(4)}(s) = 2\ddot{k}\mathbf{n}(\mathbf{r} - \mathbf{r}_0) - 2k\dot{k}\tau(\mathbf{r} - \mathbf{r}_0) - 4k\dot{k}\tau(\mathbf{r} - \mathbf{r}_0) -$$
$$- 2k^3\mathbf{n}(\mathbf{r} - \mathbf{r}_0) - 2k^2 - 2R_0\mathbf{n}_0\ddot{k}\mathbf{n} + 2R_0\mathbf{n}_0k\dot{k}\tau + 4R\mathbf{n}_0k\dot{k}\tau +$$
$$+ 2R_0\mathbf{n}_0k^3\mathbf{n},$$

$$\varphi^{(4)}(s_0) = -2k_0^2 - 2R_0\ddot{k}_0 + 2k_0^2 = 2R_0\ddot{k}_0 \neq 0.$$

Therefore, the index of a point on the osculating plane does not change sign in crossing through $s_0$.

**4.60.** $\dfrac{d\alpha}{ds} = k = \dfrac{1}{f(\alpha)}$, $ds = f(\alpha)d\alpha$,

$x = \int \cos\alpha f(\alpha)d\alpha, \quad y = \int \sin\alpha f(\alpha)d\alpha.$

**4.61.** $d\alpha/ds = 1/R$, $f'(R)dR/ds = 1/R$, $ds = Rf'(R)dR$,

$x = \int \cos[f(R)]Rf'(R)dR,$

$y = \int \sin[f(R)]Rf'(R)dR.$

**4.62.** $x = \int \cos\alpha f'(\alpha)d\alpha$, $y = \int \sin\alpha f'(\alpha)d\alpha$.

**4.63.** $x = \int \cos[f(s)]ds$, $y = \int \sin[f(s)]ds$.

**4.64.** $R = \mathbf{r} + \dfrac{\mathbf{rn}}{2k\mathbf{r}^2 + \mathbf{rn}}\{(\mathbf{rn})\mathbf{n} - (\mathbf{r}\tau)\tau\}.$

**4.65.** $\mathbf{R} = \mathbf{r} + \dfrac{[\mathbf{r}'\mathbf{r}]}{[2\mathbf{r}^2\mathbf{r}', \mathbf{r}''] + [\mathbf{r}'\mathbf{r}]\mathbf{r}'^2}\{[\mathbf{r}'\mathbf{r}][\mathbf{r}'] - (\mathbf{r}\mathbf{r}')\mathbf{r}'\}.$

**4.66.** $R = r + \dfrac{\mathbf{en}}{2k}\{\mathbf{n}(\mathbf{en}) - \tau(\mathbf{e}\tau)\}.$

If the curve is given by an equation $\mathbf{r} = \mathbf{r}(t)$, then,

$$R = \mathbf{r} + \frac{[\mathbf{r}', \mathbf{e}]}{2[\mathbf{r}', \mathbf{r}'']}\{[\mathbf{r}', \mathbf{e}]\,[\mathbf{r}'] - (\mathbf{r}', \mathbf{e})\mathbf{r}'\}.$$

If the curve is given by an equation $y = f(x)$, then

$$X = x - \frac{[m - lf'(x)]^2}{2f''(x)}f'(x) - \frac{(m - lf'(x))(l + mf'(x))}{2f''(x)},$$

$$Y = f(x) + \frac{[m - lf'(x)]^2}{2f''(x)} - \frac{(m - lf'(x))(l + mf'(x))}{2f''(x)}f'(x),$$

where $l = \{l, m\}$.

**4.67.** $(x + 1)/2 = (y - 13)/3 = z/6$, $2x + 3y + 6z - 37 = 0$.

**4.68.** $u = -1$ at the point $A$. The tangent is $(x - 3)/6 = (y + 7)/(-17) = (z - 2)/7$, and the normal plane $6x - 17y + 7z - 151 = 0$.

**4.69.** $u = 1$ at the point $A$. Since $\mathbf{r}'(1) = 0$ and $\mathbf{r}''(1) = \{2, 2, 12\} \neq 0$, the direction of the tangent is determined by this vector, or $\{1, 1, 6\}$ collinear with it. The tangent is $(x - 2)/1 = y/1 = (z + 2)/6$, and the normal plane $x + y + 6z + 10 = 0$.

**4.70.** $9x - 27y - z + 7 = 0$.

**4.71.** For the osculating plane we find the equation $cx - ay = bc - ad$ not containing the parameter $u$. Substituting the expression for $x$, $y$ in terms of $u$ in this equation, we obtain an identity, whence the curve, in fact, lies in its osculating plane.

**4.72.** The osculating plane is $6x - 8y - z + 3 = 0$, the principal normal $x = 1 - 31\lambda$, $y = 1 - 26\lambda$, $z = 1 + 22\lambda$, and the binormal $x = 1 + 6\lambda$, $y = 1 - 8\lambda$, $z = 1 - \lambda$.

**4.73.** The tangent is

$$\mathbf{r} = \{a\cos t - \lambda a\sin t,\ a\sin t + a\lambda\cos t,\ b(\lambda + t)\},$$

The normal plane

$$ax\sin t - ay\cos t - bz + b^2 t = 0,$$

The binormal

$$\mathbf{r} = \{a\cos t + \lambda b\sin t,\ a\sin t - \lambda b\cos t,\ bt + \lambda a\},$$

The osculating plane

$$bx\sin t - by\cos t + az - abt = 0,$$

The principal normal

$$\mathbf{r} = \{(a + \lambda)\cos t,\ (a + \lambda)\sin t,\ bt\}.$$

**4.74.** The tangent is $x = 1 + 2\lambda$, $y = -\lambda$, $z = 1 + 3\lambda$,
The normal plane $2x - y + 3z - 5 = 0$,
The binormal $x = 1 - 3\lambda$, $y = -3\lambda$, $z = 1 + \lambda$,
The osculating plane $3x + 3y - z - 2 = 0$,
The principal normal $x = 1 - 8\lambda$, $y = 11\lambda$, $z = 1 + 9\lambda$.

**4.75.** The tangent is

$$X = x + \lambda \begin{vmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{vmatrix},$$

$$Y = y + \lambda \begin{vmatrix} \dfrac{\partial F_1}{\partial z} & \dfrac{\partial F_1}{\partial x} \\ \dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_2}{\partial x} \end{vmatrix}$$

$$Z = z + \lambda \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{vmatrix}$$

The equation of the normal plane is

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{vmatrix} = 0.$$

**4.76.** Having chosen a convenient coordinate system, we shall write the equations of the Viviani curve in the form

$$x^2 + y^2 + z^2 = a^2, \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4},$$

or $x^2 + y^2 + z^2 = a^2$, $x^2 + y^2 - ax = 0$.

To make up the parametric equations, we put

$$x - \frac{a}{2} = \frac{a}{2} \cos t, \quad y = \frac{a}{2} \sin t.$$

Then

$$\frac{a^2}{4}(1 + \cos t)^2 + \frac{a^2}{4} \sin^2 t + z^2 = a^2, z = a \sin \frac{t}{2}$$

(sign can be omitted, since if $2\pi$ is added to $t$, then $x$ and $y$ are unaltered and $z$ changes sign). Thus,

$$\mathbf{r} = \left\{\frac{a}{2}(1 + \cos t), \frac{a}{2} \sin t, a \sin \frac{t}{2}\right\}.$$

The equation of the tangent is

$$\mathbf{r} = \left\{\frac{a}{2}(1 + \cos t) - \lambda \sin t, \frac{a}{2} \sin t + \lambda \cos t, a \sin \frac{t}{2} + \lambda \cos \frac{t}{2}\right\},$$

that of the normal plane

$$x \sin t - y \cos t - z \cos \frac{t}{2} = 0,$$

of the binormal

$$\mathbf{r} = \left\{\frac{a}{2}(1 + \cos t) + \lambda \sin\frac{t}{2}(2 + \cos t),\right.$$

$$\left.\frac{a}{2}\sin t - \lambda\cos\frac{t}{2}(1 + \cos t),\ a\sin\frac{t}{2} + 2\lambda\right\},$$

of the principal normal

$$r = \left\{\frac{a}{2}(1 + \cos t) + \lambda\left[-\cos^2\frac{t}{2}(1 + \cos t) - 2\cos t\right],\right.$$

$$\left.\frac{a}{2}\sin t - \frac{\lambda}{2}\sin t(6 + \cos t),\ a\sin\frac{t}{2} - \lambda\sin\frac{t}{2}\right\},$$

and that of the osculating plane

$$\sin\frac{t}{2}(2 + \cos t)x - \cos\frac{t}{2}(1 + \cos t)y + 2z$$

$$- \frac{a}{2}\sin\frac{t}{2}(5 + \cos t) = 0.$$

**4.77.** $s = 5at.$

**4.78.** $s = 8a\sqrt{2}.$

**4.79.** $s = 9a.$

**4.80.** $s = 10$. The curve has four cusps with $ds/dt$ changing sign at the points $t = 0, \pi/2, \pi, 3\pi/2$.

**4.81.** $\mathbf{r} = \left\{a\cos\frac{s}{\sqrt{a^2 + b^2}},\ a\sin\frac{s}{\sqrt{a^2 + b^2}},\ \frac{bs}{\sqrt{a^2 + b^2}}\right\}.$

**4.82.** $\mathbf{r} = \left\{\frac{s + \sqrt{3}}{\sqrt{3}}\cos\ln\frac{s + \sqrt{3}}{\sqrt{3}},\ \frac{s + \sqrt{3}}{\sqrt{3}}\sin\ln\frac{s + \sqrt{3}}{\sqrt{3}},\ \frac{s + \sqrt{3}}{\sqrt{3}}\right\}.$

**4.83.** $\mathbf{r} = \left\{\sqrt{\frac{2 + s^2}{2}},\ \frac{s}{\sqrt{2}},\ \ln\left(\frac{s}{\sqrt{2}} + \sqrt{\frac{2 + s^2}{2}}\right)\right\}.$

**4.84.** $\tau = \dfrac{1}{\sqrt{a^2 + b^2}}\{-a \sin t,\ a \cos t,\ b\},$

$\nu = \{-\cos t,\ -\sin t,\ 0\},$

$\beta = \dfrac{1}{\sqrt{a^2 + b^2}}\{b \sin t,\ -b \cos t,\ a\},$

$k = \dfrac{a}{a^2 + b^2}, \quad \varkappa = \dfrac{b}{a^2 + b^2}.$

**4.85.** $\tau = \dfrac{\{2t,\ -1,\ 3t^2\}}{\sqrt{1 + 4t^2 + 9t^4}},$

$\nu = \dfrac{\{1 - 9t^4,\ 2t + 9t^3,\ 3t + 6t^3\}}{\sqrt{(1 - 9t^4)^2 + (2t + 9t^3)^2 + (3t + 6t^3)^2}},$

$\beta = \dfrac{\{-3t,\ -3t^2,\ 1\}}{\sqrt{1 + 9t^2 + 9t^4}},$

$k = \dfrac{2(1 + 9t^2 + 9t^4)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}}, \quad \varkappa = \dfrac{3}{1 + 9t^2 + 9t^4}.$

**4.86.** $\tau = \dfrac{\left\{-\sin t,\ \cos t,\ \cos \dfrac{t}{2}\right\}}{\sqrt{1 + \cos^2 \dfrac{t}{2}}},$

$$\nu = \frac{\left\{-\cos^2 \dfrac{t}{2}(1 + \cos t) - 2 \cos t,\ -\dfrac{1}{2} \sin t(6 + \cos t),\ -\sin \dfrac{t}{2}\right\}}{\sqrt{\left[-\cos^2 \dfrac{t}{2}(1 + \cos t) - 2 \cos t\right]^2 + \dfrac{1}{4} \sin^2 t(6 + \cos t)^2 + \sin^2 \dfrac{t}{2}}},$$

$$\beta = \sqrt{\frac{2}{13 + 3 \cos t}}\left\{\sin \frac{t}{2}(2 + \cos t),\ -\cos \frac{t}{2}(1 + \cos t),\ 2\right\},$$

$$k = \frac{1}{a}\sqrt{\frac{13 + 3 \cos t}{2\left(1 + \cos^2 \dfrac{t}{2}\right)^3}}, \quad \varkappa = \frac{12 \cos \dfrac{t}{2}}{a(13 + 3 \cos t)}.$$

**4.87.** (1) $k = \frac{1}{4}\sqrt{1 + \sin^2 \frac{t}{2}}$, $\varkappa = \frac{\cos \frac{t}{2} (\cos t - 5)}{4(3 - \cos t)}$;

(2) $k = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$, $\varkappa = \frac{-\sqrt{2}}{(e^t + e^{-t})^2}$;

(3) $k = \frac{2t}{(1 + 2t^2)^2}$, $\varkappa = \frac{-2t}{(1 + 2t^2)^2}$;

(4) $k = \frac{\sqrt{2}}{3e^t}$, $\varkappa = -\frac{1}{3e^t}$;

(5) $k = \varkappa = \frac{1}{3(t^2 + 1)^2}$;

(6) $k = \frac{3}{25 \sin t \cos t}$, $\varkappa = \frac{4}{25 \sin t \cos t}$.

**4.88.** $y = 1$.

**4.89.** $R = \frac{x}{2}\left(\frac{x^2}{a^2} + \frac{a^2}{2x^2}\right)^2$, $r = -\frac{x}{2}\left(\frac{x^2}{a^2} + \frac{a^2}{2x^2}\right)^2$.

**4.90.** $k = \frac{\sqrt{y''^2 + z''^2 + (y'z'' - y''z')^2}}{\sqrt{(1 + y'^2 + z'^2)^{3/2}}}$,

$$\varkappa = \frac{y''z''' - y'''z''}{y''^2 + z''^2 + (y'z'' - y''z')^2},$$

$$\tau = \frac{\{1, y', z'\}}{\sqrt{1 + y'^2 + z'^2}},$$

$$\nu = \frac{\{-z'z'' - y'y'', y'' - z'(y'z'' - y'z''), y'(y'z'' - z'y'') + z''\}}{\sqrt{(z'z'' - y'y'')^2 + [y'' - z'(y'z'' - y'z'')]^2 + [z'' + y'(y'z'' - z'y'')]^2}}$$

$$\beta = \frac{\{y'z'' - y''z', -z'', y''\}}{\sqrt{(y'z'' - y''z')^2 + z''^2 + y''^2}}.$$

**4.91.** The two families of curves are:

(a) $y^2 + z^2 = \text{const}, \ xy = az;$

and

(b) $x^2 + z^2 = \text{const}, \ xy = az.$

**4.92.** Let the equation of the sphere be of the form:

$$\mathbf{r} = \{a \cos v \cos u, \ a \cos v \sin u, \ a \sin v\}.$$

Then the equation of the loxodrome is

$$u = \tan\theta \ln \tan\left(\frac{\pi}{4} + \frac{v}{2}\right),$$

where $\theta$ is the given angle.

$$\tau = \cos\theta\{-\sin v \cos u - \sin u \tan\theta, \ -\sin v \sin u + \cos u \tan\theta, \cos v\};$$

$$\nu = \frac{\cos\theta}{\sqrt{1 + \dfrac{\tan^2\theta}{\cos^2 v}}}\left\{-\cos u \cos v + \tan v \sin u \tan\theta - \frac{\cos u}{\cos v}\tan^2\theta,\right.$$

$$\left. -\frac{\sin u \cos v}{\cos v} - \frac{\sin u}{\cos v}\tan^2\theta - \tan v \tan\theta \cos u, \ -\sin v\right\};$$

$$\beta = \frac{\left\{\sin u, \ -\cos u, \ \dfrac{\tan\theta}{\cos v}\right\}}{\sqrt{1 + \dfrac{\tan^2\theta}{\cos^2 v}}};$$

$$k = \frac{\cos\theta}{a}\sqrt{1 + \frac{\tan^2\theta}{\cos^2 v}}; \qquad \varkappa = \frac{\tan\theta}{a(\cos^2 v + \tan^2\theta)}.$$

**4.93.** $v = Ce^{\frac{u \cot\theta}{\sqrt{1+k^2}}}.$

**4.97.** $\dfrac{1}{6}\dfrac{\mathbf{r}'\mathbf{r}''\mathbf{r}'''}{|[\mathbf{r}' \times \mathbf{r}'']|}$, and in the special case, $\dfrac{1}{6} k|\varkappa|.$

**4.98.** The necessary and sufficient condition is $\mathbf{e}' \neq 0$, $\boldsymbol{\varrho}'\mathbf{e}\mathbf{e}' = 0$, while the equation of the envelope

$$\mathbf{r} = \boldsymbol{\varrho} - \frac{\boldsymbol{\varrho}'\mathbf{e}'}{\mathbf{e}'^2}\mathbf{e}.$$

**4.99.** When $a = b$.

**4.101.** $\dot{\mathbf{r}} = \tau$, $\ddot{\mathbf{r}} = k\nu$, $\dddot{\mathbf{r}} = -k^2\tau + \dot{k}\nu + k\varkappa\beta$.

**4.103.** $\mathbf{e}\nu = C$, $\mathbf{e}(-k\tau + \varkappa\beta) = 0$, $\quad \mathbf{e}\tau - \dfrac{\varkappa}{k}\mathbf{e}\beta = 0$,

$$\mathbf{e}\nu k = \left(\frac{\varkappa}{k}\right)^{\cdot}\mathbf{e}\beta + \frac{\varkappa^2}{k}\mathbf{e}\nu = 0, \ \mathbf{e}\beta = C\frac{k^2 + \varkappa^2}{k\left(\dfrac{\varkappa}{k}\right)^{\cdot}}.$$

Differentiating once again, we obtain the required relation. Note that in view of the above relations, we may assume

$$\mathbf{e} = \frac{\varkappa}{k}\,\frac{k^2 + \varkappa^2}{k\left(\dfrac{\varkappa}{k}\right)^{\cdot}}\tau + \nu + \frac{k^2 + \varkappa^2}{k\left(\dfrac{\varkappa}{k}\right)^{\cdot}}\beta.$$

If the relation is held

$$\left\{\frac{k^2 + \varkappa^2}{k\left(\dfrac{\varkappa}{k}\right)^{\cdot}}\right\}^{\cdot} + \varkappa = 0,$$

then this vector is constant. This constant vector $\mathbf{e}$ forms with the vector $\nu$ an angle whose cosine equals $1/|\mathbf{e}|$ = const.

**4.104.** $\mathbf{e}\tau = 0$, $k\mathbf{e}\nu = 0$; hence either $k = 0$ (straight line) or $\mathbf{e}\nu = 0$; if $\mathbf{e}\nu = 0$, then $\mathbf{e}(-k\tau + \varkappa\beta) = 0$, whence $\varkappa = 0$ (plane line).

**4.105.** $\mathbf{e}\beta = 0$; $\varkappa\mathbf{e}\nu = 0$; hence $\varkappa = 0$, since, if the inequality $\varkappa \neq 0$ were held, we would have $\mathbf{e}\nu = 0$, $\mathbf{e}(-k\tau + \varkappa\beta) = 0, k\mathbf{e}\tau = 0$, $\mathbf{e}\tau \neq 0$. Therefore, $k = 0$, and the line is straight.

**4.106.** $\dot{\beta} = -\varkappa\nu = 0$, $\varkappa = 0$.

**4.108.** (a) Let **a** be a unit vector with a fixed direction. Then

$$\mathbf{a}\tau = \cos v \ (v = \text{const}). \tag{1}$$

We have $(\mathbf{a}\tau)^{\cdot} = \mathbf{a}\dot{\tau} = 0$. Therefore, $k\mathbf{a}\nu = 0$. Excluding the case where $k = 0$ (i.e., of straight lines), we obtain

$$\mathbf{a}\nu = 0. \tag{2}$$

Therefore, the normals are perpendicular to the fixed direction.

Conversely, if $\nu$ is perpendicular to the fixed direction, then equality (1) holds.

(b) Let $\varkappa \neq 0$. It follows from (2), with the use of the third Frenet formula, that

$$\mathbf{a}\dot{\beta} = 0,$$

whence $\mathbf{a}\beta$ = const.

Conversely, differentiating this formula, we obtain (2).

(c) Differentiating (2), we obtain

$$k\mathbf{a}\tau = \varkappa\mathbf{a}\beta,$$

whence

$$\frac{k}{\varkappa} = \frac{\mathbf{a}\beta}{\mathbf{a}\tau} = \text{const.}$$

Conversely, it follows from the first and third Frenet formulae that

$$\frac{\tau}{k} + \frac{\dot{\beta}}{\varkappa} = 0,$$

whence

$$\frac{\varkappa}{k}\dot{\tau} + \dot{\beta} = 0, \quad \frac{\varkappa}{k}\tau + \beta = \text{const} = \mathbf{a}.$$

Multiplying scalarly by $\nu$, we obtain $\mathbf{a}\nu = 0$. Therefore, condition (2) has been fulfilled.

**4.109.** Take into account that

$$\dot{\mathbf{r}}\ddot{\mathbf{r}}\dddot{\mathbf{r}}^{(4)} = k^5\left(\frac{\varkappa}{k}\right)^{\bullet},$$

and use the previous problem.

# 5 Surfaces

**5.1.** $\mathbf{r} = \varrho + v\mathbf{e}$.

**5.2.** $\mathbf{r} = v\varrho$.

**5.3.** $\mathbf{r} = \varrho + v\varrho'$.

**5.4.** $\mathbf{r} = \varrho(s) + \nu(s)\cos\varphi + \beta(s)\sin\varphi$.

**5.5.** $\mathbf{r} = \{\varphi(v)\cos u, \ \varphi(v)\sin u, \ \psi(v)\}$.

In the special case, $\mathbf{r} = \{f(v)\cos u, \ f(v)\sin u, \ v\}$.

**5.6.** $\mathbf{r} = \{(a + b\cos v)\cos u, \ (a + b\cos v)\sin u, \ b\sin v\}$.

**5.7.** Let the moving straight line coincide with the axis $Ox$ at the initial moment, and the second line in question with the axis $Oz$. Then the equation of the right helicoid is of the form

$$\mathbf{r} = \{v\cos u, \ v\sin u, \ ku\},$$

where $v$ is the distance of a point of the helicoid from its axis (i.e., the axis $Oz$), and $u$ the longitude of the point.

**5.8.** If the equation of the helix is given in the form

$$\varrho = \{a \cos u, \ a \sin u, \ bu\},$$

then the vector $\mathbf{n} = \{-\cos u, \ -\sin u, \ 0\}$ is the principal normal vector. Hence, the required equation

$$\mathbf{r} = \varrho - \lambda\mathbf{n} = \{(a + \lambda)\cos u, \ (a + \lambda)\sin u, \ bu\} =$$

$$= \{v \cos u, \ v \sin u, \ bu\}$$

is that of a right helicoid.

**5.9.** $\mathbf{r} = \varrho(s) + \lambda\{\mathbf{n}(s)\cos\varphi(s) + \mathbf{b}(s)\sin\varphi(s)\}$, where $\varphi(s)$ is an arbitrary function of the variable $s$.

**5.10.** The normal plane to the circumference $\varrho = \{a \cos u, \ a \sin u, \ 0\}$ is determined by the vectors $\mathbf{n} = \{\cos u, \ \sin u, \ 0\}$ and $\{0, \ 0, \ 1\}$. The vector lying in the normal plane and inclined at the angle $u$ to the vector $\mathbf{n}$ is $\alpha = \mathbf{n} \cos u + \mathbf{k} \sin u$. Therefore, the equation of the required surface is

$$\mathbf{r} = \{a \cos u, \ a \sin u, \ 0\} + v\alpha$$

$$= \{a \cos u + v \cos^2 u, \ a \sin u + v \sin u \cos u, \ v \sin u\}.$$

Eliminating the parameters $u$ and $v$, we find

$$x = a \cos u + \frac{\cos^2 u}{\sin u} z = \cot u(a \sin u + z \cos u),$$

$$y = a \sin u + z \cos u, \quad \frac{x}{y} = \cot u, \quad \frac{y^2}{\sin^2 u} = (a + z \cot u)^2,$$

$$y^2\left(1 + \frac{x^2}{y^2}\right) = \left(a + \frac{xz}{y}\right)^2, \quad \text{or} \quad y^2(x^2 + y^2) = (ay + xz)^2,$$

a surface of the fourth order.

**5.11.** $\mathbf{R} = \dfrac{1}{2}\{\mathbf{r}(u) + \varrho(v)\}$

**5.12.** $\mathbf{r} = \left\{v, \ a \cos u \cosh \dfrac{v}{a}, \ a \sin u \cosh \dfrac{v}{a}\right\}$, where $u$ is the longitude and $v$ the oriented distance from a point of the surface to the gorge section of the catenoid.

**5.13.** $\mathbf{r} = \left\{a \ln \tan\left(\dfrac{\pi}{4} + \dfrac{t}{2}\right) - a \sin t, \ a \cos t \cos u, \ a \cos t \sin u\right\}.$

**5.14.** The equation of the given straight line is $\mathbf{r}_1 = \{u, 0, h\}$, and that of the ellipse

$$\mathbf{r}_2 = \{a \cos v, b \sin v, 0\}.$$

Furthermore,

$$\mathbf{r}_1 - \mathbf{r}_2 = \{u - a \cos v, -b \sin v, h\}, \ u - a \cos v = 0,$$

$$\mathbf{r}_1 - \mathbf{r}_2 = \{0, -b \sin v, h\},$$

and the required equation of the conoid is

$$\mathbf{r} = \{a \cos v, b \sin v, 0\} + \lambda\{0, -b \sin v, h\}$$
$$= \{a \cos v, b(1 - \lambda)\sin v, \lambda h\}.$$

Eliminating the parameters $\lambda$ and $v$, we obtain the implicit equation of the conoid

$$\left(1 - \frac{x^2}{a^2}\right)\left(\frac{z}{h} - 1\right)^2 - \frac{y^2}{b^2} = 0.$$

**5.15.** $\mathbf{r}_1 = \{a, 0, u\}, \mathbf{r}_2 = \left\{0, v, \dfrac{v^2}{2p}\right\}, \mathbf{r}_1 - \mathbf{r}_2 = \left\{a, -v, u - \dfrac{v^2}{2p}\right\},$

$$u - \frac{v^2}{2p} = 0, \quad u = \frac{v^2}{2p}, \quad \mathbf{r}_1 - \mathbf{r}_2 = \{u, -v, 0\},$$

$$\mathbf{r} = \left\{0, v, \frac{v^2}{2p}\right\} + \lambda\{a, -v, 0\} = \left\{a\lambda, v(1 - \lambda), \frac{v^2}{2p}\right\},$$

or

$$a^2y^2 = 2pz(x - a)^2.$$

**5.16.** The parametric equations of the given circumferences are

$$\mathbf{r}_1 = \{a(1 + \cos u), 0, a \sin u\}, \ \mathbf{r}_2 = \{0, a(1 + \cos v), a \sin v\}.$$

We find

$$\mathbf{r}_1 - \mathbf{r}_2 = \{a(1 + \cos u), -a(1 + \cos v), a(\sin u - \sin v)\}.$$

We have $\sin u - \sin v = 0$, whence

(1) $v = u + 2k\pi$,
(2) $v = \pi - u + 2k\pi$.

