MATHEMATICAL TRACTS.

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MATHEMATICAL TRACTS.

PART I.

BY

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TRACT I.

ON THE BASES OF GEOMETRY WITH THE GEOMETRICAL TREATMENT OF $\sqrt{-1}$.

CONTENTS.

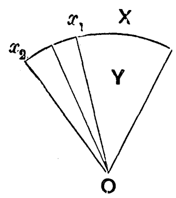
I. The Ratio of Incommensurables.

1. In arithmetic the first ideas of ratio and proportion, and the laws of passage from one set of 4 proportionals to another, ought to be learned, as preliminary to geometry; but in geometry the doctrine of incommensurables requires a special treatment, unless the learner be well grounded in the argument of infinite converging series. Repeating decimals may perhaps suffice. Another, possibly better way, is open by the introduction of Variable quantities, which will here be proposed.

2. Nothing is simpler than to imagine some geometrical quantity to vary in shape or size according to some prescribed law. This must imply at least *two* quantities varying *together*. Thus, if an equilateral triangle change the length of its *side*, its *area* also changes. If the radius of a circle increase or diminish, so does the length of the circumference. In general two magnitudes X and Y may *vary together:* they may be either the same in kind,—as the radius and circumference of a circle is each *a length;* or the two may be different in kind, say, a length and an area. In general it is a

convenient notation to suppose that when X changes to X', Y changes to Y'.

3. Again, if X receive successive additions $x_1 x_2 x_3 \ldots x_n$, the corresponding additions (if additions they be) to Y are well denoted by $y_1 y_2 y_3 \ldots y_n$. An obvious and simple case, if it occur, will deserve notice; namely, if the two variables are so regulated, that equality in the first set of additions (i.e. $x_1 = x_2 = x_3 = \ldots = x_n$) induces equality in the second set; (i.e. $y_1 = y_2 = y_3 = \ldots = y_n$). The variables X and Y are then said to *increase uniformly*. As an obvious illustration, suppose X to be the *arc* of a circle, and Y the area of the *sector* which it bounds, evidently then if the arcs x_1, x_2 are equal increments of the arc X, the sectors $y_1 y_2$ which are bounded by $x_1 x_2$ will be equal increments of Y. Then the arc X and the sector Y increase together *uniformly*.



4. We may now establish a theorem highly convenient for application in geometry, alike whether quantities are commensurable or incommensurable.

THEOREM. "If X and Y are any two connected variables, which begin *from zero* together, and increase *uniformly;* then X varies proportionably to Y. In other words, if Y become Y' when X becomes X', then X is to X' as Y is to Y'."

Proof. First, suppose X and X' commensurable, and ξ a common measure, or $X = m \cdot \xi$ (m times ξ) and $X' = n\xi$. We may then suppose X and X' made up by repeated additions of ξ. Every time that X has the increment ξ, Y will receive a uniform increment which we may call v; then Y is always the same multiple of v that X is of ξ; thus the equation $X = m\xi$ implies $Y = mv$, and $X' = n\xi$ implies $Y' = nv$. Hence $X : X' = m : n = Y : Y'$.

Next, when X' is *not* commensurate with X, yet ξ is some *sub*-multiple of X, such that $n\xi = X$, and X' contains ξ more than m times, but less than $(m+1)$ times; evidently we *cannot* have

$$X : X' = Y : Y'$$

(when the four magnitudes are presented to us) unless, as a first condition, on assuming $nv = Y$, we find Y' to contain v more than m times and less than $(m+1)$ times: and unless this condition were fulfilled, X and Y would *not* increase *uniformly*. We may therefore assume X_2X_3 on opposite sides of X', with values

$$X_2 = m\xi,\ X_3 = (m+1)\,\xi\,;$$

likewise Y_2Y_3 on opposite sides of Y', with values

$$Y_2 = mv,\ Y_3 = (m+1)\,v.$$

Then by the first case we have $X : X_2 = Y : Y_2$ and $X : X_3 = Y : Y_3$. But $X_3 - X_2 = \xi$, and $Y_3 - Y_2 = v$. Let n perpetually increase, then ξ and v perpetually lessen. X_2 and X_3 run together in X', Y_2 and Y_3 run together in Y'. Thus each of the ratios $X : X_2$ and $X : X_3$ falls into $X : X'$, and each of the ratios $Y : Y_2$, $Y : Y_3$ falls into $Y : Y'$. Inevitably then, $X : X' = Y : Y'$, even when these last are incommensurate. Q.E.D.

II. Primary ideas of the Sphere and Circle.

For the convenience of *beginners*, Postulates may be advanced concerning the straight line and the plane, as well as concerning parallel straight lines. But in the second stage of study the whole topic ought to be *treated anew* from the *beginning:* a task which is here assumed.

On Length and Distance.

Theorem. "All lengths are numerically comparable." To make this clear, it is simplest to imagine a thread indefinitely thin, flexible and inextensible. This, if applied upon any given line, will become an exact measure of its length; and if any two lines be then measured by two threads, the threads are directly comparable, shewing either that they are equal, or that one is longer than the other and how much longer. Hereby we safely assert the same fact concerning any two given lengths.

Obviously, length is *continuous* magnitude: which means, that if a point P run along from A to B, the length AP passes *through all magnitude* from zero to AB.

Theorem. Any two given points in space may be joined *either* by one path which is shorter than any other possible, *or* by several equal paths than which none other is so short. For of all possible paths joining them some must be needlessly long; yet unless there is some limit to the shortening, the distance would be *nil;* the points would not be two, but would coincide and become one.

DEF. A shortest path that joins two points in space gives a measure of their DISTANCE. The same argument applies, if the two given points and the line that joins them must lie on a given surface; or again, if two surfaces that do not touch be given, and we speak of the shortest distance of the two surfaces.

Assume a fixed point A and a second point S *so* movable as always to be *at the same distance* from it. It will be able to play all round A: therefore *its locus* will be a surface enclosing A. The solid mass enclosed is called a SPHERE (*Globe* or *Ball*) and A its CENTRE.

THEOREM. "Every point *outside* the sphere is *further* from the centre and every point *within* the sphere is *nearer* to the centre, than are the points on the surface." For if T be an exterior point, every path joining T to A must pierce the surface in some point S; therefore the path TSA is longer than SA by the interval TS. Again, if R be within the sphere, we may imagine an interior sphere whose surface is at the common distance AR from A. Then S being exterior to the new sphere, SA is longer than RA; that is, R within the sphere of S is nearer to A than is the locus of S. Q. E. D.

DEF. Two such concentric spheres enclose within their surfaces a solid called a spherical *shell.*

THEOREM. "The two surfaces are equidistant, each from the other." For if the shortest distance from a point S to the inner surface is the path SR, *symmetry* all round shews at once that if from a second point S' the shortest path will be $S'R'$, the two distances SR, $S'R'$ will be equal. Indeed it is not amiss to remark, that if any spherical surface be rigidly attached to its centre, the entire surface may glide on its own ground without disturbing its centre, because the distances SA, $S'A$ nowhere change. Hence also we may justly imagine the spherical *shell* to glide *on its own ground,* while the centre suffers no displacement, and any shortest path $S'R'$ joining the opposite sides of the shell may assume the place which was previously held by SR. *Actual superposition* thus attests equality of distance.

THEOREM. "If a spherical surface be given, its centre is determined." For if an inner point R be assumed at a given distance D from the surface, its *locus* is an *interior* continuous surface. Within this, at distance D', imagine a point R' to generate a second continuous surface, and it will be interior to the preceding and so continually. The series of surfaces must then necessarily converge towards a single point, which will be the centre of the given surface, because the sum of the distances is the same, from whichever point we calculate. The same argument proves that all the surfaces are concentric *spheres.*

Poles of a Sphere.

THEOREM. "To every point on a sphere one opposite point lies at the longest distance along the surface." For if the point P be given, and we take a point S at any distance from P *along the surface*, and suppose S to vary under the sole condition that its distance from P (along the surface) shall not change, the locus of S is a self-rejoining line enclosing P. (We call this a circle.) Next, *beyond* S, along the surface, take a new point T, which moves without changing its distance from S and from P. This generates an outer circle, cutting off a part of the surface which was beyond the circle of S. Beyond this we may similarly form a third circle, and this series of circles ever lessening the *finite* area beyond it, will necessarily converge towards a point Q on the sphere. P will then be farther from Q (along the sphere) than any of these parallel circles. We call P and Q *opposite poles* of the sphere. The distance between them is evidently the *half girth* of the sphere.

Every point on the sphere has not only its own opposite pole; but also its system of equidistant (or parallel) circles. The middle one of these (that is, the one equidistant from the two poles), is called their equator.

If in an equator whose poles are P and Q, you fix any point C, and then proceeding half round the equator fix a second point D, C and D are evidently opposite poles.

If you imagine a sphere to glide on its own ground, with centre unmoved, you may suppose P to pass over to the site held previously by Q. This carries Q to the place previously held by P. Thus the poles are exchangeable, while the sphere *as a whole* is unchanged and the same equator is attained.

THEOREM. "If P and R be any two points on a sphere that are *not* opposite poles, one equator, and one only, passes through them both."

Proof. Through P and its opposite pole Q (just as above through the poles C and D) an equator may pass. If this half equator PQ become rigid and be rigidly attached to the fixed centre A, it still may sweep over the spherical surface (without change of P or Q) until it passes through R; but after passing once through R, it does not come back to it, except in a second revolution. Q. E. D.

III. POINTS LYING EVENLY.

In Simson's Euclid, the line whose *points lie evenly* is called STRAIGHT; but the phrase "lying evenly" is not explained. We can now explain it.

When the two poles P and Q, and the centre A, all remain unchanged, nevertheless each of the parallel circles associated with P and Q can glide on their own ground. Evidently then, if P and A be fixed, this suffices to fix Q. In fact while each circle spins round its own line, Q can only spin *round itself.* Also, to fix P and Q fixes A.—These parallel circles excellently define to us the idea of rotation, which is a constrained motion, still possible, even when P, A, Q are all fixed. Now suppose that a line PMQ internal to the sphere rigidly connects P with Q. Then if the system revolve round P, A, Q, PMQ may generate a self-rejoining surface within the sphere. Again within this new surface a rigid line PNQ may connect P with Q, and the line PNQ by rotation round P, A, Q may generate a third surface interior to the preceding; and so on continually. Since there is no limit to the constant thinning of the innermost solid, we see that a mere line without thickness connects P with Q and passes through A, which line is *interior to all the solids* and during rotation remains immovable. It is called an axis, and can only turn about itself. Hence every point in this axis *lies evenly* between P and Q.

And since P and Q may represent any two points in space, we now discover that between any two there is a unique line lying evenly. This continuous line, while we talk of rotation round it, is entitled an *axis;* but ordinarily we call it simply STRAIGHT.

On the Straight Line and its "Direction."

We now infer that

1. Any two points in space can be joined by a straight line.
2. Every part of a straight line is straight.

3. A unique straight line is *determined*, when its two end-points are given.

4. Any part of a straight line, if removed, may take the place of any equal part of the same. Hence it easily follows that a straight line, *gliding along itself*, will prolong itself indefinitely far, either way, along a determinate course.

We are now able to sharpen our idea of direction. Hitherto we might say vaguely, "Imagine a path to proceed *in any direction*," that is, without particular guidance. But now we see, that if ever so short a straight line be drawn, it points to a definite prolongation beyond itself, of indefinite extent. This we entitle its *direction*. If this direction be changed, a deviation there occurs, and a sharp corner is recognized at the point of deviation. The *amount* of deviation suggests a new kind of magnitude, which will presently need attention. Now it suffices to remark on the case in which a new line AZ deviates equally from a previous line PA and from AQ the prolongation of PA. The equality is tested by imagining AZ to become an axis of a sphere. Then if P and Q revolve in the *same* circle, ZA is equally inclined to AP and to AQ. It is called perpendicular to PAQ. Evidently Z (on the sphere) is at the distance of a quarter girth from every point of the equator traced by P and Q.

IV. The Plane.

We return to the sphere. When any two poles P, Q are joined *by a straight line*, it has been seen that this passes through the centre A. The line PAQ is called a *diameter* of the sphere, and its half (AP or AQ) is called a *radius*.

Evidently all the radii of the same sphere are equal; and of different spheres the greater the radius, the greater the sphere.

If an equator $CDEC$ is midway between the poles P, Q, and D is the pole opposite to C, then as the diameter PQ, so too the diameter CD, passes through centre A. This is true, whatever point in the equator is assumed for C. Therefore CAD is a varying diameter, whose extremities trace out the equator, while the diameter traces out a surface in which the equator lies. This surface is called a Plane, and in particular is the plane of the equatorial circle.

It was seen that P and Q might exchange places, while the centre A, and the sphere's surface *as a whole*, remain unchanged. Necessarily also the plane of the equator remains unchanged. It is then symmetrical on its opposite sides, or in popular language, the plane turns *the same face* towards P as towards Q.

The axis PAQ is called perpendicular to *the plane* of the equator, being perpendicular to *every radius* of the equatorial circle.

THEOREM. "No other line but AP can be perpendicular to the plane of the equator."

Proof. For if AR be some other radius of the sphere, some one of the parallel circles, whose pole is P, passes through R, and every point of this circle is nearer to the equatorial circle than is the pole P. Therefore the distance of R from the equator is less than a quarter of the sphere's girth, a fact which shews RA *not* to be perpendicular.

THEOREM. "Through any two radii AP, AR of a sphere, that are not in the same straight line, one plane and one only may pass."

It has been seen that through P and R only one equator can pass. The plane of this equator is the plane that passes through the two radii.

Cardinal Property of the Plane.

THEOREM. "If M and N are any two points in a plane, no point in the straight line which joins M and N can *lie off* the plane on either side."

Symmetry suffices to establish this truth. Our hypothesis supplies *data* to fix what line is meant by MN, but gives no reason why any point of it should lie off the plane on one side *rather than* on the other; for the whole line is determined by merely the extreme points M, N, of which neither can guide any point towards P rather than towards Q. Thus there is *no adequate reason* for deviation towards either side.

Symmetry of data is in other mathematical topics accepted as *an adequate argument* for symmetry of results. Otherwise, "the want of *sufficient reason* for diversity" passes as refutation of alleged diversity. Therefore the argument here presented has nothing really novel.

We have now a new method of generating a plane that shall pass through two intersecting straight lines LM, MN. Along ML let a point E run, and along MN similarly a point F. Join EF while the motion of E and of F continues. Then EF (by the last

Theorem) always continues to rest on the plane LMN. This mode of generating the plane supersedes the idea of rotation. For simplicity we might suppose $\overline{ME} : \overline{MF}$ to retain a fixed ratio.

THEOREM. "A plane has no unique point or centre."

For if we start from given spherical radii AP, AR through which passes an (equatorial) plane, in AP take M arbitrarily, and in AR take N arbitrarily. Then we have seen that the *locus* of the moving line MN is our given plane. But again, in this plane take a fixed point O, and join O to *fixed* points M and N. Then from the lines OM, ON we can (as in the last) generate the very same plane, which can glide on its own ground as the sphere did; thus the point A can pass to O without changing the ground or surface as a whole. The plane is infinite, the sphere is finite; but as with the sphere, so with the plane, no point of the surface is unique.

After this, no impediment from logic forbids our passing to the received routine of Plane Geometry, until we are arrested by the difficulty of parallel straight lines, to which I proceed, after one remark on the definition *of an angle*.

Above, a sharp corner or turn was identified with deviation, or change of direction. In geometry it has the name of an angle, and we measure its magnitude by aid of the circular arc which it subtends at the centre or by the sector of that arc. But no insuperable logic forbids our estimating the magnitude of an angle by the portion of *the infinite area* which it intercepts from a plane; which indeed is suggested by a perpetual elongation of the radius of the circle whose *sector* was assumed as measure of the angle.

Monsieur Vincent in Paris (1837) adopted this definition as adequate to demonstrate the equivalent of Euclid's Twelfth Axiom without any new axiom at all. Has this method received due attention in England?

Monsieur Vincent was not the first to suggest accepting *the infinite plane* area cut off by two intersecting straight lines, as the measure of the *angle* which they enclose: but perhaps he was the first to introduce the method into a treatise on Elementary Geometry, that obtained acceptance in so high an institution as the University of France.

Two lemmas alone are wanted, and these every beginner will find natural.

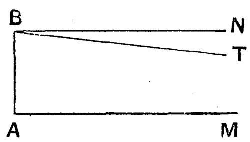
LEMMA I. "Every angle is a finite fraction of a right angle;" that is, some finite multiple of it exceeds 90°. For the circular arc which subtends it, is always some finite fraction of the quarter of the circumference.

DEF. When two straight lines AM, BN in the same plane are both perpendicular to a third straight line AB, we call that portion of the plane area which is enclosed between MA, AB and BN a BAND.

LEMMA II. Then, I say, whatever the breadth (AB) of the band, the area of the band is *less than* any finite fraction of a right angle.

Proof. Prolong AB indefinitely to X, and along it take any number of equal lengths $AB = BC = CD = DE$, &c., and through $C, D, E\ldots$ draw perpendicular to $ABCDE\ldots$ straight lines CO, DP, EQ, &c. Evidently then the successive bands are equal, by superposition. Thus, whatever multiple of the first band be *deducted* from the plane area marked off by the right angle MAX, the loss is insensible; for, as remainder, we find the area marked off still by a right angle (such as QEX, if only four bands were deducted). Any two right angles embrace areas which can be identified by superposition, and have no appreciable difference. The matter may be concisely summed up by remarking, that every band is infinite in *one* direction only,—say, horizontally—but the area embraced by any right angle is infinite in *both* directions, horizontally and also vertically. Thus it is no paradox to say, that no finite multiple of the band can, by its deduction from the area of the right angle, lessen that infinite area in our estimate. Q.E.D.

Euclid's Twelfth Axiom is now an immediate corollary; viz. If $MABN$ be any band; and, *within* the right angle NBA, any straight line BT be drawn, it can be prolonged so far as to meet the prolongation of AM. For the angle NBT is a finite fraction of a right angle, while



the band $MBAN$ is less than any finite fraction of the same; hence the angle NBT is greater than the band $MABN$, but unless BT crossed AM this would be false. Thus of necessity the two lines do cross, as we asserted.

I cannot see *any new* axiom involved in this proof: therefore I am forced to abandon several other specious methods and give it preference. Surely we may bow to the authority of the University of France in such a matter.

On the Volume of Pyramids and Cones.

