THE

GEOMETRY OF CYCLOIDS

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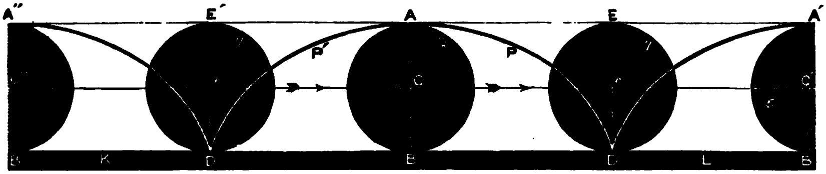
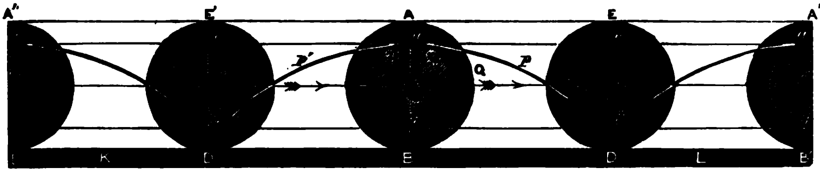
Fig. 1.

PLATE I.



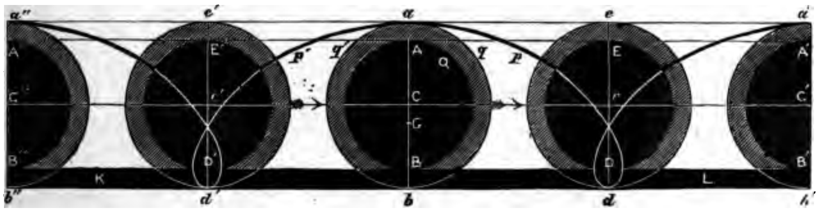
THE RIGHT CYCLOID.

Fig. 45.



THE PROLATE CYCLOID.

Fig. 46.



THE CURTATE CYCLOID.

Fig. 19.

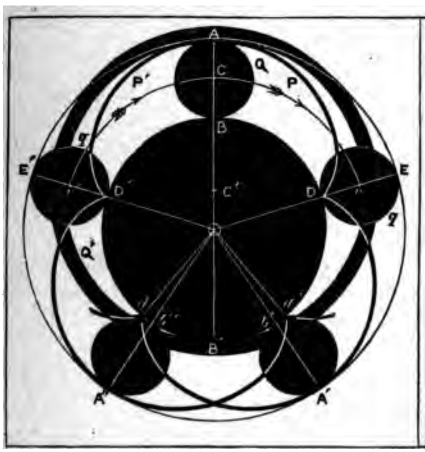
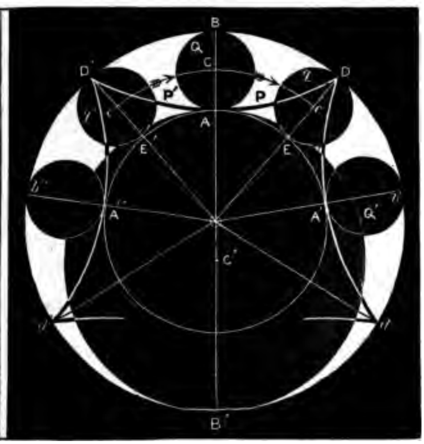


Fig. 20.



THE EPICYCLOID.

THE HYPOCYCLOID.

A TREATISE ON

THE CYCLOID

AND ALL FORMS OF

CYCLOIDAL CURVES

and on the Use of such Curves in dealing with the

MOTIONS OF PLANETS, COMETS, &c.

AND OF

MATTER PROJECTED FROM THE SUN

BY

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STAR ATLAS' 'LIBRARY STAR ATLAS' ETC.

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1878

PREFACE.

THIS WORK deals primarily with the geometry of *cycloids*, curves traced out by a point in a circle rolling on a straight line, or on or within another circle, and *trochoids* (or hoop-curves), curves traced out by a point within or without a circle so rolling.

Although the invention of the cycloid is attributed to Galileo, it is certain that the family of curves to which the cycloid belongs had been known, and some of the properties of such curves investigated, nearly two thousand years before Galileo's time, if not earlier. For ancient astronomers explained the motion of the planets by supposing that each planet travels uniformly round a circle whose centre travels uniformly round another circle. By suitably selecting radii for such circles, and velocities for the uniform motions in them, every form of epicyclic curve can be obtained, including the epicycloid and the hypocycloid. When the radius of the fixed circle is indefinitely enlarged, or, in other words, when the centre of the moving circle advances uniformly in a straight line, the curve traced out by the moving point becomes a *trochoid*, and may either be a *prolate*, a *right*, or a *curtate cycloid*, according as the velocity of the moving centre is greater, equal, or less than the velocity of the point around that centre. Lastly, if the radius of the moving circle is indefinitely enlarged, so that a straight line is carried uniformly round a centre while a point travels uniformly along the line, the curve traced out becomes a spiral of the family to which belong the spiral of Archimedes and the involute of the circle.

It is of these curves, which are all included under the general name epicyclical curves, that I treat in the present volume, though the cycloid, epicycloid, hypocycloid, and trochoid are more fully dealt with, in their geometrical aspect, than the epitrochoidal and spiral members of the epicyclic family.