In the first case, we have

$$\mathbf{r}_1 - \mathbf{r}_2 = \{a(1 + \cos u), -a(1 + \cos u), 0\} \parallel \{1, -1, 0\},$$

and thus obtain the elliptic cylinder

$$\varrho = \{a(1 + \cos u),\ 0,\ a\sin u\} + \lambda\{1,\ -1,\ 0\} =$$
$$= \{a(1 + \cos u) + \lambda,\ -\lambda,\ a\sin u\}.$$

In the second case,

$$\mathbf{r}_1 - \mathbf{r}_2 = \{a(1 + \cos u),\ -a(1 - \cos u),\ 0\},$$

and the second surface making up the given cylindroid is determined by the equation

$$\mathbf{R} = \{a(1 + \cos u),\ 0,\ a\sin u\} +$$
$$+ \lambda\{a(1 + \cos u),\ -a(1 - \cos u),\ 0\} =$$
$$= \{a(1 + \lambda)(1 + \cos u),\ -a\lambda(1 - \cos u),\ a\sin u\}.$$

Eliminating the parameters $\lambda$ and $u$, we obtain

$$z^4 + z^2[(x - y)^2 - 2a(x + y)] + 4a^2xy = 0.$$

**5.17.** $\mathbf{r}_1 = \left\{\dfrac{u^2}{2p},\ u,\ 0\right\}$, $\mathbf{r}_2 = \left\{-\dfrac{v^2}{2p},\ 0,\ v\right\}$,

$$\mathbf{r}_1 - \mathbf{r}_2 = \left\{\frac{u^2 + v^2}{2p},\ u,\ -v\right\}.$$

The condition for this vector to be collinear with the plane $y - z = 0$ is given by the relations

$$u + v = 0, \quad v = -u,\ \mathbf{r}_1 - \mathbf{r}_2 = \{u^2/p,\ u,\ u\},$$

and the required equation is the following:

$$\mathbf{r} = \left\{\frac{u^2}{2p},\ u,\ 0\right\} + v\left\{\frac{u^2}{p},\ u,\ u\right\} = \left\{\frac{u^2}{2p}(1 + 2v),\ u(1 + v),\ uv\right\}.$$

Eliminating the parameters $u$ and $v$, we obtain $y^2 - z^2 = 2px$, a hyperbolic paraboloid.

**5.18.** The equation of the axis $Oz$ is of the form: $\mathbf{r}_1 = \{0,\ 0,\ u\}$ and the equation of the given curve

$$\mathbf{r}_2 = \left\{b\cos v,\ b\sin v,\ \frac{a^3}{b^2\cos v\sin v}\right\};$$

hence

$$\mathbf{r}_2 - \mathbf{r}_1 = \left\{b\cos v,\ b\sin v,\ \frac{a^3}{b^2\cos v\sin v} - u\right\},$$

$$u = \frac{a^3}{b^2 \cos v \sin v}, \quad \mathbf{r}_2 - \mathbf{r}_1 = \{b \cos v, b \sin v, 0\},$$

$$\mathbf{r} = \left\{0, 0, \frac{a^3}{b^2 \cos v \sin v}\right\} + \lambda\{b \cos v, b \sin v, 0\}$$

$$= \left\{\lambda b \cos v, \lambda b \sin v, \frac{a^3}{b^2 \cos v \sin v}\right\}.$$

Eliminating the parameters $\lambda$ and $v$, we obtain

$$b^2 xyz = a^3(x^2 + y^2).$$

**5.19.** $(\mathbf{a} + u\mathbf{b} - \boldsymbol{\varrho})\mathbf{n} = 0, \ u = \dfrac{\mathbf{n}(\boldsymbol{\varrho} - \mathbf{a})}{\mathbf{nb}},$

$$\mathbf{a} + u\mathbf{b} - \boldsymbol{\varrho} = \frac{\mathbf{n}(\boldsymbol{\varrho} - \mathbf{a})}{\mathbf{nb}}\mathbf{b} - (\boldsymbol{\varrho} - \mathbf{a}),$$

$$\mathbf{R} = \boldsymbol{\varrho} + \lambda\left\{\frac{\mathbf{n}(\boldsymbol{\varrho} - \mathbf{a})}{\mathbf{nb}}\mathbf{b} - \boldsymbol{\varrho} + \mathbf{a}\right\}.$$

**5.20.** Take the equations of the given ellipses in the form:

$$\mathbf{r}_1 = \{a, b \cos u, c \sin u\}, \ \mathbf{r}_2 = \{-a, c \cos v, b \sin v\},$$

$$\mathbf{r}_1 - \mathbf{r}_2 = \{2a, b \cos u - c \cos v, c \sin u - b \sin v\},$$

$$c \sin u - b \sin v = 0,$$

$$\sin v = \frac{c}{b} \sin u, \ \cos v = \pm \frac{1}{b}\sqrt{b^2 - c^2 \sin^2 u},$$

$$\mathbf{r}_1 - \mathbf{r}_2 = \left\{2a, b \cos u \pm \frac{c}{b}\sqrt{b^2 - c^2 \sin^2 u}, 0\right\}.$$

The required equation is

$$\mathbf{R} = \{a, b \cos u, c \sin u\} + v\left\{2a, b \cos u \pm \frac{c}{b}\sqrt{b^2 - c^2 \sin^2 u}, 0\right\}$$

or

$$\mathbf{R} = \{a + 2av, b \cos u\} + v\left\{b \cos u \pm \frac{c}{b}\sqrt{b^2 - c^2 \sin^2 u}, c \sin u\right\}$$

**5.21.** The equation of the axis $Oz$ is $p = \{0, 0, v\}$, and we find

$$\boldsymbol{\varrho} - \mathbf{p} = \{u, u^2, u^3 - v\}, \quad u^3 - v = 0, \quad v = u^3,$$

$$\mathbf{p} = \{0, 0, u^3\}.$$

The required equation is

$$\mathbf{r} = \mathbf{p} + v(\boldsymbol{\varrho} - \mathbf{p}) = \{0, 0, u^3\} + v\{u, u^2, 0\} = \{uv, u^2v, u^3\}.$$

**5.22.** $\mathbf{r} = \{bv, av\cos u, (b + a\cos u)(1 - v) + a\sin u\}$.

**5.23.** The equations of the given straight lines are $\boldsymbol{\varrho} = \{u, 1, 1\}$ and $\mathbf{p} = \{1, v, 0\}$. The equation of the straight line passing through two arbitrary points of these straight lines is the following:

$$\mathbf{r} = \{1, v, 0\} + \lambda\{u, 1, 1\}.$$

For the point where this straight line meets the plane $xOz$, we have:

$$v + \lambda = 0, \quad \lambda = -v, \mathbf{r} = \{1, v, 0\} - v\{u, 1, 1\} = \{1 - uv, 0, -v\}.$$

This point must lie on the circumference

$$x = \cos\varphi, \quad y = 0, \quad z = \sin\varphi.$$

Therefore,

$$1 - uv = \cos\varphi, \quad v = -\sin\varphi,$$

whence

$$u = \frac{1 - \cos\varphi}{-\sin\varphi} = -\tan\frac{\varphi}{2}.$$

It remains to make up the equations of the straight line passing through the points $\left(-\tan\frac{\varphi}{2}, 1, 1\right)$ and $(1, -\sin\varphi, 0)$. Finally, we obtain:

$$\mathbf{r} = \{1, -\sin\varphi, 0\} + \psi\left(\{1, -\sin\varphi, 0\} - \left\{-\tan\frac{\varphi}{2}, 1, 1\right\}\right)$$
$$= \left\{1 + \psi\left(1 + \tan\frac{\varphi}{2}\right), -\sin\varphi - \psi(1 + \sin\varphi), -\psi\right\}.$$

**5.24.** $\mathbf{r} = \{a(\cos v - u\sin v), a(\sin v + u\cos v), b(u + v)\}$.

**5.25.** $c^2(x^2 + y^2)^2 = a^2(x^2 - y^2)(z + c)^2$.

**5.26.** We will assume that rectangular Cartesian coordinates $(\xi, \eta)$ are given on the plane $\pi$. Then the equation of the curve $\boldsymbol{\varrho} = \boldsymbol{\varrho}(u)$ can be written in coordinate form thus: $\xi = \xi(u)$, $\eta = \eta(u)$. In addition, we assume that the straight line $AB$ is the axis $z$ in space and that the axis $\eta$ of the moving plane $\pi$ slips along it. For the appropriate choice of the axes $x$, $y$ and positive directions on the coordinate axes, we have:

$$\mathbf{R}(u, v) = \{\xi(u)\cos v, \quad \xi(u)\sin v, \eta(u) + av\}.$$

**5.27.** $\mathbf{R}(u, v) = \mathbf{r}(u) + a\boldsymbol{\nu}(u)\cos v + a\boldsymbol{\beta}(u)\sin v$, where $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ are the principal normal and binormal unit vectors to the curve $\mathbf{r} = \mathbf{r}(u)$, and the points $(u, v)$ and $(u, v + 2\pi)$ regarded as identical.

**5.28.** Take the point of intersection of the normals to be the origin of the radii vectors. Then

$$\mathbf{r} \cdot \mathbf{r}_u = 0, \ \mathbf{r} \cdot \mathbf{r}_v = 0,$$

whence $\mathbf{r}^2 = \text{const}$. Therefore, the given surface is either a sphere or a part of a sphere.

**5.29.** The volume of the tetrahedron is $9a^3/2$.

**5.30.** The tangent plane is determined by the equation

$$\frac{x}{u \sin v} + \frac{y}{u \cos v} + \frac{z}{\sqrt{a^2 - u^2}} = a^2.$$

The required sum equals $a^6$.

**5.31.** The equations of the line of intersection in curvilinear coordinates are $u = u_1 \cos(v + v_1)/\cos 2v_1$ (except for the generator $v = v_1$), where $u_1$, $v_1$ are the coordinates of the point of contact. The parametric equations of the same line in Cartesian coordinates are

$$x = u_1 \frac{\cos(v + v_1)}{\cos 2v_1} \cos v, \quad y = u_1 \frac{\cos(v + v_1)}{\cos 2v_1} \sin v,$$

$$z = a \sin 2v.$$

The equation of its projection on the plane $xy$ is

$$x^2 + y^2 = \frac{u_1}{\cos 2v_1} (x \cos v_1 - y \sin v_1).$$

Since the projection is a circumference, the line itself (being a plane line) is an ellipse.

**5.32.** The equation of the tangent plane is

$$Z - xf = \left(f - \frac{y}{x} f'\right)(X - x) + (Y - y)f', \text{ or}$$

$$Z = \left(f - \frac{y}{x} f'\right)X + Yf',$$

and all the tangent planes pass through the same point, viz., the origin. Besides, it is also clear from the fact that the given equation determines a cone with vertex at the origin ($z$ being a homogeneous function in $x$ and $y$).

**5.33.** The tangent plane has the equation

$$kx \sin u - ky \cos u + vz - kuv = 0,$$

and the normal

$$\mathbf{r} = \{v \cos u + \lambda k \sin u, \ v \sin u - \lambda k \cos u, \ ku + \lambda v\}.$$

5.34. $\dfrac{X}{x} + \dfrac{Y}{y} + \dfrac{Z}{z} = 3.$

5.35. Let the equation of the curve $C$ be $\varrho = \varrho(s)$. The equation of the surface is as follows: $\mathbf{r} = \varrho + \lambda\tau$, where $\tau$ is the unit vector of the tangent to the curve $C$. We find that

$$\frac{\partial \mathbf{r}}{\partial s} = \tau + \lambda k \nu, \quad \frac{\partial \mathbf{r}}{\partial \lambda} = \tau, \quad \left[\frac{\partial \mathbf{r}}{\partial \lambda}, \frac{\partial \mathbf{r}}{\partial s}\right] = \lambda k \beta;$$

when $s = \text{const}$ (i.e., at the points of the same tangent), this vector has the same direction (for then $\beta = \text{const}$), from which it follows also that the tangent plane to such a surface at all points of the curve $C$ is the osculating plane to this curve.

5.36. The equation of the surface is

$$\mathbf{r} = \varrho + \lambda\nu, \quad \frac{\partial \mathbf{r}}{\partial s} = \tau + \lambda(-k\tau + \varkappa\beta), \quad \frac{\partial \mathbf{r}}{\partial \lambda} = \nu,$$

$$\left[\frac{\partial \mathbf{r}}{\partial s}, \frac{\partial \mathbf{r}}{\partial \lambda}\right] = (1 - \lambda k)\beta - \lambda\varkappa\tau.$$

The equation of the tangent plane is

$$(\mathbf{R} - \varrho - \lambda\nu)(\beta - \lambda k\beta - \lambda\varkappa\tau) = 0,$$

or

$$\mathbf{R}(\beta - \lambda\varkappa\tau) - \varrho(\beta - \lambda\varkappa\tau) + \lambda^2\varkappa = 0,$$

and that of the normal

$$\mathbf{R} = \varrho + \lambda\nu + \xi(\beta - \lambda\varkappa\tau).$$

5.37. $\mathbf{r} = \varrho + \lambda\beta, \ \dfrac{\partial \mathbf{r}}{\partial s} = \tau - \lambda\varkappa\nu, \ \dfrac{\partial \mathbf{r}}{\partial \lambda} = \beta,$

$$\left[\frac{\partial \mathbf{r}}{\partial s}, \frac{\partial \mathbf{r}}{\partial \lambda}\right] = -\nu - \lambda\varkappa\tau.$$

The equation of the tangent plane is

$$(\mathbf{R} - \varrho - \lambda\beta)(\nu + \lambda\varkappa\tau) = 0,$$

or

$$(\mathbf{R} - \varrho)(\nu + \lambda\varkappa\tau) = 0.$$

The equation of the normal is

$$\mathbf{R} = \varrho + \lambda\beta + \xi(\nu + \lambda\varkappa\tau).$$

**5.39.** If **a** is the direction vector of the given straight line, and the origin of the radii vectors is taken on it, then the vectors **r**, **a**, and $\left[\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right]$ lie in the same plane, while

$$\mathbf{r} \cdot \left[\mathbf{a}, \left[\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right]\right] = 0.$$

Hence,

$$\left(\mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial u}\right) \left(\mathbf{a} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) - \left(\mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial v}\right) \left(\mathbf{a} \cdot \frac{\partial \mathbf{r}}{\partial u}\right) = 0.$$

But this equality can be written as the vanishing of the functional determinant, viz.,

$$\frac{\partial \mathbf{r}^2}{\partial u} \frac{\partial}{\partial v} (\mathbf{a} \cdot \mathbf{r}) - \frac{\partial \mathbf{r}^2}{\partial v} \frac{\partial}{\partial u} (\mathbf{a} \cdot \mathbf{r}) = 0,$$

from which it follows that the entities $\mathbf{r}^2$ and $\mathbf{a} \cdot \mathbf{r}$ are in the functional dependence

$$\mathbf{r}^2 = f(\mathbf{a} \cdot \mathbf{r}).$$

Choosing the axis $Oz$ along the vector **a**, we obtain $x^2 + y^2 = f(z)$, a surface of revolution.

**5.42.** $4z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1$, the edge of regression being imaginary.

**5.43.** $x^2 + \frac{a^2}{a^2 + b^2} y^2 + z^2 = a^2$.

**5.44.** The envelope has the equation $(x^2 + y^2 + z^2 - x)^2 = x^2 + y^2$, and the edge of regression degenerates into the point (0, 0, 0).

**5.45.** Taking the equation of the parabolas in the form $y^2 = 2px$, $z = 0$ and $y^2 = 2qz$, $x = 0$, we obtain the equation of the envelope in the form $y^2 = 2px + 2qz$, i.e., a parabolic cylinder with $\sqrt{p^2 + q^2}$ as a parameter.

**5.46.** $(\mathbf{R} - \boldsymbol{\varrho})^2 = a^2$. Differentiating with respect to $s$, we obtain $(\mathbf{R} - \boldsymbol{\varrho})\boldsymbol{\tau} = 0$. Hence $\mathbf{R} - \boldsymbol{\varrho} = \lambda \mathbf{b} + \mu \boldsymbol{\nu}$. Since $(\mathbf{R} - \boldsymbol{\varrho})^2 = a^2$, $\lambda^2 + \mu^2 = a^2$, and we can put

$$\lambda = a \cos\varphi, \ \mu = a \sin\varphi$$

so that the equation of the envelope is

$$\mathbf{R} = \boldsymbol{\varrho} + a(\mathbf{b} \cos\varphi + \boldsymbol{\nu} \sin\varphi).$$

**5.47.** The edge of regression is a curve whose points are obtained by intersecting the curvature axes of the curve $\varrho = \varrho(s)$ with the corresponding spheres of the given family.

**5.48.** The equation of the family is the following:

$$(x - b\cos\varphi)^2 + y - b\sin^2\varphi + z^2 - a^2 = 0,$$

and the envelope is a torus whose equation may be obtained by eliminating $\varphi$ from the equations $F = 0$ and $\dfrac{\partial F}{\partial\varphi} = 0$; $(x^2 + y^2 + z^2 + b^2 - a^2)^2 - 4b^2(x^2 + y^2) = 0$ is a surface of the fourth order. The edge of regression when $a > b$ is reduced to the two points

$$(0,\ 0,\ \pm\sqrt{a^2 - b^2})$$

or one point $(0, 0, 0)$ if $a = b$.

**5.49.** The equation of the family is as follows

$$x^2 + y^2 + z^2 - 2u^3x - 2u^2y - 2uz = 0.$$

We shall find the envelope by eliminating $u$ from this equation and from

$$3u^2x + 2uy + z = 0.$$

Thus, the equation of the envelope is:

$$3x[9x(x^2 + y^2 + z^2) - 2zy]^2 + 2y[9x(x^2 + y^2 + z^2) - 2zy] -$$
$$- (12xz - 4y^2) + z(12xz - 4y^2)^2 = 0.$$

The edge of regression is found by adjoining another equation to the two indicated above, viz., $6ux + 2y = 0$, or $3ux + y = 0$.

Hence, $u = -y/3x$, and the equation of the edge of regression is the following

$$27x^2(x^2 + y^2 + z^2) - 4y^3 + 18xyz = 0,\ y^2 - 3xz = 0.$$

The edge of regression can also be obtained in parametric form:

$$\mathbf{r} = \left\{\frac{2u^3}{9u^4 + 9u^2 + 1},\ \frac{-6u^4}{9u^4 + 9u^2 + 1},\ \frac{6u^5}{9u^4 + 9u^2 + 1}\right\}.$$

**5.50.** $x^{2/3} + y^{2/3} + z^{2/3} = l^{2/3}$.

**5.51.** $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

**5.52.** $xyz = 2/9\ a^3$.

**5.53.** The envelope is $y^2 = 4xz$, and the edge of regression is degenerated into a point, viz., the origin.

**5.54.** The characteristic is $x = a(\cos\alpha + \alpha\sin\alpha) - z\sin\alpha$, $y = a(\sin\alpha - \alpha\cos\alpha) + z\cos\alpha$, and the edge of regression is a helix

$$x = a\cos\alpha, \quad y = a\sin\alpha, \quad z = a\alpha.$$

**5.55.** Let $\varrho = \varrho(s)$ be the equation of the given curve. The equation of the family of the osculating planes is

$$(\mathbf{r} - \varrho)\mathbf{b} = 0.$$

Differentiating with respect to $s$, we obtain $(\mathbf{r} - \varrho)\tau = 0$. The characteristic is the tangent

$$(\mathbf{r} - \varrho)\mathbf{b} = 0, \quad (\mathbf{r} - \varrho)\nu = 0.$$

The envelope is $\mathbf{r} = \varrho + \lambda\tau$, i.e., the surface formed by the tangents to the given curve. Differentiating the relation $(\mathbf{r} - \varrho)\nu = 0$ once more, we obtain $(\mathbf{r} - \varrho)\mathbf{b} = 0$. Hence, taking into account the relations

$$(\mathbf{r} - \varrho)\mathbf{b} = 0, \quad (\mathbf{r} - \varrho)\nu = 0$$

we have

$$\mathbf{r} = \varrho,$$

i.e., the edge of regression is the given curve.

**5.56.** The characteristics are the curvature axes of the given curve, and the envelope is the surface formed by the curvature axes. The edge of regression is the curve described by the centres of the osculating spheres of the given curve.

**5.57.** $\mathbf{rn}' + D' = 0$, $\mathbf{r} = \alpha\mathbf{n} + \beta\mathbf{n}' + \lambda[\mathbf{nn}']$,

$$\alpha = \mathbf{rn} = -D, \quad \beta = \frac{\mathbf{rn}'}{\mathbf{n}'^2} = -\frac{D'}{\mathbf{n}'^2}.$$

The equation of the envelope is

$$\mathbf{r} = -D\mathbf{n} - \frac{D'\mathbf{n}'}{\mathbf{n}'^2} + \lambda[\mathbf{nn}']$$

(with the parameters $u$ and $\lambda$). The characteristics are straight lines $u = \text{const}$. The edge of regression is found by solving the equations

$$\mathbf{rn} + D = 0, \quad \mathbf{rn}' + D' = 0, \quad \mathbf{rn}'' + D'' = 0$$

for $\mathbf{r}$, viz.,

$$\mathbf{r} = \frac{(\mathbf{rn})\,[\mathbf{n}'\mathbf{n}''] + (\mathbf{rn}')\,[\mathbf{n}''\mathbf{n}] + (\mathbf{rn}'')\,[\mathbf{nn}']}{\mathbf{nn}'\,\mathbf{n}''}$$

$$= \frac{D[\mathbf{n}'\mathbf{n}''] + D'[\mathbf{n}''\mathbf{n}] + D''[\mathbf{nn}']}{\mathbf{nn}'\mathbf{n}''}.$$

**5.58.** The envelope of the family of planes tangent to both parabolas. The equation of the family is

$$X\alpha - 2aY - \left(\frac{\alpha^2}{2b} + \frac{4a^3}{b\alpha}\right)Z + \frac{4a^3}{\alpha} = 0,$$

where $\alpha$ is the parameter of the family.

**5.59.** The vector of normal is $\mathbf{n} = (u + v)(\mathbf{i}\sin v - \mathbf{j}\cos v + \mathbf{k})$ is parallel to the vector $\mathbf{i}\sin v - \mathbf{j}\cos v + \mathbf{k}$, which is unaltered if the parameter $v$ remains constant. Hence, the lines $v = \text{const}$ are rectilinear generators of the surface, and $u + v = 0$ is the edge of regression, since the modulus of the vector $\mathbf{n}$ vanishes at each of the points of the surface.

**5.61.** The equation of the curve is $u = \text{const}$ and the edge of regression is

$$x = 2(a - b)u\cos^2 v,\ y = 2(a - b)u\sin^3 v,$$

$$z = 2u^2[(a - 2b)\cos^2 v + (b - 2a)\sin^2 v].$$

**5.62.** $x = 3t,\ y = -3t^2/b,\ z = -t^3/ab.$

**5.63.** The required developable surface envelops the family of planes

$$Xx + Y\sqrt{a^2 - x^2} + Z\sqrt{\frac{a^4}{b^2} - x^2} = a^2,$$

where $x$ is the parameter of the family.