The treatment of this topic in Euclid is very clumsy. It demands and it admits much improvement.

1. For parallelepipeda prove first, that if *two* such solids differ solely in the length of one edge, which we may call x in the one and a in the other, then their volumes are in the proportion of $x : a$.

2. Next, if they have a solid angle in common, but the edges round it are in one x, y, z, and in the other a, b, c, then the two volumes are in the proportion of $xyz : abc$.

3. After this it is easily shewn that parallelepipeda on *the same base* and *equal height* have equal volumes.

4. Therefore finally, that the volume of a parallelepipedon is measured by its base × its height. Cor. The same is true of any prism.

From this we proceed to approximate to the volume of a pyramid.

5. Divide the height (h) into (n) equal parts by $(n-1)$ planes all parallel to the base (B). Establish, on these $(n-1)$ bases, upright walls, and you will find you have constituted a double system of prisms, one interior to the pyramid, one exterior; the latter has the lowest prism in excess of the other system. Every base is similar to every other, by the nature of a pyramid. The volume here of every prism is $\frac{h}{n} \times$ its base, the number n and $\frac{h}{n}$ being the same for all, but the base varying.

6. The base whose distance from the vertex is $\frac{r}{n} . h$, is to the original (B) as $r^2 : n^2$; hence its area is $\frac{r^2}{n^2} . B$, which gives for the volume of the prism standing on it $\left(\frac{r^2}{n^2} . B\right) . \frac{h}{n}$. Hence the sum of the volumes of the *external* prisms is $\frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^2} . \frac{h}{n} . B$, and by omitting n^2 from the numerator of the larger fraction we obtain

the sum of volumes for the *internal* prisms. Now since $\frac{n^2 . h . B}{n^3}$ vanishes when h, B are finite and n infinite, the difference of the two systems of prisms vanishes when n is infinite. But the volume of the pyramid is less than the exterior system and greater than the interior; hence each system has the volume of the pyramid for *its limit*, when n increases indefinitely.

7. Let μ be the *unknown* numerical limit to which the fraction $\frac{1^2 + 2^2 + 3^2 + \ldots n^2}{n^3}$ approximates when n thus increases. Then the volume of the pyramid $= \mu . h . B$. Since μ is the limit of a numerical fraction, which remains the same, *whatever the form of the pyramid's base*, we shall know the value of μ for all pyramids, if we can find it in one. Meanwhile the result $V = \mu . h . B$ at once shews that pyramids with equal base and equal height have equal volume, since μ is the same for all.

8. When this theorem has been attained, we have only to divide a triangular prism into three pyramids, and instantly infer that the 3 are equal among themselves; therefore that each has a volume just $\frac{1}{3}$ of the prism, i.e. equal $\frac{1}{3}h . B$.

This, being proved of a pyramid whose base is a triangle, shews that the unknown μ is there exactly $\frac{1}{3}$.

Hence universally $\mu = \frac{1}{3}$, and volume of *every* pyramid $= \frac{1}{3}h . B$, or is equal to $\frac{1}{3}$ of the prism which has the same base and height.

Cor. Every *cone* also is one third of the *cylinder* which has the same base and height.

TRACT II.

GEOMETRICAL TREATMENT OF $\sqrt{-1}$.

1. In pure algebra, concerned with number only, the symbols + and −, denoting addition and subtraction, in an early stage needed elucidation when the mark of *minus* was doubled. It is found natural that $-(+a)$ and $+(-a)$ should both mean $-a$, but that $-(-a)$ should mean $+a$, and $(-a).(-b)$ should be $+(ab)$ surprises a beginner, and is illustrated by urging that to *subtract a debt* increases the debtor's property, and to *subtract cold* is to add heat. But as soon as we apply algebra to geometry, the symbols + and − are still better interpreted of *reverse direction;* also time past and time coming afford equally good illustration. Distinguishing *positive and negative direction* along a line, we find no mystery in the fact that to *reverse negative* direction is to make it positive, so that $-(-a)$ gives $+a$, as reasonably as $-(+a)$ gives $-a$. If we know beforehand whether a given distance is to be counted positively or negatively along a given axis, no ambiguity is incurred, and the sign + or − generally gives the needful information. For this reason some are apt to think of $+a$ and $-a$ as different numbers, instead of the same number differently *directed.* Out of this rises the learner's natural complaint when he meets $\sqrt{-1}$ or $\sqrt{-5}$. "*There is no such number: you confess it is imaginary: a proposition involving it has no sense.*" So murmurs every scrupulous and wary beginner: and the teacher's reply, "Somehow we work out useful results by $\sqrt{-1}$," sounds like saying: "Out of this nonsense useful truth is elicited."

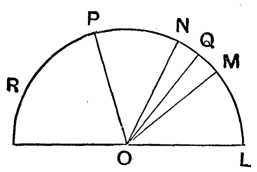
2. The *first* reply to be made is: No one ought to *desire* any number for $\sqrt{-1}$ except the unit itself; the $\sqrt{-}$ which precedes, though a double symbol, has the force of a symbol *only.* The *next* reply is decisive,—the double symbol $\sqrt{-}$ points to a *new direction* in geometry; namely, the direction *perpendicular* to $+1$ and -1. But to explain this fully, it is better to make a new beginning.

Suppose that radii issue in many directions from a fixed point in a plane, and that distances are counted along them. So long as we know along *which* radius we are to count, nothing new is involved, and of course no difficulty. Suppose one of these radii to be our ordinary *positive* axis, another to be called the m radius, and for distinction write the index m under every number to be counted along it, so that 1_m is its unit, in *length* $= 1$ as estimated absolutely. Then we deal with $a_m b_m c_m \ldots$ along the m line, and combine $a_m \pm b_m$, and interpret $5a_m$, $5b_m \pm 6c_m$, without difficulty or fear, since all are lengths to be counted along the same radius. But such a *product* as $a_m \,.\, b_n$ would need careful interpretation. In ab_n no obscurity is found, whether the a be linear or numerical. If linear, we proceed as in interpreting ab^3, though space has only three dimensions. If we put A for the value of ab^3, we attain it by the proportion $1 : a = b^3 : A$, so that A is the same *in kind* as b^3. Similarly if $A' = ab_n$, we are able to count A' along the n radius, whether a be simply numerical, or when it is linear, by aid of the ratios $1 : a = b_n : A'$. But if we proceed thus with $a_m \,.\, b_n$, using the proportion

$$1_m : a_m = b_n : A',$$

we confound $a_m b_n$ with ab_n; for $1_m : a_m$ is the same ratio as $1 : a$.

3. Mr Warren in 1826 laid a logical basis for this matter by his treatise on $\sqrt{-1}$, which I here substantially follow, and wonder that it is not found in all elementary works. He virtually distinguishes between proportionate *lengths* and proportionate *lines*. In the former, DIRECTION is not regarded; with the latter, it is essential. Thus if A, B, C, D are proportionate *lengths*, but are drawn along our radii, —viz. A along the positive axis, B along the m radius, C on the n radius and D on the p radius, we do not pronounce these *lines* proportional, unless also their directions justify it; that is, the p radius must be disposed towards the n radius, as is the m radius to the positive axis. This amounts to saying that the p line must lie on the same side of the n line, and at the same angular distance from it, as the m line compared with the positive axis. Then, if OL, OM, ON, OP be the 4 radii, and $LMNP$ a circular arc, we need that the arc PN shall $=$ arc ML before we admit that the *units* OL, OM, ON, OP are proportionate *lines*.

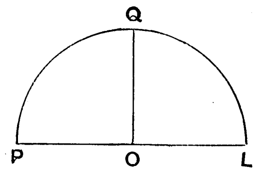


After this condition of the *directions* is fulfilled, we concede that the proportionate *lengths* counted along them are also proportionate *lines*.

If the arc $NP = \text{arc } LM$, add MN to both, then arc $PM = \text{arc } NL$. This enables us to exchange the second and third terms in the proportion, agreeably to the process called *Alternando*. Also the arc $PL = PN + NL = ML + NL$. Hence if we count from L, and call arc $LM = m$, arc $LN = n$, arc $LP = p$, the test of necessary direction is $p = m + n$.

The simplest case is, when the four proportionals become three by the second and third coalescing, as if M and N run together in Q. Then if arc $LQ = \text{arc } QP$, we have $OL : OQ = OQ : OP$. If further arc $PR = \text{arc } PQ$, then $OQ : OP = OP : OR$; and so on.

4. Apply now this to the case in which the arcs PQ and QL are both quadrants. Then OQ is the mean proportional between $OL (= 1)$ and $OP = (-1)$. The received symbol for a *mean proportion* is $\surd$, as in $OQ = \surd OP . OL$. Here then $OQ = \surd(-1 . 1) = \surd -1$. This is only a following out of *analogy* with the symbol; though, previously, $\surd$ expressed the mean proportion between *numbers*, or perhaps *lengths*, without cognizance of direction.



Now, our first care must be, to inquire whether $\surd -$ as a symbol of direction, has the same properties as when it operates on a pure number.

First, in combining factors, the order is indifferent, as $ab = ba$, and $a . (1) = a = 1 . a$. We ask, does $\surd -$ fulfil this condition? Evidently, $a . \surd -1 = \surd -1 . a$, each measuring the length a, directed along the perpendicular OQ. Similarly

$$a . \surd -b = \surd -b . a = b \surd -1 . a = ba \surd -1.$$

Next, repeat $\surd -$. We had $OQ = OL \surd -1$ or $\surd -1 . OL$. Also $OP = \surd -1 . OQ$, because $QOP = 90°$, $\therefore OP = \surd -1 . (\surd -1 . OL)$. But $OP = -OL$ or $-1 . OL$. Evidently then $\surd -1 . \surd -1$ is equivalent to -1. This further justifies the change of $\frac{1}{\surd -1}$ to $-\surd -1$.

5. But a new difficulty arises in *adding* unlike quantities, i.e. in connecting them by $+$. If along radii m and n we have two lengths a_m and b_n, what meaning can we attach to $a_m + b_n$? This urgently needs explanation. It may seem that the symbol $+$ (*plus*) receives

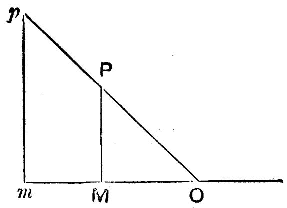
a new sense.—Now in fact when $(a+b) = zero$, the + does not strictly mean *addition;* it really expresses a *difference,* not a *sum;* but not to embarrass generalization, we call it a *sum,* and say that either a or b is negative. They may mean the very same line OL estimated in opposite directions, as OL and LO. If OL mean the line *as travelled* from O to L, and LO the same *as travelled* from L to O, the statement $OL + LO = zero$, clearly means that the *total result* of such travel is nothing; since the travel *neutralizes itself.* Thus if, instead of saying that the *sum* is zero, which gives only a *numerical* idea, I call *total result* zero, you will gain a *geometrical* idea. At this we must aim, when we deal with lines differing in direction. Evidently, if, starting from any point in the outline of a limited surface, a point travel round the circuit, until it regain its original place, we may justly say, *the total result* of such change is *zero;* and no one will suppose it to mean that the *length of the circuit* is zero. So if there be a triangle ABC, we may say, the total result of the travel $AB + BC + CA = 0$, if it be understood that each line is to be estimated in a different direction. Indeed, suppose the *lengths* of the three sides are c, a, b, then in the equation $c_m + a_n + b_p = 0$, the symbol + cannot mislead us, though its sense is evidently enlarged from sum to total result.

Again, since $AB + BC + CA = 0$ and $CA = -AC$, when direction is considered, we have $AB + BC - AC = 0$, which further justifies $AB + BC = AC$. The last is interpretable,—"Motion along two successive sides of a triangle yields *the same total result* as motion along the third side."

The word resultant characterizes mechanics, but there seems no objection to adopting it in geometry also for the total result, as distinguished from the sum.

6. In fact we have unawares made a great step forward; for the symbol $\sqrt{-}$ now enables us to express *distance in every direction.* If our parallelogram become a rectangle, and AB is the ordinary positive axis, and (as before), the *lengths* of AB, BC, CA are c, a, b, we have $BC = a\sqrt{-1}$ when direction is estimated, and $AC =$ total result of c and $a\sqrt{-1}$, or AC (an oblique line) $= c + a\sqrt{-1}$. Since c and a are independent lengths, AC may have any direction whatever.

But again, we must inquire whether the symbol +, thus extended, can be worked in the received method. First, does it fulfil the fundamental condition expressed by $A + B = B + A$? Assuming (as we must) the doctrine of parallel straight lines, and considering any rectangle whose sides are a and c, we find that $c+a\sqrt{-1}=$ the diagonal $=a\sqrt{-1}+c$ from the opposite sides. Next, does it fulfil the condition $h(a+B)=hA+hB$? The doctrine of similar triangles at once affirms it. Let $OM=x$, $MP=y$, perpendicular to it; join OP, then if OM is the positive axis, and y expresses mere length, we write



$$OP \text{ or } OM+MP=x+y\sqrt{-1}.$$

Next, along OM take $Om=hx$; that is $1:h=x:Om$ (whether h is linear or numerical). Erect mp perpendicular to Om and meeting OP in p. Then by similar triangles, $\overline{mp}=h.y$ (in length) and $Op=h.OP$. In this $h.OP=Op$ we have supposed h to be numerical.

Also $\qquad OP$ is equivalent to $x+\sqrt{-1}.y$,

and $\qquad Op$ to $Om+mp\sqrt{-1}$ or $hx+\sqrt{-1}.hy$,

that is, $\qquad h(x+\sqrt{-1}y)=hx+\sqrt{-1}.hy$,

just as if $\sqrt{-1}$ were numerical.

If further we change h from a mere number to a positive length, it affects every term of the last in the same ratio, and leaves equivalence as before.

If we have proved generally that with any factor h (provided it be counted along the *positive* axis) the product $h(x+\sqrt{-1}.y)$ is equivalent to $hx+\sqrt{-1}.hy$, the same is virtually proved, if h be changed into h_m, that is, if the numerical h be computed along an m-axis. For we may transform our hypothesis, by choosing the m-axis as positive. If hereby x, y change to x', y', we obtain a result the same *in form* as the previous result, and x', y' remain quite as general as were the x, y. Thus we may write

$$h_m.(x+\sqrt{-1}y)=h_mx+\sqrt{-1}.h_my.$$

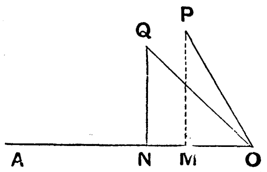
After this, we can change the oblique h_m into $a+\sqrt{-1}.b$, where a, b are along the positive axis. Now if the m-axis be perpendicular to the positive, we may write simply $h_m=k\sqrt{-1}$, where k is along the positive axis. Then $\sqrt{-1}.h_my=\sqrt{-1}.\sqrt{-1}.ky$, and since each $\sqrt{-1}$

denotes revolution of the ky through 90°, the $\sqrt{-1}\,.\,\sqrt{-1}$ shews revolution through 180°, or is equivalent to the symbol $-$. Thus

$$\begin{aligned} h_m(x+\sqrt{-1}y) &= \sqrt{-1}\,.\,k(x+\sqrt{-1}y) \\ &= \sqrt{-1}\,.\,kx+\sqrt{-1}\,.\,k\,.\,\sqrt{-1}\,.\,y \\ &= \sqrt{-1}kx-ky \end{aligned}$$

exactly *as if* $\sqrt{-1}$ were numerical. Evidently then the same holds good in multiplying out $(h+\sqrt{-1}k)$ into $(x+\sqrt{-1}y)$.

THEOREM. If $A+B\sqrt{-1}\equiv C+D\sqrt{-1}$, this implies *two* equalities, viz. $A=C$ and $B=D$. The geometrical proof appeals at once to the eye. If OA be the positive axis, and the binomials are denoted by OP and OQ, viz. $A+\sqrt{-1}B=OP$ and $C+\sqrt{-1}D=OQ$, we do not account $OP=OQ$ until they have the *same direction*, as well as the same length. This requires Q to *coincide* with P. Of course then, if PM, QN are dropt perpendicular to OA we have $OM=A$, $PM=B\sqrt{-1}$, $ON=C$, $QN=D\sqrt{-1}$, and as soon as $OP=OQ$ in our hypothesis, P coincides with Q, therefore also M with N. That is $A=C$ and $B=D$.



The reader will now see the geometrical meaning of the "imaginary roots" (so called) of a quadratic equation. As a very simple case, take first

$$x^2-16x+63=0,\ \text{which yields}\ x=8\pm 1.$$

Here both roots lie along the positive axis. But change 63 to 65, then

$$x^2-16x+65=0,\ \text{whence}\ x=8\pm\sqrt{-1}.$$

In the latter the two roots are equal radii drawn from the origin at equal angles on opposite sides, radii which terminate where the coordinate along the axis is 8, and the transverse coordinate is ± 1.

TRACT III.

ON FACTORIALS.

SUPPLEMENT II.

Extension of the Binomial Theorem.

1. THE following appeared in Cauchy's elementary treatise, as early, I think, as 1825, but without the new Factorial Notation, which adds much to its simplicity. Boole writes $x^{(2)}$ for $x(x-1)$, $x^{(3)}$ for $x(x-1)(x-2)$, and $x^{(n)}$ for $x(x-1)(x-2)\ldots(x-n+1)$, whence $x^{(n+1)} = x^{(n)}\,.\,(x-n)$. Better still it is, to place the exponent in a half-oval, since a parenthesis ought to be *ad libitum*. I propose $x^{\underset{\smile}{n}}$, $x^{\underset{\smile}{n-1}}$ which are quite distinctive. Then the Binomial Theorem is

$$(1+x)^n = 1 + n\,.\,\frac{x}{1} + n^{\underset{\smile}{2}}\,.\,\frac{x^2}{1\,.\,2} + n^{\underset{\smile}{3}}\,.\,\frac{x^3}{1\,.\,2\,.\,3} + \ldots + x^n.$$

In this notation $1\,.\,2\,.\,3\,.\,4 \ldots n$ or $n(n-1)\ldots 2\,.\,1$ is $n^{\underset{\smile}{n}}$. Gudermann for this has n'; but $(n-1)'$ is less striking to the eye than $\underline{|n}$, $\underline{|n-1}$ introduced by the late Professor Jarrett. This exhibits in the Binomial Theorem its general term, by

$$(1+x)^n = 1 + n\,.\,\frac{x}{1} + \ldots + n^{\underset{\smile}{r}}\,.\,\frac{x^r}{\underline{|r}} + \ldots + x^n;$$

of course $x^{\underset{\smile}{1}}$ is equivalent to simple x.