Ancient geometers were not very successful in their attempts to investigate any of these curves. It is strange indeed to find a mathematician even of Galileo's force so far foiled by the common cycloid as to be reduced to the necessity of weighing paper figures of the curve in order to determine its area. Pascal dealt more successfully with this and other problems. Yet he seems to have regarded their relations as of sufficient difficulty to be selected for his famous challenge to mathematicians, to try whether a priest who had long given up the study of mathematics was not a match for mathematicians at their own weapons. The argument, in so far as it was intended to prove the soundness of Pascal's faith, was feeble enough. But the failure, or partial failure, of many who attacked his problems, is noteworthy. We find, for instance, that Roberval laboured for six years over the quadrature of the cycloid, and only succeeded at last in solving it by the comparatively clumsy method indicated at p. 199, inventing a new curve for the purpose. It will be seen that in the present work this famous problem comes very early (Prop. III., pp. 5, 6), and is made to depend on the fundamental (and obvious) relation of the cycloidal ordinates. The method—which so far as I know is a new one—is extended to the epicycloid, hypocycloid, trochoid, epitrochoid, and hypotrochoid. It will be found that, in all, thirteen distinct methods of solving the problem geometrically are either given in full or indicated (seven of these methods being new so far as I know), while seven independent methods are indicated for determining the area of the epicycloid and hypocycloid (of which five are new), besides one method (see footnote, p. 50) derived from the properties of the cycloid. After the first demonstration of the area, however, those methods only are given in full which involve other useful relations.

The position of the centre of gravity of the cycloidal arc, and of the cycloidal area, has been fully dealt with geometrically in Section I. (so far as I know, for the first time). It seems to me that the treatment of such problems by geometrical methods usefully introduces the student to the use of analytical methods. For instance, Prop. XIV. is a geometrical illustration —in reality, so far as my own mathematical studies were concerned, a geometrical anticipation*—of the familiar relation

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx,$$

of the Integral Calculus.

Most of the propositions in the first three sections were established in the same manner as in this volume, in notebooks which I drew up when at Cambridge;

* I may mention, as a circumstance in which some may perhaps find encouragement and others a warning, that (owing chiefly to my liking for geometrical studies) I knew very little of the Differential Calculus, and scarcely anything of Astronomy, when I took my degree. Possibly I owe to this circumstance no small share of the pleasure derived from the study of these and other mathematical subjects since. The hurried rush made at our universities over the domain of mathematics has always seemed to me little calculated to develope a taste for mathematics, though it may not invariably destroy it when it already exists. The withdrawal of the mind during three years from other subjects of greater importance,—general literature, history, physical science, and so forth,—is still more pernicious: yet it is practically forced on those who wish for university distinctions, fellowships, and so forth.

but the proofs have been simplified and their arrangement altogether modified more than once since then. In fact anyone who compares the first two sections with recent papers of mine on the Cycloid, Epicycloid, and Hypocycloid, in the *English Mechanic*, will perceive even that in the interval since those papers were written the subject-matter has been entirely rearranged.

In defining epicycloids and hypocycloids I have made a change by which an anomaly existing in the former treatment of these curves has been removed. The definitions hitherto used run as follows:—

The $\left\{\begin{matrix}\textit{epicycloid}\\ \textit{hypocycloid}\end{matrix}\right\}$ *is the curve traced out by a point on the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the two circles being in* $\left\{\begin{matrix}\textit{external}\\ \textit{internal}\end{matrix}\right\}$ *contact.*

For this I substitute:—

The $\left\{\begin{matrix}\textit{epicycloid}\\ \textit{hypocycloid}\end{matrix}\right\}$ *is the curve traced out by a point on the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the rolling circle touching the* $\left\{\begin{matrix}\textit{outside}\\ \textit{inside}\end{matrix}\right\}$ *of the fixed circle.*

That the latter is the more correct definition is proved by the fact that, while the former leads to an altogether unsymmetrical classification of the resulting curves, the latter leads to a classification perfectly symmetrical. According to the former every epicycloid is a hypocycloid, but only some hypocycloids are epicycloids; according to the latter no epicycloid is a hypocycloid, and no hypocycloid is an epicycloid.

In the fourth section on motion in cycloidal curves I have adopted a somewhat new method of arranging the demonstrations to include cycloids, epicycloids, and hypocycloids. The proof that the cycloid is the path of quickest descent is a geometrical presentation of Bernouilli's analytical demonstration.

The section on Epicyclics was nearly complete when my attention was directed to De Morgan's fine article on Trochoidal Curves in the *Penny Cyclopædia*, the only complete investigation of any part of my subject (except a paper by Purkiss on the Cardioid) of which I have thought it desirable to avail myself. I rewrote portions of the section for the benefit of those who may already have studied De Morgan's essay, deeming it well in such cases to aim at uniformity of definition, and, as far as possible, of treatment. It will be observed, however, by those who compare Section V. with De Morgan's essay, that my treatment of the subject of epicyclics remains entirely original, and that in some places I do not adopt his views. For instance, I cannot agree with him in regarding the angle of descent as negative under any circumstances consistent with the definition of the epicyclic itself. The radius vector indeed advances and retreats in certain cases; but in every case it advances on the whole between any apocentre and the next pericentre. De Morgan has also misinterpreted the figures on p. 187, as explained, p. 186.