**5.64.** (1) $r^2(\cos^2 v\, du^2 + dv^2)$;

(2) $(a^2\sin^2 u + b^2\cos^2 u)\cos^2 v\, du^2 +$

$$+ 2(a^2 - b^2)\sin u\cos u\sin v\cos v\, du\, dv$$
$$+ \{(a^2\cos^2 u + b^2\sin^2 u)\sin^2 v + c^2\cos^2 v\}dv^2;$$

(3) $$\frac{1}{4}\left(v + \frac{1}{v}\right)^2 (a^2\sin^2 u + b^2\cos^2 u)du^2 +$$

$$+ \frac{1}{2}(b^2 - a^2)\sin u\cos u\left(v - \frac{1}{v^3}\right)du dv$$

$$+ \left\{\frac{1}{4}\left(1 - \frac{1}{v^2}\right)^2(a^2\cos^2 u + b^2\sin^2 u) + \frac{c^2}{4}\left(1 + \frac{1}{v^2}\right)^2\right\}dv^2;$$

(4) $$\frac{1}{(u + v)^4}[\{a^2(v^2 - 1)^2 + 4b^2v^2 + c^2(v^2 + 1)^2\}du^2 +$$

$$+ 2\{a^2(u^2 - 1)(v^2 - 1) - 4b^2uv + c^2(u^2 + 1)(v^2 + 1)\}dudv +$$

$$+ \{a^2(u^2 - 1)^2 + 4b^2u^2 + c^2(u^2 + 1)^2\}dv^2];$$

(5) $\frac{1}{4}\left(v - \frac{1}{v}\right)^2 (a^2 \sin^2 u + b^2 \cos^2 u)du^2 +$

$+ \frac{1}{2}(b^2 - a^2)\sin u \cos u \left(v - \frac{1}{v^3}\right) du dv +$

$+ \left\{\frac{1}{4}\left(1 + \frac{1}{v^2}\right)^2 (a^2 \cos^2 u + b^2 \sin^2 u) + \frac{c^2}{4}\left(1 - \frac{1}{v^2}\right)^2\right\} dv^2;$

(6) $(p \sin^2 u + q \cos^2 u)v^2 du^2 + 2(q - p)\sin u \cos u\, du dv$

$+ (p \cos^2 u + q \sin^2 u + v^2)dv^2;$

(7) $(p + q + 4v^2)du^2 + 2(p - q + 4uv)dudv + (p + q + 4u^2)dv^2;$

(8) $v^2(a^2 \sin^2 u + b^2 \cos^2 u)du^2 + 2(b^2 - a^2)\sin u \cos u\, du dv$

$+ (a^2 \cos^2 u + b^2 \sin^2 u + c^2)dv^2;$

(9) $(a^2 \sin^2 u + b^2 \cos^2 u)du^2 + dv^2;$

(10) $\left\{\frac{a^2}{4}\left(1 - \frac{1}{u^2}\right)^2 + \frac{b^2}{4}\left(1 + \frac{1}{u^2}\right)^2\right\} du^2 + dv^2.$

**5.65.** (1) $ds^2 + 2\tau e ds d\lambda + d\lambda^2;$

(2) $v^2 ds^2 + 2v\tau\varrho ds dv + \varrho^2 dv^2;$

(3) $\left(\tau + \lambda \frac{de}{ds}\right)^2 ds^2 + 2e\tau ds d\lambda + d\lambda^2;$

(4) $\{(1 - k \cos\varphi)^2 + \varkappa^2\}ds^2 + 2\varkappa ds d\varphi + d\varphi^2;$

(5) $\varphi^2 du^2 + \{\varphi'^2 + \psi'^2\}dv^2;$

(6) $(a + b \cos v)^2 du^2 + b^2 dv^2;$

(7) $(v^2 + k^2)du^2 + dv^2;$

(8) $\{(1 - \lambda k)^2 + \varkappa^2\lambda^2\}ds^2 + d\lambda^2;$

(9) $(1 + \lambda^2\varkappa^2)ds^2 + d\lambda^2.$

**5.66.** (1) The curves $u = \pm 1/2\, av^2$, $v = 1$ intersect at the points $A(u = 0, v = 0)$; $B(u = 1/2\, a, v = 1)$; $C(u = -1/2\, a, v = 1)$; the differentials of curvilinear coordinates on these curves being related by the formulae:

$du = avdv$ for the curve $AB$ with the equation $u = \frac{1}{2} av^2;$

$du = -avdv$ for the curve $AC$ with the equation $u = -\dfrac{1}{2}av^2$;

$dv = 0$ for the curve $BC$ with the equation $v = 1$.

Substituting these values in the first fundamental form, we obtain

$ds^2 = a^2(1/4\ v^4 + v^2 + 1)dv^2$, $\quad ds = (1/2\ v^2 + 1)dv$ for the curve $AB$;

$ds^2 = a^2(1/4\ v^4 + v^2 + 1)dv^2$, $\quad ds = (1/2\ v^2 + 1)dv$ for the curve $AC$;

$ds^2 = du^2$, $\quad ds = du$ for the curve $BC$.

It remains to evaluate the integral between the limits determined by the coordinates of the points $A$, $B$, $C$, viz.,

$$AB = AC = a\int_{v=0}^{v=1}(1/2\ v^2 + 1)dv = 7a/6, \qquad CB = \int_{u=-1/2a}^{u=1/2a} du = a.$$

Thus, the perimeter of the triangle equals $\dfrac{10}{3}a$.

(2) $\cos A = 1$, $\cos B = 2/3$, $\cos C = 2/3$, i.e.,

$A = 0$, $B = C = \cos^{-1} 2/3$.

(3) $S = a^2\left[\dfrac{2}{3} - \dfrac{\sqrt{2}}{3} + \ln(1 + \sqrt{2})\right]$.

**5.67.** $\cos\theta = \dfrac{1 - a^2}{1 + a^2}$.

**5.68.** $v = \pm\left[\sqrt{u^2 + 1} + \dfrac{1}{2}\ln\dfrac{\sqrt{u^2 + 1} - 1}{\sqrt{u^2 + 1} + 1}\right] + \text{const.}$

**5.69.** $v = \tan\theta \ln[u + \sqrt{u^2 - a^2}] + \text{const.}$

**5.70.** (1) $1/4\ (v_0^2 + \sinh^2 v_0)$;
(2) $v_0$, $\sinh v_0$, $\sqrt{2}\sinh v_0$;
(3) $\pi/2$, $\pi/4$, $\pi/4$.

**5.71.** (1) $ds^2 = \{[1 - ak(u)\cos\varphi]^2 + [a\varkappa(u)]^2\}du^2 + 2a^2\varkappa(u)dud\varphi + a^2d\varphi^2$;

(2) $\varphi(u) = -\int\varkappa(u)du$;

(3) $2\pi a|u_2 - u_1|$;

(4) $4\pi^2ab$;

(5) $2\pi^2a\sqrt{r^2 + b^2}$.

**5.72.** Consider the family of the surfaces

$$\mathbf{R}(u, v, t) = \{\tilde{\varrho}(u, t)\cos\frac{v}{t}, \tilde{\varrho}(u, t)\sin v/t, \tilde{z}(u, t)\},$$

where

$$\tilde{\varrho}(u, t) = t\varrho(u), \tilde{z}(u, t) = \int\sqrt{1 - t^2\varrho'(u)^2}du, \quad 1 \geqslant t > 0.$$

**5.73.** $\mathbf{R}(u, \varphi) = \{\sqrt{1 + u^2}\cos\varphi, \sqrt{1 + u^2}\sin\varphi, \ln(u + \sqrt{1 + u^2})\}$,

or

$$\mathbf{R}(z, \varphi) = \{\cosh z \cos\varphi, \cosh z \sin\varphi, z\}.$$

This is a catenoid, a surface of revolution of the catenary curve $x = \cosh z$.

**5.75. Hint:** The first equation determining the correspondence between points is as follows:

$$r^2 = \varrho^2 + a^2.$$

**5.78.** For the sphere $ds^2 = du^2 + R^2 \cos^2(u/R)dv^2$,

for torus $ds^2 = du^2 + \left(a + b\cos\dfrac{u}{b}\right)^2 dv^2$,

for catenoid $ds^2 = du^2 + (a^2 + u^2)dv^2$,

for pseudosphere $ds^2 = du^2 + e^{-2u/a}dv^2$.

**Hint:** $u$ is the natural parameter of the meridian.

**5.79.** $ds^2 = d\tilde{u}^2 + e^{-2\tilde{u}/a}d\tilde{v}^2$.

Putting $u = \tilde{v}$, $v = a\tilde{e}^{u/a}$, we obtain

$$ds^2 = \frac{a^2}{v^2}(du^2 + dv^2).$$

**5.80.** (a) If $a$ and $b$ are the sides containing the right angle of a right-angled spherical triangle, $c$ its hypotenuse, and $R$ the radius of the sphere, then the following relation is held

$$\cos c/\mathbf{R} = \cos a/R \cos b/R.$$

(b) Let $A$, $B$ be the angles opposite to the sides $a$ and $b$. Then

$$S = R^2(A + B - \pi/2).$$

**5.81.** $S = 2\alpha R^2$, where $R$ is the radius of the sphere.

**5.83. Hint:** Take the equation of a conic surface in the form $\mathbf{r} = v\mathbf{e}(u)$, where $|\mathbf{e}(u)| = 1$, and compare its first fundamental form with the quadratic form of the plane in polar coordinates.

**5.86.** (1) $R(du^2 + \cos^2 u\, dv^2)$;

(2) $\dfrac{ac}{\sqrt{a^2 \sin^2 u + c^2 \cos^2 u}}(du^2 + \cos^2 u\, dv^2)$;

(3) $\dfrac{-ac}{\sqrt{a^2 \sinh^2 u + c^2 \cosh^2 u}}(du^2 - \cosh^2 u\, dv^2)$;

(4) $\dfrac{ac}{\sqrt{a^2 \cosh^2 u + c^2 \sinh^2 u}}(du^2 + \sinh^2 u\, dv^2)$;

(5) $\dfrac{2}{\sqrt{1 + 4u^2}}(du^2 + u^2 dv^2)$;

(6) $R dv^2$;

(7) $\dfrac{ku}{\sqrt{1 + k^2}} dv^2$;

(8) $b du^2 + \cos u (a + b \cos u) dv^2$;

(9) $-\dfrac{1}{a}(du^2 - a^2 dv^2)$;

(10) $-a \cot u (du^2 - \sin^2 u\, dv^2)$.

**5.87.** $-2a du dv/\sqrt{u^2 + a^2}$.

**5.88.** $-\dfrac{a}{u^2 + a^2} du^2 + a dv^2$.

**5.89.** $\dfrac{2a^3}{\sqrt{x^4 y^4 + a^6(x^2 + y^2)}}\left[\dfrac{y}{x} dx^2 + dx dy + \dfrac{x}{y} dy^2\right]$.

**5.90.** (1) $\dfrac{(\varrho' x'' - \varrho'' x') du^2 + \varrho x' d\varphi^2}{(x')^2 + (\varrho')^2}$;

(2) $K = \dfrac{x'(\varrho' x'' - \varrho'' x')}{\varrho[(x')^2 + (\varrho')^2]^2}$.

$K > 0$ if the convexity of the meridian is directed from the axis of rotation; $K < 0$ if the convexity of the meridian is directed towards the axis of rotation; $K = 0$ if the meridian has a point of inflexion or if it is orthogonal to the axis of rotation (when $\varrho \neq 0$).

(3) $K = -1$, when $x \neq 0$; $K$ is undetermined, when $x = 0$;

(4) $H = \dfrac{\varrho(\varrho' x'' - \varrho'' x') + x'[(x')^2 + (\varrho')^2]}{2\varrho[(x')^2 + (\varrho')^2]^{3/2}}$;

(5) $\varrho(x) = \dfrac{1}{a}\cosh a(x - x_0)$,

where $x_0$ and $a > 0$ are arbitrary constants (catenoid).

**5.91.** (a) $K = 0$;

(b) $H = -\dfrac{\varkappa}{2kv}$.

**5.92.** $K = -\dfrac{\partial_{uu}\sqrt{G}}{\sqrt{G}}$.

**5.93.** $K = -1$.

**5.94.**

$$K = -\frac{1}{(\partial_x F)^2 + (\partial_y F)^2 + (\partial_z F)^2}\begin{vmatrix} \partial_{xx}F & \partial_{xy}F & \partial_{xz}F & \partial_x F \\ \partial_{yx}F & \partial_{yy}F & \partial_{yz}F & \partial_y F \\ \partial_{zx}F & \partial_{zy}F & \partial_{zz}F & \partial_z F \\ \partial_x F & \partial_y F & \partial_z F & 0 \end{vmatrix}$$

**5.95.** $K = \dfrac{rt - s^2}{(1 + p^2 + q^2)^2}$,

$$H = \frac{(1 + p^2)t + (1 + q^2)r - 2pqs}{2(1 + p^2 + q^2)^{3/2}},$$

where

$$p = \partial_x z, \quad q = \partial_y z, \quad r = \partial_{xx} z, \quad s = \partial_{xy} z, \quad t = \partial_{yy} z.$$

**5.96.** $\pm\dfrac{x^2 + y^2 + a^2}{a}$.

**5.97.** $\dfrac{1}{R_1} = 0, \quad \dfrac{1}{R_2} = -\dfrac{1}{(u + v)\sqrt{2}}$.

**5.98.** $K = -\dfrac{1}{(2u^2 + 1)^2}, \quad H = -\dfrac{2(1 + u^2)}{(2u^2 + 1)^{3/2}}$.

**5.99.** $K = -\dfrac{4}{9(u^2 + v^2 + 1)^4}, \quad H = 0$.

**5.102.** $\mathbf{r} = \varrho(s) + u\beta(s)$,

$$K = -\frac{\varkappa^2}{(1 + u^2\varkappa^2)^2}, \quad H = \frac{k + k\varkappa^2 u^2 - u\dfrac{d\varkappa}{ds}}{(1 + u^2\varkappa^2)^{3/2}}.$$

**5.103.** $K = -\dfrac{\varkappa^2}{[(1 - ku)^2 + u^2\varkappa^2]^2}$,

$$H = \frac{u^2(\dot{k}\varkappa - k\dot{\varkappa}) + u\varkappa}{2[(1 - ku)^2 + u^2\varkappa^2]^{3/2}}.$$

**5.104.** $\mathbf{r}^* = \mathbf{r} + a\mathbf{m}$, $\mathbf{r}_u^* = \mathbf{r}_u + a\mathbf{m}_u$, $\mathbf{r}_v^* = \mathbf{r}_v + a\mathbf{m}_v$,

$$E^* = \mathbf{r}^{*2}_u = \mathbf{r}_u^2 + 2a\mathbf{r}_u\mathbf{m}_u + a\mathbf{m}_u^2 = E - 2aL + a^2(2HL - EK).$$

Similarly,

$$F^* = (1 - a^2K)F + 2a(aH - 1)M,$$
$$G^* = (1 - a^2K)G + 2a(aH - 1)N,$$
$$L^* = aEK + (1 - 2aH)L,$$
$$M^* = M - a(2MH - FK),$$
$$N^* = N - a(2NH - GK).$$

**5.105.** $K^* = \dfrac{K}{1 - 2aH + a^2K}$.

**5.106.** $H^* = \dfrac{H - aK}{1 - 2aH + a^2K}$.

**5.107.** $\mathbf{r}^* = \mathbf{r} + \dfrac{H}{K}\mathbf{m}$.

**5.108.** $K^* = 4H^2 = \text{const}$.

**5.109.** $H^* = -\dfrac{1}{2}\sqrt{K}$.

**5.111.** $\dfrac{du^2}{a^2 + b^2 + u^2} - \dfrac{dv^2}{a^2 + b^2 + v^2} = 0$.

**5.112.** $v = \pm \ln[u + \sqrt{u^2 + a^2}] = \text{const}$.

**5.114.** $u = \text{const}$, $v = \text{const}$.

**5.115.** (1) $K = -\dfrac{k\cos\varphi}{a(1 - ak\cos\varphi)}$;

(2) $H = -(1 - ak\cos\varphi)^2$;

(3) $u = \text{const}$, $\varphi = \text{const}$.

**5.116.** $u = \text{const}$, $\varphi = \varphi_0 - \int \varkappa(u)du$.

**5.117.** The lines of curvature are $u \pm v = \text{const}$,

$$K = -\frac{4}{(3u^2 + 3v^2 + 1/3)^4}, \qquad H = 0.$$

**5.122.** (1) The rectilinear generators are $y/x = \text{const}$;

(2) $\dfrac{1}{x^2} - \dfrac{1}{y^2} = \text{const}.$

**5.123.** (1) The rectilinear generators are $y = \text{const}$;

(2) $x^2 y = \text{const}.$

**5.124.** The equation of the asymptotic lines is

(1) $u = \text{const}$;

(2) $u^5 = v^2(C - \sqrt{u})^2.$

**5.125.** $v = \text{const}$,

$$\frac{\cos^2 u \cos v}{(1 + \cos u)^2} = \text{const}.$$

**5.126.** $\boldsymbol{\beta} = \pm\mathbf{n}$ on the asymptotic line. Therefore, $\varkappa^2 = \left(\dfrac{d\mathbf{n}}{ds}\right)^2$. We select a coordinate system $(u, v)$ in a special way so that the following conditions may be fulfilled at the point $u = u_0$, $v = v_0$ under consideration: (1) the lines $u = \text{const}$ and $v = \text{const}$ have principal directions; (2) $E(u_0, v_0) = G(u_0, v_0) = 1$. Then $F(u_0, v_0) = 0$ due to the orthogonality of the principal directions and $\mathbf{n}_u = -k_1\mathbf{R}_u$, $\mathbf{n}_v = -k_2\mathbf{R}_v$ by the Rodrigues theorem. Therefore,

$$\left(\frac{d\mathbf{n}}{ds}\right)^2 = \left(\mathbf{n}_u \frac{du}{ds} + \mathbf{n}_v \frac{dv}{ds}\right)^2 = \frac{k_1^2 du^2 + k_2^2 dv^2}{ds^2}.$$

Let $\varphi$ be the angle between the line $v = v_0$ and asymptotic direction. Then with respect to this direction $du/ds = \cos\varphi$, $dv/ds = \sin\varphi$ because $E = G = 1$, $F = 0$. On the other hand, we find from the Euler formula that

$$k_1 \cos^2\varphi + k_2 \sin^2\varphi = 0.$$

Finally, we have

$$\varkappa^2 = \left(\frac{d\mathbf{n}}{ds}\right)^2 = (-k_1k_2 \sin^2\varphi - k_1k_2 \cos^2\varphi) = -K.$$

**5.127.** $\varkappa/(1 + u^2\varkappa^2)$.

**5.128.** $\dfrac{\varkappa}{(1 - ku)^2 + u^2\varkappa^2}$.

**5.129.** $\dfrac{u + \sqrt{u^2 + a^2 + b^2}}{v + \sqrt{v^2 + a^2 + b^2}} = \text{const};$

$[u + \sqrt{u^2 + a^2 + b^2}]\,[v + \sqrt{v^2 + a^2 + b^2}] = \text{const.}$

**5.143.** Assume that the rectilinear generators are parallel to the axis $Oz$. Then the equation of the surface can be written in the form

$$\mathbf{r} = f(u)\mathbf{e}_1 + \varphi(u)\mathbf{e}_2 + v\mathbf{e}_3,$$

where $u$ is the natural parameter of the directing line. We will seek the equation of the geodesic in the form

$$v = v(u). \tag{*}$$

Then

$$\mathbf{N} = [\mathbf{r}_u, \mathbf{r}_v] = \varphi'\mathbf{e}_1 - f'\mathbf{e}_2,$$

$$d\mathbf{r} = (f'\mathbf{e}_1 + \varphi'\mathbf{e}_2 + v'\mathbf{e}_3)du,$$

$$d^2\mathbf{r} = (f''\mathbf{e}_1 + \varphi''\mathbf{e}_2 + v''\mathbf{e}_3)du^2,$$

and the equation for determining the geodesic lines is

$$\begin{vmatrix} \varphi' & -f' & 0 \\ f' & \varphi' & v' \\ f'' & \varphi'' & v'' \end{vmatrix} = 0,$$

or

$$(\varphi'^2 + f'^2)v'' - (\varphi'\varphi'' + f'f'')v' = 0.$$

Since $\varphi'^2 + f'^2 = 1$, we have

$$\varphi'\varphi'' + f'f'' = \frac{1}{2}(\varphi'^2 + f'^2)' = 0.$$

Thus, $v'' = 0$ and $v = c_1u + c_2$. The vector equation of the family of geodesics is

$$\mathbf{r} = f(u)\mathbf{e}_1 + \varphi(u)\mathbf{e}_2 + (c_1u + c_2)\mathbf{e}_3,$$

whence

$$\cos\theta = \cos(\widehat{\mathbf{r}_u, OZ}) = \frac{c_1}{\sqrt{1 + c_1^2}}.$$

Therefore, the geodesics found are generalized helices. Besides, the rectilinear generators are also geodesics which were not included in the general solution, since their equation cannot be represented in the form (*).

**5.144.** Let the equation of a developable surface be in the form

$$\mathbf{r} = \varrho(s) + u\tau(s),$$

and let $\theta$ be the angle at which a geodesic intersects a rectilinear generator $s = \text{const}$. If $k$ is the curvature of the curve $\mathbf{R} = \varrho(s)$, then the differential equation of the geodesic is

$$\frac{du}{ds} - uk\cot\theta + 1 = 0.$$

It is a linear differential equation of the first order integrable by quadratures.

**5.145.** The equations of the geodesics are

$$\mathbf{r} = \left\{ \frac{C\cos v}{\sin\dfrac{C_1 \pm v}{\sqrt{2}}}, \frac{C\sin v}{\sin\dfrac{C_1 \pm v}{\sqrt{2}}}, \frac{C}{\sin\dfrac{C_1 \pm v}{\sqrt{2}}} \right\}.$$

**5.146.** $v = C_1 \pm \displaystyle\int \frac{C\,du}{\sqrt{(u^2 + h^2)(u^2 + h^2 - C^2)}}$.

**5.147.** Consider the equation of a cone in the form $\mathbf{r} = u\rho(v)$ and assume that $|\rho| = 1$, $|\rho'| = 1$. Then the equations of the geodesics are of the form

$$\mathbf{r} = \frac{C_1}{\sin(C - v)}\rho(v).$$

**5.150.** Great circumferences of the sphere.

**5.154.** By the Meusnier theorem, the curvature radius $R$ of the curve $\gamma$ at a certain point equals the projection of the geodesic curvature radius $R_g$ $(= 1/k_g)$ onto the osculating plane of the curve $\gamma$, i.e., $R = |R_g \cos\theta|$; the vector $\mathbf{e} = [\mathbf{t}, \mathbf{m}]$ is the unit vector lying in the tangent plane to the surface, and orthogonal to $\gamma$; $\mathbf{n}$ is the unit vector of the principal normal to the curve $\gamma$;

$$|\cos\theta| = |\mathbf{e}\mathbf{n}|, \quad k_g = k|\cos\theta| = k|\mathbf{e}\mathbf{n}| = |\mathbf{e}\dot{\mathbf{t}}| = |\dot{\mathbf{t}}\mathbf{t}\mathbf{m}| = |\mathbf{m}\dot{\mathbf{r}}\ddot{\mathbf{r}}|.$$

**5.155.** $k_g = u/(u^2 + a^2)$.

**5.158.** Considering $v$ as a function of $u$ along a geodesic, we obtain the following differential equation for the geodesics:

$$2(\varphi + \psi)\frac{d^2 v}{du^2} = -\frac{d\varphi}{du}\left(\frac{dv}{du}\right)^3 + \frac{d\psi}{du}\left(\frac{dv}{du}\right)^2 - \frac{d\varphi}{du}\frac{dv}{du} + \frac{d\psi}{du},$$

or

$$(\varphi + \psi)du^2 d(dv^2) = (du^2 + dv^2)(d\psi du^2 - d\varphi dv^2),$$

whence

$$d\left(\frac{\psi du^2 - \varphi dv^2}{du^2 + dv^2}\right) = 0.$$

Integrating this relation, we obtain the required equations.

**5.159.** $\pi + \sigma/R_0^2$.

**5.160.** $\pi - a^2\sigma$.

**5.161.** $\sigma^* = \pi/2$.

**5.162.** $\sigma^* = \int\limits_{v_1}^{v_2} dv \int\limits_{u_1}^{u_2} |B_{uu}(u, v)|\, du$.

**5.164.** $s(\rho) = 2\pi \sinh \rho$; $k_g(\rho) = \coth \rho \to 1$ as $\rho \to +\infty$; $\Pi(\rho) = 2\pi \cosh \rho$; $\Pi(\rho) \to +\infty$ as $\rho \to +\infty$. On the Euclidean plane, $s(\rho) = 2\pi\rho$; $k_g(\rho) = 1/\rho \to 0$ as $\rho \to +\infty$; $\Pi(\rho) = 2\pi$.

**5.165.** First, we establish that the metrics defined on $P_1$ and $P_2$ have the same curvature $K = -1$. Then we introduce semi-geodesic coordinates $(\xi, \eta)$ on the plane $P_2$ so that:

(1) the geodesics are the lines $\eta = \text{const}$ and $\xi = \text{const}$;

(2) $\xi$ is the natural parameter of the line $\eta = 0$;

(3) $\eta$ is the natural parameter of the line $\xi = 0$.

Then $ds^2 = d\xi^2 + B^2(\xi, \eta)d\eta^2$, with $B_{\xi\xi}(\xi, \eta) = B(\xi, \eta)$, $B(0, \eta) = 1$, $B_\xi(0, \eta) = 0$. It follows from these equalities that $B(\xi, \eta) = \cosh \xi$.

**5.166.** $d\mathbf{n}^2 = 2H(\mathbf{n}, d^2\mathbf{R}) - K ds^2$.

**5.167.** Apply the Frenet formula $\dot{\mathbf{n}} = -k\mathbf{t} + \varkappa\mathbf{b}$. For the geodesic line $(\mathbf{m} = \mathbf{n})$, we have

$$\left(\frac{d\mathbf{n}}{ds}\right)^2 = \left(\frac{d\mathbf{m}}{ds}\right)^2 = k^2 + \varkappa^2.$$

On the other hand,

$$\left(\frac{d\mathbf{m}}{ds}\right)^2 = 2H \cdot \mathrm{II} - K \cdot \mathrm{I},$$

where I and II are the first and second fundamental forms of the surface. When $H = 0$, we have $\left(\frac{d\mathbf{m}}{ds}\right)^2 = -K$, and therefore, $k^2 + \varkappa^2 = -K$.

**5.168.** If the equations of a surface of revolution are written in the form

$$x = \varphi(u) \cos v, \quad y = \varphi(u) \sin v, \quad z = u,$$

then the vanishing of the mean curvature implies that

$$1 + \varphi'^2 - \varphi\varphi'' = 0.$$

Putting $p = d\varphi/du$ and considering $\varphi$ as a new variable, we obtain

$$1 + p^2 - \varphi p \frac{dp}{d\varphi} = 0, \quad \frac{d\varphi}{\varphi} = \frac{1}{2} d(\ln (1 + p^2)),$$

whence

$$c^2\varphi^2 = 1 + p^2.$$

With respect to the original variables,

$$d\varphi/\sqrt{c^2\varphi^2 - 1} = du,$$

$$(1 + u^2)f'(u) = a, \quad f'(u) = a/(1 + u^2).$$

Integrating this equation, we get

$$f(u) + b = z + b = a \tan^{-1} u.$$

Therefore,

$$u = \tan (z + b)/a, \quad y/x + \tan (z + b)/a,$$

which is an implicit equation of the right helicoid

$$x = \xi \cos \eta, \quad y = \xi \sin \eta, \quad z = a\eta - b.$$

**5.170.** Let the coordinate lines coincide on the surface $S$ with the lines of curvature. Then

$$\mathbf{r}_u^* = (1 - ak_1)\mathbf{r}_u, \quad \mathbf{r}_v^* = (1 - ak_2)\mathbf{r}_v.$$

Therefore, the coefficients of the first fundamental forms of the surfaces $S$ and $S^*$ are related by the formulae

$$E^* = (1 - ak_1)^2 E, \quad G^* = (1 - ak_2)^2 G, \quad F^* = F = 0.$$

Hence,

$$d\sigma^* = (1 - ak_1)(1 - ak_2) d\sigma,$$

and

$$\lim_{a \to 0} \frac{d\sigma - d\sigma^*}{2a\,d\sigma} = \lim_{a \to 0} \left(\frac{k_1 + k_2}{2} + \frac{1}{2} a k_1 k_2\right) = \frac{k_1 + k_2}{2} = H.$$

**5.171.** Let $S$ be a minimal surface, and $S^*$ a surface parallel to it, the distance between them along the normal being equal to $a$. As it follows from the previous problem, the corresponding elements of the areas of the surfaces $S^*$ and $S$ are related by the formula

$$d\sigma^* = (1 + a^2K)d\sigma,$$

where $K$ is the Gaussian curvature of the surface $S$. Therefore,

$$\iint_D d\sigma^* = \iint_D d\sigma + a^2 \iint_D K d\sigma.$$

Since $K \leqslant 0$ on the minimal surface,

$$\iint_D d\sigma^* \leqslant \iint_D d\sigma.$$

**5.174.** Take the axis of the cylinder to be the axis $Oz$ and place the axis $Ox$ in the sectional plane. Then the equations of the cylinder assume the form

$$x = a \cos t, \quad y = a \sin t, \quad z = u,$$

and the equation of the sectional plane is

$$z = Ay.$$

Cut the cylinder along a generator intersecting the axis $Ox$, and place it on the plane $xOz$. Since after the superposition, the part of the abscissa is played by the length of an arc of the perpendicular section of the cylinder $s = at$, the equation of the required line is

$$z = aA \sin \frac{s}{a}.$$

i.e., a sine curve.