The *Exponent* (of a power) is already distinguished from an *Index*. In a Factorial $x^{\underset{\smile}{r}}$ for $x(x-1)(x-2)\ldots(x-\overline{r-1})$ one may call r (which must be integer) the *Numero*, as stating the number of factors.

2. If m, n, p be all positive integers, and $p = m + n$, then

$$(1+x)^m . (1+x)^n = (1+x)^p,$$

or, in condensed expansion,

$$\left\{1 + \Sigma m^{\underset{\smile}{r}} \frac{x^r}{\underline{|r}} + x^m\right\} . \left\{1 + \Sigma n^{\underset{\smile}{r}} . \frac{x^r}{\underline{|r}} + x^n\right\} = \left\{1 + \Sigma p^{\underset{\smile}{r}} . \frac{x^r}{\underline{|r}} + x^p\right\}.$$

This equation being of the $(m+n)^{\text{th}}$ or p^{th} degree of x, and being true for values of x indefinite in number (therefore in more than p values), must be equal term by term for every power of x. Now when we multiply any two such series,

$$(1 + M_1 x + M_2 x^2 + M_3 x^3 + \text{etc.}) \text{ by } (1 + N_1 x + N_2 x^2 + \text{etc.}),$$

we have a product of the same form

$$1 + P_1 x + P_2 x^2 + P_3 x^3 + \dots$$

by the routine of multiplication, in which

$$P_1 = M_1 + N_1; \quad P_2 = M_2 + M_1 N_1 + N_2; \quad P_3 = M_3 + M_2 N_1 + M_1 N_2 + N_3;$$

and the law of the indices is so visible, that we get generally

$$P_r = M_r + M_{r-1} N_1 + M_{r-2} N_2 + \dots\dots + M_1 N_{r-1} + N_r.$$

This being true for all series of this form may be applied to the three series $(1+x)^m$, $(1+x)^n$, $(1+x)^p$, and at once it yields to us the result

$$\frac{p^{\underset{\smile}{r}}}{\underline{|r}} = \frac{m^{\underset{\smile}{r}}}{\underline{|r}} + \frac{m^{\underset{\smile}{r-1}}}{\underline{|r-1}} \cdot \frac{n}{1} + \frac{m^{\underset{\smile}{r-2}}}{\underline{|r-2}} \cdot \frac{n^{\underset{\smile}{2}}}{\underline{|2}} + \frac{m^{\underset{\smile}{r-3}}}{\underline{|r-3}} \cdot \frac{n^{\underset{\smile}{3}}}{\underline{|3}} + \dots + \frac{n^{\underset{\smile}{r}}}{\underline{|r}};$$

an equation true for more values of m and n than are counted by the integer r, therefore it is true also when m and n are arbitrary and fractional. Write x for m, h for n, and $x+h$ for p, and you have an *extension* of the ordinary $(x+h)^r$. For, this latter may be written

$$\frac{(x+h)^r}{\underline{|r}} = \frac{x^r}{\underline{|r}} + \frac{x^{r-1}}{\underline{|r-1}} \cdot \frac{h}{1} + \frac{x^{r-2}}{\underline{|r-2}} \cdot \frac{h^2}{\underline{|2}} + \frac{x^{r-3}}{\underline{|r-3}} \cdot \frac{h^3}{\underline{|3}} + \dots + \frac{h^r}{\underline{|r}};$$

with *exponents* replacing the *numeros* of the preceding.

NOTE. The reader must carefully observe in that which follows, that the upper *index* of P, Q, A, B, C, is *not* an *exponent*.

Powers in Series of Factorials.

3. Since $x^{\underset{\smile}{2}} = x(x-1) = x^2 - x,$

of which the general law is

$$x^{\underset{\smile}{r}}\,(x-r) = x^{\underset{\smile}{r+1}},$$

conversely $x^2 = x^{\underset{\smile}{2}} + x.$

Again $x^{\underset{\smile}{3}} = x(x-1)(x-2) = x^3 - 3x^2 + 2x,$

$$\therefore\; x^3 = x^{\underset{\smile}{3}} + 3x^2 - 2x.$$

But $3x^2 = 3x^{\underset{\smile}{2}} + 3x,$

$$\therefore\; x^3 = x^{\underset{\smile}{3}} + 3x^{\underset{\smile}{2}} + 3x.$$

Evidently we can thus in succession obtain x^4, x^5,... and generally x^n in series of $x^{\underset{\smile}{n}},\; x^{\underset{\smile}{n-1}},\; x^{\underset{\smile}{n-2}},\ldots\; x^{\underset{\smile}{2}},\; x.$

Since only $x^{\underset{\smile}{n}}$ contains x^n, its coefficient must be 1. In general, with unknown coefficients P, dependent on n, but not involving x, we may write

$$x^n = P_0^n . x^{\underset{\smile}{n}} + P_1^{n-1} x^{\underset{\smile}{n-1}} + P_2^{n-2} x^{\underset{\smile}{n-2}} + \ldots + P_{n-2}^2 x^{\underset{\smile}{2}} + P_{n-1}^1 x \ldots (a).$$

Here the lower index denotes the place of the term in the series; the sum of upper and lower index $= n$, the exponent of x^n; and the upper index is the same as the *numero* of its term. We have also seen that $P_0^n = 1$, whatever n may be. It remains to calculate P_r^{n-r}. Multiply the left member of (a) by x, and on the right multiply the successive terms by the *equivalents* of x, viz.

$$(x-n)+n,\; (x-n+1)+(n-1),\; (x-n+2)+(n-2), \text{ etc.,}$$

and apply to each term the formula $x^{\underset{\smile}{r}} . (x-r) = x^{\underset{\smile}{r+1}}$. Then

$$\left.\begin{aligned} x^{n+1} = {} & P_0^n . x^{\underset{\smile}{n+1}} + P_1^{n-1} . x^{\underset{\smile}{n}} + P_2^{n-2} x^{\underset{\smile}{n-1}} + \ldots + P_{n-2}^2\, x^{\underset{\smile}{3}} + P_{n-1}^1\, x^{\underset{\smile}{2}} \\ & + n . P_0^n . x^{\underset{\smile}{n}} + (n-1)\, P_1^{n-1} . x^{\underset{\smile}{n-1}} + (n-2)\, P_2^{n-2}\, x^{(n-2)} + \ldots \\ & \qquad + 2P_{n-2}^2\, x^{\underset{\smile}{2}} + 1P_{n-1}^1\, x \end{aligned}\right\}(b).$$

But if in (a) we write $(n+1)$ for n, we have

$$x^{n+1} = P_0^{n+1} x^{\underset{\smile}{n+1}} + P_1^n x^{\underset{\smile}{n}} + P_2^{n-1} x^{\underset{\smile}{n-1}} + \ldots + P_{n-1}^2\, x^{\underset{\smile}{2}} + P_n^1 x \ldots (c),$$

and we cannot be wrong in *identifying* (b) and (c); that is, in equating the coefficients belonging to every particular *numero* (r). At the right hand end $P_n^1 = 1P_{n-1}^1$, coefficients of x; i.e. since when $n = 2$, $x^2 = x^{\underset{\smile}{2}} + x$, so that $P_1^1 = 1$,

$$\therefore\; P_2^1 = 1,\; P_3^1 = 1;$$

and universally $P^1_n = 1$, just as $P^n_0 = 1$. Also in general we find

$$P^r_{n+1-r} = rP^r_{n-r} + P^{r-1}_{n+1-r},$$

otherwise, $$P^r_p = rP^r_{p-1} + P^{r-1}_p,$$

if $$p = n + 1 - r.$$

This enables us to fill in the vacancies of a table, beginning from

1	1	1	1
1	P^2_1	P^3_1	P^4_1
1	P^2_2	P^3_2	P^4_2
1	P^2_3	P^3_3	P^4_3
1	P^2_4	P^3_4	P^4_4

one horizontal row and one vertical, each consisting of units. Each P is computed from the P *above* it, multiplied by its upper index (which is the number of its column) + its companion *to the left* in the same row. Thus to form the second *row* from

1	1	1	1	1
1	2 . (1) + 1 = 3	3 . (1) + 3 = 6	4 . (1) + 6 = 10	5 . (1) + 10 = 15

Evidently this second row is

$$1;\ 1 + 2;\ 1 + 2 + 3;\ 1 + 2 + 3 + 4; \ldots$$

of which the general term is $P^n_1 = \frac{1}{2}n \, . \, (n + 1)$. Similarly from the second row we form the third, working *from left to right,* and the law is manifest.

1	3	6	10	15	21
1	2 . (3) + 1 = 7	3 . (6) + 7 = 25	4 . (10) + 25 = 65	5 . (15) + 65 = 140	6 . (21) + 140 = 266

Hence a table to any extent required can be made, such as is here presented.

	P^1	P^2	P^3	P^4	P^5	P^6
0	1	1	1	1	1	1
1	1	3	6	10	15	21
2	1	7	25	65	140	266
3	1	15	90	350	1050	2646
4	1	31	301	1701	6951	
5	1	63	966	7770	42,525	
6	1	127	3025	34,105	246,730	
7	1	255	9330			

When this table is used solely to evaluate the coefficients of (a), the indices of $P^n_0, P^{n-1}_1, P^{n-2}_2 \ldots$ warn us that the numbers will be taken out *diagonally*. Thus for x^6 we take out (beginning from P^6_0 at the top on right hand) 1, 15, 65, 90, 31, 1.

It will be observed that the *second column* is 2^1-1, 2^2-1, 2^3-1 and in general $P^2_r = 2^{r+1}-1$.

Such is the Table of *Direct* Factorials.

The letter P being almost appropriated for the Legendrian functions, I see an advantage in superseding P^n_r by $\left| {}^{n}\diagup{}_{r} \right|$. Necessarily n and r are both integers, n index of the column, $r+1$ index of the row.

Collect the results for P

$$P^n_0 = 1\,;\ P^1_n = 1\,;\ P^n_1 = \tfrac{1}{2}n\,(n+1)\,;\ P^2_n = 2^{n+1}-1\,;$$

and in general $$P^r_p = rP^r_{p-1} + P^{r-1}_p,$$

which yield identically

$$x^n = \left| {}^{n}\diagup{}_{0} \right| . \, x^{\underset{\smile}{n}} + \left| {}^{n-1}\diagup{}_{1} \right| x^{\underset{\smile}{n-1}} + \left| {}^{n-2}\diagup{}_{2} \right| x^{\underset{\smile}{n-2}} + \text{etc.}$$

Factorials in Series of Powers.

4. This is a mere problem of common multiplication,

$$x^{\underset{\smile}{n}} = x\,(x-1)\,(x-2)\ldots(x-n+1),$$

yet the factors being special and their combinations often recurring, the work of one computer may avail for many after him. We may assume

$$x^{\underset{\smile}{n}} = Q_0^n x^n - Q_1^{n-1} x^{n-1} + Q_2^{n-2} x^{n-2} - \text{etc.} \ldots \pm Q_{n-1}^1 x \ldots\ldots (a);$$

where, as before, obviously $Q_0^n = 1$. Also dividing by x, and then making $x = 0$, you have

$$Q_{n-1}^1 = \pm 1 . 2 . 3 \ldots (n-1) = \pm \underline{|n-1}.$$

Since $$(x-n)\, x^{\underset{\smile}{n}} = x^{\underset{\smile}{n+1}},$$

multiply (a) by $x - n$,

$$\left.\begin{aligned} \therefore\ x^{\underset{\smile}{n+1}} = Q_0^n x^{n+1} - Q_1^{n-1} x^n + Q_2^{n-2} x^{n-1} - \text{etc.} \ldots \pm Q_{n-1}^1 x^2 \\ - nQ_0^n x^n + nQ_1^{n-1} x^{n-1} - \text{etc.} \ldots \pm nQ_{n-2}^2 x^2 \mp nQ_{n-1}^1 x \end{aligned}\right\} (b).$$

Also in (a) write $n+1$ for n;

$$\therefore\ x^{\underset{\smile}{n+1}} = Q_0^{n+1} x^{n+1} - Q_1^n x^n + Q_2^{n-1} x^{n-1} - \text{etc.} \ldots \mp Q_n^1 x \ldots\ldots (c).$$

But (b) and (c) ought to be identical. We have anticipated the remarks that $Q_0^{n+1} = Q_0^n = 1$, and $Q_n^1 = nQ_{n-1}^1$, inasmuch as

$$Q_n^1 = 1 . 2 . 3 \ldots n; = \underline{|n}.$$

But generally, $$Q_{r+1}^{n-r} = nQ_r^{n-r} + Q_{r+1}^{n-r-1};$$

or, if $n - r = m$, $$Q_{r+1}^m = (m+r)\, Q_r^m + Q_{r+1}^{m-1}.$$

As before, this enables us to continue the table, when the first row and first column are known. To compare our formula with that of the first table, we may write it

$$Q_p^r = (r+p-1)\, Q_{p-1}^r + Q_p^{r-1}.$$

In fact, the first row is unity as before. The first column is

$$1,\ 1,\ 1.2,\ 1.2.3,\ 1.2.3.4,$$

when $$r = 0,\ Q_1^m = mQ_0^m + Q_1^{m-1},$$

also in the former table, when

$$n = r,\ P_1^r = rP_0^r + P_1^{r-1}.$$

Hence the second row is the same in the new table as in the old.

To compute the third row from the second:

1	3	6	10	15	21
1 . 2	4 . (3) + 1 . 2 = 11	4 . (6) + 11 = 35	5 . (10) + 35 = 85	6 . (15) + 85 = 175	$Q_p^r = (r+p-1)\, Q_{p-1}^r + Q_p^{r-1}$

The multiplier $r+(p-1)$ combining upper and lower index of its Q distinguishes the Q table from the P table: thus

	11	35	85
1 . 2 . 3	4 . (11) + 1 . 2 . 3 = 50	5 . (35) + 50 = 225	6 . (85) + 225 = 735
1 . 2 . 3 . 4	5 . (50) + 2 . 3 . 4 = 274	6 . (225) + 274 = 1624	etc.

or indeed $$Q_p^{r+1} = (r+p)\,Q_{p-1}^{r+1} + Q_p^r.$$

Inverse Factorials.

	Q^1	Q^2	Q^3	Q^4	Q^5	Q^6	Q^7
0	1	1	1	1	1	1	1
1	1	3	6	10	15	21	
2	1 . 2	11	35	85	175		
3	1 . 2 . 3	50	225	735			
4	$\underline{\lvert 4}$	274	1624	6769			
5	$\underline{\lvert 5}$	1764	13,132				
6	$\underline{\lvert 6}$						

These too are used diagonally for $x^{\underset{\smile}{n}}$. Thus

$$x^{\underset{\smile}{7}} = x^7 - 21x^6 + 175x^5 - 735x^4 + 1624x^3 - 1764x^2 + \underline{\lvert 6}\,.\,x.$$

Again it seems better to supersede

Q_r^n by $\left|{}_{r}\diagdown^{n}\right|$, then $x^{\underset{\smile}{n}} = \left|{}_{0}\diagdown^{n}\right| x^n - \left|{}_{1}\diagdown^{n-1}\right| x^{n-1} + \left|{}_{2}\diagdown^{n-2}\right| x^{n-2} + \text{etc.}$

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5. Even in Arithmetic we are driven upon "recurring decimals," and learn that an infinite series may tend to a unique finite limit. Nor can Elementary Algebra fail to recognize, from

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \dots + x^{n-1};$$

that when x is numerically less than 1, with n indefinitely increasing, the series $1 + x + x^2 + \ldots + x^n$ tends to the limit $\frac{1}{1-x}$.

After this it quickly follows (by Cauchy's process now perhaps universal), from Binomial Theorem with *n positive integer,* that $\left(1+\frac{1}{n}\right)^n$ with n infinite, has for limit

$$1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \text{etc.} = 2{\cdot}7182818\ldots$$

which we call ϵ, and that $\left(1+\frac{a}{n}\right)^n$ has for limit

$$1 + \frac{a}{1} + \frac{a^2}{1\,.\,2} + \frac{a^3}{1\,.\,2\,.\,3} + \text{etc.}\ldots$$

which also $= \epsilon^a$. On this we need not here dwell: but it will presently be assumed. Now let us propound

Factorials with Negative Numero.

6. Analogy suggests to define $x^{\overset{-2}{\smile}}$ as meaning $\frac{1}{x(x+1)}$: of course $x^{\overset{-1}{\smile}}$ will be identical with x^{-1}. Then $x^{\overset{-3}{\smile}}$ will stand for

$$[x(x+1)(x+2)]^{-1},$$

and $x^{\overset{-n}{\smile}}$ for $\qquad [x(x+1)(x+2)\ldots(x+n-1)]^{-1}$.

Hence $\qquad x^{\overset{-n-1}{\smile}} = (x+n)^{-1}\,.\,x^{\overset{-n}{\smile}}$.

Now $\qquad x^{\overset{-2}{\smile}} = (x+1)\,x^{\overset{-1}{\smile}}$,

and when x is > 1, we have, in descending powers

$$x^{\overset{-2}{\smile}} = x^{-2} - x^{-3} + x^{-4} - x^{-5} + \text{etc.}$$

Also $\qquad x^{\overset{-3}{\smile}} = (x+3)^{-1}\,.\,x^{\overset{-2}{\smile}}$

of the two factors here on the right, each can take the form of a series descending in powers of x. So then their product, the equivalent of $x^{\overset{-3}{\smile}}$. By like reasoning we claim a right to assume with coefficients independent of x,

$$x^{\overset{-n-1}{\smile}} = A_0^n x^{-n-1} - A_1^n x^{-n-2} + A_2^n x^{-n-3} - \text{etc.} \;\ldots\ldots\ldots\; (a),$$

and our task is to discover the coefficients when n is given.

First make $n = 1$,

$$\therefore \; x^{\underset{\smile}{-2}} = A_0^1 x^{-2} - A_1^1 x^{-3} + A_2^1 x^{-4} - A_3^1 x^{-5} + \text{etc.}$$

But from the series already obtained for $x^{\underset{\smile}{-2}}$ we see that every coefficient of the last is 1; or in general $A_r^1 = 1$. This gives the first *column* of our table. Also universally A_0^n obviously $= 1$; which fixes the first *row* of the table.