In two respects this treatise has gained from my study of De Morgan's essay. In the first place, I had not originally intended to devote a section to the equations of cycloidal curves. Secondly, and chiefly, I was led, by the study of the very valuable illustrations engraved by Mr. Henry Perigal for Prof. De Morgan's article, to cancel all the drawings which I had constructed to illustrate Section V., and to apply to Mr. Perigal for permission to use his mechanically traced curves. A study of Plates II., III., and IV., and of other figures illustrating Section V., will show how much the work has gained by the change. For figs. 119 to 122, and two of those of Plate IV., also mechanically drawn, I am indebted to Mr. Boord. I may add, to show the value of these illustrations, that Prof. De Morgan, in his 'Budget of Paradoxes,' says that without Mr. Perigal's 'diagrams direct from the lathe,' his article on Trochoidal Curves 'could not have been made intelligible.' Yet even those cuts, and many others added to them in this volume, will give the reader but inadequate ideas of the immense number, variety, and beauty of the sets of diagrams published by Mr. Perigal himself, in his 'Contributions to Kinematics.' In these the curves are shown white on a black background, and hundreds of varieties at once instructive and ornamental are presented for study and comparison. Even for the mere patterns thus formed, and apart from their mathematical interest, these sets of diagrams possess great value. (See further the note, pp. 193–195.)

The portions of Section V. relating to planetary motions, and the concluding section relating to the graphical use of cycloidal curves for determining the motion of bodies in elliptical orbits under gravity and of matter projected from the sun, will be useful, I trust, to students of astronomy. In some respects cycloidal curves are even more closely related to astronomy than the conic sections. If planets and comets travel approximately in ellipses about the sun, and moons in ellipses about their primaries, the planets' paths, relatively to our earth regarded as at rest, are epicyclic curves; while the cycloid and its companion curves supply an effective construction for dealing with Kepler's famous problem relating to the motion of a body in an ellipse round an orb in the focus attracting according to the law of gravity.

A treatise such as this is rather intended to afford the means of solving such problems as may be suggested to the student than of supplying examples. I have, however, added a collection of about 150 examples. All except those to which a name is appended are original. They are, in fact, a selection from among those which occurred to me as the work proceeded. Many which I had intended to present as riders have ultimately been worked into the text among the corollaries and scholia. If these had been included as examples, the total number would have amounted to about 300; but it seemed to me better in their case to indicate the nature of the proof.

RICH. A. PROCTOR.

LONDON: *December*, 1877.

P.S.—As the last sheets are receiving their latest corrections for press, I receive, through Mr. Boord's kindness, the eight figures on p. 256. Of these, figs. 154, 158 represent orthoidal, figs. 155, 159 cuspidate, and figs. 156, 160 centric epicyclics; while fig. 157 is a transcentric, and fig. 161 a loop-touching epicyclic.

Errata.

On p. 59, line 11, for 'Area ABD,' read 'Area OBD.'
" 129, " 17, " 'D,' read 'O.'

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THE

GEOMETRY OF CYCLOIDS.

SECTION I.

THE RIGHT CYCLOID.

NOTE.—*Any curve traced by a point on the circumference of a circle which rolls without sliding upon either a straight line or a circle in the same plane is called a cycloid, but the term is usually limited to the right cycloid, and will be so employed throughout this work.*

DEFINITIONS.

The *right cycloid* is the curve traced by a point on the circumference of a circle which rolls without sliding upon a fixed straight line in the same plane.

The rolling circle is called the *generating circle*; the point on the circumference the *tracing point.* Similar terms are employed for all the curves dealt with in this work.

Let AQB (fig. 1, Plate I.) be the rolling circle, KL the fixed straight line. Let the centre of the rolling circle move along the line c' C c parallel to KL through C the centre of AQB, in the direction shown by the arrow. Then it is manifest that at regular intervals the tracing point will (i.) coincide with the line KL, as at D′ and D (E′q'D′ and E q d being corresponding positions of the generating circle), and (ii.) will be at its greatest distance from KL, as at A (AQB being the corresponding position of the generating circle), this distance being the diameter of AQB, so that ACB the diameter through the tracing point is at right angles to KL. It is clear also from the way in which the curve is traced out that the parts AP′D′ and APD are similar and equal. Therefore ACB is called the *axis* of the cycloidal curve; D′D is the *base*; and A the *vertex*. The points D′ and D are called the *cusps*. The radius CA drawn to the tracing point is called the *tracing radius*, the diameter through the tracing point the *tracing diameter*. The radius of the generating circle may be conveniently represented by the symbol R. Where the tracing diameter coincides with the axis, the generating circle is said to be *central*, and AQB so placed is called the *central generating circle*. A diameter to the generating circle parallel to ACB, that is perpendicular to D′D, is said to be *diametral*. The line c' C c is called the *line of centres*.

The complete cycloid consists of an infinite number of equal cycloidal arcs; but it is often convenient to speak of the cycloidal arc D′AD as the cycloid.

It is clear that if D′E′ and DE be drawn perpendicular to D′D, the semi-cycloidal arcs on either side of D′E′ and DE are symmetrical with respect to these lines. Therefore D′E′ and DE may conveniently be called *secondary axes*.

A straight line E′AE through A parallel to D′D manifestly touches the cycloid at A; for there is one position, and one only, of the generating circle (between D′E′ and DE) which brings the tracing point to the distance AB from D′D. E′AE is called the *tangent at the vertex.*

PROPOSITIONS.