**5.175.** The general equation of the motion of a point across the surface is of the form

$$m \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + R\mathbf{m} - \mu |R| \mathbf{t},$$

where $\mathbf{F}$ is an external force, $R$ the normal reaction of the surface, $\mu$ the coefficient of friction, $\mathbf{t}$ the unit tangent vector to the trajectory, and $\mathbf{m}$ the unit vector of the normal to the surface. Since

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2s}{dt^2}\mathbf{t} + \left(\frac{ds}{dt}\right)^2 \frac{d\mathbf{t}}{ds},$$

when $\mathbf{F} = 0$, the equation assumes the form

$$m\left(\frac{d^2 s}{dt^2}\mathbf{t} + \left(\frac{ds}{dt}\right)^2 \frac{d\mathbf{t}}{ds}\right) = R\mathbf{m} - \mu |R| \mathbf{t}.$$

Multiplying it scalarly by $[\mathbf{t}, \mathbf{m}]$, we obtain

$$\mathbf{tm}\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{r}}{ds}\mathbf{m}\frac{d^2\mathbf{r}}{ds^2} = 0,$$

i.e., the point moves along a geodesic.

**5.177.** Take a semi-geodesic coordinate system on the surface. Then

$$ds^2 = du^2 + G(u, v)\,dv^2.$$

On the line $u = 0$, we have $\sqrt{G}\,|_{u=0} = 1$. Besides, we obtain from the equation of geodesic lines that

$$\left.\frac{\partial\sqrt{G}}{\partial u}\right|_{u=0} = 0.$$

In the semi-geodesic coordinate system,

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}.$$

If $K = 0$, then

$$\frac{\partial^2\sqrt{G}}{\partial u^2} = 0,$$

and the solution of this equation satisfying the initial conditions indicated above is $\sqrt{G} = 1$. Therefore, for all surfaces of zero Gaussian curvature the first fundamental form can be reduced to the form

$$ds^2 = du^2 + dv^2;$$

hence, all of them are locally isometric to each other.

If $K = \frac{1}{a^2}$ ($a$ = const), then

$$\sqrt{G} = \cos\frac{u}{a}, \quad ds^2 = du^2 + \cos^2\frac{u}{a}\,dv^2.$$

If $K = -\frac{1}{a^2}$ ($a$ = const), then

$$ds^2 = du^2 + \cosh^2\frac{u}{a}\,dv^2.$$

**5.180.** The surface $S$ can be obtained by bending the hemisphere so that the two halves of its boundary circumference may overlap each other, and then glue the surface along these semi-circumferences, from which it follows that the geodesics on the surface $S$ not passing through its singular points (ends of meridians) become closed after traversing around the surface twice (i.e., after increasing $\varphi$ by $4\pi$).

**5.182.** It follows from the formula

$$\iint_D K d\sigma + \int_L k_g ds = 2\pi$$

when $k_g = 0$ that

$$\iint_D K d\sigma = 2\pi.$$

And this equality cannot be valid if $K \leqslant 0$ at all points of the surface.

# 6
# Manifolds

**6.1.** As the atlas of charts, the sets $U_k^{\pm}$ determined by the inequality $U_k^+ = \{x_k > 0\}$, $U_k^- = \{x_k < 0\}$ should be taken. As the coordinate functions, all Cartesian coordinates except $x_k$ should be taken in the chart $U_k^{\pm}$.

**6.2.** Notice that $T^2$ is homeomorphic to the Cartesian product $S^1 \times S^1$, and reduce the problem to the previous when $n = 1$.

**6.3.** Any neighbourhood $U$ of the origin 0 can be split into at least 4 connected components, while discarding the point 0, which is impossible on a manifold.

**6.4.** The sphere $S^n$ is a compact space.

**6.5.** (a) Yes. (b) No.

**6.6.** The space $\mathbf{R}P^n$ is the set of collections $(x_0 : x_1 : \ldots : x_n)$, where $x_i \in \mathbf{R}$, $\Sigma x_i^2 \neq 0$, with the equivalence relation $(x_0 : x_1 : \ldots : x_n) \sim (\lambda x_0 : \lambda x_1 : \ldots : \lambda x_n)$. Introduce a real analytic structure on $\mathbf{R}P^n$. To this end, cover $\mathbf{R}P^n$ by a set of $n + 1$ charts. Consider the collections $(x_0 : x_1 : \ldots : x_n)$ such that $x_i \neq 0$. The set of such collections can be naturally considered identical with $\mathbf{R}^n$, viz.,

$$(x_0 : x_1 : \ldots : x_n) \to \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

It is easy to see that the definition of this correspondence is correct. It remains to consider the functions of transition from the $i$-th chart to the $j$-th. Let $x_k^{(i)}$ be the $k$-th coordinate of the collection $(\lambda_0 : \lambda_1 : \ldots : \lambda_n)$ in the $i$-th chart, and $x_l^{(j)}$ the $l$-th coordinate in the $j$-th, respectively (let, for simplicity, $i < j$). Then

$$x_1^{(i)} = \frac{x_1^{(j)}}{x_i^{(j)}}, \ldots, x_i^{(i)} = \frac{x_{i+1}^{(j)}}{x_i^{(j)}}, \ldots, x_j^{(i)} = \frac{1}{x_i^{(j)}}, \ldots$$

Thus, the transition functions are not only smooth, but also real and analytic.

**6.7.** See Problem 6.6.

**6.8.** The atlas consists of one chart with coordinate functions $(x_1, \ldots, x_n)$.

**6.9.** Represent the elements of the group $SO(2)$ as rotations of the plane through a certain angle about the origin. The group $O(2)$ is homeomorphic to the union of two replicas of $S^1$.

**6.10.** Represent the elements of the group $SO(3)$ as rotations of the space about a certain axis through a certain angle.

**6.11.** The groups $GL(n, \mathbf{R})$, $GL(n, \mathbf{C})$ are open sets of the space of all matrices.

**6.12.** A cylinder.

**6.17.** Use the rule for differentiating a function of a function.

**6.19.** 1.

**6.20.** Apply the implicit function theorem.

**6.22.**

$$U_i = Rx_i/\sqrt{1 + \Sigma x_k^2}.$$

**6.23.** Use Problem 6.22.

**6.24.** $y = x^3$.

**6.25.** Use the function $y = e^{-1/x^2}$.

**6.26.** Use the function from the previous problem.

**6.27.** Let $\{U_2\}$ be an atlas of charts sufficiently fine for the following conditions to be fulfilled: if $A \cap U_\alpha \neq \varnothing$, then $U_\alpha \subset U$. Let $\{\varphi_\alpha\}$ be a partition of unity, subordinate to the covering $\{U_\alpha\}$.

**6.28.** Use the average operation

$$g_\varepsilon(y) = \int_{|x-y|<\varepsilon} f(x)dx.$$

Put $f = \sum_\alpha \varphi_\alpha$, where the summation is over all indices $\alpha$ for which $A \cap U_\alpha \neq \varnothing$.

**6.30.** The composite of smooth mappings is a smooth mapping.

**6.31.** Write formulae explicitly expressing the coordinates of the normal in terms of local coordinates on the torus.

**6.32.** Homogeneous coordinates on the straight line depend smoothly on local coordinates on the sphere and local coordinates on $\mathbf{R}P^2$ are expressed in terms of homogeneous coordinates.

**6.33.** Coordinate functions are a special case of a smooth function in a manifold.

**6.34, 6.35, 6.36.** Use the implicit function theorem.

**6.37.** Use Problem 6.36 and partition of unity.

**6.38.** Using local coordinates, calculate the rank of the Jacobian matrix of the mapping.

**6.39.** Use the properties of the rank of the product of two matrices.

**6.40, 6.41, 6.42.** Consider the group $GL(n, \mathbf{R})$ of all square matrices of order $n$ with non-zero determinants. Each matrix from $GL(n, \mathbf{R})$ can be associated with a vector from the space $\mathbf{R}^{n^2}$, while the mapping $\det: \mathbf{R}^{n^2} \to \mathbf{R}$ is a continuous function. Therefore, the group $GL(n, \mathbf{R})$ is an open subset in $\mathbf{R}^{n^2}$. Any open subset is a smooth manifold.

**Definition.** A linear group is said to be algebraic if it can be singled out of the group $GL(n, \mathbf{R})$ by some set of algebraic, i.e., polynomial, relations among matrix elements.

**Theorem.** Any algebraic linear group $G$ is a Lie group.

**Proof.** Consider the space $M(n, \mathbf{R})$ of all square matrices of order $n$ with elements in $\mathbf{R}$. Let $J$ be the ideal, formed by all polynomials vanishing on $G$, of the ring $S$ of polynomials on $M(n, \mathbf{R})$. By the Hilbert theorem [5], there exist polynomials $f_1, \ldots, f_\sigma \in J$ (forming the ideal $J$) such that any polynomial $f \in J$ can be represented in the form $f = \Sigma f_\alpha g_\alpha$, where $g_\alpha \in S$ ($\alpha = 1, \ldots, \sigma$). Let $\rho$ be the rank of the Jacobian matrix $(\partial f_\alpha / \partial x_{ij})$ (having $\sigma$ rows and $n^2$ columns) when $(x_{ij}) = E$.

**Lemma.** The rank of the matrix $(\partial f_\alpha / \partial x_{ij})$ equals $\rho$ at all points of the group.

Let $A \in G$. For any polynomial $f \in S$, we put $(Af)(X) = f(A^{-1}X)$, $X \in M(n, \mathbf{R})$. The transformation $f \to Af$ is an automorphism of the ring $S$, transforming the ideal $J$ into itself. Therefore, the polynomials $Af_1, \ldots, Af_\sigma$ are generators of the ideal $J$ as well as of $f_1, \ldots, f_\sigma$. We have

$$Af_\alpha = \sum_\beta f_\beta g_{\alpha\beta}, \quad f_\alpha = \sum_\beta (Af_\beta) h_{\alpha\beta},$$

where $g_{\alpha\beta}, h_{\alpha\beta} \in S$. At the points of the group $G$, we have

$$\frac{\partial (Af_\alpha)}{\partial x_{ij}} = \sum_\beta \frac{\partial f_\beta}{\partial x_{ij}} g_{\alpha\beta}, \quad \frac{\partial f_\alpha}{\partial x_{ij}} = \sum_\beta \frac{\partial (Af_\beta)}{\partial x_{ij}} h_{\alpha\beta},$$

from which it follows that the ranks of the matrices $(\partial(Af_\alpha)/\partial x_{ij})$ and $(\partial f_\alpha/\partial x_{ij})$ are equal at all the points of the group $G$. On the other hand, the rank of the matrix $(\partial(Af_\alpha)/\partial x_{ij})$ at the point $A$ equals the rank of the matrix $(\partial f_\alpha/\partial x_{ij})$ at the point $E$, i.e., equals $\rho$. Therefore, the rank of the matrix $(\partial f_\alpha/\partial x_{ij})$ at the point $A$ equals $\rho$, and thus the lemma has been proved.

Now, let $\Delta$ be a minor of order $\rho$, not vanishing at $E$, of the functional matrix $(\partial f_\alpha/\partial x_{ij})$. Assume, for definiteness, that it is contained in the first $\rho$ rows. It has been proved that all the minors bordering it are identically equal to zero on the group $G$, i.e., belong to the ideal $J$. Similarly to the proof of the classical matrix rank theorem, we obtain

$$\Delta \frac{\partial f_\alpha}{\partial x_{ij}} \equiv \sum_{\beta=1}^{\rho} \frac{\partial f_\beta}{\partial x_{ij}} g_{\alpha\beta} (\mathrm{mod} J), \tag{1}$$

where $g_{\alpha\beta} \in S$, $\alpha = 1, \ldots, \sigma$. Consider the set $\tilde{G}$ of matrices satisfying the equations $f_1 = \ldots = f_\rho = 0$. It is obvious that $G \subset \tilde{G}$. By the implicit function theorem, there exists a neighbourhood $U$ of the unit matrix in $M(n, \mathbf{R})$ such that the intersection $\tilde{G} \cap U$ can be given parametrically, the number of parameters being equal to $d = n^2 - \rho$. We can see to it that $\Delta$ does not vanish on $U$, and the range of parameters is connected. Let $(x_{ij}(t))$ be a curve in $\tilde{G} \cap U$, with $(x_{ij}(0)) = E$. When $\alpha = 1, \ldots, \rho$, $df_\alpha/dt = 0$ along this curve. We obtain from (1)

$$\Delta \frac{df_\alpha}{dt} = \sum_\beta h_{\alpha\beta} f_\beta,$$

where $\alpha$ is any index. The unique solution of this system, with the initial conditions $f_\alpha(0) = 0$, is the zero solution. Therefore, $f_\alpha = 0$ for all $\alpha$, i.e., $\tilde{G} \cap U = G \cap U$, which shows that $G$ is a Lie group.

The group $SO(n)$ in question is a Lie group (consequently, a smooth manifold), since it is an algebraic group. Its relations are

$$\sum_j a_{ij} a_{kj} = \delta_{ik}, \det(a_{ij}) = 1.$$

The dimension of the group $SO(n)$ equals $a(n-1)/2$. Consider the case of the groups $U(n)$ and $SO(n)$. Every linear transformation over $\mathbf{C}$ with the matrix $A$ can be treated as a linear transformation over $\mathbf{R}$. This transformation will have the following matrix with respect to the basis $e_1, \ldots, e_n, ie_1, \ldots, ie_n$ (where $e_1, \ldots, e_n$ are basis vectors in $\mathbf{C}^n$):

$$\begin{pmatrix} \mathrm{Re}\, A & \mathrm{Im}\, A \\ -\mathrm{Im}\, A & \mathrm{Re}\, A \end{pmatrix}.$$

The group $GL(n, \mathbf{C})$ is a subgroup of the group $GL(2n, \mathbf{R})$. Therefore, the theorem proved can be applied to the groups of linear transformations of $\mathbf{C}^n$. Note that $U(n)$ is the group of linear transformations of $\mathbf{C}^n$, preserving the hermitian metric. The group $SU(n) \subset U(n)$, $\det A = 1$ if $A \in SU(n)$. The condition for unitarity is written with respect to the orthonormal basis thus:

$$\sum_k a_{ik}\bar{a}_{jk} = \delta_{ij}.$$

In the passage to real coordinates, we obtain symmetric relations for the matrix elements. It follows from what has been considered previously that the group $U(n)$ is an algebraic group. The group $SU(n)$ is also algebraic and singled out of the group $U(n)$ by the additional equation $\det A = 1$.

**6.43.** Calculate the rank of the Jacobian matrix of the mapping.

**6.45.** All the roots are of multiplicity one.

**6.47.** Since $M^n$ is compact, it can be covered with a finite number of charts each of which is homeomorphic to the open ball $D^n$. Let there be a covering of charts $V_\alpha \approx D^n$ (where $x_\alpha^1, \ldots, x_\alpha^n$ are local coordinates). Thereby, each point $x \in M^n$ is put into correspondence with the collection of its coordinates in $D^n$. This is a smooth mapping $f$. Extend $f$ to the whole manifold $M^n$ by constructing a new covering with the charts $W_\alpha$, $V_\alpha \subset W_\alpha$, and also constructing the functions $f_\alpha(x) = 0$ when $x \notin W_\alpha$, $f_\alpha(x) = 1$ on $V_\alpha$, $0 \leqslant f_\alpha \leqslant 1$. Suppose there were $k$ charts in the covering. Then, to each point of the manifold, we assign the $nk$-dimensional vector

$$x \to \{f_1^{(1)}(x), \ldots, f_n^{(i)}(x), \ldots, f_n^{(k)}(x)\}, \quad f_\alpha^{(i)}(x) = f_\alpha(x)x_\alpha^i.$$

Under such a mapping, one-to-one correspondence is not achieved at those points $x, y$ which belong to $W_\alpha \setminus V_\alpha$, since here we smoothen the functions in an arbitrary manner. To eliminate this defect, construct another covering $U_\alpha \subset W_\alpha$. Perform the same constructions for the coverings $W_\alpha$ and $U_\alpha$ as for $V_\alpha$ and $W_\alpha$ to obtain the collection of functions $g_\alpha^{(i)}$. Then the correspondence

$$x \to \{g_1^{(1)}(x), \ldots, g_n^{(k)}(x)\} \in \mathbf{R}^{2nk}$$

is one-to-one.

**6.49.** Verify that the neighbourhood of $S^n \subset \mathbf{R}^{n+1}$ is diffeomorphic to $S^n \times \mathbf{R}^1$, and apply the method of induction.

**6.50.** Apply the Sard lemma.

**6.51.** Use the two-dimensional surface classification.

**6.52.** Prove, at first, that the manifold $M^n$ can be immersed into $\mathbf{R}^{2n}$. It is known that any compact, smooth manifold $M^n$ can be embedded in $\mathbf{R}^N$, where $N$ is a sufficiently large number. We assume then that

$M^n \hookrightarrow \mathbf{R}^N$. We shall decrease the dimension of $N$ by projecting $M^n$ along a certain vector $\xi$ onto its orthogonal complement. Under these projections $p_\xi$, there may appear points at which the smoothness of $p_\xi$ is disturbed, i.e., such points at which the tangent space $T_x(M^n)$ contains a vector $\eta$ parallel to the vector $\xi$.

Thus, for the projection $p_\xi$ to be a smooth mapping at every point, the vector $\xi$ should be selected so as not to belong to the tangent space (at any point $x \in M^n$). The tangent space $T_x(M^n)$ is diffeomorphic to $\mathbf{R}^n$, but since we are interested in the directions of the vectors only, the manifold of directions is $\mathbf{R}P^{n-1}$. The dimension of all the tangent directions is not more than $n + (n - 1) = 2n - 1$. If $N - 1 > 2n - 1$, then a direction independent of all the directions of the tangent spaces to the manifold $M$ (i.e., not contained in them) can be chosen in the space $\mathbf{R}P^{N-1}$. Select one of such directions and project along it. This procedure is repeated while the inequality $2n - 1 < N - 1$ is fulfilled. Finally, when $2n - 1 = N - 1$, this reasoning will not hold for the first time, since the existence of a convenient direction no longer follows from the dimension inequalities. Thereby, it has been proved that $N$ can be lowered to $2n$ (at any rate), i.e., the manifold $M^n$ is immersed into the space $\mathbf{R}^{2n}$. Let us now prove the existence of the embedding $M^n \hookrightarrow \mathbf{R}^{2n+1}$. For the immersion, it is sufficient to ban all the directions (as projected) in the tangent spaces to the manifold. Now, we have to ban possible self-intersections which can arise under a projection, i.e., ban all the chords of the form $(x, y)$ parallel to the projecting direction $P\xi$. The space of chords is the space of pairs $(x, y)$, where $x \in M^n, y \in M^n$. Consider the mapping $f: (x, y) \to \mathbf{R}P^{N-1}$. The mapping $f$ is not smooth, since singular points appear on the diagonal $\Delta$ in $M^n \times M^n$. Restrict the mapping $f$ to $M^n \times M^n \setminus \Delta$, where $\Delta = \{(x, x) : x \in M^n\}$, $\dim(M^n \times M^n \setminus \Delta) = 2n$. Now $f$ is a smooth mapping. Consider the image $f(M^n \times M^n \setminus \Delta) \subset \mathbf{R}P^{N-1}$. Close this image in $\mathbf{R}P^{N-1}$; $\dim \operatorname{Im} f$ is unaltered under this operation (note that under the closure operation, the directions from the tangent spaces are necessarily taken into account). Furthermore, if $N - 1 > 2n$, then we project, similarly to the investigation of immersion, along any direction $\xi$ not belonging to $\operatorname{Im} f$. Thereby, we decrease the dimension of the space by unity and continue the process while $N - 1 > 2n$. When $N - 1 = 2n$, a "free" direction may not exist.

**6.53.** The zero-dimensional compact manifold consists of a finite number of points.

**6.54.** Since the point $y$ is a zero-dimensional manifold, its tangent space vanishes.

**6.55.** In this case, the condition for transversality is equivalent to the subspaces $TM_1$ and $TM_2$ generating the whole tangent space of the ambient manifold.

**6.56.** Use the implicit function theorem.

**6.57.** Transversally in all the cases.

**6.58.** $a \neq 1$.

**6.59.** Construct linearly independent vector fields which are normal to the fibres under a certain metric, and then construct the required homeomorphisms by means of motions along the integral curves.

**6.60-6.62.** Calculate the rank of the Jacobian matrices of the mappings in local coordinates.

**6.69.** (a), (b), (d), (e): orientable;

(c) orientable when $n$ is odd; non-orientable when even.

**6.70.** Represent the Klein bottle as a square whose opposite sides are identified, and transfer the basis consisting of the tangent vectors along the midline.

**6.71.** A manifold is said to be orientable if there exists a collection of charts such that the Jacobians of all the transition functions are positive (i.e., there exists at least one such collection of charts for the manifold). Let $\varphi_{ij}$ be a transition function of variables $z^1, \ldots, z^n$, and $\partial\varphi_{ij}/\partial\bar{z}^\alpha \equiv 0$. Let $A$ be the Jacobian of the transition function $\varphi_{ij}$, and $A = (a_{ij})$. The mapping $A$ can be considered as a linear operator $\mathbf{C}^n \to \mathbf{C}^n$. The realification of the mapping $f: GL(n, \mathbf{C}) \to GL(2n, \mathbf{R})$ will assume the form $f(A) = \mathbf{R}_A = \begin{pmatrix} B & -D \\ D & B \end{pmatrix}$, where $A = B + iD$. We take the basis $e_1, \ldots, e_n, ie_1, \ldots, ie_n$ in $\mathbf{R}^{2n}$. Let us prove the formula $\det \mathbf{R}_A = |\det A|^2$ by induction on $n$. When $n = 1$, for $A = a + bi$, we obtain

$$|\det A|^2 = a^2 + b^2, \mathbf{R}_A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \det \mathbf{R}_A = a^2 + b^2.$$

Let the statement be proved for $k \leqslant n$. We now prove it for $k = n + 1$. Reduce the operator $A$ to the Jordan normal form (the determinant remains unaltered):

$$A = \begin{pmatrix} a_1 + ib_1 & & \varepsilon_1 & & 0 \\ & \ddots & & \ddots & \vdots \\ 0 & & a_{n+1} + ib_{n+1} & & \varepsilon_n \end{pmatrix}, \quad \text{where} \quad \varepsilon_i = \{0, 1\},$$

$$R_A = \left\{ \begin{matrix} a_1 & & \varepsilon_1 & 0 & -b_1 & 0 \\ 0 & \ddots a_{n+1} & & \ddots \varepsilon_n & 0 & \ddots -b_{n+1} \\ b_1 & 0 & a_1 & & \varepsilon_1 & 0 \\ 0 & \ddots b_{n+1} & 0 & \ddots a_{n+1} & & \ddots \varepsilon_n \end{matrix} \right\} |\det A|^2$$

$$= (a_{n+1}^2 + b_{n+1}^2) D_n,$$

where

$$D_n = \left| \det \begin{pmatrix} a^2 + ib_1 & & & \varepsilon_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & a_n + ib_n & & & \varepsilon_{n-1} \end{pmatrix} \right|^2.$$

Let us calculate $\det^R A$ by expanding it along the last row:

$$\det^R A = (-1)^{3n+3} b_{n+1} \det \begin{pmatrix} a_1\varepsilon_1 & & -b_1 & \\ & \ddots & & \ddots \\ b_1 & & a_1\varepsilon_1 & \\ & \ddots & & \ddots \end{pmatrix}$$

$$+ a_{n+1} \det \begin{pmatrix} a_1\varepsilon_1 & & -b_1 & \\ & \ddots & & \ddots \\ b_1 & & a_1\varepsilon_1 & \\ & \ddots & & \ddots \end{pmatrix} = (-1)^{3n+3} b_{n+1}(-b_{n+1})(-1)^{3n+3} D_n$$

$$+ a_{n+1}^2 D_n = (b_{n+1}^2 + a_{n+1}^2) D_n.$$

Thus, we have introduced a collection of charts such that in changing the coordinates (the change being smooth)

$$(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n) \to (x^1, \ldots, x^n, y^1, \ldots, y^n)$$

(realification) the Jacobians of all the transition functions are positive.

**6.74.** We obtain from the existing classification of two-dimensional, closed, differentiable manifolds (which are orientable), that all of them are spheres with $g$ handles, i.e., surfaces of genus $g$. Each of such manifolds is the Riemann surface of a certain polynomial $\sqrt{P_n(z)}$ without multiple roots, where $g = \left[n - \dfrac{1}{2}\right]$. The function $w = \sqrt{P_n(z)}$ is complex and analytic; therefore, by taking $z$ and $w$ as coordinate patches, we obtain an atlas with a complex and analytic transition function.