Next, multiply equation (a) by $(x + n)$, and you get

$$\left.\begin{aligned} x^{\underset{\smile}{-n}} = x^{-n} - A_1^n x^{-n-1} + A_2^n x^{-n-2} - \text{etc.} \dots \pm A_r^n x^{-n-r} \mp \dots \\ + nx^{-n-1} - n\,.\,A_1^n x^{-n-2} + \text{etc.} \dots \mp nA_{r-1}^n x^{-n-r} \mp \dots \end{aligned}\right\} \dots (b).$$

But in (a) we may write n for $n + 1$, which gives

$$x^{\underset{\smile}{-n}} = x^{-n} - A_1^{n-1} x^{-n-1} + \dots \pm A_r^{n-1} x^{-n-r} \mp \dots\dots\dots\dots (c).$$

Now (b) and (c) must be identical, hence

$$A_1^n = n + A_1^{n-1} = nA_0^n + A_1^{n-1},$$

and generally $$A_r^n = nA_{r-1}^n + A_r^{n-1}.$$

But this is exactly the law of P in Art. 3, only there we had r, p for what are here n, r. Now as the first row and first column are here, as there, unity, and the law of continuation the same, the whole table will be the same, and we may write P of Art. 3 in place of this A. Thus we get

$$x^{\underset{\smile}{-n}} = x^{-n} - P_1^{n-1} x^{-n-1} + P_2^{n-1} x^{-n-1} - P_3^{n-1} x^{-n-3} + \text{etc.} \dots$$

with the *same* values of P as before. But in the last equation we no longer take the P's *diagonally*, but *vertically*, down the column, as the same upper index $(n - 1)$ above every P denotes; thus

$$x^{\underset{\smile}{-3}} = x^{-3} - 3x^{-4} + 7x^{-5} - 15x^{-6} + 31x^{-7} - \text{etc.} \dots$$

To verify, multiply by $x + 2$. But for convergence x ought to exceed 2. So

$$x^{\underset{\smile}{-4}} = x^{-4} - 6x^{-3} + 25x^{-4} - 90x^{-5} + \text{etc.} \dots$$

and for convergence, x ought to exceed 3. Evidently in the series for $x^{\underset{\smile}{-n}}$, x ought to exceed $(n - 1)$.

7. Assume now the *Inverse* Problem, to develop x^{-n} in series of Factorials.

With unknown coefficients B independent of x, we start from

$$x^{-n-1} = x^{\underset{\smile}{-n-1}} + B_1^n x^{\underset{\smile}{-n-2}} + B_2^n x^{\underset{\smile}{-n-3}} + \dots\dots\dots\dots (a).$$

Multiply the lefthand by x, and the successive terms on the right by the equivalents of x, viz.

$$(x+n)-n,\ (x+n+1)-(n+1),\ (x+n+2)-(n+2), \text{ etc.}$$

Observe that $\quad (x+p)\,x^{(-p-1)} = x^{(-p)}$;

$$\left.\begin{array}{l} \therefore\ x^{-n} = x^{\underbrace{(-n)}} + B_1^n . x^{\underbrace{-n-1}} + \ldots + B_r^n . x^{\underbrace{-n-r}} + \ldots \\ -n . x^{\underbrace{(-n-1)}} - (n+1)\, B_1^n . x^{\underbrace{-n-2}} - \ldots - (n+r) . B_r^n x^{\underbrace{-n-r-1}} - \ldots \end{array}\right\} \ldots (b).$$

But in (a) we may write n for $n+1$, which gives

$$x^{-n} = x^{\underbrace{-n}} + B_1^{n-1} x^{\underbrace{-n-1}} + B_2^{n-1} x^{\underbrace{-n-2}} + \ldots + B_r^{n-1} . x^{\underbrace{-n-r}} + \ldots\ldots (c).$$

Identify (b) with (c),

$$\therefore\ B_1^n = n + B_1^{n-1};$$

and generally $\quad B_r^n = (n+r-1)\, B_{r-1}^n + B_r^{n-1}$;

the same formula as for Q in Art. 4, only n and r here standing for what there was r and p. Also since $B_0^n = 1$, the top row is unity, here as there.

We may further prove that the first column of our B's is the same as the first column of the Q's. For

$$x^{-2} = x^{\underbrace{-2}} + B_1^1 x^{\underbrace{-3}} + B_2^1 x^{\underbrace{-4}} + \ldots \text{ \&c.}$$

Multiply by x on the left, also by $(x+1)-1$, $(x+2)-2$, $(x+3)-3$, for the successive terms on the right; then

$$\left.\begin{array}{l} x^{-1} = x^{\underbrace{-1}} + B_1^1 x^{\underbrace{-2}} + B_2^1 x^{\underbrace{-3}} + \ldots \\ -1\, x^{\underbrace{-2}} - 2B_1^1 x^{\underbrace{-3}} - 3B_2^1 x^{\underbrace{-4}} - \ldots \end{array}\right\}.$$

Obviously $x^{-1} = x^{\underbrace{-1}}$; and the other terms must annihilate themselves, making $B_1^1 - 1 = 0$; $B_2^1 - 2B_1^1 = 0$; $B_3^1 - 3B_2^1 = 0$; etc.,

or $\quad B_1^1 = 1,\ B_2^1 = 2B_1^1 = 1 . 2$;

$$B_3^1 = 3B_2^1 = 1 . 2 . 3, \text{ and } B_n^1 = 1 . 2 . 3 \ldots (n-1)\, n.$$

Thus $B_2^1 = Q_2^1$, $B_3^1 = Q_3^1$, etc. exact. Therefore the B table is the same as the Q table *throughout.*

Here also we take the Q's vertically, to obtain x^{-n} in series of factorials as $\quad x^{-4} = x^{\underbrace{-4}} + 6x^{\underbrace{-5}} + 35x^{\underbrace{-6}} + 225x^{\underbrace{-7}} + \text{etc.}$

In general $x^{-n} = x^{\underbrace{-n}} + Q_1^{n-1} x^{\underbrace{-n-1}} + Q_2^{n-1} x^{\underbrace{-n-2}} + \ldots + Q_r^{n-1} . x^{\underbrace{-n-r}} + \text{etc.}$; but special inquiry is needed concerning convergence. Apparently it converges more rapidly than

$$x^{-2} + (x+1)^{-2} + (x+2)^{-2} + \text{etc.}$$

8. To develope $(\epsilon^x - 1)^n$ or $\left\{\frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \text{etc.}\right\}$ in powers of x^n. We may assume for this the form

$$M_n x^n + M_{n+1} x^{n+1} + \ldots + M_r x^r + \text{etc.},$$

where M_n manifestly $= 1$, and if r is less than n, $M_r = 0$. Now by the Binomial Theorem where n is a positive integer

$$(\epsilon^x - 1)^n = \epsilon^{nx} - \frac{n}{1}\epsilon^{(n-1)x} + \frac{n^{\underset{\smile}{2}}}{\lfloor 2} \cdot \epsilon^{(n-2)x} - \text{etc.} \pm \frac{n}{1}\epsilon^x \mp 1,$$

out of which we have to pick up the partial coefficients of x^r which make up M_p observing that

$$\epsilon^{px} = 1 + \frac{px}{1} + \frac{p^2x^2}{\lfloor 2} + \frac{p^3x^3}{\lfloor 3} + \text{etc.}$$

in which we have to make p successively $n, n-1, n-2, \ldots$ When we make $p = n$, the only term that here concerns us is $\frac{n^r x^r}{\lfloor r}$. When $p = n-1$, we get $-\frac{n}{1} \cdot \frac{(n-1)^r \cdot x^r}{\lfloor r}$. When $p = n-2$, we have

$$+\frac{n^{\underset{\smile}{2}}}{\lfloor 2} \cdot \frac{(n-2)^r \cdot x^r}{\lfloor r},$$

and so on. All have $\lfloor r$ in the denominator. Hence

$$M_r \cdot \lfloor r = n^r - \frac{n}{1}(n-1)^r + \frac{n^{\underset{\smile}{2}}}{\lfloor 2} \cdot (n-2)^r - \text{etc.} \ldots \text{to } n \text{ terms},$$

of which the last is $\pm \frac{n}{1} \cdot 1^2$.

Let N be to $(n+1)$ what M is to n; then, with r the same for both,

$$N_r \cdot \lfloor r = (n+1)^r - \frac{n+1}{1} \cdot (n)^r$$
$$+ \frac{(n+1)^{\underset{\smile}{2}}}{\lfloor 2}(n-1)^r - \frac{(n+1)^{\underset{\smile}{3}}}{\lfloor 3} \cdot (n-2)^r + \ldots \text{to } (n+1) \text{ terms.}$$

Add this to the preceding, coupling every pair that with the same value of p has $(n-p)^r$ as factor;

$$\therefore \lfloor r \cdot (M_r + N_r) = (n+1)^r - \left[\frac{n+1}{1} - 1\right](n)^r + \left[\frac{n+1}{2} - 1\right] \cdot \frac{n}{1} \cdot (n-1)^r$$
$$- \left[\frac{n+1}{3} - 1\right] \frac{n \cdot n-1}{1 \cdot 2}(n-2)^r$$
$$+ \left[\frac{n+1}{4} - 1\right] \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}(n-3)^r - \text{etc.}$$

Observe that $\frac{n+1}{1} - 1 = n$, $\frac{n+1}{2} - 1 = \frac{n-1}{2}$, $\frac{n+1}{3} - 1 = \frac{n-2}{3}$, and so on;

$$\therefore \ \lfloor r \, . \, (M_r + N_r) = (n+1)^n - \frac{n}{1}(n)^r + \frac{n \, . \, n-1}{1 \, . \, 2}(n-1)^r - \text{etc.},$$

or, in shorter notation,

$$= (n+1)^r - \frac{n}{1} \, . \, (n)^r + \frac{n^{\underset{\smile}{2}}}{\lfloor 2} \, . \, (n-1)^r - \frac{n^{\underset{\smile}{3}}}{\lfloor 3} \, . \, (n-2)^2 + \text{etc. to } (n+1) \text{ terms.}$$

But in N_r change r to $r+1$ and you have

$$\lfloor r+1 \, . \, N_{r+1} = (n+1)^{r+1} - \frac{n+1}{1}(n)^{r+1} + \frac{n+1 \, . \, n}{1 \, . \, 2}(n-1)^{r+1}$$

$$- \frac{n+1 \, . \, n \, . \, n-1}{1 \, . \, 2 \, . \, 3} \, . \, (n-2)^{r+2} + \text{etc.} \dots \text{ to } (n+1) \text{ terms}$$

$$= (n+1)\left\{(n+1)^r - \frac{1}{1}(n)^{r+1} + \frac{n}{1 \, . \, 2}(n-1)^{r+1} - \frac{n \, . \, n-1}{1 \, . \, 2 \, . \, 3}(n-2)^{r+1} + \text{etc.} \dots\right\}$$

$$= (n+1)\left\{(n+1)^r - \frac{n}{1}(n)^r + \frac{n \, . \, n-1}{1 \, . \, 2}(n-1)^r \right.$$

$$\left. - \frac{n \, . \, n-1 \, . \, n-2}{1 \, . \, 2 \, . \, 3}(n-2)^r + \text{etc.}\right\}$$

$= (n+1) \, . \, \lfloor r \, (M_r + N_r)$, from above. Divide by $(n+1) \, . \, \lfloor r+1$;

$$\therefore \ \frac{N_{r+1}}{n+1} = \frac{M_r + N_r}{r+1} \quad \dots\dots\dots\dots\dots\dots\dots \ (a).$$

Our r always exceeds n. We simplify the notation somewhat by assuming as coefficients after dividing by x^n,

$$\left(\frac{\epsilon^x - 1}{x}\right)^n = 1 + \frac{C_1^n x}{n+1} + \frac{C_2^n x^2}{n+1 \, . \, n+2} + \frac{C_3^n x^3}{n+1 \, . \, n+2n+3} + \dots$$

$$+ \frac{C_p^n x^p}{n+1 \, . \, n+2 \dots n+p} + \text{etc.}$$

Multiply by x^n. The general term becomes

$$\frac{C_p^n \, . \, x^{n+p}}{(n+1)(n+2)\dots(n+p-1)(n+p)}.$$

Put $n+p=r$, to identify this term with our previous $M_r \cdot x^r$;

$$\therefore\ M_r = \frac{C^n_{r-n}}{n+1\,.\,n+2\,.\,n+3\ldots(r-1)\,.\,r};$$

so that $N_r = \dfrac{C^{n+1}_{r-n-1}}{n+2\,.\,n+3\ldots r-1\,.\,r}$ with one less factor in denominator: but when both n and r in M_r are increased by 1, $r-n$ undergoes no change; and $N_{r+1} = \dfrac{C^{n+1}_{r-n}}{n+2\,.\,n+3\ldots r-1\,.\,r\,.\,r+1}$.

Introduce these values into (a) and multiply by
$(n+1)\,(n+2)\ldots r\,.\,(r+1)$, $\therefore\ C^{n+1}_{r-n} = C^n_{r-n} + (n+1)\,.\,C^{n+1}_{r-n-1}$ ...... (b).

Change $r-n$ to p and $n+1$ to n, $\therefore\ C^n_p = C^{n-1}_p + nC^n_{p-1}$; the same law as for the P's in Art. 3.

Also when $n=1$,

$$\left(\frac{\epsilon^x-1}{x}\right)^1 = 1 + \frac{x}{1\,.\,2} + \frac{x^2}{1\,.\,2\,.\,3} + \ldots + \frac{x^r}{1\,.\,2\ldots(r+1)} + \text{etc.}$$

But we assumed as equivalent

$$1 + \frac{C^1_1 x}{2} + \frac{C^1_2 x^2}{2\,.\,3} + \frac{C^1_3 x^3}{2\,.\,3\,.\,4} + \ldots + \frac{C^1_r x^r}{2\,.\,3\ldots(r+1)} + \ldots$$

which identifies C^1_r with 1, for all values of r. Just so $P^1_r = 1$ in its first column. Also evidently $C^n_0 = 1$; just as $P^n_0 = 1$, in the whole first row. Thus, with first column and first row identical with those of P and the same law of continuation, the whole table is the same. Finally then we obtain

$$\left(\frac{\epsilon^x-1}{x}\right)^n = 1 + \frac{P^n_1\,.\,x}{n+1} + \frac{P^n_2\,.\,x^2}{n+1\,.\,n+2} + \frac{P^n_3 x^3}{n+1\,.\,n+2\,.\,n+3} + \text{etc.}\ldots(c)$$

in which the increasing numerators are pulled down by increasing denominators.

The general term may be written $n^{\smile p}\,.\,P^n_p\,.\,x^p$, or equally

$$\frac{P^n_p\,.\,x^p}{(n+p)^{\smile p}}.$$

[The course of analysis here pursued forestalls that of $\Delta^n\,.\,0^r$ to the learner.]

Thus

$$\left(\frac{\epsilon^x-1}{x}\right)^n = 1 + \frac{\boxed{{}^{n}\!\diagup_{1}}\;x}{n+1} + \frac{\boxed{{}^{n}\!\diagup_{2}}\;x^2}{n+1\,.\,n+2} + \frac{\boxed{{}^{n}\!\diagup_{3}}\;x^3}{n+1\,.\,n+2\,.\,n+3} + \text{etc.}$$

9. To investigate $\log(1+x)$ in series, with no aid from $(1+x)^n$ except in the elementary case of n being a positive integer, and no aid from the Higher Calculus.

Let $y=\epsilon^x-1$, then $x=\log(1+y)$. Also

$$y=\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\text{etc.}\ldots$$

by elementary algebra.

From the last we see that for minute values of x, as a *first* approximation, $y=x$, and $x^2=y^2$. As a second, $y=x+\frac{1}{2}x^2, =x+\frac{1}{2}y^2$, $\therefore\ x=y-\frac{1}{2}y^2$. *Whatever* the series in powers of x, x can be thus *reverted* into powers of y, at least *within certain limits.* Hence *we may assume* with unknown coefficients called A,

$$x \text{ or } \log(1+y)=y-A_2y^2+A_3y^3-A_4y^4+\text{etc.}\ldots\ldots (a),$$

which (with the appropriate numerical values of $A_2A_3A_4\ldots$) will be an *identical* equation.

Consequently, writing $y+z$ for y,

$$\log(1+y+z)=(y+z)-A_2(y+z)^2+A_3(y+z)^3+\text{etc.}\ldots (b).$$

Subtract (a) from (b) developing the powers

$$(y+z)^2,\ (y+z)^3,\ (y+z)^4,\ldots$$

then $\log(1+y+z)-\log(1+y)=z-A_2(2yz+z^2)$

$$+A_3(3y^2z+3yz^2+z^3)-A_3(4y^3z+6y^2z^2+4yz^2+z^4)+\text{etc.}\ldots (c).$$

But the left hand $=\log.\dfrac{1+y+z}{1+y}$ or $\log\left(1+\dfrac{z}{1+y}\right)$ which again, by (a), if we write $\dfrac{z}{1+y}$ for y, has for equivalent

$$\left(\frac{z}{1+y}\right)-A_2.\left(\frac{z}{1+y}\right)^2+A_3\left(\frac{z}{1+y}\right)^3-\text{etc.}\ldots$$

of which the first term alone contains the simple power of z, and there its whole coefficient is $\left(\dfrac{1}{1+y}\right)$. This then must be the sum of the partial coefficients of z found in (c).

That is $\dfrac{1}{1+y}=1-2A_2y+3A_3y^2-4A_4y^3+\text{etc.}$

But $\dfrac{1}{1+y}=1-y+y^2-y^3+\text{etc.}$ *when* y *is numerically less than* 1.

Hence $2A_2 = 1$, $3A_3 = 1$, $4A_4 = 1, \ldots nA_n = 1$, or in general $A_n = \dfrac{1}{n}$.

Finally then, $\log(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \text{etc.} \ldots$ while $y^2 < 1$.

N.B. This furnishes the means of computing logarithms in Elementary Algebra, before knowing the Binomial Theorem with negative or fractional exponent. If Differentiation or "Derivation," as far as $\phi(x) = x^n$, $\phi'(x) = nx^{n-1}$, be admitted, this equation at once proves that when $\phi(z)$ means $\log(z)$, $\phi'(z) = \dfrac{1}{z}$: a process which eases the next Article.