PROP. I.—*The base of the cycloid is equal to the circumference of the generating circle.*

This is manifest from the way in which the curve is traced out; for every point of the generating circle AQB (fig. 1) is brought successively into rolling contact with the base D′D; so that necessarily

D′D = circumference of the circle AQB.

Cor. 1. BD′ = BD = semicircular arc AQB.

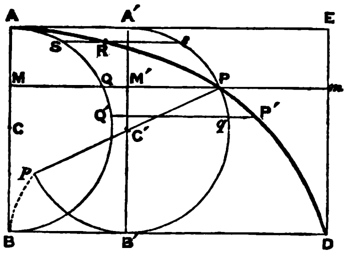
Cor. 2. Drawing D′E′ and DE square to D′D and c' C c parallel to D′D,

Area E′D = 2 area AD = 4 area CD
= 4 rect. under CB, BD
= 4 rect. under CB, arc AQB
= 4 times area of generating circle AQB.

PROP. II.—*If through P, a point on the cycloidal arc APD (fig. 2), the straight line PQM be drawn parallel to the base BD, cutting the central generating circle in Q and meeting the axis AB in M; then QP = arc AQ.*

Let A′PB′ be the position of the generating circle when the tracing point is at P, C′ its centre, A′C′B′ diametral, cutting MP in M′. Draw the tracing diameter PC′p. Then MQ = M′P ; MM′ = QP ; and arc AQ = arc A′P. Now, since PC′p is the tracing diameter, p is the point which had been at B when the

FIG. 2.



tracing point was at A; hence the arc p B′ = BB′, for every point of pB′ has been in rolling contact with BB′. But

Arc pB′ = arc A′P = arc AQ; and BB′ = MM′ = QP.
Wherefore, QP = arc AQ.

Cor. 1. PM = arc AQ + MQ.

Cor. 2. Since BD = arc AQB = arc AQ + arc QB, BD > PM; wherefore the whole arc APD lies on the left of DE, perpendicular to BD.

Cor. 3. Let MP produced meet DE in m. Then

$$Pm = Mm - PM = \text{arc } AQB - \text{arc } AQ - MQ$$
$$= \text{arc } QB - MQ.$$

Cor. 4. Arc $A'P = BB'$; and arc $PB' = B'D$.

Cor. 5. If through P', a point on the arc PD, $P'qQ'$ be drawn parallel to BD, meeting AQB in Q' and cutting $A'PB'$ in q; then $Q'P' = \text{arc } A'q$, and $QP = \text{arc } A'P$; wherefore

$qP' \, (= Q'P' - Q'q = Q'P' - QP) = \text{arc } A'q - \text{arc } A'P$;
that is, $qP' = \text{arc } Pq$.

Cor. 6. If through R, a point on the arc AP, sRS parallel to BD meet the arcs AQB, $A'PB'$ in S and s, then

$$Ss = QP = A'P; \text{ and } SR = \text{arc } A's;$$

wherefore $Rs = \text{arc } sP = \text{arc } SQ$.

Prop. III.—*The area $D'AD$ (fig. 1, Plate I.) between the cycloid and its base is equal to three times the area of the generating circle.*

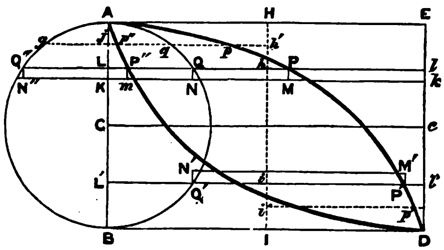
A, B, D, E, C, &c. (fig. 3), representing the same points as in the preceding proposition; take $CL = CL'$ on AB, and draw LPl, $L'P'l'$ parallel to BD, cutting the cycloid in P and P', and the central generating circle in Q and Q', respectively. Complete the elementary rectangles PN, $P'N'$, Lk, of equal width, $(PM = P'M')$. Then

$QP = \text{arc } AQ$, and $Q'P' = \text{arc } AQ' = \text{arc } BQ$; therefore $QP + Q'P' = \text{semicircle } AQB = Ll$; and the two rectangles NP and $N'P'$ are together equal to the rectangle L*k*. Taking all such pairs of rectangular elements as NP and N′P′, it follows that in the limit area AQBDP = rectangle CE = circle AQB. (Prop. I. Cor. 2.)

Hence the area between the cycloid and its base (= 2AQBDP + circle AQB) = three times the area of the generating circle. Q.E.D.

Another proof.—Let AP″D be a cycloidal arc having A as cusp, D as vertex, and DE as axis. Let

Fig. 3.



*l*L cut AP″D in P″ and be produced to meet the circle AQB in Q″. Then

LP = arc AQ + LQ; and LP″ = arc AQ − LQ
(Prop. II. Cor. 2). Wherefore
P″P = LP − LP″ = 2LQ = Q″Q; and the elementary
area P*m* = the elementary area Q″N.

Taking all such elementary rectangles, we have in the limit area AP″DP = circle AQB = rectangle CE. Hence, taking these equals from the rectangle BE, it follows that the equal areas ABDP″ and APDE are together equal to the rectangle CD, that is, to the circle AQB. Therefore AP″DB = the semicircle AQB; APDB = three times the semicircle AQB; and the area between the cycloid and the base = three times the generating circle.