**6.75.** Obviously, a complex structure can be introduced only on even-dimensional manifolds. Let $\Gamma$ be a group operating on $\mathbf{C}^n \setminus (0)$ and generated by the transformation $z \to 2z$. Consider the factor space relative to $z \to 2z$. It carries a complex structure induced by the structure of the space $\mathbf{C}^n \setminus (0)$ and is homeomorphic to $S^{2n-1} \times S^1$. Therefore, $S^{2n-1} \times S^1$ also has a complex structure. There exists a fibration $S^{2n-1} \times S^{2n-1} \to \mathbf{C}P^{n-1} \times \mathbf{C}P^{n-1}$ with the fibre $F = S^1 \times S^1 = T^2$. The fibre and base space have a complex structure (proved). A complex structure on $\mathbf{C}P^n$ can be defined by means of a form which is the restriction of the hermitian form on $\mathbf{C}^n$ to the sphere $S^{2n-1}$:

$$dS^2 = \Sigma dz^k d\bar{z}^k - (\Sigma z^k d\bar{z}^k)(\Sigma \bar{z}^k dz^k).$$

This form is obtained from the form on $\mathbf{C}P^{n-1}$, since the former is invariant with respect to the action of $S^1$, where $S^{2n-1} \xrightarrow{S^1} \mathbf{C}P^{n-1}$. We define the action of $S^1$ thus:

$$(z^0, \ldots, z^{n-1})e^{i\alpha} \to (e^{i\alpha}z^0, \ldots),$$

$$z^k \to \omega^k = e^{i\alpha} z^k, d\omega = e^{i\alpha}(dz^k + i\alpha z^k d\alpha), d\bar{\omega}^k = e^{-i\alpha}(d\bar{z}^k - i\alpha \bar{z}^k d\alpha),$$

$$\Sigma d\omega^k d\bar{\omega}^k = \Sigma dz^k d\bar{z}^k + i\alpha(\Sigma(z^k d\bar{z}^k - \bar{z}^k dz^k))d\alpha + \alpha^2 d\alpha,$$

$$\Sigma \omega^k d\bar{\omega}^k = \Sigma z^k d\bar{z}^k - i\alpha d\alpha, \quad \Sigma \bar{\omega}^k d\omega^k = \Sigma \bar{z}^k dz^k + i\alpha d\alpha.$$

Therefore,

$$\Sigma d\omega^k d\bar{\omega}^k - (\Sigma \omega^k d\bar{\omega}^k)(\Sigma \bar{\omega}^k d\omega^k) = \Sigma dz^k d\bar{z}^k - (\Sigma z^k d\bar{z}^k)(\Sigma \bar{z}^k dz^k).$$

Let $z^k$ be the coordinates of the first factor $S^{2n-1}$, and $z'^j$ of the second factor $S^{2n-1}$. Let

$$V_{kj} = \{(z, z') \in S^{2n-1} \times S^{2n-1} : z^k z'^j \neq 0\}.$$

The sets $V_{kj}$ form an open covering of the space $S^{2n-1} \times S^{2n-1}$. Introduce complex coordinates

$$_k\omega^r = \frac{z^r}{z^k}, \quad _j\omega'^s = \frac{z'^s}{z'^j},$$

$$t_{kj} = \frac{1}{2\pi i}(\ln z^k + \gamma \ln z'^j)$$

on each set $V_{kj}$, where $\gamma$ is a vector from $\mathbf{C}^n$, and $t_{kj}$ are determined modulo 1. Therefore, $t_{kj}$ is a point of the torus $T(1, \gamma)$ obtained by gluing together the opposite sides of the parallelogram constructed on the vectors 1 and $\gamma$. Thus, we have $2n + 1$ coordinates in $V_{kj}$ which determine the mapping $f\colon V_{kj} \to \mathbf{C}^{2n} \times T(1, \gamma)$. The mapping $f$ is a homeomorphism. The quantities $_k\omega^r$, $_j\omega'^s$, $t_{kj}$ uniquely determine the coordinates $z^k$ and $z'^j$. In fact,

$$\sum_{r \neq k} {}_k\omega^r \, {}_k\bar{\omega}^r = \frac{1}{z^k \bar{z}^k} \sum_{k \neq r} z^r \bar{z}^r = \frac{1}{z^k \bar{z}^k} \sum_r z^r \bar{z}^r - 1 = \frac{1}{|z^k|^2} - 1.$$

The quantities $|z'^j|$ are determined similarly (uniquely). Besides,

$$\ln z^k = \ln|z^k| + i \arg z^k,$$

whence

$$2\pi t_{kj} = -i(\ln|z^k| + \gamma \ln|z'^j|) + \arg z^k + \gamma \arg z'^j.$$

If $|z^k|$ and $|z'^j|$ are known, then $\arg z^k$ and $\arg z'^j$ can be found ($\gamma$ has been chosen so that $\operatorname{Im} \gamma \neq 0$). Consequently, $z^k$ and $z'^j$ are also determined uniquely. The transition function in $V_{kj} \cap V_{uv}$ is complex and analytic because it is determined by the formulae:

$$_u\omega^r = \frac{_k\omega^r}{_k\omega^u}, \quad _v\omega'^s = \frac{_j\omega'^s}{_j\omega'^v},$$

$$t_{uv} = t_{kj} + \frac{1}{2\pi i}(\ln {}_k\omega^u + \gamma \ln {}_j\omega'^v).$$

Thereby, a complex structure has been introduced on $S^{n-1} \times S^{2n-1}$. This construction introduces a complex structure on any product $S^{2p-1} \times S^{2q-1}$, where $p, q > 0$ and can be different.

**6.76.** If two piecewise smooth paths $c_0, c_1 : [\alpha, \beta] \to M^n$ are freely homotopic, then, in traversing them, the orientations are either both altered or both unaltered. There is the shortest periodic geodesic in each free homotopy class of paths on $M^n$. We can now prove the required statement. It is clear that it suffices to give the proof for a connected manifold $M^n$. Let $p \in M^n$. It suffices to prove that in traversing any smooth loop with the origin and end at the point $p$, the orientation in the space $T_pM^n$ is unaltered.

Assume the contrary, i.e., that there is a smooth, closed path with the origin and end at the point $p$, and, in traversing it, the orientation in $T_pM^n$ is altered. Then there exists a non-trivial periodic geodesic $c : [0, 1] \to M^n$ which is freely homotopic to this path and shortest in its free homotopy class. Let $c(0) = c(1) = q \in M^n$. Then the parallel displacement along the geodesic $c$ induces an automorphism $\tau$ reversing the orientation on the subspace $M_q^\perp \subset T_qM$ orthogonal to the vector $\dot{c}(0)$. Since $c$ is a geodesic, $\tau$ is an orthogonal automorphism, and since $\dim M_q^\perp = n - 1 = 2k$, there exist two-dimensional Euclidean spaces $E_1, \ldots, E_k$ invariant with respect to $\tau$ such that $M_q^\perp = E_1 \oplus \ldots \oplus E_k$. It is clear that

$$\det \tau = \prod_i \det \tau | E_i = -1,$$

and then the relation $\det \tau / E_i = -1$ is fulfilled for a certain $i$ so that $\tau$ reverses the orientation on $E_i$. But then $\tau$ has a non-zero fixed vector $u$, i.e., $\tau u = u \neq 0$. Now let $Y$ be a parallel vector field along $c$, for which $Y(0) = Y(1) = u$. Then there exists an open interval $Y \in \mathbf{R}$ containing zero such that $\varepsilon Y(t)$ lies in the domain of the exponential mapping exp on $M^n$ for all $\varepsilon \in I, t \in [0, 1]$. We define $V : [0, 1] \times I \to M^n$ by the equality $V(t, \varepsilon) = \exp(\varepsilon Y(t))$. Let $L(\varepsilon)$ be the length of the curve $V(t, \varepsilon)$. Then, since $c$ is a geodesic, $L'(0) = 0$. It follows from $Y$ being parallel that $Y' = 0$ and $<Y, c> = 0$. Since $\varepsilon \to V(t, \varepsilon)$ is a geodesic for any $t \in [0, 1], L''(0) = -\int_0^1 <R(Y, \dot{c})\dot{c}, Y> dt$, where $R$ is the Riemann tensor of the manifold $M^n$. If follows from the curvature along the geodesic $c$ being positive that $L''(0) < 0$, and, therefore, $L$ has a relative maximum on $c$, i.e., $c$ is not shortest, which is a contradiction.

**6.78.** If $A$ is a complex Jacobian matrix, then the real Jacobian matrix is

$$\begin{pmatrix} \operatorname{Re} A & \operatorname{Im} A \\ -\operatorname{Im} A & \operatorname{Re} A \end{pmatrix}.$$

**6.79.** Use Problem 6.78.

**6.80.** Changing the coordinates $z_2 = 1/z_1$, we obtain that the atlas consists of two charts.

# 7
# Transformation Groups

**7.1.** Use the theorem on the existence and uniqueness of a solution to a system of ordinary differential equations of the first order.

**7.2.** Construct a vector field such that one of the trajectories may join the points $x_0$ and $x_1$ and that it may be trivial outside a certain neighbourhood of this trajectory.

**7.4.** The ratio of the coordinates of the field $\xi$ should be a rational number.

**7.6.** Select an atlas of charts so fine that each orbit may intersect an arbitrary chart at no more than one point.

**7.7.** The action of the group $\mathbf{Z}_2$ on the sphere $S^n$ should be given by the formula $x \to -x$.

**7.8.** The action of $S^1$ on $S^{2n+1} \subset \mathbf{C}$ is given by the formula

$$(\lambda, x) \to \lambda x, \quad \lambda \in S^1 \subset \mathbf{C}^1.$$

**7.9.** Use the differential of the mapping determining the action of an element of the group $G$.

**7.18.** Fix an orthonormal coordinate 3-frame $\{e_1, \bar e_2, \bar e_3\}$ in $\mathbf{R}^3$. An arbitrary state of the described system is uniquely determined by a point $x \in S^2$ and the velocity vector $\bar v(x) \in T_x(S^2)$, where $|v(x)| = C = \text{const} = 0$. It is obvious that the mapping $\bar x \to x$, $\bar v(x) \to \bar v(x)/c$ is a homeomorphism, $x$ is the unit vector in $\mathbf{R}^3$ emanating from the point 0, and $v(x)$ is a unit vector in $\mathbf{R}^3$. Shift the origin of $v(x)$ to the point 0. This transformation is the identity on the vectors $\bar x$ and $v(x)$, and $\bar x$ and $\bar v$ are orthogonal. Let $\bar y$ be a vector in $\mathbf{R}^3$ such that $|\bar y| = 1$, it is orthogonal to $\bar x$ and $\bar v$, and the system $\{\bar e_1, \bar e_2, \bar e_3\}$ is oriented in the same sense as $\{\bar x, \bar v, \bar y\}$. Obviously, the mapping $x, v \to \{x, v, y\}$ is a homeomorphism. All systems $\{\bar x, \bar v, \bar y\}$ are in one-to-one and continuous correspondence with the matrices associated with the linear transformations in $\mathbf{R}^3$, which map the orthonormal coordinate 3-frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ into the orthonormal coordinate 3-frame $\{\bar{x}, \bar{v}, \bar{y}\}$. These matrices form the group $SO(3) : A \in SO(3) \Rightarrow AA^t = E$, and $\det A = +1$. Thus, the space of the states of the system under consideration is homeomorphic to the manifold $SO(3)$. Any orthogonal transformation of $\mathbf{R}^3$ preserving the orientation is a rotation about a certain axis in a plane perpendicular to it through an angle $\varphi$, where $-\pi < \varphi < \pi$.

Therefore, all elements of the group $SO(3)$ are in one-to-one and continuous correspondence with the points of a ball of radius $\pi$ in $\mathbf{R}^3$ whose diametrically opposite boundary points are considered to be identical. It remains to show that the ball glued in this manner is homeomorphic to $\mathbf{R}P^3$. In fact, $\mathbf{R}P^3 = S^3/\mathbf{Z}_2$ where $S^3$ is standardly embedded in $\mathbf{R}^4$. Therefore, $\mathbf{R}P^3$ can be considered as a hemisphere of $S^3$ placed in the region with $x^1 \geqslant 0$ and with the diametrically opposite boundary points considered to be identical:

$$S^3 \cap \{x^1 = 0\} = \{(0, x^2, x^3, x^4) \in \mathbf{R}^4 : (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$$

which is homeomorphic to the sphere $S^2$ of radius $\pi$.

**7.21.** Let us prove that if $A \in SU(2)$, then

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbf{C}.$$

Let $\begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix} \in SU(2)$. Then

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1}$$

$$\alpha\gamma + \beta\bar{\delta} = 0 \tag{2}$$

$$|\gamma|^2 + |\delta|^2 = 1 \tag{3}$$

$$\det A = \alpha\delta - \beta\gamma = 1. \tag{4}$$

Substituting $\alpha = -\beta\delta/\bar{\gamma}$ from (2) in (4), we obtain $-\dfrac{\beta}{\gamma}(|\delta|^2 + |\gamma|^2) = 1$, or $\gamma = -\bar{\beta}$, whence $\alpha = \bar{\delta}$. On the other hand,

$$Sp(1) = \{q: (q_1 q, q_2 q) = (q_1, q_2), q_1, q_2 \in Q\},$$

where $(q_1, q_2) = \operatorname{Re} q_1 \bar{q}_2$. It is easy to see that $|q| = 1$, since, for $q_1 = q_2 = 1$, we have $|1 \cdot q|^2 = 1$, i.e., $|q| = 1$. Conversely, if $|q| = 1$, then $(q_1 q, q_2 q) = (q_1, q_2)$. Thus, $Sp(1)$ consists of quaternions of length 1, i.e., $S^3 \subset Q = \mathbf{R}^4$. Further,

$$q = a + ib + jc + kd = (a + ib) + j(c - id) = z_1 + jz_2,$$

where $z_1$ and $z_2 \in \mathbf{C} \subset Q$ and $|z_1|^2 + |z_2|^2 = |q|^2$. If $|q| = 1$, then $|z_1|^2 + |z_2|^2 = 1$. Let us arrange for a homomorphism $\varphi$ of $Sp(1)$ into $SU(2)$, viz., $\varphi(q) = \varphi(z_1 + jz_2) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$, $|z_1|^2 + |z_2|^2 = 1$. It is obvious that the element $\varphi(g)$ belongs to $SU(2)$. It is easy to verify that $\varphi$ is an isomorphism.

**7.23.** It suffices to show for the group $G = SL(2, R)/\{\pm E\}$ that any element of the group $G$ can be joined to $\pm E$. Let $\lambda_1, \lambda_2$ be the eigenvalues of the matrix $A$. Then either (a) $\lambda_1, \lambda_2 \in R$, $\lambda_2 = \lambda_1^{-1}$, since $\det A = \pm 1$, or (b) $\lambda_2 = \bar{\lambda}_1$, $\lambda_1 = e^{i\varphi}$, $\lambda_2 = e^{-i\varphi}$. Consider case (a). With respect to the basis consisting of eigenvectors, the matrix $A$ has the form $A' = CAC^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. We can assume that $\lambda > 0$. Construct a path $\gamma : I \to G$

$$\gamma(t) = \begin{pmatrix} \lambda(1 - t) + t & 0 \\ 0 & (\lambda(1 - t) + t)^{-1} \end{pmatrix}.$$

Consider case (b). There exists a basis on the plane with respect to which $A$ is of the form

$$A' = CAC^{-1} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

Construct a path $\gamma: I \to G$

$$\gamma(t) = \begin{pmatrix} \cos(1 - t)\varphi & -\sin(1 - t)\varphi \\ \sin(1 - t)\varphi & \cos(1 - t)\varphi \end{pmatrix}.$$

**7.25.** Consider a model of the Lobachevski plane $L^2$ in the upper half-plane ($\operatorname{Im} z > 0$ in the complex notation). The metric is of the form $ds^2 = (dx^2 + dy^2)/y^2$, or, in complex terms, $ds^2 = dz d\bar{z}/(\bar{z} - z)^2$. Consider the linear fractional transformations of $\mathbf{C}^1$ into $\mathbf{C}^1$, keeping the upper half-plane fixed (i.e., transform it into itself). These are transformations of the form

$$G = \left\{ w = \frac{az + b}{cz + d}\,;\, a, b, c, d \in \mathbf{R},\, ad - bc = 1 \right\}.$$

This transformation class preserves the metric, but there are other transformations preserving it. E.g., the transformation $w = -\bar{z}$ which is, evidently, a motion, but does not belong to the group $G$, at least because it is not an analytic function. Similarly, it is easy to verify that the whole class of transformations of the form

$$H = \{ w = (a\bar{z} + b)/(c\bar{z} + d);\, a, b, c, d \in \mathbf{R},\, ad - bc = -1 \}$$

preserves the metric. The group of motions of the Lobachevski plane consists of transformations of forms $G$ and $H$ only. In fact, $G \cup H$ is a group, $G \cup H = \mathfrak{G}$ acts on the Lobachevski plane $L^2$ transitively. Consider a subgroup $S$ of the group $\mathfrak{G}$ (i.e., subgroup of transformations keeping a point $i$ fixed) and a certain motion $h : L^2 \to L^2$, $h(i) = i$. We prove that $h \in S$. We shall show that the motion keeping $i$ fixed is fully determined by its action in the tangent plane at the point $i$. Let $h, g$ be two motions, and $h_* = g : T_iL^2 \to T_iL^2$. Then the transformations $h$ and $g$ act on the geodesics passing through $i$ in the same way; therefore, coincide on them, and since any point of $L^2$ can be joined to $i$ with a geodesic, $h = g$ on $L^2$. It remains to establish that, for any $\alpha \in O(2)$, there exists an element $g \in \mathfrak{G}$ such that $g_* = \alpha$. Let $g(z) = (az + b)/(cz + d) \in \in \mathfrak{G}$, $g_* : \mathbf{R}^2 \to \mathbf{R}^2$. The differential $g_*$ is realified multiplication by $g'(i)$, where $g'(z)$ denotes the derivative with respect to complex variable $z$, viz., $g'(t) = 1/(ci + d)^2$. Let

$$a = \cos(-\varphi/2),\ b = -\sin(-\varphi/2),\ c = \sin(-\varphi/2),\ d = \cos(-\varphi/2).$$

Then $g'(i) = \cos\varphi + i\sin\varphi$, i.e., it is a rotation of $\mathbf{C}^1$ through the angle $\varphi$. In the case of symmetry, consider the transformation $w = -\bar{z}$ whose differential is a symmetry, and then apply a linear fractional transformation which carries out a rotation. Now, let $h$ be an arbitrary motion and $h(i) = z_0$. Due to the transitivity of the group $\mathfrak{G}$, there exists an element $g \in \mathfrak{G}$ such that $g(z_0) = i$. The motion $g \cdot h \in S$, i.e., $g \cdot h(i) = i$. The subgroup $G$ is a connected subgroup containing the identity element. The transformation $w = -\bar{z}$ does not belong to $G$. Therefore, the group $G$ is a subgroup of index 2 of the group $\mathfrak{G}$.

**7.27.** Consider the cases $n = 2k + 1$ and $n = 2k$. The group $O(n)$ is disconnected and is the disjoint union of two path-connected components, viz., $O^+(n)$, i.e., the collection of matrices with det $= +1$, and $O^-(n)$, i.e., the collection of matrices with det $= -1$. When $n = 2k + 1$, the unit matrix $E \in O^+(n)$, and the matrix $-E \in O^-(n)$. Consider a discrete normal subgroup $H$ of $O(n)$. The element $ghg^{-1} \in H$ for any $h \in H$, and any $g \in O(n)$. If $g \in O^+(n)$, then $g$ can be joined to $E$ with a continuous path $\varphi(t)$ so that $\varphi(0) = g$, $\varphi(1) = E$. If $g = O^-(n)$, then the elements $g$ and $-E$ can be joined with a continuous path $\psi(t)$ so that $\psi(0) = g$, $\psi(1) = -E$. It is possible to construct two mappings $M(t)$ and $N(t)$ such that $M(t) = \varphi(t)h\varphi^{-1}(t)$, and $N(t) = \psi(t)h\psi^{-1}(t)$ for $g \in O^+(n)$ and $g \in O^-(n)$, respectively. Then

$$\begin{cases} M(0) = ghg^{-1} = \tilde{h}, \\ M(1) = h, \end{cases} \qquad \begin{cases} N(0) = ghg^{-1} = \tilde{h}, \\ N(1) = h. \end{cases}$$

The elements $\tilde{h}$ and $h$ are joined with a continuous path which lies in $H$ wholly. Since $H$ is discrete, we obtain that $h = \tilde{h}$, i.e., $ghg^{-1} = h$ for any $h \in H$, $g \in O(n)$.

**(The Schur Lemma.)** Let $\rho^1 : G \to GL(V_1)$, $\rho^2 : G \to GL(V_2)$ be two irreducible representations of a group $G$, and let $f$ be a linear mapping of the space $V_1$ into the space $V_2$ such that $\rho^2(S)f = f\rho^1(S)$ for each $S \in G$. Then (a) if $\rho^1$ and $\rho^2$ are not isomorphic, then $f = 0$, (b) if $V_1 = V_2$, $\rho^1 = \rho^2$, then $f$ is a homothety (i.e., multiplication by a certain number).

The mapping $\rho: O(n) \to O(n) \subset GL(\mathbf{R}^n)$ is an irreducible representation, since reflections with respect to each of the axes can be considered. These are matrices of the form

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & -1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix},$$

where $-1$ is placed at $(i, i)$. All such transformations are contained in the group $O(n)$. Collectively, they keep fixed only the point $(0, 0, \ldots, 0)$. Besides $\mathbf{R}^n$ and 0, there are no other invariant subspaces in $\mathbf{R}^n$. Applying the Schur lemma and using $ghg^{-1} = h$ for any $h \in H$ we obtain that $h$ is a scalar matrix. But there are only two scalar matrices in $O(n)$, viz., $E$ and $-E$. It is they that make up the discrete normal subgroup of $O(n)$.

Consider the case $n = 2k$. The matrices $E$ and $-E \in O^+(n)$ are a subgroup of a discrete normal subgroup $H$ of $O(n)$. We show that $H$ contains no other elements. $O^+(n)$ contains no other elements from $H$, since $H \cap O^+(n)$ is a discrete normal subgroup of $O^+(n)$, but only the group $\pm E$ can be that in $O^+(n)$. The reasoning is similar to the previous. We prove that $O^-(n)$ contains no elements from $H$. Assume that $h \in O^-(n) \cap H$. Then $ghg^{-1} = h$ for any $g \in O^+(n)$. The matrix $h$ can be reduced to block triangular form with an odd number of eigenvalues $-1$ by a certain orthogonal transformation of the basis with determinant $+1$. If $\dim \mathbf{R}^n > 2$, then, by an even number of transpositions of the basis vectors, the diagonal elements can be interchanged. Generally speaking, we will obtain a new matrix then, i.e., $ghg^{-1} \neq h$. E. g., we interchange $-1$ and the block; if there is no block, then we interchange $-1$ and the block formed by $+1$, $+1$. In the case where $n = 2$, we have only two kinds of matrices to which any matrix from $O^-(n)$ can be reduced by an orthogonal transformation of the variable, viz.,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Therefore, there are no elements from the discrete normal subgroup of $O^-(n)$.

**7.28. Theorem.** Any group of motions of finite order $N$ in $\mathbf{R}^3$ is isomorphic (assuming that the action has no kernel) to one of the following groups: $C_N$, a cyclic group; $D_N$, a dihedral group; $T$, the tetrahedral group; $W$, the hexahedral (octahedral) group; and $P$, the dodecahedral (icosahedral) group.

**Proof.** Let $\Gamma$ be a finite rotation group of order $N$. Consider the fixed points (poles) of all transformations from $\Gamma$ different from the identity transformation. Let the multiplicity of a pole $p$ (number of transformations from $\Gamma$, leaving $p$ fixed) be equal to $\nu$. The number of operations different from the identity transformation $I$ and leaving the pole $p$ fixed equals $\nu - 1$. Let $\{g\}$ be the set of points into which the pole $p$ is transformed under the action of elements from the group $\Gamma$. Then $\{g\}$ is an orbit consisting of points equivalent to each other. The number of points $g$ equivalent to $p$ equals $N/\nu$. In fact, the multiplicity of $g$ also equals $\nu$. The transformation $L_i \in \Gamma$ reduces $p$ into $g_i$ $(i = 1, \ldots, n)$. Let $S_1, \ldots, S_\nu$ be transformations leaving the point $p$ fixed, and

$$\Gamma = \{S_1 \cdot L_1, \ldots, S_\nu \cdot L_1; S_1 \cdot L_2, \ldots, S_\nu \cdot L_2; S_1 \cdot L_n, \ldots, S_\nu \cdot L_n\}.$$

All these transformations are different, each element $g \in \Gamma$ is contained in this set, and $|\Gamma| = N$, i.e., $N = n\nu$ for any orbit $N = n_e = \nu_e$, where $c$ is a certain orbit. Consider all pairs $(S, p)$, where $S \in \Gamma$ are fixed on $p$, and $S \neq I$. The number of such pairs equals, on the one hand, $2(N - 1)$, and, on the other hand, $\sum_c (\nu_c - 1)n_c$, viz.,

$$2(N - 1) = \sum_c n_c(\nu_c - 1),\ 2 - 2/N = \sum_c (1 - 1/\nu_c),\ N \geqslant 2$$

(because if $N = 1$, we shall have a trivial group). Therefore, $\nu_c \geqslant 2$, and, from the evident relations, we get that $2 \leqslant c \leqslant 3$. The following cases are possible:

1. $c = 2$. Then $2/N = 1/\nu_1 + 1/\nu_2$, $2 = N/\nu_1 + N/\nu_2 = n_1 + n_2$, $n_1 = n_2 = 1$.

Each of the two classes of equivalent poles consists of one pole of multiplicity $N$, i.e., we have obtained a cyclic group of order $N$ of rotations about one axis.