10. PROBLEM. To develop $\left[\log . \overline{1+x}\right]^n$ in a series of powers of x.

That this is possible, when x^2 is < 1, the preceding Article shows. We may then assume, with unknown coefficients $\lambda_1^n, \lambda_2^n, \lambda_3^n \ldots$ depending on n alone, where the upper index is *not* an exponent,

$$z = (\log . \overline{1+x})^n = x^n - \frac{\lambda_1^n x^{n+1}}{n+1} + \frac{\lambda_2^n x^{n+2}}{n+1 . n+2} - \text{etc.} \ldots$$
$$\pm \frac{\lambda_r^n . x^{n+r}}{n+1 . n+2 \ldots n+r} \mp \text{etc.} \ldots\ldots (a).$$

If $\log \overline{1+x} = u$, $z = u^n$, $dz = nu^{n-1} du$, and $du = \dfrac{dx}{1+x}$. [It is hardly worth while to *disguise* Differentials by a more elaborate and tedious Algebra.]

Differentiate both sides of (a), then drop the common factor dx:

$$\text{hereby, } \frac{n(\log \overline{1+x})^{n-1}}{1+x} = nx^{n-1} - \lambda^n . x^n + \frac{\lambda_2^n\, x^{n+1}}{n+1} - \text{etc.} \ldots$$
$$\pm \frac{\lambda_r^n . x^{n+r-1}}{n+1 . n+2 \ldots (n+r-1)} \mp \text{etc.}$$

Multiply by $1+x$, and divide by n,

$$\left.\begin{aligned} \therefore (\log \overline{1+x})^{n-1} = x^{n-1} - \lambda_1^n . \frac{x^n}{n} + \frac{\lambda_2^n . x^{n+1}}{n . n+1} - \text{etc.} \\ \pm \frac{\lambda_r^n . x^{n+r-1}}{n . n+1 \ldots (n+r-1)} \mp \text{etc.} \\ + x^n - \frac{\lambda_1^n . x^{n+1}}{n} + \text{etc.} \mp \frac{\lambda_{r-1}^n . x^{n+r-1}}{n . n+1 \ldots (n+r-2)} \pm \text{etc.} \end{aligned}\right\} \ldots\ldots (b).$$

But writing $(n-1)$ for n in equation (a) we get

$$(\log \overline{1+x})^{n-1} = x^{n-1} - \frac{\lambda_1^{n-1} \,.\, x^n}{n} + \frac{\lambda_2^{n-1} \,.\, x^{n+1}}{n \,.\, n+1} - \text{etc.} \ldots$$

$$\pm \frac{\lambda_r^{n-1} \,.\, x^{n+r-1}}{n \,.\, n+1 \ldots (n+r-1)} \mp \text{etc.} \ldots\ldots (c).$$

Identifying (b) with (c), we obtain $\lambda_1^n = n + \lambda_1^{n-1}$; and generally $\lambda_r^n = (n+r-1)\,\lambda_{r-1}^n + \lambda_r^{n-1}$ or $\lambda_{r+1}^n = (n+r)\,\lambda_r^n + \lambda_{r+1}^{n-1}$, the same law as of Q in Art. 4. Also evidently $\lambda_0^n = 1$ whatever n may be, as $Q_0^n = 1$. Thus the top row is 1. Again by (a) making $n=1$,

$$\log(1+x) = x - \frac{\lambda_1^1 x^2}{2} + \frac{\lambda_2^1 x^3}{2 \,.\, 3} - \frac{\lambda_3^1 x^4}{2 \,.\, 3 \,.\, 4} + \text{etc.}$$

But this is known to be $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{etc.}$, whence

$$\lambda_1^1 = 1,\ \lambda_2^1 = 2,\ \lambda_3^1 = 2 \,.\, 3,\ \lambda_4^1 = 2 \,.\, 3 \,.\, 4, \ldots \text{ just as } Q_n^1 = \underline{|n}.$$

Thus the first column also of λ agrees with that of Q. In short then, the two tables are the same. Finally:

$$(\log \overline{1+x})^n = x^n - \frac{Q_1^n \,.\, x^{n+1}}{n+1} + \frac{Q_2^n \,.\, x^{n+2}}{n+1 \,.\, n+2} - \frac{Q_3^n \,.\, x^{n+3}}{n+1 \,.\, n+2 \,.\, n+3} + \text{etc.},$$

with coefficients already known.

The analogy to the series of Art. 8 deserves notice.

$$\left(\frac{\log . \overline{1+x}}{x}\right)^n = 1 - \frac{\boxed{1 \diagdown n}\; x}{n+1} + \frac{\boxed{2 \diagdown n}\; x^2}{n+1 \,.\, n+2} - \frac{\boxed{3 \diagdown n}\; x^3}{n+1 \,.\, n+2 \,.\, n+3} + \text{etc.}$$

TRACT IV.

ON SUPERLINEARS.

SOME *Apology* may seem needful from me, since Dr Todhunter in his volume on *Higher Algebraic Equations* has treated the same subject under the name of "Determinants." Of course I do not pretend to add to him, nor indeed he to Mr Spottiswoode, who carries off all merit on this subject. I read the details with much admiration as treated by the latter, but found his notation by *accents* very dazzling to the eye, in so much as to make it hard to know by sweeping over half a page, what was the meaning of the formula presented to one. Also I found the chief strain in argument and chief liability to error, to turn upon the question, whether this or that resulting term would require a *plus* or a *minus*. By reasoning from linear functions in my own way, though less direct, I was able to avoid this danger and lessen fatigue to my brain. When I exchanged words with my then colleague in University College, London, the late Professor De Morgan, on the question, *why* this topic was not admitted into common Algebra, he replied, that it was *too difficult* for beginners. I have since thought that he might not have so judged, if some of the arguments were otherwise treated; and in fact I have found, that some whom I supposed to speak with authority thought my slight change of method easier to learners. At the same time I must add, that on the very rare occasions in which I have tried to teach an elementary class of mathematics, a mode of reasoning which to one pupil was easier, to another seemed less satisfactory. Perhaps every teacher ought to have "two strings to his bow."

1. Equations of the first degree are called simple, but when three or more letters (x, y, z...) are involved, complexity arises with much danger of error, even when there is no difficulty of principle. To solve two equations of the form $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$ *is always the same process.* If we could always certainly *remember* the solutions

$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}, \quad y=\frac{c_2a_1-c_1a_2}{a_1b_2-a_2b_1},$$

and never confound the indices, nor mistake between + and −, this alone might have much value. The modern method, due eminently to the genius of William Spottiswoode, is quite adapted to Elementary Algebra; but its vast range of utility cannot there be guessed.

First study the denominator $D=a_1b_2-a_2b_1$. It arises from the left-hand of the equation $\begin{cases} a_1x+b_1y=c_1, \\ a_2x+b_2y=c_2 \end{cases}$ in which the four letters stand in square, as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. Here D is the difference of the two products *formed diagonally,* and the diagonal which slopes downward *from left to right* is accepted as the *positive* diagonal. This is a cardinal point. Remember it, and you will not go wrong on + and −. Understand then, that in square with vertical sides

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} \text{ means } MQ-PN.$$

Of course then, so does $\begin{vmatrix} M & P \\ N & Q \end{vmatrix}$, which exchanges rows into columns.

But if you change the order of the rows, or the order of the columns, into $\begin{vmatrix} P & Q \\ M & N \end{vmatrix}$ or $\begin{vmatrix} N & M \\ Q & P \end{vmatrix}$ you change the result to $PN-NQ$, i.e. you change D to $-D$.

After this is fixed in the mind, it is easy to remember the *common denominator* above, viz. $a_1b_2-a_2b_1$, in the form $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. Call it C. Then the *numerator* of x is obtained from C by changing the column a_1a_2 into the column c_1c_2, making $A=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$.

So B for the numerator of y is found by changing the column b_1b_2 into c_1c_2, yielding $B=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Finally, $x=\frac{A}{C}$, $y=\frac{B}{C}$, without mistake.

Observe, if the equations be presented in the form

$$a_1x+b_1y+c_1=0\,;\quad a_2x+b_2y+c_2=0\,;$$

this is equivalent to changing the signs of c_1 and c_2, which does not affect C, but exactly reverses the signs of A and B. Previously we had

$$(x : A)=(1 : C)=(y : B),$$

or $$(x : y : 1)=(A : B : C)=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} : \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

But from the two new equations $\left.\begin{matrix} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \end{matrix}\right\}$ you get

$$x : y : z=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

in circular order.

We may also present the solution as follows:

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

2. Simple equations are often called *Linear*, by a geometrical metaphor. If a quantity u is so dependent on $x, y, z\ldots$ that however the values of these may vary, yet always $u=ax+by+cz+\ldots$ (where $a, b, c\ldots$ are *numerical*), then u is called a linear *function* of $x, y, z\ldots$ its constituents. [One might have expected u to be called a *dependent* or a *resultant:* but for mysterious reasons of their own, the French have adopted the strange word function; and it cannot now be altered.] It is convenient now to set forth a few properties of linear functions. We here suppose the function to have no absolute (constant) term.

I. To multiply every constituent by any number (m), multiplies the function by that number.

[For, if $u=ax+by+cz+\ldots$, then

$$mu=amx+bmy+cmz+\ldots.]$$

II. If two linear functions have the same number of constituents and these have the same coefficients [as $U=ax+by+cz$,

$U_1 = ax_1 + by_1 + cz_1$]; you will add the functions if you join the constituents in pairs [for here

$$U + U_1 = a(x + x_1) + b(y + y_1) + c(z + z_1);$$

whatever the number of constituents].

3. Observing now that $\begin{vmatrix} a & M \\ b & N \end{vmatrix}$ or $aN - bM$ is a linear function of N and M, we see that to multiply a *column* MN by m multiplies the function by m. Or $\begin{vmatrix} a, & mM \\ b, & mN \end{vmatrix} = m \begin{vmatrix} a & M \\ b & N \end{vmatrix}$.

The same is true if we multiply a *row;* for we may regard a and M as the variables and b and N as constant; then

$$\begin{vmatrix} ma, & mM \\ b, & N \end{vmatrix} = m \begin{vmatrix} a & M \\ b & N \end{vmatrix}.$$

This leads to the remark, that our function ought to be called *superlinear* rather than linear; for it is open to us to suppose constituents alternately constant or variable.

4. Next, by making a column or row binomial, we can sometimes blend two superlinear tablets into one. Thus

$$\begin{vmatrix} A & x \\ B & y \end{vmatrix} + \begin{vmatrix} C & x \\ D & y \end{vmatrix},$$

having the second column the same, yield,

$$(Ay - Bx) + (Cy - Dx) = (A + C)\,y - (B + D)\,x = \begin{vmatrix} A + C, & x \\ B + D, & y \end{vmatrix}.$$

Here the column which was in both, remains as before, but the other columns are added and make a binomial. Evidently the same process holds, if a *row*, instead of a column, is the same in both.

Conversely, when a *given* column is binomial, we can resolve the tablet into *two* tablets; and when each column is binomial, we can resolve the tablets into four.

Thus $\begin{vmatrix} A + x, & C + z \\ B + y, & D + v \end{vmatrix} = \begin{vmatrix} A, & C + z \\ B, & D + v \end{vmatrix} + \begin{vmatrix} x, & C + z \\ y, & D + v \end{vmatrix}$

by a *first* process. By a second, each of these tablets becomes two, giving as result,

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} + \begin{vmatrix} A & z \\ B & v \end{vmatrix} + \begin{vmatrix} x & C \\ y & D \end{vmatrix} + \begin{vmatrix} x & z \\ y & v \end{vmatrix}.$$

5. THEOREM. If one column (or row) is *identical with* the other, the tablet = zero. For obviously $\begin{vmatrix} A & A \\ B & B \end{vmatrix}$, by definition, is $AB - BA$ or *zero*.

COR. Equally the tablet = zero, if one column (or row) be *proportional* to the other. Thus if $A : B = C : D$, this proportion yields $AD = BC$; hence

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} \text{ or } \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \text{ meaning } AD - BC, \text{ vanishes.}$$

6. This sometimes usefully simplifies a tablet. Thus

$$\begin{vmatrix} A \pm mC, & C \\ B \pm mD, & D \end{vmatrix}$$

is resolvable by Art. 4 into $\begin{vmatrix} A & C \\ B & D \end{vmatrix} \pm \begin{vmatrix} mC, & C \\ mD, & D \end{vmatrix}$. But the second tablet is zero, because its two columns are proportional. Hence the given tablet simply $\equiv \begin{vmatrix} A & C \\ B & D \end{vmatrix}$.

COR. Hence an important inference. The value of a tablet is not changed if to one column (or row) you make addition proportional to the other column (or row); nor again if you subtract instead of adding.

Further, if in the original equations of Art. 1, the absolute terms c_1, c_2 become zero, the solution for x and y is simply $x = 0$ and $y = 0$. This is indeed the only general solution, unless also the denominator vanish; which makes $x = \frac{0}{0}$, $y = \frac{0}{0}$; deciding nothing as to the value of x or y. In that case we can equate the two values of $\frac{y}{x}$; viz. $\frac{a_1}{b_1} = -\frac{y}{x} = \frac{a_2}{b_2}$. Conversely, this shows $a_1 b_2 \equiv a_2 b_1$, equivalent to $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$. The last is the *condition* which provides that the two equations shall be mutually *consistent*, though x and y do not vanish. It results *from eliminating* x and y, leaving their value arbitrary.

7. Begin from the problem of three simple equations,

$$\begin{Bmatrix} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{Bmatrix}.$$

These present three equations to be fulfilled by only two disposable quantities, viz. $\frac{x}{z}$ and $\frac{y}{x}$, if they are all divided by z. The three equations are not certainly self-consistent. If x, y, z be *eliminated*, an equation of condition will remain. Our first business is to investigate it.

From the *two first* equations treated as at the end of Art. 1, but abandoning *circular order*, we have

$$x : y : z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

in which $-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ now replaces $+\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ of Art. 1.

In the third given equation, substitute for xyz the three quantities now proved proportional to them, and you get

(1) $$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Call it $V_3 = 0$. Then V_3 is linear in $a_3b_3c_3$. This is the condition that the three equations shall be compatible. It is seen to result from eliminating x, y, z. Professor De Morgan wished to call these tablets *Eliminants*. Why Gauss entitled them Determinants, no one explains. Spottiswoode apparently introduced the excellent notation

(2) $$V_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then V_3 is linear in a_3, b_3, c_3, that is, in its third row. Hence also in any row. V_3 is superlinear of the third order.

for the value written above. The coefficient of a_3 is obtained in (1) by obliterating the constituents as last written that are in the same *column* or row as a_3, thus reducing V_3 to $\begin{vmatrix} \dots & b_1 & c_1 \\ \dots & b_2 & c_2 \\ a_3 & \dots & \dots \end{vmatrix}$:

then we see the tablet by which a_3 must be multiplied.

The same process is used with b_3 and c_3, producing

$$\begin{vmatrix} a_1 & \cdots & c_1 \\ a_2 & \cdots & c_2 \\ \cdots & b_3 & \cdots \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & \cdots \\ a_2 & b_2 & \cdots \\ \cdots & \cdots & c_3 \end{vmatrix}.$$

Finally the signs of the terms of V_3 are alternate

$$+ \; - \; +,$$

just as in the bilinear $\quad V_2 = a_1b_2 - a_2b_1.$

Evidently V_3 is formed of six terms, three positive and three negative; each term having three factors, but in no term is any factor combined with another of its own row or its own column.

In $c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ the term $a_1b_2c_3$ (the *diagonal* sloping down *from left to right*) is positive as before. Thus when V_3 is given in contracted form, we can expand it into three apparent binomials.

8. In the three given equations you may exchange the position of x and y; then by eliminating y, x, z you obtain

$$U_3 = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = 0;$$

but you cannot infer that $U_3 = V_3$. In fact, to exchange the a column with the b column reverses the sign of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. Thus it changes V_3 into

$$b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}.$$

That is, to exchange the first and second columns just *reverses the sign* of V_3. The same effect follows from exchanging any two contiguous columns or *rows*. Thus generally

$$\begin{vmatrix} A & D & G \\ B & E & H \\ C & F & J \end{vmatrix} = - \begin{vmatrix} D & A & G \\ E & H & B \\ F & C & J \end{vmatrix} = + \begin{vmatrix} D & G & A \\ E & H & B \\ F & J & C \end{vmatrix}.$$

Again, $$\begin{vmatrix} A & D & G \\ B & E & H \\ C & F & J \end{vmatrix} = - \begin{vmatrix} B & E & H \\ A & D & G \\ C & F & J \end{vmatrix} = + \begin{vmatrix} B & E & H \\ C & F & J \\ A & D & G \end{vmatrix}.$$

Observe, that if three binomials of V_3 are expressed by

$$ma_3 + na_2 + pa_1$$

the multipliers m, n, p contain nothing of the column a_1, a_2, a_3, therefore V_3 is linear in these three constituents. Evidently it is linear in regard to any column; as we before saw, as to any row.

9. THEOREM. Further, *I say*, To exchange rows and columns does not alter the value of V_3. In proof, multiply the three equations *given* in Art. 7, by disposable numbers m, n, p, and so assume m, n, p that when the three products are added together the coefficients of y and z may vanish. There will remain $(ma_1 + na_2 + pa_3)\,x = 0$, and as we do not admit $x = 0$, we have three equations connecting m, n, p; viz.

$$\left.\begin{array}{l} a_1 m + a_2 n + a_3 p = 0 \\ b_1 m + b_2 n + b_3 p = 0 \\ c_1 m + c_2 n + c_3 p = 0 \end{array}\right\},$$

and that these may be compatible, we need

$$S_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0,$$

by eliminating m, n, p.

Here S_3 is nothing but V_3 with rows changed to columns and columns to rows, retaining the same positive diagonal $a_1 b_2 c_3$. Every learner will easily find by developing V_3 and S_3 that they are identical: but there is an advantage here in general argument applicable to higher orders. $S_3 = 0$ and $V_3 = 0$, being each a condition of compatibility of the previous equations, must contain *the same relation of the constituents.* S_3 is a linear function of its column $a_1 b_1 c_1$; so is V_3 a linear function of its row $a_1 b_1 c_1$. But $S_3 = 0$, $V_3 = 0$ being derivable one from the other, there is no possible relation but $S_3 = \mu V_3$ in which μ must be free from $a_1 b_1 c_1$. But the same arguments will prove that μ is free from $a_2 b_2 c_2$, also from $a_3 b_3 c_3$. Therefore μ is *wholly numerical.* Make $b_1 = 0$, $c_1 = 0$ and this will not affect μ. But this makes

$$V_3 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \text{ and } S_3 = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

But that the two minor tablets are identical was implied in their definition. Hence, on this assumption for b_1 and c_1, we find $V_3 = S_3$ or $\mu = 1$. This then is the *universal* value of μ_1 or $V_3 = S_3$ in all cases. Q.E.D.