Cor. 1. Rectangle Al = area AQP + area BQ′P′D.

Cor. 2. Rectangle Cl = area QPP′Q′.

Cor. 3. If AE and BD be bisected in H and I, and HI cut PQ and P′Q′ in h and i; then if, as in the figure, P and P′ are on the same side of HI,

$$Ph + P'i = Ph + P''h = P''P = Q''Q = 2LQ.$$

If P falls between AB and HI, as at p, then, completing the construction indicated by the dotted lines,

$$p'i' - p\,h' = p''h' - p\,h' = p''p = g\,q = 2\,qj.$$

That is, if two points are taken on the cycloidal arc equidistant from Cc, the sum or difference of the perpendiculars from these points upon HI will be equal to the chord of the generating circle formed by either perpendicular produced, according as the points on the cycloid are on the same or on opposite sides of HI. This relation will be found useful hereafter in determining the centre of gravity of the cycloidal area.

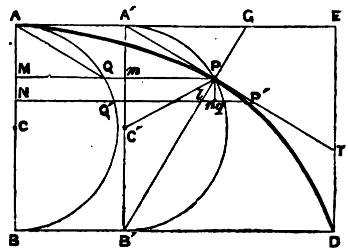
Cor. 4. When the tracing point is at P, the generating circle passes through P″; for its chord through P parallel to AE = QQ″ = PP″.

Cor. 5. Area AQ″Q = area AP″P; and area AQ″P″ = area AQP. The latter relation, established independently (by showing that QP = Q″P″), leads to a third demonstration of the area.

PROP. IV.—*If P (fig. 4) is a point on the cycloidal arc APD, A′PB′ the generating circle when the tracing point is at P, A′C′B′ diametral, then PB′ is the normal and A′P is the tangent to the cycloid at the point P.*

Since, when the tracing point is at P, the generating circle A′PB′ is turning round the point B′, the direction of the motion of the tracing point at P must be

FIG. 4.



at right angles to B′P; wherefore PB′ is the normal and A′P is the tangent at the point P.

Another demonstration.—The objection may be raised against the preceding proof, that, by the same reasoning, B′ would be proved to be the centre of curvature at P, which is not the case. Although the objection is not really valid, an independent proof may conveniently be added.

Take P′ a point near to P, and draw PQM, P′Q′N parallel to BD, cutting AQB in Q and Q′, and P′Q′N cutting A′PB′ in q. Join PC′. Then q P′ = arc Pq (Prop. II. Cor. 4), and ultimately PqP′ is an isosceles triangle, whose equal sides Pq and qP′ are respectively perp. to the equal sides C′P and C′B′ of the isosceles triangle PC′B′; wherefore the third side PP′ is perp. to the third side PB′.* That is, PB′ is the normal at P, and therefore PA′ the perp. to PB′ is the tangent at P.

Cor. 1. If Pn be drawn perp. to P′N, then the figure PP′n is in the limit similar to the triangles A′B′P, A′Pm, PB′m (m being the point in which A′B′ and PM intersect).

Cor. 2. If B′P cut P′N in l, the triangle lP′P is similar to the four triangles named in Cor. 1.

Cor. 3. Triangles P q l, P q P′ are similar respectively to triangles PC′A′ and PC′B′; and $l\,q = q$ P′.

Cor. 4. AQ is parallel to the tangent at P.

Cor. 5. If AQ prod. meet P′N in r, QQ′ ultimately = Q′ r.

SCHOL.—A tangent may be drawn to the cycloid from any point on the curve. For if we draw PQ parallel to BD, the tangent PA′ is parallel to AQ. To draw a tangent from any point A′ on the tangent at vertex, we draw A′B′ perp. to base, and the semicircle A′PB′ on ADB′ intersects APD in the point P such that A′P is tangent to APD.

* Thus, let the triangle P q P′ be turned in its own plane round the point P till Pq coincides with PC′—that is, through one right angle; the other sides q P′ and PP′ will also have been turned through a right angle, therefore q P′ will be parallel to C′ B′, and q P′ being equal to q P, P′ will fall on B′P (for any parallel to C′B′ will cut off an isosceles triangle from B′PC′); hence B′PP′ is the angle through which PP′ has been turned, and is therefore a right angle.

PROP. V.—*If PQ (fig. 5), parallel to the base of cycloid APD, and above the line of centres C c, meets the central generating circle in Q, and QN, PM are perpendicular to C c,*

$$\text{Area } AhQP + \text{rect. } QM = \text{rect. } CF$$

(F being the point in which NQ produced meets the tangent at the vertex AT).

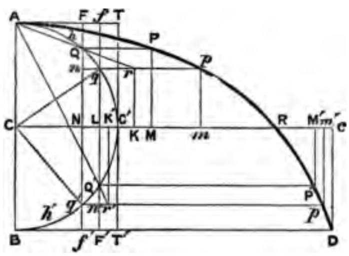
If P′Q′ be a parallel to the base below the line of centres Q′L, P′M′, perpendicular to C c,

$$\text{Area } AhQ'P' - \text{rect. } Q'M' = \text{rect. } CF'$$

(F′ being the point in which LQ′ produced meets the base BD).