2. $c = 3$. Then $1/\nu_1 + 1/\nu_2 + 1/\nu_3 = 1 + 2/N$, $\nu_1 \leqslant \nu_2 \leqslant \nu_3$. At least one $\nu_i = 2$. Let $\nu_1 = 2$. Then $1/\nu_2 + 1/\nu_3 = 1/2 + 2/N$. The numbers $\nu_1$, $\nu_2$ cannot be greater than or equal to 4, i.e., $\nu_2 = 2$ or 3. (a) $\nu_1 = \nu_2 = 2$, $N = 2\nu_3$, $\nu_3 = n$, a dihedral group $D_N$; (b) $\nu_1 = 2$, $\nu_2 = 3$, $1/\nu_3 = 1/6 + 2/N$. Then the following cases are possible: $\nu_3 = 3$,

$N = 12$, the tetrahedral group $T$; $\nu_3 = 4, N = 24$, the hexahedral group $W$; $\nu_3 = 5, N = 60$, the dodecahedral group $P$.

The dodecahedral group $T$ contains two classes of poles with four poles of multiplicity 3, $|T| = 1 + 4 \cdot 2 + 1 \cdot 3 = 12$. The generators and relations of the group $T$ are $abc = adb = acd = bdc = 1$ ($a, b, c, d$ being rotations about all four vertices through the angle $2\pi/3$), $a^3 = b^3 = c^3 = d^3 = 1$. Let $e, f, g$ be rotations about the axes $l_i$, and $ef = fe = g$. To $T$, reflections may be added. Let $h$ be an improper rotation, and $he = eh$, $a^i h = ha^{3-i}$ $(i = 0, 1, 2)$. With all the improper motions added, we obtain $|T| = 24$. The cube and octahedron possess the same group of motions $|W| = 1 + 3 \cdot 3 + 1 \cdot 6 + 4 \cdot 2 = 24$. We have one class of six poles of order 4 (the vertices of the cube), eight poles of order 3 (the centres of the faces), and twelve poles of order 2 (the midpoints of the edges) for $W$. $T$ is a subgroup of $W$. This is obvious from geometry (the tetrahedron can be inscribed in the cube). The relations are $a^4 = b^3 = c^2 = d^2 = 1$ (where $d$ is a reflection); $a^i d = da^{4-i}$, $b^i d = db^{3-i}$, $cd = dc$, $ac = b$, $|P| = 60$ are only proper motions. With the reflections added, we obtain $|P| = 120$. The subgroup of proper motions in $P$ is isomorphic to $A_5$. It has twelve poles of order 5 (the vertices of the icosahedron), twenty poles of order 3 (the centres of the faces) and thirty poles of order 2 (the midpoints of the edges). This group is commutative only partly. The relations in the dodecahedral group are

$$abcde = 1, \quad bkef^{-1}i^{-1} = 1, \quad aidk^{-1}h^{-1} = 1,$$

$$ci^{-1}g^{-1}eh = 1, \quad bh^{-1}f^{-1}dg = 1, \quad ag^{-1}k^{-1}cf = 1,$$

or

$$bce = 1, bkei^{-1} = 1, i = k, ci^{-1}g^{-1}e = 1, b = g^{-1}, g^{-1}k^{-1}c = 1.$$

Eliminating $g$ and $k$, we get

$$bce = 1, \quad biei^{-1} = 1, \quad ci^{-1}be = 1, \quad bic^{-1} = 1.$$

It follows from the relations $bce = 1$ and $biei^{-1} = 1$ that $i = cb$, and from $biei^{-1} = 1$, $ci^{-1}be = 1$ and $i = cb$ that $bebc^{-1}b^2c^{-1} = 1$ and $cb^{-1}c^{-1}bc^{-1}b^{-1} = 1$. From all the relations obtained, we deduce the statement of the complete non-commutativity of the group $P$, i.e., $P = [p, p]$. Another variant of the corepresentation is

$$bc = cb^{-1}b^{-1}cb^{-1}, \quad cb = bcb^{-1}cbb, \quad a^5 = b^3 = abab = 1,$$

where $a$ is the rotation about the axis of order 5, $a = (12345)$, $b$ the rotation about the axis of order 3, $b = (452)$ and $P = A_5$.

**7.45.** Let $G$ be a finite group operating effectively on $\mathbf{R}^n$, i.e., if $xg = x$, $x \in \mathbf{R}^n$, $g \in G$, then $g = e$. The group $\mathbf{Z}_k$ generated by the element $g$ also operates effectively on $\mathbf{R}^n$. Consider the space $x = G/\mathbf{Z}_k$, where $x \sim y$ if $y = g^l x$, $g \in \mathbf{Z}_k$; $\pi_1(X) = \mathbf{Z}_k$, $\pi_i(X) = \pi_i(\mathbf{R}^n) = 0$ when $i > 1$, since $\mathbf{R}^n \to X$ is a covering map. Therefore, $X$ is homotopy equivalent to $K(\mathbf{Z}_k, 1)$, i.e., to a lens space. But the homology $K(\mathbf{Z}_k, 1)$ is different from 0 in an infinite number of dimensions, whereas $X$ has no cells of dimensions greater than $n$. Thus, substantiating the statements: (a) if a discrete group $G$ acts on $\mathbf{R}^n$ without fixed points, and $X_G = \mathbf{R}^n/G$ is the set of orbits, then the natural mapping $p: \mathbf{R}^n \to X_G$ is a covering map; (b) $\pi_1(X_G) = G$ (the proofs of statements (a) and (b) are left to the reader), we complete the proof.

# 8
# Vector Fields

**8.2.** (a) $\sqrt{15}/5$; (b) $3\sqrt{21}/7$; (c) $\sqrt{3}/3 \cdot l^3$; (d) $-2/5$.
**8.3.** $3\sqrt{2}/5$.
**8.4.** $1/4$.
**8.5.** (a) 0; (b) $2\dfrac{\sqrt{3}}{3}(\sqrt{2} + 3)$; (c) 0; (d) $-2$; (e) $\pi a^2/\sqrt{a + R^2}$.
**8.7.** $1/r^2$.
**8.8.** 1.
**8.9.** $(\operatorname{grad} f, \operatorname{grad} g)$.

**8.21.** (a) $(0, x, y - x)$;
(b) $(0, 0, y^2 - 2xz)$;
(c) $(0, e^x - xe^y, 0)$;
(d) $(0, 3x^2, 2y^3 - 6xz)$;
(e) $(0, -x(x + y^2), x^3 + y^3)$;
(f) $(0, xz^2 + yze^{x^2}, -2xyz)$;
(g) $(\sin xz/x, 0, -\sin xz/y)$;
(h) $(xz/(x^2 + y^2), yz/(x^2 + z^2), -1)$.

**8.22.** Use the existence and uniqueness theorem for a solution of a system of ordinary differential equations.
**8.23.** Investigate the action of the Poisson bracket on the product of two smooth functions.

**8.26.** Consider the case, where $\xi = \partial/\partial x^1$.

**8.33.** Let $z_0 = x_0 + iy_0$ be a singular point, and

$$f(z) = u(z) = iv(z), \qquad \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} = 0.$$

Since

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} = -\left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)} = 0,$$

$f_z'(z_0) = 0$. Let $f_z'(z_0) = 0$, i.e.,

$$\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = 0,$$

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial x}(z_0) = -\frac{\partial u}{\partial y}(z_0) = 0.$$

Then grad Re $f(z_0) = 0$.

**8.31.** Represent the sphere $S^3$ as the group of quaternions of unit length.

**8.34.** We shall seek the integral curves only in the half-plane lying over the straight line $AB$. The level curves for the function $f(x)$ are the arcs of the circumferences for which the line-segment $AB$ is a chord. The vector grad $f(x)$ is orthogonal to the level curve. Therefore, the vector orthogonal to it is tangent to the level curve, i.e., a circumference, and all the arcs of the circumferences described are the integral curves of the flow $v_1(x)$.

**8.35.** (a) The integral curves of the vector field grad (Re $z^n$) are the level curves of the conjugate function Im $z^n = r^n \sin n\varphi$. The unique singular point of the field $v =$ grad(Re $z^n$) is $z = 0$, since $f'(z) = 0$ only at this point. The point $z = 0$ is the case of a degenerate saddle point. Let us give a small perturbation to the function $z^n \to \prod_{i=1}^{n} (z - \varepsilon_i)$.

Then the singular point splits into $n - 1$ non-degenerate saddle points of the second order. Consider the behaviour of the integral curves near to one of the singular points. Expanding the function in Taylor's series, viz.,

$$f(z) = f(a_i) + f'(a_i)(z - a_i) + \ldots, \text{ where } f'(a_i) = 0,$$

we see that the expansion starts from a term of the second order, since $f''(a_i) \neq 0$ (non-degenerate critical point); $f''(a_i) \neq 0$ if and only if all $a_i$

are multiples of $f'(a_i)$, but this is not true, since all $\varepsilon_i$ are different. Therefore, we have, near to the point $a_i$, that $f(z) = k(z - a_i)^2 + o(z^2)$, i.e., the saddle point is non-degenerate. If the equality $f''(a_i) = 0$ were valid, then we would have the case of a degenerate singular point (i.e., saddle point of the third or higher order).

(b) $f(z) = z + 1/z$;

$$\mathrm{Re}\,(f(z)) = \rho \cos\varphi + \cos\varphi/\rho = (\rho + 1/\rho)\cos\varphi;$$

$$\mathrm{Im}\,(f(z)) = \rho \sin\varphi - \sin\varphi/\rho = (\rho - 1/\rho)\sin\varphi;$$

$$(z = \rho e^{i\varphi}).$$

A singular point is at the origin, since the function $1/z$ is discontinuous. The derivative of the function $f(z)$ equals $1 - 1/z^2$, i.e., the singular points are $z = 1, z = -1$. Both points are non-degenerate. Consider the integral curves for $\mathrm{Im}\,f(z)$, emanating from and returning to the singular points, i.e., the separatrices $(\rho - 1/\rho)\sin\varphi = c$ ($c = 0$ at the point $(1, 0)$). Thus, $(\rho - 1/\rho)\sin\varphi = 0$, whence $\varphi = K\pi$, or $\rho = 1$. Hence, we find the separatrices viz., the unit circumference consisting of two separatrices, and the real axis consisting of four separatrices. In the case of $\mathrm{grad}[\mathrm{Re}(f(z))]$, the separatrices are given similarly, by the equation $(\rho + 1/\rho)\cos\varphi = 2$, and have the shape of two loops tangent to each other.

(c) $f(z) = z + 1/z^2$. Consider $\mathrm{grad}[\mathrm{Re}(f(z))]$. The integral curves of this flow are the level curves of the function $\mathrm{Im}(f(z))$

$$\mathrm{Im}(f(z)) = y - \frac{2xy}{(x^2 + y^2)^2} = r \sin\varphi - \frac{\sin 2\varphi}{r^2}$$

(in polar coordinates on $\mathbf{R}^2$). Similarly, we seek the level curves of the function $\mathrm{Re}(f(z))$:

$$\mathrm{Re}(f(z)) = r\cos\varphi + \frac{\cos 2\varphi}{r^2}.$$

(d) $f(z) = z + 1/(z - 2)$. The singular points are $z = 1$, $z = 2$, $z = 3$. In a neighbourhood of the point $z = 1$, $f'(z) = -2(z - 1)$ (the first term in Taylor's expansion). This is a singular point, a saddle. Similarly, the singular point $z = 3$ is also a saddle point. In a neighbourhood of the point $z = 2$, the expansion in Laurent's series of $f'$ is $f'(z) = -1/(z - 2)^2 + \ldots$. Therefore, the integral curves of this flow in a neighbourhood of the point $z = 2$ have the form of the integral curves of the flow $\mathrm{grad}[\mathrm{Re}(1/(z - 2))]$.

(e) $f(z) = z^3(z - 1)^{100}(z - 2)^{900}$. The singular points, i.e., zeroes of the derivative $f'(z)$, are the following: $z_1 = 0$, a saddle point of the second order; $z_2 = 1$, a saddle point of the 100-th order; $z_3 = 3$, a saddle point of the 900-th order; $z_{4,5} \approx (1109 \pm 1093)/2006$, non-degenerate singular points.

Locally, in a neighbourhood of each singular point, the integral curves play the role of saddle points (degenerate or non-degenerate at the points $z_4 \approx 0.008$, $z_5 \approx 1.09$) of the corresponding order.

(f) $f(z) = 1 + z^4(z^4 - 4)^{44}(z^{44} - 44)^{444}$. In a neighbourhood of $z = 0$, $f(z)$ can be replaced by $\tilde{f}(z) = 1 + 4^{44}4^{444}z^4$. The qualitative behaviour of the curves in a neighbourhood of the point $z = 0$ is nevertheless unaltered. But adding a constant does not change the form of the trajectories. Therefore, the function $f_1(z) = cz^4$, $c = 4^{44} \cdot 4^{444}$, where $z \approx 0$, can be considered. The equations of the trajectories are of the form $c\rho^4 \cos 4\varphi = k$. The point $z = 0$ is a degenerate singular point, which, after a slight disruption, splits into four non-degenerate points. More precisely, $g(z) = c(z - \varepsilon_1)(z - \varepsilon_2)(z - \varepsilon_3)(z - \varepsilon_4)$ is a slight disruption of the function $f(z)$.

(g) $f(z) = 1/100 \ln[(z - 2i)/(z - 4)]^3$. At the points $z = 2i$ and $z = 4$, we have logarithmic singularities. Apart from them, there are no other singular points.

(h) $f(z) = 1/(z^2 + 2z - 1)$. To simplify the notation, we perform a translation $w = z + 1$. Then $f(w) = 1/(w^2 - 2)$. The singular points are $w = \sqrt{2}$, $w = -\sqrt{2}$. The singular points of grad[Re($f(z)$)] coincide with the zeroes of $f'(w)$, i.e., $w = 0$, a singular point (it is non-degenerate, since $f''(w) \neq 0$).

(i) $f(z) = 2/z + 21 \ln z^2$. The singular points are $(0, 0)$, $(1/21, 0)$. The separatrices are the curves $21\varphi = \sin \varphi/2$ and the axis $x$ from 0 to $+\infty$.

(j) $f(z) = z^5 + 2 \ln z$. The singular points are $z = 0$, $z^5 = -2/5$, the vertices of a pentagon. At these points, $f(z) \sim kz^2$, $k \neq 0$.

(k) $f(z) = 2 \ln(z - 1)^2 - 4/3 \ln(z + 10i)^3$. The singular points are $z = 1$, $z = -10i$, $f'(z) \neq 0$ for all $z$.

(l) $f(z) = 1/z^3 - 1/(z - i)^3$. The singular points are $z = 0$, $z = i$ (the poles of the third order). Differentiating,

$$f'(z) = -\frac{3}{z^4} + \frac{1}{(z - i)^4} = 0.$$

We obtain four other points:

$$z = i/(\sqrt[4]{3} - i), z = i/(\sqrt[4]{3} + 1),$$
$$z = (\sqrt[4]{3}i + \sqrt[4]{3})/(1 + \sqrt{3}), z = (\sqrt[4]{3}i - \sqrt[4]{3})/(1 + \sqrt{3}).$$

The integral curves behave at infinity as the integral curves of the flow grad[Re($1/z^4$)], i.e., as the integral curves of a pole of order four.

**8.36.** Let the flow $v = (P, Q)$ be irrotational, i.e.,

$$\operatorname{rot} \bar{v} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \equiv 0, \quad \text{or} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Let us find a function $f$ such that $P = \partial f/\partial x$, $Q = \partial f/\partial y$. To this end, we integrate the first relation with respect to $x$ between 0 and $x$, viz., $f(x, y) = \int_0^x P dx + g(y)$. To find $g(y)$, we integrate the latter relation with respect to $y$, viz.,

$$Q(x, y) = \frac{\partial f}{\partial y}(x, y) = \int_0^x \frac{\partial P}{\partial y} dx + g'(y)$$

$$= \int_0^x \frac{\partial Q}{\partial x} dx + g'(y) = Q(x, y) - Q(0, y) + g'(y).$$

Thus, $Q(x, y) = Q(x, y) - Q(0, y) + g'(y)$. Therefore, $g'(y) = Q(0, y)$, i.e.,

$$g(y) = \int_0^y Q(0, y) dy + C.$$

Consequently,

$$f(x, y) = \int_0^x P(x, y)\, dy + \int_0^y Q(0, y)\, dy + C.$$

Let $\gamma_1$ and $\gamma_2$ be two paths from (0, 0) to the point $(x, y)$ in the plane $(x, y)$. Consequently, if $\operatorname{rot} \bar{v} = 0$, then

$$\int_{\gamma_1} Pdx + Qdy = \int_{\gamma_2} Pdx + Qdy.$$

Therefore,

$$f(x, y) = \int_{\gamma} Pdx + Qdy + C,$$

where $\gamma$ is an arbitrary path from (0, 0) to $(x, y)$.

Besides, let the flow be incompressible, i.e.,

$$f(x, y) = \int_{\gamma} P\, dx + Q\, dy + C.$$

Consider the flow

$$v' = (Q, P), \qquad \operatorname{rot} v' = 0.$$

Therefore, the field $v'$ is potential. Thus, there exist functions $a(x, y)$ and $b(x, y)$ such that $\bar{v} = \operatorname{grad} a(x, y)$, $v' = \operatorname{grad} b(x, y)$. Since $\operatorname{div} v = \operatorname{div} v' = 0$, we obtain that $a(x, y)$ and $b(x, y)$ are harmonic functions, i.e., $\Delta a \equiv \Delta b \equiv 0$. Consider the function $f = a + ib$. It is complex and analytic, since the Cauchy-Riemann equations are valid, viz.,

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} = P(x, y), \qquad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x} = Q(x, y).$$

Such a function $f$ is called a complex potential of the flow.

**8.39. Hint:** $d\varphi_2$ is homotopic to $d\varphi_1$.

**8.47.** Consider the differential equation in $\mathbf{R}^4$

$$\dot{x} = Ax, \quad \text{where} \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad x \in \mathbf{R}^4.$$

The required set consists of the integral curves of this equation, which belong to the sphere. It is clear that $x(t) = e^{At}x(0)$. If $\mathbf{R}^4$ is regarded as $\mathbf{C}^2$, then the integral curve passing through a point $(z_1, z_2) \in S^3$ is of the form $(e^{it}z_1, e^{it}z_2)$, since $e^{At}$ may be written in the complex notation as follows: $\begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix}$. Let us assign to this trajectory the point $(z_1 : z_2)$ belonging to $\mathbf{C}P^1$. The definition of this correspondence is correct because any other pair $(z_3 : z_4)$ lying on the same trajectory differs from $(z_1 : z_2)$ only by the factor $e^{it}$, and therefore, determines the same point of $\mathbf{C}P^1$. It remains to note that the mapping is one-to-one and continuous.

# 9
# Tensor Analysis

**9.1.** (a) (0, 1); (b) (0, 2); (c) (1, 1); (d) (0, 2).
**9.5.** If $k = \dim V$, then $\dim V_n^m = k^{(n+m)}$.
**9.14.** $\operatorname{grad} f = -\dfrac{1}{\sqrt{x^2 + y^2 + z^2}}\,(x, y, z)$.
**9.25.** (a) Use Problem 9.22 while replacing the sphere by the cone.
(b) The meridians and the equator are geodesic lines.
(c) Apply (a) and (b).
**9.28.** Perform covariant differentiation with respect to the parameter $\alpha$ which parametrizes the family of curves.

**9.39. Hint:** The integral curves of the left-invariant vector field $X$ are left translations of a one-parameter subgroup, i.e., geodesics. Therefore, we may assume that $X = \dot\gamma$, where $\dot\gamma$ is the vector field of the velocities of the geodesic $\gamma(t)$. Since, by the definition of a geodesic, $\nabla_{\dot\gamma}(\dot\gamma) = 0$, for any left-invariant field $X$, we have $\nabla_X(X) = 0$. In particular, $\nabla_{X+Y}(X + Y) = 0$, i.e., $\nabla_X Y + \nabla_Y X = 0$. On the other hand, $\nabla_X Y - \nabla_Y X = [X, Y]$. In fact,

$$X^i\nabla_i Y^k = Y^i\nabla_i X^k = X^i\left(\frac{\partial Y^k}{\partial x^i} + Y^p\Gamma^k_{pi}\right) - Y^i\left(\frac{\partial X^k}{\partial x^i} + X^p\Gamma^k_{pi}\right)$$

$$= X^i\frac{\partial Y^k}{\partial x^i} - Y^i\frac{\partial X^k}{\partial x^i} + X^iY^p(\Gamma^k_{pi} - \Gamma^k_{ip}) = X^i\frac{\partial Y^k}{\partial x^i} - Y^i\frac{\partial X^k}{\partial x^i},$$

since $\Gamma^k_{pi} = \Gamma^k_{ip}$ (the connection is symmetric). The required statement follows from the system: $\nabla_X Y + \nabla_Y X = 0$ and $\nabla_X Y - \nabla_Y X = [X, Y]$.

**9.41. Hint:** The invariant definition of a curvature tensor is of the form $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z$. Since $\nabla_X Y = 1/2[X, Y]$, we obtain

$$R(X, Y)Z = \frac{1}{4}([X, [Y, Z]] - [Y, [X, Z]]) - \frac{1}{2}[[X, Y], Z].$$

Taking into account the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0,$$

we get

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z].$$

# 10
# Differential Forms, Integral Formulae, De Rham Cohomology

**10.6.** (a) $-2(z+1)dx \wedge dy \wedge dz$; (b) $yzdx \wedge dz + xzdy \wedge dz$; (c) $6ydx \wedge \wedge dy \wedge dz$; (d) 0; (e) 0; (f) 0; (g) $df \wedge dg$; (h) 0.

**10.7.** Reduce the problem to the case of constant coefficients.

**10.12.** (a) $-4\pi/15$; (b) $-\pi/3$; (c) $4\pi R^3/3$; (d) 0.

**10.13.** (a) $\left(2(\rho + \cos\varphi), -\left(2\sin\varphi + \dfrac{e^z}{\rho}\cos\varphi\right), e^{-z}\sin\varphi\right)$.

**10.14.** (a) $(2r\cos\theta, -r\sin\theta, 0)$;

(b) $\left(-\dfrac{2\cos\theta}{r^3}, -\dfrac{\sin\theta}{r^3}, 0\right)$.

**10.15.** (a) $2 + z/\rho\cos\varphi - e^\varphi \sin z$;

(b) $\dfrac{\varphi}{\rho}\tan^{-1}\rho + \dfrac{\varphi}{1+\rho^2} - (z^2 + 2z)e^z$.

**10.17.** $4r - \dfrac{2}{r}\cos^2\varphi\cot\theta + \dfrac{1}{r(r^2+1)\sin\theta}$.

**10.19.** (a) $\rho + \varphi + z + c$; (b) $\dfrac{1}{2}(\rho^2 + \varphi^2 + z^2) + c$; (c) $\rho\varphi z + c$; (d) $e^\beta \sin\varphi + z^2 + c$; (e) $\rho\varphi\cos z + c$.

**10.20.** **(a)** $r\theta + c$; (b) $r^2 + \varphi + \theta + c$; (c) $\dfrac{1}{2}(r\varphi^2 + \theta^2) + c$; (d) $r\cos\varphi\sin\theta + c$; (e) $e^r\sin\theta + \ln(1+\varphi^2) + c$.

**10.21.** $4\pi R^2$.

**10.22.** (a) 1; (b) $\pi^2$; (c) $2\pi R$; (d) 0; (e) $-2\pi R^4$; (f) $\pi$.

**10.23.** (a) $\pi^2$; (b) 1; (c) $\dfrac{\pi}{4} + \dfrac{\sqrt{2}}{2} - 1$; (d) $\pi$; (e) 0; (f) 0.

**10.24.** (a) $24\pi$; (b) $\pi/2$.

**10.25.** (a) $4\pi$; (b) $\dfrac{2\pi}{3}R^3$; (c) $4\pi R^4$; (d) $2\pi R^3$; (e) $\dfrac{\pi}{2}R^4 - \dfrac{R^5}{3}$.

**10.30.** (a) $H^1(S^1) = \mathbf{R}^1$; (b) $H^2(S^2) = \mathbf{R}^1$; (c) $H^k(RP^2) = 0, k \geqslant 1$; (d) $H^1(T^2) = R^2$; $H^2(T^2) = \mathbf{R}^1$; (e) $\dim H^k(T^n) = \binom{n}{k}$; (f) $\dim H^1 = = k$, where $k$ is the number of points excluded.

# 11
# General Topology

**11.1.** Note that the open ball $B^n$ and punctured sphere $S^n$ are homeomorphic. We prove the statement by induction on the dimension of the complex. If the dimension $n = 0$, then the statement is evident. Let the statement be held for all numbers less than $n$. Then, by the inductive hypothesis, the $(n-1)$-dimensional skeleton $K^{n-1}$ of the complex in question is embeddable in the Euclidean space $\mathbf{R}^N$. This means that continuous real functions $f_1(x), \ldots, f_N(x)$ are given on $K^{n-1}$ such that $(f_1(x), \ldots, f_N(x)) \neq (f_1(y), \ldots, f_r(y))$ when $x \neq y$. Let $e_j^n$ $(j = 1, \ldots, k)$ be all $n$-dimensional cells of our complex. Then the functions $f_i(x)$ are defined on the boundary of each cell $e_j^n$ (we denote it by $\dot{e}_j^n$). Let $e_j^n$ be homeomorphic to the interior $B^n$ of the closed ball $D^n$. We may assume then that the functions $f_i(x)$ are given on $D^n \setminus B^n$. Their continuity is preserved, but they may not be one-to-one now. Let us extend these functions from $D^n \setminus B^n$ to $B^n$ (i.e., from $\dot{e}_j^n$ to $e_j^n$) as follows. Let $z \in B^n$, and $z \neq 0$. We put $f_i(z) = |z| f_i(z/|z|)$. If $z = 0$, then we set $f_i(z) = 0$. Thus, we have extended the functions $f_i$ to continuous functions on the whole complex $K$. Now, we define

$$g_1^j(x), \ldots, g_{n+1}^j(x).$$

We put $g_s^j(x) \equiv 0$ $(s = 1, \ldots, n+1)$ outside $e_j^n$ and

$$(g_1^j(x), \ldots, g_n^j(x), g_{n+1}^j(x))$$

$$= \left(\frac{x_1}{|x|} \sin \pi |x|, \ldots, \frac{x_n}{|x|} \sin \pi |x|, \cos \pi |x| + 1\right) \text{ on } e_j^n.$$

We define $F : K \to \mathbf{R}^{N+k(n+1)}$ by the equality

$$F(x) = (f_1(x), \ldots, f_N(x); g_1^1(x), \ldots, g_{n+1}^1(x), \ldots, g_1^k(x), \ldots$$
$$\ldots, g_{n+1}^k(x)).$$

The mapping $F$ is thus one-to-one, and the statement proved.