10. In the developed value of V_3 (Art. 7) if $b_3=0$ and $c_3=0$, you get simply $V_3=a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ which does not contain a_1 or a_2. These two constituents are made *wholly inefficient* by the vanishing of b_3 and c_3, and may be changed to zero. Thus

$$\left|\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \hline a_3 & 0 & 0 \end{array}\right| = \left|\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ \hline a_3 & 0 & 0 \end{array}\right|. \text{ So } \left|\begin{array}{c|cc} a_1 & b_1 & c_1 \\ \hline 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{array}\right| = \left|\begin{array}{c|cc} a_1 & 0 & 0 \\ \hline 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{array}\right|.$$

When a major square is divided into two minor squares and two complemental rectangles, the vanishing of *one* rectangle obliterates the other, and the greatest tablet has only the two squares for its factors.

11. Since V_3 is a linear function of every row and of every column, we can argue as in the Second Order or Bilinears, that to multiply any column, or any row, by m, multiplies V_3 by m; and if a column or row consist of Binomials, we resolve the tablet into two. Conversely two V_3s which differ only in a single row or single column can be joined into one universally,

$$\begin{vmatrix} A+m, & A' & A_0 \\ B+n, & B' & B_0 \\ C+p, & C' & C_0 \end{vmatrix} = \begin{vmatrix} A & A' & A_0 \\ B & B' & B_0 \\ C & C' & C_0 \end{vmatrix} + \begin{vmatrix} m & A' & A_0 \\ n & B' & B_0 \\ p & C' & C_0 \end{vmatrix}.$$

12. Further, if two contiguous columns (or rows) be identical, the $V_3=$zero. For to exchange them changes V_3 to $-V_3$. Yet the exchange leaves V_3 exactly what it was before. Hence $V_3=-V_3$. This can only be when $V_3=0$.

It follows that if a column (or row) be *proportional* to a contiguous column (or row), the tablet is zero. For instance, if a_1, b_1, c_1 are proportional to a_2, b_2, c_2, we may assume $a_1=ma_2$, $b_1=mb_2$, $c_1=mc_2$, then

$$V_3=\begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This gives

$$V_3=m\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and the last tablet is zero, because its first and second row are identical.

What has been said in this Article of two *contiguous* rows or columns is evidently true of *any two* rows or any two columns, since exchange of contiguous rows (or columns) does but multiply the tablet by -1.

13. The same argument as before in the Second Order now proves that a tablet V_3 is not changed in value, if any row (or column) receives increase or decrease proportional to some other row (or column). Thus we have shown in V_3 the same properties as those enunciated in V_2.

14. NEW PROBLEM: to solve for x, y, z in the three equations

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \\ a_3x + b_3y + c_3z + d_3 = 0. \end{cases}$$

Assume $d_1 = A_1x$, $d_2 = A_2x$, $d_3 = A_3x$. Then

$$\left.\begin{array}{l} (a_1 + A_1)\,x + b_1y + c_1z = 0 \\ (a_2 + A_2)\,x + b_2y + c_2z = 0 \\ (a_3 + A_3)\,x + b_3y + c_3z = 0 \end{array}\right\}.$$

Eliminate the x, y, z here visible; then

$$\begin{vmatrix} a_1 + A_1 & b_1 & c_1 \\ a_2 + A_2 & b_2 & c_2 \\ a_3 + A_3 & b_3 & c_3 \end{vmatrix} = 0.$$

The last tablet may be resolved into two, namely:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} A_1 & b_1 & c_1 \\ A_2 & b_2 & c_2 \\ A_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Multiply the former of these by x, and, as an equivalence, multiply each constituent of the first column of the latter by x,

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x + \begin{vmatrix} A_1x & b_1 & c_1 \\ A_2x & b_2 & c_2 \\ A_3x & b_3 & c_3 \end{vmatrix} = 0.$$

In the first column of the last restore for A_1x, A_2x, A_3x their values d_1, d_2, d_3. Then if

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ you have } Dx+\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}=0,$$

which solves for x. By perfectly similar steps

$$Dy+\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_2 & c_3 \end{vmatrix}=0\,;\ Dz+\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0.$$

These are easy to remember: each suggests the other, by entire symmetry. The *method* succeeds in Higher Orders.

Cor. If we make $x=\dfrac{\xi}{u},\ y=\dfrac{\eta}{u},\ z=\dfrac{\zeta}{u},$

$$\therefore\ \begin{cases} a_1\xi+b_1\eta+c_1\zeta+d_1u=0, \\ a_2\xi+b_2\eta+c_2\zeta+d_2u=0, \\ a_3\xi+b_3\eta+c_3\zeta+d_3u=0. \end{cases}$$

Then from

$$-Dx\equiv\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix};\ Dy=\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix};\ -Dz=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

you obtain the proportion

$$\xi : \eta : \zeta : u$$

$$=\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} : -\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} : -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

15. Problem. To eliminate x from the two equations

$$\begin{Bmatrix} Px^2+Qx+R=0 \\ px^2+qx+r=0 \end{Bmatrix},$$

where P, Q, R, p, q, r may involve y or other quantities. First, put $x^2=X$, then we have

$$\left.\begin{matrix} PX+Qx+R=0 \\ pX+qx+r=0 \end{matrix}\right\}.$$

Eliminate X; i.e. solve for x;

$$\therefore\ \begin{vmatrix} P & Q \\ p & q \end{vmatrix}x+\begin{vmatrix} P & R \\ p & r \end{vmatrix}=0 \ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, writing in the original

$$\begin{cases}(Px+Q)\,x+R=0\\(px+q)\,x+r\;=0\end{cases}$$

eliminate x, as if $Px+Q$ and $px+q$ were ordinary coefficients. Then

$$\begin{vmatrix} Px+Q, & R \\ px+q, & r \end{vmatrix}=0,$$

which expanded, gives

$$\begin{vmatrix} Px & R \\ px & r \end{vmatrix}+\begin{vmatrix} Q & R \\ q & r \end{vmatrix}=0,\ \text{or}\ \begin{vmatrix} P & R \\ p & r \end{vmatrix}x+\begin{vmatrix} Q & R \\ q & r \end{vmatrix}=0\ldots\ldots(2).$$

Eliminate x between (1) and (2), which gives

$$\left\|\begin{matrix} \begin{vmatrix} P & Q \\ p & q \end{vmatrix} & \begin{vmatrix} P & R \\ p & r \end{vmatrix} \\ \begin{vmatrix} P & R \\ p & r \end{vmatrix} & \begin{vmatrix} Q & R \\ q & r \end{vmatrix} \end{matrix}\right\|=0;$$

that is, $$\begin{vmatrix} P & Q \\ p & q \end{vmatrix}\cdot\begin{vmatrix} Q & R \\ q & r \end{vmatrix}-\begin{vmatrix} P & R \\ p & r \end{vmatrix}^2=0,$$

the result required.

16. Problem. To eliminate x from the two equations,

$$\begin{cases}ax^3+3bx^2+3cx+d=0,\\ax^2+2bx+c\qquad\;=0.\end{cases}$$

First, multiply the last by x and subtract;

$$\therefore\ bx^2+2cx+d=0.$$

Compare the two last with the two equations of Art. 15,

$$\therefore\ P=a,\ Q=2b,\ R=c,\ p=b,\ q=2c,\ r=d.$$

Hence the elimination of x yields

$$\begin{vmatrix} a & 2b \\ b & 2c \end{vmatrix}\cdot\begin{vmatrix} 2b & c \\ 2c & d \end{vmatrix}-\begin{vmatrix} a & c \\ b & d \end{vmatrix}^2=0,\ \text{or}\ \begin{vmatrix} a & b \\ b & c \end{vmatrix}\cdot\begin{vmatrix} b & c \\ c & d \end{vmatrix}=\tfrac{1}{4}\begin{vmatrix} a & c \\ b & d \end{vmatrix}^2.$$

This is the condition of *two equal roots* in the given cubic.

17. By conducting elimination from three given simple equations by a second process, we attain relations which were not easy to anticipate. Assume the three equations of Art. 7. Eliminate y

twice, (1) from the two first; (2) from the second and third. Hence we get

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} x = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix};$$

from which, by eliminating x, we obtain $U=0$, if we define U by the equation

$$U = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} . \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} . \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} .$$

But, otherwise eliminated, we find, as the condition of compatibility, $V_3 = 0$.

Therefore $U=0$ contains the *same relation* of the constituents as $V_3=0$. By inspection, we see that U, equally with V_3, is linear in a_1, b_1, c_1. Since then U and V vanish together, we have necessarily $U = \mu . V_3$, in which μ does *not* involve a_1, b_1 nor c_1. To determine μ, suppose $a_1 = 1$, $b_1 = 0$, $c_1 = 0$,

$$\therefore \; U = \begin{vmatrix} a_1 & 0 \\ a_2 & b_2 \end{vmatrix} . \begin{vmatrix} b_2 & c_2 \\ b_3 & b_3 \end{vmatrix} = b_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

and
$$V_3 = \begin{vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 1 . \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix};$$

so that in this particular case $U = b_2 V_3$, or $\mu = b_2$. This then is the value of μ for *all* values of a_1, b_1, c_1, or universally $U = b_2 . V_3$, while all the nine constituents are arbitrary. Q.E.D.

18. To remember this important equation, write the square trilinear larger and mark out its minor square. The factor b_2 is in the centre.

a_1	b_1	c_1
a_2	b_2	c_2
a_3	b_3	c_3

for $U = b_2 . V_3$.

By interchanging rows or columns without altering the value of V_3, fresh relations are obtained. Indeed no one constituent can claim the central place for itself.

Fourth Order.

19. To eliminate x, y, z, u from four given equations each of the form $a_1x + b_1y + c_1z + d_1u = 0$, we have now much facility from the Cor. to Art. 14. First, eliminate from the three first equations and get the *proportions* of x, y, z, u. Next, insert these proportionate values in the fourth equation whereby you entirely eliminate all the four, and obtain an equation $V_4 = 0$; if V_4 *stand for*

$$a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} - b_4 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This *developed* form of V_4 can always be recovered (by attention to the simple rule given for developing V_3) from the conciser or *un*developed form

$$V_4 = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Evidently in the definition V_4 is a linear function of $a_4\ b_4\ c_4\ d_4$.

So then it must be of any other *row*, the order of the given equations being arbitrary. Also the *first* term of V_4 as defined is free from $a_1a_2a_3$; each of the *others* is linear in $a_1a_2a_3$. Therefore V_4 in square is linear as to its first column; so then must it be as to every column.

Being thus linear, if one column (or row) of such quadrilinear tablet is binomial, it may be split into two tablets by the same process as in the third order. Likewise to multiply any column (or row) by m multiplies the whole tablet by m.

To exchange first and second column, exchanges the first and second term of V_4, but *reverses* their signs. It reverses the sign of the third term, also of the fourth. Thus to exchange the first and second column reverses the sign of V_4. Evidently then the same must happen by exchanging *any two* contiguous columns. The same argument applies concerning any two contiguous *rows*.

The reasoning of Art. 12 concerning V_3 now applies to V_4, showing that the tablet vanishes, if one column (or row) be proportional to another column (or row). From this it further follows (as concerning V_3 in Art. 13, and concerning V_2 in Art. 6), that V_4 is not changed in value, if any row (or column) receive increase or decrease in proportion to some other row (or column).

20. That V_4 is not altered by exchanging rows and columns, is generally proved by elaborate inspection of the separate terms when V_4 is resolved into 24 elements each of the form $a_1 b_2 c_3 d_4$, no two factors of the same row or column, and showing that the + or − of the term is never altered. It is, no doubt, a perfect demonstration, and more elementary than mine; but I find the less obvious argument of Art. 9 the easier for all the higher orders. Multiply the given equations by $m_1 n_1 p_1 q_1$ and assign to these multipliers the condition that from the *sum* of the equations thus multiplied y, z and u shall disappear.

There will remain $(ma_1 + na_2 + pa_3 + qa_4)\,x = 0$. But our hypothesis forbids $x = 0$, hence we have four equations to determine $mnpq$, viz.

$$\begin{aligned} a_1 m + a_2 n + a_3 p + a_4 q &= 0, \\ b_1 m + b_2 n + b_3 p + b_4 q &= 0, \\ c_1 m + c_2 n + c_3 p + c_4 q &= 0, \\ d_1 m + d_2 n + d_3 p + d_4 q &= 0. \end{aligned}$$

When we eliminate $mnpq$ the result, which we may call $S_4 = 0$, shows S_4 differing from V_4 only in the exchange of rows with columns. Each of them is linear in $a_1 b_1 c_1 d_1$. Each involves the same relations between the constituents. The equation $S_4 = 0$ must be deducible from $V_4 = 0$. The only possible relation, making S_4 and V_4 vanish together, has the form $S_4 = \mu V_4$, in which μ is independent of $a_1 a_2 a_3 a_4$. But symmetry proves μ equally independent of every other column; therefore μ is *numerical*. To find it, we may make the constituents on the positive diagonal all $= 1$, and all the other constituents vanish. Then both S_4 and $V_4 = a_1 b_2 c_3 d_4 = 1^5$. Universally then, $\mu = 1$, or $S_4 = V_4$. Therefore V_4 is not altered by exchanging rows with columns.

Evidently this argument holds, however high the order of the Tablet, if the successive definitions follow the same law.

21. We can now solve when four given simple equations connecting $xyzu$ have on the left side absolute terms $e_1e_2e_3e_4$, with *zero* (as before) on the right. We proceed as in Art. 14. Let

$$e_1 = A_1x, \quad e_2 = A_2x, \quad e_3 = A_3x, \quad e_4 = A_4x.$$

Then our equation becomes

$$\left.\begin{array}{l}(a_1+A_1)\,x+b_1y+c_1z+d_1u=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (a_4+A_4)\,x+b_4y+c_4z+d_4u=0\end{array}\right\}.$$

Eliminate x, y, z, u, then

$$\begin{vmatrix} a_1+A_1 & b_1 & c_1 & d_1 \\ a_2+A_2 & b_2 & c_2 & d_2 \\ a_3+A_3 & b_3 & c_3 & d_3 \\ a_4+A_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

The first column being binomial, we can resolve this tablet into two. Then multiply the left tablet by x, and the first column of the second also by x; whence

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} x + \begin{vmatrix} A_1x & b_1 & c_1 & d_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_4x & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

In the second tablet we now replace its column by its value $e_1\,e_2\,e_3\,e_4$.

Thus we have solved for x. By perfectly similar steps we solve for y, for z, and for u.

Finally, if

$$M = \begin{vmatrix} b_1 & c_1 & d_1 & e_1 \\ \cdots & \cdots & \cdots & \cdots \\ b_4 & c_4 & d_4 & e_4 \end{vmatrix}, \quad N = \begin{vmatrix} a_1 & c_1 & d_1 & e_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_4 & c_4 & d_4 & e_4 \end{vmatrix}, \quad P = \begin{vmatrix} a_1 & b_1 & d_1 & e_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_4 & b_4 & d_4 & e_4 \end{vmatrix},$$

$$Q = \begin{vmatrix} a_1 & b_1 & c_1 & e_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_4 & b_4 & c_4 & e_4 \end{vmatrix}, \quad R = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

we have $\quad x : y : z : u : 1 = M : -N : P : -Q : R.$

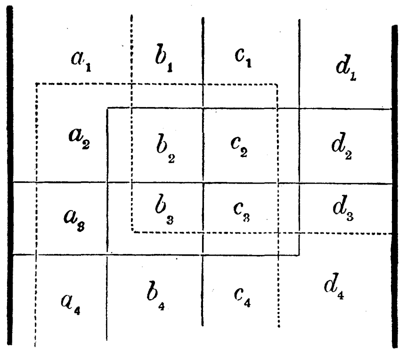
22. Take our four equations as in Art. 19. From the three first and also from the three last eliminate both y and z; whence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = - \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}; \text{ and } \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} x = - \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

To eliminate x from these two, we have

$$U_4 = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ \cdots & \cdots & \cdots \\ a_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} b_2 & c_2 & d_2 \\ \cdots & \cdots & \cdots \\ b_4 & c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ \cdots & \cdots & \cdots \\ a_4 & b_4 & c_4 \end{vmatrix} . \begin{vmatrix} b_1 & c_1 & d_1 \\ \cdots & \cdots & \cdots \\ b_3 & c_3 & d_3 \end{vmatrix},$$

which we may remember by



Thus $U_4 = 0$ and $V_4 = 0$ express the same condition of the constituents for reconciling the four equations.

Inspection shows that U_4, like V_4, is linear in $a_1b_1c_1d_1$. Put $U_4 = \mu V_4$, and μ will be free from these four. Assume then $a_1 = 1$, $b_1 = c_1 = d_1 = 0$, and it will not affect μ. But it makes

$$V_4 = 1 . \begin{vmatrix} b_2 & c_2 & d_2 \\ \cdots & \cdots & \cdots \\ b_4 & c_4 & d_4 \end{vmatrix} \text{ and } U_4 = 1 . \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} . \begin{vmatrix} b_2 & c_2 & d_2 \\ \cdots & \cdots & \cdots \\ b_4 & c_4 & d_4 \end{vmatrix};$$

that is, $$\mu = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

whence U_4 generally $= \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} . V_4$. This could not have been foreseen. By varying the order of the elements, we have other results.

Observe that $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is the square in the centre of V_4. The trilinears in U are squares cut from the four corners of V_4; those of the first term in U_4 being from the positive diagonal.

23. By aid of Art. 21 we readily proceed to the Fifth Order, with the same law of continuous formation; whence in every Order we have these same properties.

(1) To exchange rows and columns does not affect the value of V_n.

(2) V_n is a linear function of any one row, or any one column.

(3) If a row or column be binomial, the V_n may be split into $V'_n \pm V''_n$.

(4) To multiply a row or column by m, multiplies V_n by m.

(5) To exchange any row (or column) with a contiguous row (or column) changes V_n to $-V_n$.

(6) If one row (or column) is identical with or proportional to another row (or column), the $V_n = \text{zero}$.

(7) V_n is not altered in value when a row (or column) receives increase or decrease proportioned to another row or column.

(8) If V_n be divided along the diagonal, so as to fall into four parts, two squares and two rectangular complements, the vanishing of one complement makes the other wholly ineffective.