Take p a point near to P, and let pn perp. to QN cut arc AQQ′ in q; join AQ and produce to meet pn

FIG. 5.



in r; draw fq L, rK, pm perp. to Cc, and join Cq.

Then in passing from P to p, area AhQP + rect. QM is increased by PpmM and diminished by QqLN, or in the limit, increased by rect. Mp or Nr (since QrpP is a parallelogram, Prop. IV. Cor. 4) and diminished by rect. Nq; wherefore total increase = rect. Lr. But $nq : qQ$ ($=qr$, Prop. IV. Cor. 5) $:: qL : Cq(=NF)$,

$$\therefore \text{ rect. under } nq, NF = \text{rect. under } qr, qL;$$

$$\text{that is, rect. } Nf = \text{rect. } Lr,$$

or incrt. of rect. CF = incrt. of (area $AhQP$ + rect. QM).

But these areas start together from nothing, at A,

$$\therefore \text{ rect. } AhQP + \text{rect. } QM = \text{rect. } CF.$$

Cor. 1. Area $AQC'RP$ = square CT = square CT', TCT' being the tangent to $AC'B$ at C' on the line of centres.

Again, making a similar construction for the second case (for convenience in figure $Q'q'$ is so taken that Qq' and qQ' are perp. to Cc), we have ultimately

$$\text{decrement of area } (AhQ'P' - Q'M') = Lq' + P'm'$$

$$= \text{rect. } Lq' + \text{rect. } n'K' \text{ (ultimately)} = \text{rect. } Nr'.$$

But since $n'q' : Q'q'$ ($= q'r'$) $:: q'N : Cq'$ ($= Nf'$),

$$\text{rect. under } n'q', Nf' = \text{rect. under } q'r', q'N;$$

that is, rect. NF' = rect. Nr', or

decrt. of rect. CF' = decrt. of area ($AhQ'P' - Q'M'$).

But these areas begin together from the equal areas $AQC'R$ and square CT',

$$\therefore \text{ area } AhQ'P' - \text{rect. } Q'M' = \text{rect. } CF'.$$

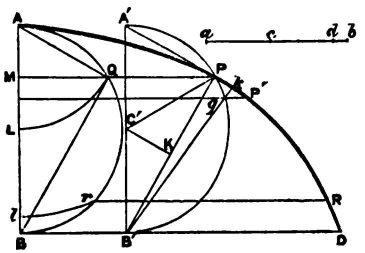
Cor. 2. Area $AC'BDR$ = rect. $CBDc$ = generating circle, so that we have here a new demonstration of the area.

PROP. VI.—*If from P a point on the cycloid APD (fig. 6) PQ drawn parallel to the base, meets the generating circle in Q,*

arc AP = 2 chord AQ.

With the same construction as in Prop. IV., join AQ and B′*q*; produce B′*q* to meet PP′ in *k*; and draw C′K perpendicular to B′P. Then ultimately,

FIG. 6. (Join A′*q*.)



qk is perpendicular to PP′, and the triangle P*q*P′ is isosceles;

∴ PP′ = 2*k*P ultimately.

But PP′ is ultimately the increment of the cycloidal arc AP; and P*k* is ultimately the increment of the chord A′P (for A′*q* = A′*k* ultimately). Hence the increment of the cycloidal arc AP = twice the increment of the chord A′P or of the chord AQ. Therefore, since the arc and chord begin together at A,

Arc AP = 2 chord AQ.

Cor. 1. Arc APD = 2 AB = 4R, and the entire cycloidal arc from cusp to cusp = 4AB = 8R.

Cor. 2. Since the square on AQ = rect. AB.AM, sq. on st. line equal to arc AP = 4 rect. AB. AM, and we have,

$$\text{Arc AP} = 2\sqrt{2\text{R}} \,.\, \sqrt{\text{AM}},$$

that is, $\text{Arc AP} \propto \sqrt{\text{AM}}$.

Cor. 3. Arc AP : arc PD :: AL : LB.

PROP. VII. PROB.—*To divide the arc of a cycloid into parts which shall be in any given ratio.*

Let a straight line *ab* (fig. 6) be divided into any parts in the points *c* and *d*: it is required to divide the arc APRD in the same ratio.

Divide AB in L and *l* so that

$$\text{AL} : \text{L}l : l\text{B} :: ac : cd : db.$$

With centre A and radius AL and A*l*, describe circular arcs LQ, *l r*, meeting the semicircle AQB in Q and *r*. Through Q, *r*, draw QP, *r*R, parallel to BD. Then

$$\text{Arc AP} = 2\text{AQ} = 2\text{AL}; \text{ and arc AR} = 2\text{A}l.$$

Therefore

$$\text{Arc PR} = 2\text{L}l; \text{ and similarly arc RD} = 2l\text{B}.$$

Therefore

$$\text{Arc AP} : \text{arc PR} :: \text{arc RD} :: \text{AL} : \text{L}l : l\text{B} :: ac : cd : db;$$

or the arc APD has been divided in the points P and R in the required ratio.

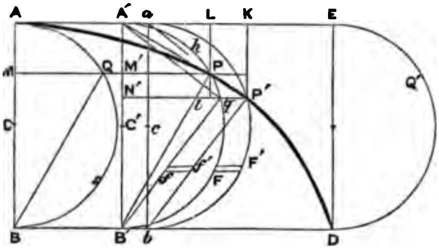
Similarly may the arc APD be divided into any number of parts, bearing to each other any given ratios.