**11.4.** A section of the Klein bottle by a plane should be considered so that there may be two Möbius strips. Then this plane should be lifted (while discarding one Möbius strip), thereby carrying out the boundary circumference deformation represented. When this circumference becomes free of self-intersections and turns into the standardly embedded circumference, it should be glued to the two-dimensional disk. Considering the surface obtained in lifting the plane and which is the trace of the circumference deformed, we get an embedding of $\mathbf{R}P^2$ in $\mathbf{R}^3$.

**11.5.** The set of the points of self-intersection is homeomorphic to the wedge of three circumferences $S^1 \vee S^1 \vee S^1$. The vertex of this wedge is a triple self-intersection point, and any of its points different from the vertex is double.

**11.6.** The boundary $M^2$ of the normal tubular neighbourhood of radius $\varepsilon$ constructed is, obviously, projected onto $\mathbf{R}P^2$ (two endpoints of the normal line-segment are sent to its centre lying on $\mathbf{R}P^2$). Thus, $M^2$ is a smooth, two-dimensional, compact, and closed manifold and a two-sheeted covering of the projective plane. If we prove that this manifold is connected, then we shall prove thereby that it is a two-dimensional sphere, since $S^2$ is the unique two-sheeted connected covering of $\mathbf{R}P^2$.

To establish the connectedness, it suffices to consider two points on $M^2$ which are the endpoints of the same normal line-segment, and find the path on $M^2$ joining these two points. To construct such a path, it suffices to consider a point $T$ on $\mathbf{R}P^2$ which is the centre of the line-segment under consideration, and take on $\mathbf{R}P^2$ a closed path starting and ending at the point $T$ and such that the orientation of the two-frame slipping along the path and always tangent to $\mathbf{R}P^2$, changes in moving along it. Then, by adding to this frame a third vector orthogonal to $\mathbf{R}P^2$ and considering the trace of this vector which is cut out by the frame in moving continuously along the closed path, we obtain the continuous path on $M^2$ joining the two selected points.

**Note.** The embedding of the two-dimensional sphere in Euclidean three-dimensional space helps to prove a remarkable topological fact, viz., the possibility of "turning the two-dimensional sphere in $\mathbf{R}^3$ inside out". This task is outside the scope of our course, and we confine ourselves to a short sketch only. The embedding of $S^2$ indicated is such that admits interchanging the exterior and interior of the two-dimensional sphere while remaining in the regular embedding class. In fact, it suffices to consider a smooth deformation of the two-dimensional sphere along the normal vector field determined by the normal line-segments described above. In doing so, the interior and exterior surfaces of the sphere interchange.

**11.7.** Consider a vector space over $\mathbf{R}$ with a basis of the power of the continuum. We introduce the following topology on it. Consider the "cube" $B = \{x: -1 < x_\alpha < 1 \text{ for all } \alpha\}$, where $x_\alpha$ are the coordinates of the vector $x$, and the cross-section $B$ is of finite codimension, viz.,

$$B_{\alpha\beta\ldots\delta} = B \cap \{x_\alpha = 0, x_\beta = 0, \ldots, x_\delta = 0\}.$$

We call the sets $B_{\alpha\beta\ldots\delta}$ the neighbourhoods of the point 0. It is obvious that the point 0 has no countable base for the neighbourhoods in such a topology, i.e., the space constructed does not satisfy the first countability axiom, nor does it satisfy the second, since the first axiom is a corollary to the first.

**11.8.** Consider the mapping $F : S^2 \to \mathbf{R}^2$, $x \to (f(x), g(x))$, where $F(S^2) \subset \mathbf{R}^2$ is the image of the sphere. The image $F(S^2)$ is a set symmetric about the point $(0, 0)$, since if $(a, b) \in F(x)$, then $(-a, -b) = F(\tau x)$. Assume that $(0, 0) \in F(S^2)$ and project the plane with the exclusion of the origin onto the unit circumference. In polar coordinates, where this projection can be written in the form $h(re^{i\varphi}) = e^{i\varphi}$. Then $h(F(S^2))$ is a certain centrally symmetric set on the unit circumference $S^1$, where $h(F(S^2))$ is the image of the connected set $S^2$ under a continuous mapping $h \cdot F$. Therefore, it is also connected. It is obvious that a connected, centrally symmetric set on $S^1$ must coincide with $S^1$. Further, $h(F(S^2))$ must be 1-connected as the image of the 1-connected set $S^2$, which is contrary to the equality $h(F(S^2)) = S^1$.

**11.9.** As the space $X$, take a space $l_2$ whose elements are sequences of real numbers $x = (x_1, x_2, \ldots, x_n, \ldots)$ satisfying the condition $\|x^2\| =$

$= \sum_{n=1}^{\infty} |x_n|^2 < \infty$. As the space $Y \subset X$, take a sphere in $X$, i.e., the set

of $x$ such that $\|x\|^2 = 1$. Consider a sequence of points $x_i$ in $Y$, so that unity is at place $i$, and the other coordinates are zeroes. This infinite seguence has no limit point, since $\|x_i - x_j\| = \sqrt{2}$ for all $i, j$. Therefore, $Y$ is not a compactum.

**11.11.** Let $e_1, \ldots, e_s$ be the vertices of the complex $K$. Take the points $e_1', \ldots, e_s'$ in general position in $\mathbf{R}^{2n+1}$, i.e., any $j$ points are linearly independent when $j \leqslant 2n + 2$. To each skeleton

$$T = |e_{i_0} \ldots e_{i_r}| \in K,$$

we assign the simplex

$$T' = |e_{i_0}' \ldots e_{i_r}'| \in \mathbf{R}^{2n+1}.$$

This simplex exists, since due to the points $e_1', \ldots, e_s'$ being in general position in $\mathbf{R}^{2n+1}$ and inequality $r \leqslant n$, the points $e_{i_0}', \ldots, e_{i_r}'$ are linearly independent. The simplexes form a complex isomorphic to the complex $T'$, since to each vertex, there corresponds one and only one vertex from $K$.

The complex $K$ is a triangulation. To prove this; it suffices to show that no two simplexes $T_i', T_j' \in K'$ intersect. Let $e_{i_0}', \ldots, e_{i_p}'$ be the vertices of $T_i'$, and $e_{j_0}', \ldots, e_{j_q}'$ those of $T_j'$ (some of the vertices may be common).

Let $e_{k_0}', \ldots, e_{k_r}'$ be all the points which are the vertices of at least one of the simplexes $T_i'$ and $T_j'$. The number of these points $r + 1$ satisfies the inequality

$$r + 1 \leqslant (p + 1) + (q + 1) \leqslant (n + 1) + (n + 1) = 2n + 2.$$

Since the points $e_1', \ldots, e_s'$ are in general position in $\mathbf{R}^{2n+1}$, the points $e_{k_0}', \ldots, e_{k_r}'$ are the vertices of a certain degenerate simplex $T_0$ of dimension not higher than $2n + 1$. The simplexes $T_i'$ and $T_j'$ are two faces of $T_0$, and therefore, do not intersect each other if different.

**11.19.** The main fact to prove is to show that any two fibres $\Omega(x_0, x)$ and $\Omega(x_0, y)$ are homeomorphic for any points $x$ and $y$ from the space $X$. We assume here that $X$ is a connected manifold, $x$ and $y$ two points from $X$, and $\Omega(x_0, x)$ the space of continuous paths from the base point $x_0$ to $x$. We should be able to assign to each path $\gamma$ from $x_0$ to $x$, a path $\gamma'$ from $x_0$ to $y$ so that this correspondence may define a fibre homeomorphism. On joining $x$ and $y$ with a path $S$ we consider a tubular neighbourhood $U(s)$ of the path $S$ and define a diffeomorphism $\varphi_t : U(s) \to U(s)$ which is identity outside $U(s)$ and sends the point $x$ to a point $s(t) \in S$, where $0 \leqslant t \leqslant 1, s(0) = x, s(1) = y$. As to the rest, the diffeomorphism $\varphi_t$ is arbitrary. The family $\{\varphi_t\}$ $(0 \leqslant t \leqslant 1)$ determines a homotopy in the manifold $X$. In constructing the homotopy, we have used the fact that $X$ is a manifold. Now, we define the homotopy

$$f : \Omega(x_0, x) \to \Omega(x_0, y), \quad f(\gamma(t)) = \varphi_t \gamma(t).$$

It is easy to verify that this mapping establishes a homeomorphism between the spaces $\Omega(x_0, x)$ and $\Omega(x_0, y)$.

**11.22.** The existence of a convergent sequence in any infinite sequence of points on a finite-dimensional sphere follows, e.g., at least from this fact being valid for infinite-dimensional sequences on the unit line-segment. Each vector is specified by a set of $n + 1$ real numbers (coordinates of the vector). It is clear that we use here the finiteness of the number of coordinates. An infinite sphere is non-compact: it is easy to find a sequence from which a convergent one cannot be singled out. E. g., the endpoints of the unit vectors of an orthonormal frame can be taken as such a sequence. Since the distance between any two of such (non-coincident) points equals $\sqrt{2}$, a convergent sequence cannot be singled out.

**11.34.** No. If a topological, metric and compact space is connected, then it is not necessarily path-connected, the well-known example being the set of points on the plane $(x, y)$ specified as follows:

$$\left\{y = \sin \frac{1}{x}\right\} \cup \{(x = 0; -1 \leqslant y \leqslant 1)\}.$$

# 12
# Homotopy Theory

**12.2. Axiom (W).** If $K$ is a $CW$-complex, then the set $F \subset K$ is closed if and only if, for all cells $e_i^q$, the full inverse image $(f_i^q)^{-1}(F) \subset B^q$ is closed in $B^q$. Assume that there are two topologies in the space $X$: $\{U_\alpha\}$ and $\{V_\beta\}$. We will say that $\{V_\beta\} \geqslant \{U_\alpha\}$ (stronger) if for any point $x \in X$ and any $V_{\beta_0} \ni x$, there is $U_{\alpha_0} \ni x$ such that $U_{\alpha_0} \subset V_{\beta_0}$.

Assume that apart from the topology determined by axiom $(W)$, there is another topology $\{U_\alpha\}$ in the $CW$-complex. Take an arbitrary point $x \in K$, i.e., a point belonging to the $CW$-complex, and $U_{\alpha_0} \ni x$. A neighbourhood is the union of mutually disjoint open intersections $(e_i^q \cap U_{\alpha_0})$. Consider the full inverse image $(f_i^q)^{-1}(U_{\alpha_0})$. It is open in $B^q$ (this follows from the continuity of the mappings $f_i^q$). Therefore, for the complement $(K \setminus U_{\alpha_0})$, the full inverse image $(f_i^q)^{-1}(K \setminus U_{\alpha_0})$ is closed in $B^q$ for all $e_i^q$. It follows from axiom $(W)$ that $(K \setminus U_{\alpha_0})$ is closed in $K$. Therefore, $U_{\alpha_0}$ is open in the topology determined by axiom $(W)$, i.e., $U_{\alpha_0}$ belongs to the system of open sets determined by $(W)$.

**12.21.** The Klein bottle.

**12.26.** Let $\alpha_1, \alpha_2 \in H(X', Y)$, $\alpha_1 \sim \alpha_2$. This means that there exists a homotopy $F: X' \times I \to Y$ such that $F(x, 0) = \alpha_1(x)$, $F(x, 1) = \alpha_2(x)$. Put $F' = F \cdot \varphi$. Then $F' : X \times I \to Y$, $F'(x, 0) = F(h(x), 0) = = \alpha_1(h(x))$, $F'(x, 1) = F(h(x), 1) = \alpha_2(h(x))$. Therefore, $\alpha_1 \cdot h \sim \alpha_2 \cdot h$.

**12.28.** Let $S^\infty = \lim\limits_{n \to \infty} S^n$, where $S^{n+1}$ is the suspension of $S^n$. The sphere $S^\infty$ so defined is a $CW$-complex. Consider $\alpha \in \pi_i(S^\infty)$ and $f \in \alpha$ : $f: S^i \to S^\infty$, $f$ sending the base point in $S^i$ to the base point of $S^\infty$. Let $f: K \to L$ be a continuous mapping of a complex $K$ into a complex $L$, the map being cellular on the subcomplex $K_1 \subset K$. Then there exists a map $g: K \to L$ such that (a) $f$ is homotopic to $g$; (b) $g$ is cellular on $K$; (c) $f|_{K_1} \equiv g|_{K_1}$; (d) the homotopy connecting $f$ and $g$ is the identity on $K_1$. Therefore, there exists a mapping homotopic to $f$ which transforms $S^i$ into the $i$-dimensional skeleton of $S^\infty$, i.e., into $S^i$, but $S^i \subset S^{i+1} \subset S^\infty$. Consequently, $g: S^i \to S^{i+1}$. Since $\pi_i(S^n) = 0$ when $i < n$, any mapping $f \in \alpha \in \pi_i(S^\infty)$ is homotopic to the mapping sending the whole of $S^i$ to the base point of $S^n$ (i.e., constant mapping). This means that the mapping $f: S^i \to S^\infty$ is homotopic to constant. The mapping $f$ has been chosen arbitrarily. Therefore, $\pi_i(S^\infty) = 0$.

If $X$ and $Y$ are cell complexes and a mapping $f: X \to Y$ induces the isomorphism of all the homotopy groups, then $f$ is a homotopy equivalence. We take the mapping $S^\infty \to *$ as $f$. The isomorphism of the homotopy groups is induced, since all of them are zero. Therefore, the sphere $S^\infty$ is homotopy equivalent to a point, and $S^\infty$ contractible to a point.

**12.30.** Let $p^{-1}(x_0) = F_0, p^{-1}(x_1) = F_1$, and $\varphi_0 : F_0 \to X$ an embedding. Then $p \cdot \varphi_0 : F_0 \to x_0 \in Y$. Join $x_0$ to $x_1$ with a path, i.e., arrange for a homotopy between the mappings of the fibre $F_0$ into $x_0$ and $x_1$, viz., $\psi_t : F_0 \to Y, \psi_t(F_0) = \gamma(t)$, where $\gamma$ is our path. Then it follows from the covering homotopy axiom that there exists a covering homotopy (family of mappings $\varphi_t : F_0 \to X$) such that $(p \cdot \varphi_t)(F_0) = \gamma(t)$, i.e., $(p \cdot \varphi_1)(F_0) = \gamma(1) = x_1$, from which it follows that $\varphi_1(F_0) \subset F_1$. Thus, we have constructed the mapping $_\gamma\varphi_1 : F_0 \to F_1$ by means of the path $\gamma$. We now prove that $_\gamma\varphi_1$ depends only on the homotopy class of the path $\gamma$, i.e., if $\gamma_1$ is homotopic to $\gamma_2$ then $_{\gamma_1}\varphi_1$ is homotopic to $_{\gamma_2}\varphi_1$. Note that the constructed mapping $F_0 \to F_1$ does not depend on the choice of a covering homotopy in the sense that any two such mappings are homotopic. In fact, let $\varphi_t$ and $\xi_t$ cover $\psi_t$. Then the mapping $\varphi_1 : F_0 \to F_1$ is homotopic to $\varphi_0 : F_0 \to F_0, \varphi_0 = \psi_0$ whereas the latter is homotopic to $\psi_1 : F_0 \to F_1$. Now, let the family of paths $\gamma_t$ be given. We shall show that $_{\gamma_0}\varphi_1$ is homotopic to $_{\gamma_1}\varphi_1$. We have the mapping

$$_{\gamma_0}\varphi : F_0 \times I \to X; \qquad (p \cdot {_{\gamma_0}\varphi})(F_0 \times I) = \gamma_0.$$

In $Y$, there exists a homotopy of $\gamma_0$ into $\gamma_1$ which can be covered by a mapping $\Phi : (F_0 \times I) \times I \to X$ such that $\Phi|_{(F_0 \times I) \times 0} = {_{\gamma_0}\varphi}$ and $\Phi|_{(F_0 \times I) \times 1} = f_t$ with $(p \cdot f)(F_0 \times I) = \gamma_1$. Therefore, the mapping $f_t$ can be taken as a covering map for all $\gamma_1, f_1 = {_{\gamma_1}\varphi_1}$. Then $\Phi|_{(F_0 \times 1) \times I}$ is a homotopy between $_{\gamma_1}\varphi_1$ and $_{\gamma_2}\varphi_1$. Note furthermore that the mapping $_{(-\gamma)}\chi_1 : F_1 \to F_0$ can be constructed similarly by means of the path $(-\gamma)$. It remains to prove that the mapping $_{(-\gamma)}\chi_1 \circ {_\gamma\varphi_1}) : F_0 \to F_0$ is homotopic to the identity. But this mapping can be considered as the one induced by the path $\gamma + (-\gamma)$ which is, evidently, homotopic to a mapping into a point.

**12.37.** Let $S^k \times S^{n-k}$ be a cell complex with only four cells, viz., $e^0$, $e^k$, $e^{n-k}$, $e^n$. Consider the mapping $f : S^k \times S^{n-k} \to S^n$, where $f(e^0) = *$ is a certain point in $S^n$. We take the latter as the zero-dimensional cell in $S^n$.

By the cellular approximation theorem, there exists a map $g : S^k \times S^{n-k} \to S^n$ which is cellular and homotopic to $f$, the equality $f(e^0) = g(e^0)$ being held on $e^0$ and the whole homotopy connecting $f$ and $g$ coinciding on $e^0 \subset f$. Since $S^n$ consists of only two cells, viz., the zero-dimensional $(*)$ and $n$-dimensional, then, under the mapping $g$, the cells $e^k$ and $e^{n-k}$ are transformed into a point on $S^n$. We obtain that the mapping $g$ may not be equal to a constant only on the $n$-dimensional cell.

Therefore, all the mappings $S^k \times S^{n-k} \to S^n$ differ only by a certain mapping of the $n$-dimensional cell into $S^k \times S^{n-k}$, and then into $S^n$, which transforms the whole boundary into a point on $S^n$ (due to the path-connectedness of $S^n$, the choice of a point is immaterial). But they are mappings of type $S^n \to S^n$ (more exactly, a one-to-one correspondence can be established between $\pi(S^k \times S^{n-k}, S^n)$ and $\pi(S^n, S^n)$).

**12.49.** Let $(x^1, \ldots, x^n) \to (x^1, \ldots, x^n, 0)$ be the standard embedding $\mathbf{R}^n \to \mathbf{R}^{n+1}$ (in the form of a hyperplane). Consider two points $A = (0, \ldots, 0, 1)$ and $B(0, \ldots, 0, -1)$ in $\mathbf{R}^{n+1}$ and construct cones $C_A M$ and $C_B M$ with vertices at the points $A$ and $B$, respectively, and a common base $H \subset \mathbf{R}^n$. Then any deformation of the subset $\mathbf{R}^n \setminus H$ in $\mathbf{R}^n$ can be extended to a deformation of the subset $\Sigma(\mathbf{R}^n \setminus H)$ in $\mathbf{R}^{n+1}$.

**12.50.** Assume the contrary, viz., let $\mathrm{cat}(M^n) < l(M^n; G)$, i.e., that there exists a covering of $M^n$ by closed sets $X_1, \ldots, X_k, k < l(M^n; G)$ each of which contracts on $M^n$ to a point. Due to Poincaré duality, $H_k(M^n; G) \cong H^{n-k}(M^n; G)$, to the cocycles $x_1, \ldots, x_l$ there correspond cycles $y_1, \ldots, y_l$ and to the product $h = x_1 \wedge \ldots \wedge x_l$ (of the cocycles $x_1, \ldots, x_l$) there corresponds the cycle $\alpha = y_1 \cap \ldots \cap y_l$ which is the intersection of all the cycles $y_1, \ldots, y_l$. Since the Poincaré duality operator $D$ is an isomorphism, the intersection $y_1 \cap \ldots \cap y_l = \alpha$ is different from zero (i.e., the cycle $\alpha$ is not homologous to zero). Since each subset $X_i$ $(1 \leqslant i \leqslant k)$ is contractible on $M^n$ to a point, $H^*(M^n; X_i) = H^*(M^n)$ (where $* > 0$). Therefore, the cycle $y_i \in H_*(M^n)$ can be assumed to be homologous to the cycle $\bar{y}_i \in H_*(M^n; X_i)$, i.e., the carrier of the cycle $\bar{y}_i$ lies in $M^n \setminus X_i$ $(1 \leqslant i \leqslant k)$. Hence, it follows that the intersection $\bar{y}_1 \cap \ldots \cap \bar{y}_k$ (homologous to the intersection $y_1 \cap \ldots \cap y_k$) lies in the complement of (the union) $X_1 \cup \ldots \cup X_k$; the more so, $\bar{y}_1 \cap \ldots \cap \bar{y}_k \cap \ldots \cap \bar{y}_l \subset M^n \setminus (X_1 \cup \ldots \cup X_k) = \varnothing$, since $X_1, \ldots, X_k$ forms a covering of $M^n$. Since the intersection of the carriers of the cycles $\bar{y}_1 \cap \ldots \cap \bar{y}_l = \varnothing$, the corresponding product of the cocycles $x_1 \wedge \ldots \wedge x_l = 0$, which contradicts the condition that $x_1 \wedge \ldots \wedge x_l \neq 0$, and the theorem is thus proved.

**12.51.** Consider the fibration $(E, p, x)$, where $E$ is the space of all paths of the space $X$ starting at the point $x_0$, and $p$ the mapping associating each path with its endpoint. The total space $E$ is considered here in the compact-open topology. The fibre of this fibration is the space $\Omega X = \Omega_{x_0}$ of all loops of the space $X$ at the point $x_0$. It is easy to see that the space $E$ is contractible on itself to a point (each path is contractible on itself to the point $x_0$). Therefore, $\pi_n(E) = 0$, and the homotopy sequence of this fibration

$$\ldots \to \pi_{n+1}(E) \to \pi_{n+1}(X) \to \pi_n(\Omega_{x_0}) \to \pi_n(E) \to \ldots$$

generates the isomorphism $\pi_n(\Omega_{X_0}) \approx \pi_{n+1}(X)$. In particular, $\pi_1(\Omega_{X_0}) \approx \pi_2(X)$. The group $\pi_n(X)$ is Abelian when $n \geqslant 2$.

**12.52. Definition.** A space $X$ is said to be contractible if the identity mapping $X \to X$ is homotopic to the mapping $X \to X$ sending all $X$ to a point.

**Definition.** A space $X$ is said to be 1-connected if $\pi_1(X) = 0$.

Since $X$ is contractible, there exist $\varphi_t \colon X \to X$, $\varphi_0$ being the identity mapping $X \to X$, and $\varphi_1$ the mapping $X \to x_0 \in X$. Since the definition of a fundamental group does not depend on a base point (up to isomorphism), let $\gamma : I \to X$ be an arbitrary path on $X$, $\gamma(0) = \gamma(1) = x_0$, $\delta(\tau) \equiv x_0$, $\delta : I \to X$. The same homotopy $\varphi_t : X \to X$ stipulates that the loops $\gamma$ and $\delta$ are homotopic. Thus, any two paths on $X$ are homotopic, i.e., $\pi_1(X) = 0$.

**12.53.** We prove that (a) any element from $\pi_1(B^1_A)$ (where $B^1_A$ is the wedge of circumferences) is representable as the finite product of elements $\eta_\alpha^{-1}$ and $\eta_\alpha$, where $\eta_\alpha \in \pi_1(B^1_A)$ is the class of the mapping $i_\alpha$ (which is the standard embedding); (b) such a representation is unique up to cancelling the factors $\eta_\alpha$ and $\eta_\alpha^{-1}$ placed in a row.

(a) This is equivalent to $\pi_1(B^1_A)$ being a free group with the generators $\eta_\alpha$, $\alpha \in A$. Consider the mapping $f \colon S^1 \to B^1_A$. Represent each circumference $S^1$ and $S^1_\alpha \in B^1_A$ as the sum of three one-dimensional simplexes $P$, $Q$, $R$ and $P_\alpha$, $Q_\alpha$, $R_\alpha$. By the simplicial approximation theorem, the mapping $f$ is homotopic to a simplicial mapping $F$ of a certain subdivision of the complex $S^1$ into $B^1_A$. Multiply the mapping $F$ on the right by a homotopy $\varphi_t$, where $\varphi_0$ is the identity mapping, $\varphi_1$ transforms $P_\alpha$, $R_\alpha$ into a base point and stretches $Q_\alpha$ to the whole of $S^1_\alpha$. We obtain a mapping $\tilde{F}$ homotopic to the original. The mapping $\tilde{F}$ either transforms each of the equal parts, into which $S^1$ is divided, into a point or winds it round one of $S^1_\alpha$, $\alpha \in A$. The class of such a mapping in $\pi_1(B^1_A)$ is the product of elements of the form $\eta_\alpha$, $\eta_\alpha^{-1}$, and $e$ (identity element of the fundamental group), i.e., the constant mapping class.

(b) The product $\eta_{\alpha_1}^{\varepsilon_1} \ldots \eta_{\alpha_k}^{\varepsilon_k}$ ($\varepsilon_s = \pm 1$, $k \geqslant 1$), which has no $\eta_\alpha$ and $\eta_\alpha^{-1}$ in a row, is not equal to the unit element in $\pi_1(B^1_A)$, i.e., there exist no relations in $\pi_1(B^1_A)$. Under the covering map $\pi : T \to X$, the inverse image of each point $\pi(x) = D$ is found to be in one-to-one correspondence with the cosets of the group $\pi_1(X)$ relative to the subgroup $\pi_*(\pi_1(T))$. In particular, if $x_1, x_2 \in T$, $x \in X$, $\pi(x_1) = \pi(x_2) = x$, and $S$ any path from $x_1$ to $x_2$, then the loop $\pi(S)$ with vertex at the point $x$ is not homotopic to zero: otherwise, $x_1 = x_2$. Let $\eta = \eta_{\alpha_1}^{\varepsilon_1} \ldots \eta_{\alpha_k}^{\varepsilon_k}$, where $\eta_{\alpha_i}^{\varepsilon_i}$ is a loop traversed in the direction of a circumference of the wedge according to the sign of $\varepsilon_i$. Take $k + 1$ replicas of the wedge, and place them one over another. We take $\eta_{\alpha_i}$ in the first and second wedges, cut out a line-segment in both replicas, and join their ends "crosswise", while extending the projection $\pi$ to them. Similarly, we join the second wedge to the third by using $\eta_{\alpha_2}^{\varepsilon_2}$, etc. If there are two identical letters in the word $\eta$ one after the other, then two line-segments of the same circumference should be cut out. In doing so, the second operation precedes the first if $\varepsilon_i = 1$, and follows it otherwise. We obtain a $(k + 1)$-sheeted covering of $B_A^1$, the path $\eta$ being covered by a path starting at the lower point, and ending at the upper. This loop is not homotopic to zero.