24. To prove the last universally, it suffices to prove it for the fifth order.

Call the two squares P, S and the complements Q, R. Then if one of the complements, as Q, have all its constituents *zero, I say*, R is ineffective, and $V = P \,.\, S$, just as if R also had all its constituents zero. For every term of V_5 when fully expanded, has the form $a_m b_n c_p d_q e_r$, where $mnpqr$ are taken from 1, 2, 3, 4, 5 and no two are the same. Hence $a_4 b_4 c_4$ and $a_5 b_5 c_5$ (the constituents of R) are necessarily multiplied by one or other of the zeros $(d_1 e_1 d_2 e_2 d_3 e_3)$ in Q, and all the products vanish. Consequently, if P is of the third order and S of the second, the V in question is equivalent to

$$V_5 = \begin{vmatrix} a_1 & b_1 & c_1 & \vdots & 0 & 0 \\ a_2 & b_2 & c_2 & \vdots & 0 & 0 \\ a_3 & b_3 & c_3 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & d_4 & e_4 \\ 0 & 0 & 0 & \vdots & d_5 & e_5 \end{vmatrix}$$

and might arise from two equations separately, yielding $P=0$, and $S=0$. In fact $V=P.S$.

If Spottiswoode did not plant the first germ of this very valuable theory, he first investigated the laws and exhibited its vast power.

TRACT V.

Introduction to Tables I. and II.

To these four Elementary Tracts I have added two Numerical Tables, solely because their compilation and verification is elementary.

Table I. gives values of A^{-n} to 20 decimal places. Here A means the series 2, 3, 4, ... up to 60, and the odd numbers from 61 to 77; and n means 1, 2, 3, ... continued until A^{-n} is about to vanish. To verify, use the formula

$$A^{-1} + A^{-2} + A^{-3} + \dots A^{-m} + \frac{A^{-m}}{A-1} = \frac{1}{A-1}.$$

The reader may convince himself how searching is this test, by applying it, for instance, to A^{-n} when $A = 37$ or when $A = 71$. Only in the case of 2^{-n} and 3^{-n}, where the Tablets give only *odd* values of n, we must apply the formula

$$(A^{-1} + A^{-3} + A^{-5} + \dots + A^{-2m-1}) = \frac{A - A^{-2m-1}}{A^2 - 1}.$$

Table II. has values of x^n with 12 decimal places, where x means ·02, ·03, ·04 up to ·50 and n is continued from 1, 2, 3,... until x^n is insignificant. The formula of verification is (with m any integer less than r)

$$(x^m + x^{m+1} + x^{m+2} + \dots + x^r).(1-x) = x^m - x^{r+1}.$$

I compiled this table while working at Spence's integral

$$\int_0 \log(1+x)\frac{dx}{x},$$

but it has much wider use.

One who is sagely incredulous of printed tables can verify *for himself* any tablet which he is disposed to use with much greater ease than he could compose the tablet. Thus, too, he would detect any error from miscopying or misprinting, against which I can least give a guarantee.

n	2^{-n} (n odd).	n	3^{-n} (n odd).
1	·5	1	·33333 33333 33333 33333
3	·125	3	·03703 70370 37037 03703
5	·03125	5	411 52263 37448 55967
7	·00781 25	7	45 72473 70827 61774
9	·00195 3125	9	5 08052 63425 29086
11	48 82812 5	11	...... 56450 29269 47676
13	12 20703 125	13	...... 6272 25474 38630
15	3 05175 78125	15	...... 696 91719 37625
17	...... 76293 94531 25	17	...... 77 43524 37514
19	...... 19073 48632 8125	19	...... 8 60391 59724
21	...... 4768 37158 20312	21	.... 95599 06636
23	...... 1192 09289 55078	23	.... 10622 11848
25	...... 298 02322 38769	25	.... 1180 23539
27	...... 74 50580 59692	27	.... 131 13726
29	...... 18 62645 14923	29	.... 14 57081

n	4^{-n}	n	4^{-n}
1	·25	17	·00000 00000 58207 66091
2	·0625	18	.... 14551 91523
3	·01562 5	19	.... 3637 97880
4	·00390 625	20	.... 909 49470
5	97 65625	21	.... 227 37367
6	24 41406 25	22	.... 56 84342
7	6 10351 5625	23	.... 14 21085
8	1 52587 89062 5	24	.... 3 55271
9	...... 38146 97265 625	25	 88818
10	...... 9536 74316 40625	26	 22204
11	...... 2384 18579 10156	27	 5551
12	...... 596 04644 77539	28	 1388
13	...... 149 01161 10385	29	 347
14	...... 37 25290 29846	30	 87
15	...... 9 31322 57461	31	 22
16	...... 2 32830 64365	32	 5
		33	 1

n	5^{-n}	n	6^{-n}
1	·2	1	·16666 66666 66666 66666
2	·04	2	·02777 77777 77777 77777
3	·008	3	462 96296 29629 62962
4	·00160	4	77 16049 38271 60494
5	·00032	5	12 86008 23045 26749
6	·00006 4	6	2 14334 70507 54458
7	·00001 28	7	...... 35722 45084 59076
8	 256	8	...... 5933 74180 76513
9	 05120	9	...... 992 29030 12752
10	 01024	10	...... 165 38171 68792
11	...... 00204 8	11	...... 27 53661 97499
12	...... 40 96	12	...... 4 59393 65799
13	...... 8 192	13	.... 76565 60966
14	...... 1 63840	14	.... 12760 93494
15	 32768	15	.... 2126 82249
16	.... 6553 6	16	.... 354 47041
17	.... 1310 72	17	.... 59 07840
18	.... 262 144	18	.... 9 84640
19	.... 52 42880	19	.... 1 64106
20	.... 10 48576	20	 27351
21	.... 2 09715	21	 4558
22	 41943	22	 759
23	 8388	23	 126
24	 1677	24	 21
25	 335	25	 3
26	 67		
27	 13		
28	 2		

n	7^{-n}	n	7^{-n}
1	·14285 71428 57142 85714	12	·00000 00000 72247 61581
2	·02040 81632 65306 12245	13	.... 10321 08797
3	291 54518 95043 73178	14	.... 1474 44114
4	41 64931 27863 39025	15	.... 210 63445
5	5 94990 18266 19861	16	.... 30 09063
6	...... 84998 59752 31409	17	.... 4 29866
7	...... 12142 65678 90201	18	 61409
8	...... 1734 66525 55743	19	 8774
9	...... 247 80932 22249	20	 1253
10	...... 35 40133 17464	21	 179
11	...... 5 05733 31066	22	 25
		23	 4

n	8^{-n}	n	9^{-n}
1	·125	1	·11111 11111 11111 11111
2	·01562 5	2	·01234 56790 12345 67901
3	·00195 3125	3	137 17421 12482 85322
4	24 41406 25	4	15 24157 90275 87258
5	3 05175 78125	5	1 69350 87808 43029
6	...... 38146 97265 625	6	...... 18816 76423 15892
7	...... 4768 37158 20312	7	...... 2090 75158 12877
8	...... 596 04644 77539	8	...... 232 30573 12542
9	...... 74 50580 59692	9	...... 25 81174 79171
10	...... 9 31322 57461	10	...... 2 86797 19908
11	...... 1 16415 32182	11	.... 31866 35545
12	.... 14551 91523	12	.... 3540 70616
13	.... 1818 98940	13	.... 393 41179
14	.... 227 37367	14	.... 43 71242
15	.... 28 42171	15	.... 4 85693
16	.... 3 55271	16	 53966
17	 44409	17	 5996
18	 5551	18	 666
19	 694	19	 74
20	 87	20	 8
21	 11	21	 0,9
22	 1		

n	11^{-n}	n	12^{-n}
1	·09090 90909 09090 90909	1	·08333 33333 33333 33333
2	826 44628 09917 35537	2	694 44444 44444 44444
3	75 13148 00901 57776	3	57 87037 03703 70370
4	6 83013 45536 50707	4	4 82253 08641 97531
5	...... 62092 13230 59153	5	...... 40187 75720 16461
6	...... 5644 73930 05377	6	...... 3348 97976 68038
7	...... 513 15811 82307	7	...... 279 08164 72336
8	...... 46 65073 80209	8	...... 23 25680 39361
9	...... 4 24097 61837	9	...... 1 93806 69946
10	.... 38554 32894	10	.... 16150 55829
11	.... 3504 93899	11	.... 1345 87985
12	.... 318 63082	12	.... 112 15665
13	.... 28 96644	13	.... 9 34639
14	.... 2 63331	14	 77886
15	 23939	15	 6490
16	 2176	16	 541
17	 198	17	 45
18	 18	18	 4
19	 1		

n	13^{-n}	n	14^{-n}
1	·07692 30769 23076 92307	1	·07142 85714 28571 42857
2	591 71597 63313 60947	2	510 20408 16326 53061
3	45 51661 35639 50842	3	36 44314 86880 46647
4	3 50127 79664 57757	4	2 60308 20491 46189
5	...... 26932 90743 42904	5	...... 18593 44320 81870
6	...... 2071 76211 03300	6	...... 1328 10308 62991
7	...... 159 36631 61792	7	...... 94 86450 61642
8	...... 12 25894 73984	8	...... 6 77603 61546
9	.... 94299 59537	9	.... 48400 25825
10	.... 7253 81503	10	.... 3457 16130
11	.... 557 98577	11	.... 246 94009
12	.... 42 92198	12	.... 17 63858
13	.... 3 30169	13	.... 1 25989
14	 25397	14	 8999
15	 1594	15	 643
16	 150	16	 46
17	 11	17	 3

n	15^{-n}	n	17^{-n}
1	·06666 66666 66666 66666	1	·05882 35294 11764 70588
2	444 44444 44444 44444	2	346 02076 12456 74740
3	29 62962 96296 29629	3	20 35416 24262 16161
4	1 97530 86419 75308	4	1 19730 36721 30362
5	...... 13168 72427 98360	5	...... 7042 96277 72375
6	...... 877 91495 19891	6	...... 414 29192 80728
7	...... 58 52766 34659	7	...... 24 37011 34160
8	...... 3 90184 42311	8	...... 1 43353 60833
9	.... 26012 29487	9	.... 8432 56519
10	.... 1734 15299	10	.... 496 03324
11	.... 115 61020	11	.... 29 17842
12	.... 7 70734	12	.... 1 71638
13	 51382	13	 10096
14	 3425	14	 594
15	 228	15	 35
16	 15	16	 2
17	 1		

For 16^{-n} look to 4^{-2n}.

n	18^{-n}	n	19^{-n}
1	·05555 55555 55555 55555	1	·05263 15789 47368 42105
2	308 64197 53086 41975	2	277 00831 02493 07479
3	17 14677 64060 35665	3	14 57938 47499 63551
4	...... 95259 86892 24204	4	...... 76733 60394 71766
5	...... 5292 21494 01345	5	...... 4038 61073 40619
6	...... 294 01194 11186	6	...... 212 55845 96874
7	...... 16 33399 67288	7	...... 11 18728 73519
8	.... 90744 42627	8	.... 58880 45975
9	.... 5041 35701	9	.... 3098 97156
10	.... 280 07539	10	.... 163 10376
11	.... 15 55974	11	.... 8 58441
12	 86443	12	 45181
13	 4802	13	 2378
14	 267	14	 125
15	 15	15	 6

n	21^{-n}	n	22^{-n}
1	·04761 90476 19047 61904	1	·04545 45454 54545 45454
2	226 75736 96145 12471	2	206 61157 02479 33884
3	10 79796 99816 43451	3	9 39143 50112 69722
4	...... 51418 90467 44926	4	...... 42688 34096 03169
5	...... 2448 51927 02139	5	...... 1940 37913 45599
6	...... 116 59615 57245	6	...... 88 19905 15709
7	...... 5 55219 78916	7	...... 4 00904 77987
8	.... 26439 03758	8	.... 18222 94454
9	.... 1259 00179	9	.... 828 31566
10	.... 59 95246	10	.... 37 65071
11	.... 2 85488	11	.... 1 71139
12	 13594	12	 7779
13	 647	13	 353
14	 31	14	 16
15	 1		

n	23^{-n}	n	24^{-n}
1	·04347 82608 69565 21739	1	·04166 66666 66666 66666
2	189 03591 68241 96597	2	173 61111 11111 11111
3	8 21895 29053 99852	3	7 23379 62962 96296
4	...... 35734 57784 95646	4	...... 30140 81790 12345
5	...... 1553 67729 78071	5	...... 1255 86741 25514
6	...... 67 55118 68612	6	...... 52 32780 88563
7	...... 2 93700 81244	7	...... 2 18032 53690
8	.... 12769 60054	8	.... 9084 68903
9	.... 555 20002	9	.... 378 52871
10	.... 24 13913	10	.... 15 77203
11	.... 1 04953	11	 65716
12	 4563	12	 2738
13	 198	13	 114
14	 8	14	 4

For 25^{-n} look to 5^{-2n}.

n	26^{-n}	n	27^{-n}
1	·03846 15384 61538 46153	1	·03703 70370 37037 03703
2	147 92899 40828 40236	2	137 17421 12482 85322
3	5 68957 66954 93855	3	5 08052 63425 29086
4	...... 21882 98729 03609	4	...... 18816 76423 15892
5	...... 841 65335 73216	5	...... 696 91719 37625
6	...... 32 37128 29739	6	...... 25 81174 79171
7	...... 1 24504 93451	7	.... 95599 06636
8	.... 4788 65133	8	.... 3540 70616
9	.... 184 17890	9	.... 131 13726
10	.... 7 08380	10	.... 4 85693
11	 27245	11	 17982
12	 1048	12	 666
13	 40	13	 25
14	 1		

n	28^{-n}	n	29^{-n}
1	·03571 42857 14285 71428	1	·03448 27586 20689 65517
2	127 55102 04081 63265	2	118 90606 42092 74673
3	4 55539 35860 05831	3	4 10020 91106 64644
4	...... 16269 26280 71637	4	...... 14138 65210 57401
5	...... 581 04510 02558	5	...... 487 53972 77841
6	...... 20 75161 07234	6	...... 16 81171 47512
7	.... 74112 89544	7	.... 57971 43011
8	.... 2646 88912	8	.... 1999 01483
9	.... 94 53175	9	.... 68 93154
10	.... 3 37613	10	.... 2 37695
11	 12057	11	 8196
12	 430	12	 283
13	 15	13	 9

n	31^{-n}	n	32^{-n}
1	·03225 80645 16129 03226	1	·03125
2	104 05827 26326 74298	2	97 65625
3	3 35671 84720 21751	3	3 05175 78125
4	...... 10828 12410 32960	4	...... 9536 74316 40625
5	...... 349 29432 59127	5	...... 298 02322 38769
6	...... 11 26755 89004	6	...... 9 31322 57461
7	.... 36346 96419	7	.... 29103 83045
8	.... 1172 48271	8	.... 909 49470
9	.... 37 82202	9	.... 28 42171
10	.... 1 22007	10	 88818
11	 3936	11	 2775
12	 127	12	 86
13	 4	13	 2

n	33^{-n}	n	34^{-n}
1	·03030 30303 03030 30303	1	·02941 17647 05882 35294
2	91 82736 45546 37282	2	86 50519 03114 18685
3	2 78264 74107 46584	3	2 54427 03032 77020
4	...... 8432 26488 10502	4	...... 7483 14795 08147
5	...... 255 52317 82136	5	...... 220 09258 67888
6	...... 7 74312 66125	6	...... 6 47331 13761
7	.... 23464 02004	7	.... 19039 15110
8	.... 711 03091	8	.... 559 97503
9	.... 21 54639	9	.... 16 46985
10	 65292	10	 48441
11	 1978	11	 1425
12	 59	12	 42
13	 2	13	 1

n	35^{-n}	n	37^{-n}
1	·02857 14285 71428 57142	1	·02702 70270 27027 02702
2	81 63265 30612 24489	2	73 04601 89919 64938
3	2 33236 15160 34985	3	1 97421 67295 12566
4	...... 6663 89004 58142	4	..... 5335 72089 05745
5	...... 190 39685 84518	5	...... 144 20867 27182
6	...... 5 43991 02415	6	...... 3 89753 16951
7	.... 15542 60069	7	.... 10533 86945
8	.... 444 07430	8	.... 284 69917
9	.... 12 68783	9	.... 7 69457
10	 36251	10	 20796
11	 1036	11	 562
12	 29	12	 15
13	 0,8		

For 36^{-n} look to 6^{-2n}.

n	38^{-n}	n	39^{-n}
1	·02631 57894 73684 21052	1	·02564 10256 41025 64102
2	69 25207 75623 26870	2	65 74621 05923 73438
3	1 82242 30937 45444	3	1 68580 05023 68549
4	...... 4795 85024 66985	4	...... 4322 56539 06886
5	...... 126 20658 54394	5	...... 110 83501 00176
6	...... 3 32122 59326	6	...... 2 84192 33338
7	.... 8740 06824	7	.... 7286 98290
8	.... 230 00179	8	.... 186 84571
9	.... 6 05268	9	.... 4 79091
10	 15928	10	 12284
11	 419	11	 315
12	 11	12	 8

n	41^{-n}	n	42^{-n}
1	·02439 02439 02439 02439	1	·02380 95238 09523 80952
2	59 48839 97620 46401	2	56 68934 24036 28118
3	1 45093 65795 62107	3	1 34974 62477 05431
4	...... 3538 86970 62490	4	...... 3213 68154 21558
5	...... 86 31389 52744	5	...... 76 51622 71942
6	...... 2 10521 69579	6	...... 1 82181 49332
7	.... 5134 67550	7	.... 4337 65460
8	.... 125 23599	8	.... 103 27749
9	.... 3 05453	9	.... 2 45899
10	 7450	10	 5854
11	 182	11	 139
12	 4	12	 3

n	43^{-n}	n	44^{-n}
1	·02325 58139 53488 37209	1	·02272 72727 27272 72727
2	54 08328 82639 26447	2	51 65289 25619 83471
3	1 25775 08898 58754	3	1 17392 93764 08715
4	...... 2925 00206 94389	4	...... 2668 02131 00198
5	...... 68 02330 39404	5	...... 60 63684 79549
6	...... 1 58193 73009	6	...... 1 37811 01808
7	.... 3678 92395	7	.... 3132 06859
8	.... 85 55637	8	.... 71 18337
9	.... 1 98968	9	.... 1 61780
10	 4627	10	 3677
11	 107	11	 84
12	 2	12	 2

n	45^{-n}	n	46^{-n}
1	·02222 22222 22222 22222	1	·02173 91304 34782 60869
2	49 38271 60493 82716	2	47 25897 92060 49149
3	1 09739 36899 86282	3	1 02736 91131 74981
4	...... 2438 65264 44139	4	...... 2233 41111 55977
5	...... 54 19228 09869	5	...... 48 55241 55565
6	...... 1 20427 29108	6	...... 1 05548 72947
7	.... 2676 16202	7	.... 2294 53759
8	.... 59 47026	8	.... 49 88125
9	.... 1 32156	9	.... 1 08438
10	 2937	10	 2357
11	 65	11	 51
12	 1	12	 1

n	47^{-n}	n	48^{-n}
1	·02127 65957 44680 85106	1	·02083 33333 33333 33333
2	45 26935 26482 57130	2	43 40277 77777 77777
3	...... 96317 77159 20364	3	...... 90422 45370 37037
4	...... 2049 31428 91923	4	...... 1883 80111 88271
5	...... 43 60243 16849	5	...... 39 24585 66422
6	.... 92771 13124	6	.... 81762 20136
7	.... 1973 85385	7	.... 1703 37919
8	.... 41 99689	8	.... 35 48706
9	 89355	9	 73931
10	 1901	10	 1540
11	 40	11	 32