Prop. VIII.—*With the construction of Prop. IV.*

Area APB′B : sectorial area A′B′Ph
:: area PB′D : segment PFB′ :: 3 : 1.

Let a P′b (fig. 7) be the position of the tracing circle when the tracing point is at P′ near to P, on the

Fig. 7. (Join a P′.)



side remote from A; $a\,c\,b$ diametral. Join bP′, draw P′$q\,l$ parallel to BD meeting A′PB′ in q and PB′ in l, join q B′, which is parallel to b P′, because q P′ = P q = B′b. Then ultimately P′ q = $q\,l$ (Prop. IV. Cor. 3), wherefore parallelogram $q\,b$ = twice the triangle $l\,q$ B′ and trapezium l P′ b B′ = 3 times the triangle $l\,q$ B′: that is, ultimately (when the triangle l PP′ vanishes compared with l P′b B′), the elementary area

B′PP′ b = 3 times the elementary area PB′q
= 3 (area A′B′$q\,h$ – area A′B′P h)
= 3 (area $a\,b$P′ – area A′B′P).

Thus the increment of the area ABB′P = 3 times the increment of the area A′B′P, and the decrement of area PB′D = 3 times the decrement of the area PFB′. But the areas ABB′P and A′B′P commence together, and the areas PB′D and PFB′ end together, as P passes from A to D. Hence

ABB′P = 3 times sectorial area A′B′P *h*.
Area PB′D = 3 times the segment PFB′

and

Area APB′B : sectorial area A′B′P *h*
:: area PB′D : segment PFB′ :: 3 : 1.

Cor. 1. Area PFB′D = 2 segment PFB′. This is easily proved independently. For any elementary parallelograms *ff′* and FF′ (having sides parallel to BD), are manifestly equal; wherefore area *q* F *b* F′ = parallelogram *q b* = twice triangle B′ *q l* = (ultimately) twice the decrement of segment *b*F′ P′.

Cor. 2. Area AQBB′P (BQ straight) = 2 sectorial area AQB.

Cor. 3. Area Q*s*BDP = 2 seg. Q*s*B + par. PB = 2 seg. Q*s*B + rect. BM′.

Schol.—Prop. VIII. affords another proof of the relation established in Prop. III. The first corollary, established independently, gives another proof.

PROP. IX.—*With the same construction as in the preceding propositions.*

Area APA′ = segment A′hP.

Join PA′, *q* A′, and P′*a*. Then A′PP′ is ultimately a diameter of the parallelogram A′*a*P′*q*, and the ultimate triangle A′PP′*a* is equal to the triangle A′PP′*q*, or in the limit to the triangle A′P*q*. But A′PP′*a* is the increment of the area APA′, and A′P*q* is the increment of the segment A′*h*P. Since these areas then begin together and have constantly equal increments, they are constantly equal. Therefore

Area APA′ = segment A′*h*P.

Cor. 1. Draw PL, PM′M perp. to AE, AB respectively, PM intersecting AB in M′. To each of the equal areas APA′ and A′*h*P add the equal triangles A′PL and A′MP. Then the area APL = area A′*h* PM′ = area AQM. This may be proved independently. For drawing P′K, P′N′ perp. to AE, A′B′, we see that A′PP′ is ultimately a diameter of the rectangle N′K, and therefore the rectangles PK and PN′, being complements to rectangles about the diameter, are equal: or ultimately the increment of the area APL = increment of the area A′*h* PM′; wherefore, since these areas begin together, area APL = area A′*h* PM′ = area AQM.

Cor. 2. Area AQP = rect. ML − 2 area AMQ.

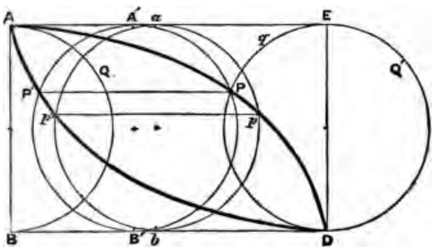
Cor. 3. Area Q*s*BDP = circ. AQB − area AQP
= circle AQB − rect. ML + 2 area AMQ
= 2 (semicircle AQB + area AMQ) − rect. ML.

Cor. 4. Area AA′h P = 2 area AA′P = 2 segment A′hP. This may be proved independently, in the same way as Cor. 1, Prop. VIII. Area A′a P′q h, ultimately equal to the area A′aP′Ph, is shown to be equal to the area of the parallelogram A′aP′q, that is, to twice the area A′PP′a or A′PP′q (the ultimate increments of AA′P, A′ h P, respectively).

Schol.—Prop. IX. and Cor. 1 and 4 (established independently) afford three new demonstrations of the area of the cycloid. For they severally show that area APDE = semicircle DQ′E, on DE as diameter; and since BE = twice the generating circle, the area APDB = 3 times the semicircle AQB.

It will be noticed that the area AEQ′DP = area A s BDP. This, which may easily be proved independently, affords yet another proof of the area of the cycloid. Thus let APD, AP′D (fig. 8) be cycloidal

Fig. 8.