**12.54.** Let $f: Y_1 \to Y_2$ and $g: Y_2 \to Y_1$ be two homotopy equivalences, i.e., $g \cdot f \sim \mathrm{Id}_{Y_1}; f \cdot g \sim \mathrm{Id}_{Y_2}$. We define the mappings $f_*: \pi_1(Y_1) \to \to \pi_1(Y_2)$ and $g_*: \pi_1(Y_2) \to \pi_1(Y_1)$ (if $\alpha: S^1 \to Y_1$, $\alpha \in \alpha \in \pi_1(Y_1)$, then $f^\alpha$ is the class of the loop $f \cdot \alpha : S^1 \to Y_2$). Since $f_* \cdot g_* = (f \cdot g)$, $f_* \cdot g_*: \pi_1(Y_2) \to \pi_1(Y_2)$ and $g_* \cdot f_*: \pi_1(Y_1) \to \pi_1(Y_1)$ are isomorphisms, whence $\pi_1(Y_1) = \pi_1(Y_2)$.

**12.55.** Let $\pi_1(X) * \pi_1(Y)$ be the free product of $\pi_1(X)$ and $\pi_1(Y)$. Let $\hat{X}\hat{Y}$ be two universal coverings of $X$ and $Y$, respectively. Let $x_0$ be a base point of $X$, $Y$ and the wedge $X \vee Y$. We construct the following covering: taking $\hat{X}$, we consider $p^{-1}(x_0)$, where $p: \hat{X} \to X$ is a covering map, and glue $\hat{Y}$ at each point $x_0^i \in p^{-1}(x_0)$. We identify $x_0^{i'}$ with $x_0^i$, where $x_0^i$ is a certain point from $p_1^{-1}(x_0)$ and $p_1: \hat{Y} \to Y$ a covering map. At each remaining point from $p_1^{-1}(x_0)$, and to each replica of $\hat{Y}$ "glued", we glue $\hat{X}$ in this manner, etc. The projection $p'': (X \vee Y) \to X \vee Y$ is defined in a natural manner, viz., each replica of $\hat{Y}$ is mapped into $Y$ via $p'$, and each replica of $\hat{X}$ into $X$ via $p$. It is obvious that the space obtained is a covering of $X \vee Y$. Consider the fundamental group $X \vee Y$, points $t_1$ and $t_2$ in $X \vee Y$ such that $t_1, t_2 \in (p'')^{-1}(x_0)$, and a path $\dot{\alpha}$. Under the projection $p''$, this path will be transformed into a certain loop $\alpha$ representing the class of $\dot{\alpha}$ in $\pi_1(X \vee Y)$. Note that it follows from the construction of the covering map and $\hat{X}$ and $\hat{Y}$ being 1-connected that the path from $t_1$ to $t_2$ is unique up to homotopy.

Let $\tilde{\alpha} \in \pi_1(X \vee Y)$ be decomposed in terms of the generators $\tilde{c}_i \in \in \pi_1(X)$ and $\tilde{b}_j \in \pi_1(Y)$, i.e., $\tilde{\alpha} = \tilde{c}_{i_1}^{\varepsilon_1}\tilde{b}_{j_1}^{\sigma_1}\tilde{c}_{i_2}^{\varepsilon_2} \ldots \tilde{b}_{j_n}^{\sigma_n}$. Then this *representation is unique up to the relations in* $\pi_1(X)$ and $\pi_1(Y)$. In fact, let $\tilde{\beta} = \tilde{c}_{i_1}^{\varepsilon_1} \tilde{b}_{j_2}^{\sigma_1} \ldots \tilde{c}_{i_n}^{\varepsilon_n}\tilde{b}_{j_n}^{\sigma_n} \sim 1$, where 1 is the constant loop at the point $x_0$ and not all $\varepsilon_k$ and $\sigma_s$ equal zero (we take a reduced word). Then $\tilde{\beta}$ can be realized as a path in $X \vee Y$ which, as it is obvious from the form of the covering map, is not closed; therefore, $\tilde{\beta} \approx 1$. Thus, we have obtained that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. The same result follows from the van Kampen theorem on expressing the fundamental group of a complex in terms of the fundamental groups of its subcomplexes and their intersections.

**12.56. Definition.** If $K$ is a knot, then the fundamental group $\pi_1(\mathbf{R}^3 \setminus K)$ is called the knot group.

Let us find the corepresentation of this group. Consider the upper (or lower) corepresentation of the trefoil knot. Let $PK$ be its projection. The points $K_i$ ($i = 1, \ldots, 6$) divide the knot into two alternating classes of closed, connected arcs, viz., the class of overpasses and the class of underpasses. Let $A_1, A_2, A_3$ be the overpasses, $B_1, B_2$ $B_3$ the underpasses, and $F_3$ the free group with the generators $x, y, z$. We call a path $v$ in $\mathbf{R}^2$ simple if it is the union of a finite number of closed straight line-segments, its origin and endpoint do not belong to $PK$, and it meets $PK$ in a finite number of points which are not the vertices either of $PK$ or $v$. We associate each path $v$ with $v^{\#} \in F_3$: $v^{\#} = x_{i_1}^{\varepsilon_1} \ldots x_{i_l}^{\varepsilon_l}$, where $x_{i_k}$ are the generators of the free group, $\varepsilon_k = 1$ or $\varepsilon_k = -1$ depending on how $v$ passes under $A_{i_k}$. The upper corepresentation of the group $\pi_1(\mathbf{R}^3 \setminus K)$ is of the form

$$(x, y, z; r_1, r_2, r_3), \tag{1}$$

where $r_i = v_i^{\#}$ are the relations. The upper corepresentation determined by formula (1) is known to be the corepresentation of $\pi_1(\mathbf{R}^3 \setminus K)$. The loops $v_1$, $v_2$, $v_3$ around the overpasses ($x, y, z$ are the generators) satisfy the equalities

$$v_1^{\#} = x^{-1}yzy^{-1}, \quad v_2^{\#} = y^{-1}zxz^{-1}, \quad v_3^{\#} = z^{-1}xyx^{-1}.$$

We have obtained the corepresentation $(x, y, z; x = yzy^{-1}, y = zxz^{-1}, z = xyx^{-1})$. Substitute $z = xyx^{-1}$, then

$$\pi_1(\mathbf{R}^3 \setminus K) = (x, y; x = yxyx^{-1}y^{-1}, y = xyx). \tag{2}$$

Thus, $\pi_1(\mathbf{R}^3 \setminus K) = (x,\ y;\ xyx = yxy)$. It is impossible to untie the trefoil knot, since its type is different from the trivial knot type. If two knots $K'$ and $K''$ have the same type, then their complementary spaces possess coincident fundamental groups. The group $G = (x,\ y;\ xyx = yxy)$ is not the infinite, cyclic group $Z$. In fact, a homomorphism $\theta : G \to S_3$ can be constructed, where $S_3$ is generated by the cycles (12) and (23).

Let $K'$ and $K''$ be two connected subcomplexes of a connected $n$-dimensional, and simplicial complex $K$, each simplex from $K$ belonging to at least one of these subcomplexes. Their intersection $D = K' \cap K''$ is neither empty nor connected. Let $F$, $F'$, $F''$, $F_D$ be the fundamental groups of the complexes $K$, $K'$, $K''$, $D$. We take $0 \in D$ as the starting point of the closed paths. Then each closed path of the complex $D$ is, at the same time, a path of the complexes $K'$ and $K''$. We refer here to the well-known van Kampen theorem. The group $F$ is obtained from the free product $F' \times F''$ if each pair of elements of $F'$ and $F''$ corresponding to the same element of $F_D$ are identified, i.e., assuming these elements to be equal, we thereby add relations to the generators of the groups $F'$ and $F''$.

We now find the fundamental group of the "helical" knot defined as follows. Draw generators on the lateral surface of a circular cylinder at the distance of $2\pi/m$ from each other, and then rotate the upper base through $2\pi n/m$. Then, identifying the bases, add one point at infinity ($\infty$) and thereby turn $\mathbf{R}^3$ into $S^3$. Remove from $S^3$ all the points belonging to the tubular neighbourhood of the knot. We obtain a polyhedron $K$, the complement of the knot. Divide $S^3$ into two parts by the torus which contains the "helical" knot. The complex $K$ is then divided into two solid tori each of which has been stripped of the knot tubular neighbourhood on the surface. We take one solid torus as $K'$, and the other (with the point at infinity) as $K''$. The fundamental group $F'$ (resp. $F''$) of the polyhedron $K'$ (resp. $K''$) is a free group with one generator $A$ (resp. $B$). The generator $A$ can be represented as the midline of the solid torus of the polyhedron $K$ (the same be done with $B$). The intersection $D$ of both the solid tori is a twisted annulus. The fundamental group $D$ is also free with one generator which we take to be the midline of the annulus. The group $F' \odot F''$ is a free group with the generators $A$ and $B$. For an appropriate orientation of the paths $A$ and $B$, the path $C$ considered as an element of the group $F'$ equals $A^m$, and as an element of the group $F''$, it equals $B^n$. We obtain the relation $A^m = B^n$. Thus, the corepresentation of the group $\pi_1(S^3 \setminus \gamma)$ is $\{A, B; A^2 = B^3\}$, where $\gamma$ is the trefoil knot.

The two corepresentations of the fundamental group of the trefoil knot obtained are equivalent. We leave the verification of this proposition to the reader.

**12.58.** We choose the point 0 belonging to $W$ as the starting point of the closed paths. Then each closed path of the complex $W$ is, at the same time, a path of the complexes $Z$, $Y$, i.e., to each element of the group $\pi_1(W)$ there correspond an element of the group $\pi_1(X)$ and an element of the group $\pi_1(Y)$. We represent $Z$, $Y$, $W$ as simplicial complexes. Join each vertex of $X$ to 0 with a path. If the vertex lies in $W$, then the path may be drawn in $W$ wholly (because of the connectedness). A simplex of an arbitrary dimension of the complex $X$ belongs either to $Z$ (but not to $Y$) or $Y \setminus Z$ or $Y \cap Z$. The set of all simplexes is thus divided into three disjoint subsets $\bar{Z}$, $\ddot{Y}$, $\dddot{W}$. The generators $a_i$ of the group $\pi_1(X)$ can be put into one-to-one correspondence with the edges of the complex $X$. In accordance with that simplicial complex to which this edge belongs ($\bar{Z}$, $\ddot{Y}$ or $\dddot{W}$), we redesignate $a_i$ into $z_i$, $y_i$ or $w_i$, respectively. Thus, $\pi_1(X)$ has as its generators those of the fundamental groups $\pi_1(Y)$ and $\pi_1(Z)$ (the generators of $\pi_1(W)$ being included in those of $\pi_1(Z)$ and $\pi_1(Y)$). The relations in the group $\pi_1(X)$ are in one-to-one correspondence with the edges and triangles of the complex $X$. Since the complex $X$ has been subdivided into three subsets, these relations also get into three classes. Let us write out the relations:

$$\varphi_j(w_i, z_i) = 1 \text{ (in } \bar{Z}), \tag{1}$$

$$\varphi_j'(w_i, y_i) = 1 \text{ (in } \bar{Y}), \tag{2}$$

$$\psi_j(w_i) = 1 \text{ (in } \bar{W}). \tag{3}$$

Relations (3) are defining relations for the group $\pi_1(W)$, relations (2) and (3) for the groups $\pi_1(Y)$ and $\pi_1(W)$, relations (1) and (3) for the groups $\pi_1(Z)$ and $\pi_1(W)$, and relations (1), (2), (3) for the group $\pi_1(X)$. These relations can be rewritten in the following manner:

$$\varphi_j(w_i', z_i) = 1, \qquad \psi_j(w_i') = 1, \tag{1'}$$

$$\varphi_j'(w_i'', y_i) = 1, \qquad \psi_j(w_i'') = 1, \tag{2'}$$

$$w_i' = w_i''. \tag{3'}$$

Relations (1′) and (2′) determine the free product of the groups $\pi_1(Z)$ and $\pi_1(Y)$, and (3′) implies that these elements of the groups $\pi_1(Z)$ and $\pi_1(Y)$ corresponding to the same element $w_i$ of the group $\pi_1(W)$ must be identified. In proof, we have used the fact that $W$ is connected, since otherwise the statement derived for the group $\pi_1(X)$ is incorrect. E.g., $Z = Y = I$ (a line-segment), $W = S^0$, $X = S^1$, $\pi_1(X) = Z$, $\pi_1(Z) = \pi_1(Y) = e$.

**12.78.** It is known that for any subgroup $G \subset \pi_1(X)$, there exists a covering map $p: \bar{X}_G \to X$ such that $\operatorname{Im} \pi_*(\pi_1(\hat{X}_G)) = G$. Introduce multiplication on $\hat{X}_G$. Let $\hat{e} \in p^{-1}(e)$, where $e$ is the unit element in $X$, and $\hat{x}, \hat{y} \in \hat{X}_G$. Join $\hat{e}$ to $\hat{x}$ and $\hat{y}$ with paths $\hat{x}_t$ and $\hat{y}_t: \hat{x}_0 = e, \hat{x}_1 = \hat{x}$, $\hat{y}_0 = e, \hat{y}_1 = \hat{y}$. Let $p(\hat{x}) = x$, and $p(\hat{y}) = y$. Then $x$ and $y$ are joined to $e$ with the paths $p(\hat{x}_t) = x_t$ and $p(\hat{y}_t) = y_t$, respectively. These two paths can be multiplied together in $X$, i.e., we can consider the path $z_t = x_t \times y_t$ which joins $e$ to the point $z_1 = z = xy$. Let $z_t$ be liftable to $\hat{z}_t$ in $X_G$, and $\hat{x} \times \hat{y} = \hat{z}_1$. It remains to verify the correctness of the definition. The following statement can be proved. Let $X$ be a groupoid with identity, and $\alpha, \beta \in \pi_1(X, e)$. Then $\alpha\beta = \alpha \times \beta$, meaning multiplication in $\pi_1(X, e)$ on the left-hand side, and multiplication in $X$ on the right-hand.

We omit the proof leaving it to the reader. The correctness of the definition follows from this statement immediately.

**12.79.** When $p > 0$ and $q > 0$, for any $n < p + q - 1$, the isomorphism holds: $\pi_n(S^p \vee S^q) \approx \pi_n(S^p) + \pi_n(S^q)$. Since the pair $(S^p \times S^q, S^p \vee S^q)$ is a relative $(p + q)$-dimensional cell, it follows from proposition 1 (see below) that $\pi_m(S^p \times S^q, S^p \vee S^q) = 0$ when $m < p + q$. Therefore,

$$\pi_m(S^p \vee S^q) = \pi_m(S^p) + \pi_m(S^q).$$

If for the triple $(X, A, x_0)$, the pair $(X, A)$ is a relative $n$-dimensional cell, then $\pi_m(X, A, x_0) = 0$ when $0 < m < n$. The proof is left to the reader.

**12.81.** It follows from the Freudenthal theorem that the excision homomorphism $\pi_m(U, S^n) \to \pi_m(S^{n+1}, V)$ is an isomorphism when $m < 2n$, and an epimorphism when $m = 2n$, where $U$ and $V$ are the north and south hemispheres of $S^{n+1}$. We find $\pi_3(D^2, \partial D^2)$:

$$\ldots \to \pi_n(\partial D^2) \to \pi_n(D^2) \to \pi_n(D^2, \partial D^2) \to \pi_{n-1}(\partial D^2) \ldots \to .$$

When $n = 3$, we have $\pi_3(\partial D^2) = \pi_3(S^1) = 0$, $\pi_3(D^2) = 0$, $\pi_2(\partial D^2) = 0$. Hence, $\pi_3(D^2) \approx \pi_3(D^2, \partial D^2) = 0$. From the exact sequence, $\pi_3(S^2) = \pi_3(S^2, D^2) = \mathbf{Z}$.

**12.82.** The proof follows from the exact homotopy sequence of the Serre fibration.

**12.83.** The proof follows from the cellular representation of the projective space, and from the investigation of the standard covering map.

**12.85.** Let us prove that $\pi_1(\mathbf{C}P^n) = 0$, where $\mathbf{C}P^n$ is a cell complex having one cell in each even dimension, i.e., having no one-dimensional cells. By the theorem on the fundamental group of a cell complex with one zero-dimensional cell, we obtain that $\pi_1(\mathbf{C}P^n) = 0$. Further, the sphere $S^{2n+1}$ fibres over $\mathbf{C}P^n$ with the fibre $S^1$. In fact, let $S^{2n+1} \subset \mathbf{C}^{n+1}$ (standard embedding). The point $(z_1, \ldots, z_{n+1}) \in S^{2n+1}$ if and only if $\Sigma |z_i|^2 = 1$. Further, $\mathbf{C}P^n = \{(z_1, \ldots, z_{n+1})$ up to multiplication by $\lambda$, i.e., $\lambda(z_1, \ldots, z_{n+1}) = (z_1, \ldots, z_{n+1})\}$. Set the mapping $p : S^{2n+1} \to \mathbf{C}P^n$, $p(z_1, \ldots) = (z_1, \ldots)$. It is continuous, and its image is the whole space $\mathbf{C}P^n$. Over the point from $\mathbf{C}P^n$, the following set of points from $S^{2n+1}$ "hangs": let $(z_1, \ldots, z_{n+1}) \in \mathbf{C}P^n$, then

$$f^{-1}(z_1, \ldots, z_{n+1}) = \{e^{i\varphi}(z_1, \ldots, z_{n+1})\} \subset S^{2n+1},$$

where $f^{-1}$ is the full inverse image, and $0 \leqslant \varphi \leqslant 2\pi$. In fact, $e^{i\varphi_1}(z_1, \ldots, z_{n+1})$ and $e^{i\varphi_2}(z_1, \ldots, z_{n+1})$ are the same point in $\mathbf{C}P^n$, but if $\varphi_1 \neq \varphi_2$, then these are distinct points in $S^{2n+1}$. Therefore, the mapping $p : S^{2n+1} \to \mathbf{C}P^n$ is a fibre map. It now remains to apply the exact homotopy sequence of the fibre map.

**12.86.** The statements follow from the cellular approximation theorem.

**12.87.** Let $P_X : X \times Y \to Y$, $P_Y : X \times Y \to Y$ be two projections. Set the homomorphism $\varphi : \pi_i(X \times Y) \to \pi_i(X) \oplus \pi_i(Y)$; viz.,

$$\varphi(\alpha) = (p_{X*}\alpha, p_{Y*}\alpha).$$

(a) $\varphi$ is a homomorphism: $\varphi(\alpha + \beta) = (p_{X*}(\alpha + \beta), p_{Y*}(\alpha + \beta)) = = (p_{X*}\alpha, p_{Y*}\alpha) \oplus (p_{X*}\beta, p_{Y*}\beta) = \varphi(\alpha) \oplus \varphi(\beta)$.

(b) $\varphi$ is a monomorphism. Let $\varphi(\alpha) = 0$, i.e., $P_{X*}\alpha = 0, P_{Y*}\alpha = 0$. Therefore, $\psi_X = P_X \circ \alpha : S^n \to X$ is homotopic to a constant mapping, i.e., there exists $\psi_{Xt} : S^n \to X$ such that $\psi_{X0} = \psi_X, \lambda\psi_{X1} = *$. Then we set the homotopy $\Phi_t$ thus: $\Phi_t(\alpha) = (\psi_{Xt}(\alpha), p_{Y*}(\alpha))$ when $t = 0$, $\Phi_t(\alpha) = \alpha$ when $t = 1$. $\Phi_t(\alpha)$ is a mapping of $S^n$ into $(*) \times Y \subset X \times Y$, $(*, p_{Y*}(\alpha)) \in \pi_n((*) \times Y) = \pi_n(Y)$. But $p_{Y*}(\alpha)$ is a contractible spheroid, i.e., $\alpha$ is contractible.

(c) $\varphi$ is an epimorphism. Let $\beta \in \pi_n(X), \gamma \in \pi_n(Y)$. Then $\alpha = (\beta, \gamma)$ is transformed into $\beta \oplus \gamma$ under $\varphi$.

Let there be two universal coverings: $E_1 \xrightarrow{p_1} X$, $E_2 \xrightarrow{p_2} Y$. Consider the mapping $p_1 \times p_2 : E_1 \times E_2 \to X \times Y$, $(p_1 \times p_2)(e_1 \times e_2) = = (p_1 e_1 \times p_2 e_2)$. We assert that this is a covering. The proof is left to the reader.

Let $\gamma_1 \in \pi_1(X, x_0), \gamma_2 \in \pi_1(Y, y_0), \alpha_1 \in \pi_n(X, x_0), \alpha_2 \in \pi_n(Y, y_0)$, and a homotopy $F_t$ along the path $\gamma_1$ of the spheroid $\alpha_1$ be given so that $F_0(\alpha_1) = \alpha_1, F_1(\alpha_1) = \gamma_1[\alpha_1]$, $F_t(\alpha_1) \in \pi_n(X, \gamma_1(t))$. Similarly, $\Phi(t)$: $\Phi_0(\alpha_2) = \alpha_2, \Phi_1(\alpha_2) = \gamma_2[\alpha_2], \Phi_t(\alpha_2) \in \pi_n(Y, \gamma_2(t))$. Define a homotopy along the loop $\gamma = (\gamma_1 \oplus \gamma_2)(t)$ into $X \times Y$ of the spheroid $\alpha = (\alpha_1 \oplus \alpha_2)$ as $F_t(\alpha_1) \times \Phi_t(\alpha_2)$. Then $F_1(\alpha_1) \times \Phi_1(\alpha_2) = = \gamma_1[\alpha_1] \oplus \gamma_2[\alpha_2]$. Thus, $[\gamma_1 \oplus \gamma_2][\alpha_1 \oplus \alpha_2] = \gamma_1[\alpha_1] \oplus \gamma_2[\alpha_2]$. But since any loop $\gamma$ and any spheroid from $X \times Y$ are of the forms $\gamma_1 \oplus \gamma_2$ and $\alpha_1 \oplus \alpha_2$ for certain $\gamma_1 \in \pi_1(X), \gamma_2 \in \pi_1(Y)$, $\alpha_1 \in \pi_n(X)$, $\alpha_2 \in \pi_n(Y)$, then the action of $\pi_1(X \times Y)$ on $\pi_n(X \times Y)$ has been fully determined.

**12.89.** Use the Hopf map $S^3 \to S^2$. It is arranged as follows:

$$S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\} \in \mathbf{C}^2, S^2 = \mathbf{C}P^1,$$

i.e.,

$$S^2 = \{(z_1, z_2) : (\lambda z_1, \lambda z_2) \sim (z_1, z_2)\}.$$

We obtain the fibre map $S^3 \to S^2$. The exact sequence may be written for this fibre map, viz.,

$$\ldots \to \pi_i(S^1) \to \pi_i(S^3) \to \pi_i(S^2) \to \pi_{i-1}(S^1) \to \ldots$$

From the properties of the sequence, $\pi_i(S^3) = \pi_i(S^2)$ when $i \geqslant 3$. By the Freudenthal theorem, the homomorphism $\pi_{i-1}(S^{n-1}) \to \pi_i(S^n)$ is an epimorphism when $i \leqslant 2n - 2$, and an isomorphism when $i < 2n - 2$, i.e., the homomorphisms

$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3)$$

are isomorphisms. Therefore, $\pi_3(S^3) = \mathbf{Z}$. Since $\pi_i(S^3) = \pi_i(S^2)$ $(i \geqslant 3)$, $\pi_3(S^2) = \mathbf{Z}$.

# 15
# Simplest Variational Problems

**15.1.** It suffices to consider the Euler equations for the action functional and write these equations in local coordinates. In doing so, use explicit formulae for the Christoffel symbols.

**15.2.** The Euler equations for both functionals should be written out, and then the behaviour of the functionals when changing the parameter (i.e., time) on the extremal solutions considered. The required statement follows from the length functional being invariant under parameter changes and the action functional being not invariant.

**15.3.** The proof is reduced to the direct computation: the Euler equation in Cartesian coordinates should be written out, and the explicit formulae for the mean curvature used (the mean curvature is calculated for the graph of a smooth function).

**15.4.** The proof is similar to that in Problem 15.3. This analogy is based on the codimension of the graph of the function being equal to unity in both problems. Therefore, the mean curvature tensor is given by one function only (by the mean curvature itself).

**15.5.** Use the classical inequality

$$\left(\int_a^b fg\,dt\right)^2 \leqslant \left(\int_a^b f^2dt\right)\left(\int_a^b g^2dt\right).$$

**15.6.** Square the integrands and compare them.

**15.7.** Use local Beltrami-Laplace equation theory and write the equation in a curvilinear coordinate system. It follows from the theory that conformal coordinates can be always locally introduced on a two-dimensional surface (given by analytic functions), while adding the condition that the mean curvature equals zero transforms these coordinates into harmonic.

**15.8.** For example, the function $\vec{r}(u, v) = (u, v, u^2 - v^2)$.

**15.9.** (c) The calculations performed in point (b) are of local character, which enables us to carry them out in a neighbourhood of each point on the Kähler manifold. On the contrary, Stokes' formula is valid for any smooth manifold (recall that the Kähler exterior 2-form is closed).

**15.10.** By the implicit function theorem, one of the coordinates, e.g., $X^n$, can be expressed (on the level surface $F_0 =$ const) as a smooth function of the other coordinates. Substitute this function in the Euler equation for the extremal of the functional $J$.

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