For 49^{-n} look to 7^{-2n}.

n	51^{-n}	n	52^{-n}
1	·01960 78431 37254 90196	1	·01923 07692 30769 23076
2	38 44675 12495 19415	2	36 98224 85207 10059
3	...... 75385 78676 37636	3	...... 71119 70869 36732
4	...... 1478 15268 16424	4	...... 1367 68670 56475
5	...... 28 98338 59143	5	...... 26 30166 74163
6	.... 56830 16846	6	.... 50580 12965
7	.... 1114 31703	7	.... 972 69480
8	.... 21 84935	8	.... 18 70567
9	 42842	9	 35972
10	 840	10	 692
11	 16	11	 13

n	53^{-n}	n	54^{-n}
1	·01886 79245 28301 88679	1	·01851 85185 18518 51852
2	35 59985 76005 69598	2	34 29355 28120 71331
3	...... 67169 54264 25841	3	...... 63506 57928 16136
4	...... 1267 34986 11808	4	...... 1176 04776 44743
5	...... 23 91226 15317	5	...... 21 77866 23051
6	.... 45117 47459	6	.... 40330 85612
7	.... 851 27310	7	.... 746 86769
8	.... 16 06176	8	.... 13 83088
9	 30305	9	 25613
10	 572	10	 474
11	 11	11	 9

n	55^{-n}	n	56^{-n}
1	·01818 18181 81818 18181	1	·01785 71428 57142 85714
2	33 05785 12396 69421	2	31 88775 51020 40816
3	...... 60105 18407 21262	3	...... 56942 41982 50729
4	...... 1092 82152 85841	4	...... 1016 82892 54477
5	...... 19 86948 23379	5	...... 18 15765 93829
6	.... 36126 33152	6	.... 32424 39175
7	.... 656 84239	7	.... 579 00699
8	.... 11 94259	8	.... 10 33941
9	 21714	9	 18463
10	 386	10	 330
11	 7	11	 6

n	57^{-n}	n	58^{-n}
1	·01754 38596 49122 80701	1	·01724 13793 10344 82758
2	30 77870 11388 11942	2	29 72651 60523 18668
3	...... 53997 72129 61613	3	...... 51252 61388 33082
4	...... 947 32844 37923	4	...... 883 66575 66087
5	...... 16 61979 72595	5	...... 15 23561 64932
6	.... 29157 53905	6	.... 26268 30429
7	.... 511 53577	7	.... 452 90179
8	.... 8 97431	8	.... 7 80865
9	 15744	9	 13463
10	 276	10	 232
11	 5	11	 4

n	59^{-n}	n	61^{-n}
1	·01694 91525 42372 88136	1	·01639 34426 22950 81967
2	28 72737 71904 62511	2	26 87449 61031 98065
3	...... 48690 46981 43432	3	...... 44056 55098 88493
4	...... 825 26220 02431	4	...... 722 23854 08008
5	...... 13 98749 49194	5	...... 11 83997 60787
6	.... 23707 61851	6	.... 19409 79685
7	.... 401 82404	7	.... 318 19339
8	.... 6 81058	8	.... 5 21628
9	 11543	9	 8551
10	 196	10	 140
11	 3	11	 2

n	63^{-n}	n	65^{-n}
1	·01587 30158 73015 87301	1	·01538 46153 84615 38461
2	25 19526 32905 01385	2	23 66863 90532 54438
3	...... 39992 48141 34942	3	...... 36413 29085 11607
4	...... 634 80129 22777	4	...... 560 20447 46333
5	...... 10 07621 09885	5	...... 8 61853 03789
6	.... 15993 98569	6	.... 13259 27751
7	.... 253 87279	7	.... 203 98888
8	.... 4 02973	8	.... 3 13829
9	 6396	9	 4828
10	 101	10	 75

n	67^{-n}	n	69^{-n}
1	·01492 53731 34328 35821	1	·01449 27536 23188 40579
2	22 27667 63198 93072	2	21 00399 07582 44066
3	...... 33248 77062 67061	3	...... 30440 56631 62957
4	...... 496 25030 78613	4	...... 441 16762 77724
5	...... 7 40672 10128	5	...... 6 39373 37358
6	.... 11054 80748	6	.... 9266 28077
7	.... 164 99712	7	.... 134 29392
8	.... 2 46264	8	.... 1 94629
9	 3675	9	 2821
10	 55	10	 40

n	71^{-n}	n	73^{-n}
1	·01408 45070 42253 52213	1	·01369 86301 36986 30137
2	19 83733 38623 28903	2	18 76524 67629 94933
3	...... 27939 90684 83506	3	...... 25705 81748 35547
4	...... 393 51981 47655	4	...... 352 13448 60761
5	...... 5 54253 26023	5	...... 4 82376 00832
6	.... 7806 38395	6	.... 6607 89052
7	.... 109 94907	7	.... 90 51905
8	.... 1 54858	8	.... 1 23999
9	 2181	9	 1699
10	 31	10	 23

n	75^{-n}	n	77^{-n}
1	·01333 33333 33333 33333	1	·01298 70129 87012 98701
2	17 77777 77777 77777	2	16 86625 06324 84384
3	...... 23703 70370 37037	3	...... 21904 22160 06291
4	...... 316 04938 27160	4	...... 284 47041 03977
5	...... 4 21399 17695	5	...... 3 69442 09142
6	.... 5618 65569	6	.... 4797 94924
7	.... 74 91541	7	.... 62 31103
8	 99887	8	 80923
9	 1339	9	 1051
10	 18	10	 14

Table II. Powers of ·02, ·03, ·04, … up to ·50, useful to compute $A_1x + A_2x^2 + A_3x^3 + \&c.$, when x does not exceed $\frac{1}{2}$.

(Twelve Decimals.)

n	$(\cdot02)^n$	$(\cdot03)^n$	$(\cdot04)^n$	n
2	·0004	·0009	·0016	2
3	·00 00 08	…… 27	…… 64	3
4	…… 0016	…… 0081	…… 0256	4
5	…… …… 32	…… 2 43	…… 10 24	5
6	…… …… 0064	…… …… 0729	…… …… 4096	6
7	…… …… 01	…… …… 0022	…… …… 164	7
8			…… …… 6	8

n	$(\cdot05)^n$	$(\cdot06)^n$	$(\cdot07)^n$	n
2	·0025	·0036	·0049	2
3	1 25	2 16	3 43	3
4	…… 0625	…… 1296	…… 2401	4
5	…… 31 25	…… 77 76	…… 168 07	5
6	…… 1 5625	…… 4 6656	…… 11 7649	6
7	…… …… 781	…… …… 2799	…… …… 8235	7
8	…… …… 39	…… …… 168	…… …… 576	8
9	…… …… 2	…… …… 10	…… …… 40	9
10			…… …… 3	10

n	$(\cdot08)^n$	$(\cdot09)^n$	$(\cdot11)^n$	n
2	·0064	·0081	·0121	2
3	5 12	7 29	13 31	3
4	…… 4096	…… 6561	1 4641	4
5	…… 327 05	…… 590 49	…… 1610 51	5
6	…… 26 2144	…… 53 1441	…… 177 1561	6
7	…… 2 0971	…… 4 7830	…… 19 4871	7
8	…… …… 1678	…… …… 4305	…… 2 1436	8
9	…… …… 134	…… …… 387	…… …… 2358	9
10	…… …… 11	…… …… 35	…… …… 259	10
11	…… …… 1	…… …… 3	…… …… 28	11
12			…… …… 3	12

n	$(\cdot 12)^n$	$(\cdot 13)^n$	$(\cdot 14)^n$	n
2	·0144	·0169	·0196	2
3	17 28	21 97	27 44	3
4	2 0736	2 8561	3 8416	4
5	...... 2488 32	...... 3172 93	...... 5378 24	5
6	...... 298 5984	...... 482 6809	...... 752 9536	6
7	...... 35 8318	...... 62 7485	...... 105 4135	7
8	...... 4 2988	...... 8 1573	...... 14 7579	8
9	 5160	...... 1 0604	...... 2 0661	9
10	 619	 1378	 2892	10
11	 74	 179	 405	11
12	 8	 23	 57	12
13	 1	 3	 8	13
14			 1	14

n	$(\cdot 15)^n$	$(\cdot 16)^n$	$(\cdot 17)^n$	n
2	·0225	·0256	·0289	2
3	38 75	40 96	49 13	3
4	5 0625	6 5538	8 3521	4
5	...... 7593 75	1 0485 76	1 4198 57	5
6	...... 1139 0625	...... 1677 7216	...... 2413 7569	6
7	...... 170 8594	...... 268 4354	...... 410 3387	7
8	...... 25 6289	...... 42 9497	...... 69 7576	8
9	...... 3 8443	...... 6 8719	...... 11 8588	9
10	 5766	...... 1 0995	...... 2 0160	10
11	 864	 1759	 3427	11
12	 130	 281	 583	12
13	 19	 45	 99	13
14	 3	 7	 17	14
15		 1	 3	15

n	$(\cdot 18)^n$	$(\cdot 19)^n$	$(\cdot 20)^n$	n
2	·0324	·0361	·04	2
3	58 32	68 59	·008	3
4	10 4976	13 0321	·0016	4
5	1 8895 68	2 4760 99	·0003 2	5
6	...... 3401 2224	...... 4704 5881	 64	6
7	...... 612 2200	...... 893 8717	 128	7
8	...... 110 1996	...... 169 8356	 0256	8
9	...... 19 8359	...... 32 2688	...... 0051 2	9
10	...... 3 5705	...... 6 1311	...... 0010 24	10
11	 6427	...... 1 1649	...... 0002 048	11
12	 1157	 2213	 4096	12
13	 208	 421	 819	13
14	 37	 80	 164	14
15	 7	 15	 33	15
16	 1	 3	 1	16

n	$(\cdot 21)^n$	$(\cdot 22)^n$	$(\cdot 23)^n$	n
2	·0441	·0484	·0529	2
3	92 61	·0108 48	·0121 67	3
4	19 4481	23 4236	27 9841	4
5	4 0841 01	5 1536 32	6 4363 43	5
6	...... 8576 6121	1 1337 9904	1 4803 5889	6
7	...... 1801 0885	...... 2494 3579	...... 3404 8254	7
8	...... 378 2286	...... 548 7587	...... 783 1098	8
9	...... 79 4280	...... 79 4280	...... 180 1153	9
10	...... 16 6799	...... 26 5599	...... 41 4265	10
11	...... 3 5028	...... 5 8432	...... 9 5281	11
12	 7356	...... 1 2855	...... 2 1914	12
13	 1545	 2828	 5040	13
14	 324	 622	 1159	14
15	 68	 137	 267	15
16	 14	 30	 61	16
17	 3	 6,6	 14	17
18		 1,6	 3	18

n	$(\cdot 24)^n$	$(\cdot 25)^n$	$(\cdot 26)^n$	n
2	·0576	·0625	·0676	2
3	·0138 24	·0156 25	·0175 76	3
4	33 1776	·0039 0625	·0045 6976	4
5	7 9626 24	9 7656 25	·0011 8813 76	5
6	1 9110 2976	2 4414 0625	3 0891 5776	6
7	...... 4586 4714	...... 6103 5156	...... 8031 8102	7
8	...... 1100 7531	...... 1525 8789	...... 2088 2706	8
9	...... 264 1807	...... 381 4697	...... 542 9504	9
10	...... 63 4034	...... 95 3674	...... 141 1671	10
11	...... 15 2168	...... 23 8418	...... 36 7034	11
12	...... 3 6520	...... 5 9604	...... 9 5429	12
13	 8765	...... 1 4901	...... 2 4811	13
14	 2103	 3725	 6451	14
15	 505	 931	 1677	15
16	 121	 233	 436	16
17	 29	 58	 113	17
18	 7	 14	 29	18
19	 1,7	 4	 7	19
20		 1	 2	20

n	$(\cdot 27)^n$	$(\cdot 28)^n$	$(\cdot 29)^n$	n
2	·0729	·0784	·0841	2
3	·0196 83	·0219 52	·0243 89	3
4	·0053 1441	·0061 4656	·0070 7281	4
5	·0014 3489 07	·0017 2103 68	·0020 5111 49	5
6	3 8742 0489	4 8189 0304	5 9482 3321	6
7	1 0460 3532	1 3492 9285	1 7249 8763	7
8	...... 2824 2954	...... 3778 0200	...... 5002 4641	8
9	...... 762 5597	...... 1057 8456	...... 1450 7146	9
10	...... 205 8911	...... 296 1968	...... 420 7072	10
11	...... 55 5906	...... 82 9351	...... 122 0051	11
12	...... 15 0095	...... 23 2218	...... 35 3817	12
13	...... 4 0525	...... 6 5021	...... 10 2607	13
14	...... 1 0942	...... 1 8026	...... 2 9756	14
15	 2954	 5098	 8629	15
16	 797	 1427	 2502	16
17	 215	 400	 726	17
18	 58	 112	 210	18
19	 16	 31	 61	19
20	 5	 9	 18	20
21	 1	 2	 5	21
22			 1	22

n	(·30)^n	(·31)^n	(·32)^n	*n*
2	·09	·0961	·1024	2
3	·027	·1207 91	·0327 68	3
4	·0081	·0092 3521	·0104 3576	4
5	·0024 3	·0028 6291 51	33 5544 32	5
6	7 29	8 8750 3681	10 7374 1824	6
7	2 1870	2 7512 6141	3 4359 7384	7
8	 6561	...... 8528 9104	1 0995 1163	8
9	...... 1968 3	...... 2643 9622	...... 3518 4372	9
10	...... 0590 49	...... 819 6283	...... 1125 8999	10
11	...... 0177 147	...... 254 0848	...... 360 2880	11
12	...... 53 1441	...... 78 7663	...... 115 2921	12
13	...... 15 9432	...... 24 4175	...... 36 8935	13
14	...... 4 7830	...... 7 5694	...... 11 8059	14
15	...... 1 4349	...... 2 3465	...... 3 7779	15
16	 4304	 7274	...... 1 2089	16
17	 1291	 2255	 3868	17
18	 387	 699	 1238	18
19	 116	 217	 396	19
20	 35	 67	 127	20
21	 10	 21	 40	21
22	 3	 6	 13	22
23		 2	 4	23
24			 1	24

n	$(·33)^n$	$(·34)^n$	$(·35)^n$	n
2	·1089	·1156	·1225	2
3	·0359 37	·0393 04	·0428 75	3
4	·0118 5927	·0133 6336	·0150 0625	4
5	39 1353 93	45 4334 24	52 5218 75	5
6	12 9146 7969	15 4480 4416	18 3826 5625	6
7	4 2618 4430	5 2523 3501	6 4339 2969	7
8	1 4064 0862	1 7857 9390	2 2518 7539	8
9	...... 4641 1484	...... 6071 6993	...... 7881 5639	9
10	...... 1531 5790	...... 2064 3777	...... 2758 5473	10
11	...... 505 4210	...... 701 8884	...... 965 4916	11
12	...... 166 7889	...... 238 6421	...... 337 9220	12
13	...... 55 0403	...... 81 1383	...... 118 2727	13
14	...... 18 1633	...... 27 5870	...... 41 3954	14
15	...... 5 9939	...... 9 3796	...... 14 4884	15
16	...... 1 9780	...... 3 1890	...... 5 0709	16
17	 6527	...... 1 0843	...... 1 7748	17
18	 2154	 3686	 6212	18
19	 710	 1253	 2174	19
20	 234	 426	 761	20
21	 77	 145	 266	21
22	 25	 49	 93	22
23	 8	 17	 33	23
24	 3	 6	 11	24
25	 1	 2	 4	25
26			 1	26

n	$(\cdot36)^n$	$(\cdot37)^n$	$(\cdot38)^n$	n
2	·1296	·1369	·1444	2
3	·0466 56	·0506 53	·0548 72	3
4	·0167 9616	·0187 4161	·0208 5136	4
5	60 4661 76	69 3439 57	79 2351 68	5
6	21 7678 2336	25 6572 6409	30 1093 6384	6
7	7 8364 1641	9 4931 8771	11 4415 5826	7
8	2 8211 0991	3 5124 7945	4 3477 9314	8
9	1 0155 9957	1 2996 1740	1 6521 6101	9
10	...... 3656 1584	...... 4808 5844	...... 6278 2118	10
11	...... 1316 3170	...... 1779 1762	...... 2385 7205	11
12	...... 473 8381	...... 658 2952	...... 906 5738	12
13	...... 170 5817	...... 243 5692	...... 344 4980	13
14	...... 61 4094	...... 90 1206	...... 130 9092	14
15	...... 22 1074	...... 33 3446	...... 49 7455	15
16	...... 7 9587	...... 12 3375	...... 18 9033	16
17	...... 2 8651	...... 4 5649	...... 7 1833	17
18	...... 1 0314	...... 1 6890	...... 2 7296	18
19	 3713	 6249	...... 1 0373	19
20	 1337	 2312	 3942	20
21	 481	 855	 1498	21
22	 173	 316	 569	22
23	 62	 117	 216	23
24	 22	 43	 82	24
25	 8	 16	 31	25
26	 3	 6	 12	26
27	 1	 2	 4	27
28			 1,7	28