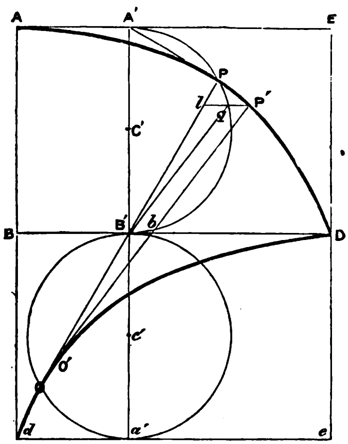


arcs, placed as in Prop. III.; A′PB′P′ and $a\,p\,b\,p'$ adjacent positions of the tracing circle. Then, Prop. III. Cor. 4, P′P and $p'p$ are both parallel to BD. Hence ultimately area A′$a\,p$P = area A′$a\,p'$P′; but these are the increments of the areas AA′P, and AA′P′, which commence together. Hence area AA′P = area AA′P′, wherever P and P′ may be. Wherefore (taking P to D) area AEQ′DP = area AE*q*DP′ = area AQBDP. Therefore the arc APD divides the area AEQ′DBQ into two equal parts. But area AEQ′DBQ = area AEDB = twice the generating circle. Hence area AQBDP = area APDQ′E = the generating circle; area APDB = 3 the semicircle AQB; and area AEDP = semicircle AQB.

PROP. X.—*The radius of curvature at P (fig. 9) is equal to twice the normal PB′.*

With so much of the construction of fig. 7 as is con-

FIG. 9. (For O′, O read *o*, *o*′; and join *o*′*a*′.)



tained in fig. 9, produce P′*b*, which is parallel to *q* B′, to meet PB' produced in o'. Then since ultimately $l P' = 2\, l q$; $l o'$ ultimately $= 2\, l B'$. So that if the normals at the adjacent points P and P′, intersect ultimately (when P′ moves up to P) in o (which, therefore, is the centre of curvature at P),

Rad. of curvature $P\,o = 2$ normal PB'.

Cor. The radius of curvature diminishes from the vertex, where it has its maximum length, to the cusp, where the radius vanishes or the curvature becomes infinite.

PROP. XI.—*The evolute of the cycloid APD (fig. 9) is an equal cycloid D o d, having its vertex at D, and its cusp d on AB produced to d so that $Bd = AB$.*

Complete the rectangle $DBde$, produce $A'B'$ to a', and join $o\,a'$. Then in the triangles $A'B'P$ and $a'B'o$ the sides $A'B'$, $B'P$, are equal to the sides $a'B'$, $B'o$, each to each, and enclose equal angles; therefore, the triangles are equal in all respects, and the angle $a'o\,B'$ ($=$ the angle $B'PA'$) is a right angle. Hence a circle described on $B'a'$ as diameter will pass through o. Again, in the equal circles $A'B'P$ and $a'B'o$, the angles $A'B'P$ and $a'B'o$ at the circumference are equal. Therefore the arc $o\,a' =$ the arc $PA' = BB'$ (Prop. II. Cor. 4) $= d\,a'$. Wherefore o is a point on a cycloid having $d\,e$ for base, a cusp at d, and $B'o\,a'$ as tracing circle. Since $d\,e = BD =$ arc $B'o\,a'$, $D\,e$ is the axis and D is the vertex of the evolute cycloid.

Cor. $o P = 2\, o B' =$ arc $o\,D$ (Prop. VI.); so that, if a string coinciding with the arc $d\,o\,D$ and fastened at d be unwrapped from this arc, its extremity will always lie on the cycloid APD, which may, therefore, be traced out in this way as the involute of the arc doD.

Prop. XII.—*If APD (fig. 9) be a semi-cycloidal arc, d o D its evolute, and o B′P the radius of curvature at any point P on APD, cutting the base BD in B′, then the area APB′B = three times the area d BB′o.*

If $P'o'$ be a contiguous radius of curvature cutting BD in b, and $P'l$ parallel to BD meet PB′ in l; then in the limit $o\,l = 2\,o\,B'$, and therefore the area of the ultimate triangle $o\,l\,P' = 4$ times the area of the ultimate triangle $o\,B'b$; or ultimately the area $B'l\,P'b = 3$ times the area $o\,B'b$. But these areas are the elementary increments of the areas APB′B and $d\,BB'o$, which begin together from AB d. Wherefore the area APB′B = 3 times the area $d\,BB'o$.

Cor. 1. Area ABD = 3 times area d BD = 3 times area AED = $\frac{3}{4}$ rect. BE = 3 times the generating circle. We have here another demonstration of the area.

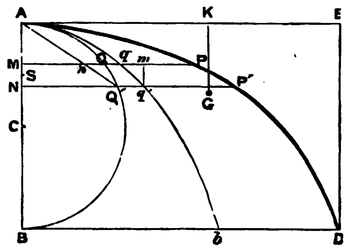
Cor. 2. Area $o\,B'D = \frac{1}{3}$ area B′DP = segm. P q B′ (Prop. VIII.). This may be proved independently; for triangle $o\,B'\,b$ = triangle $B'\,l\,q$ = (ultimately) triangle B′P q; but triangles $o\,B'\,b$, B′P q, are decrements of area $o\,B'D$ and segment P q B′ which end together at D; $\therefore o\,B'D$ = seg. P q B′.

Hence, $d\,B'D = \frac{1}{2}$ generating circle. We have here, then, yet another demonstration of the area.

Prop. XIII.—*If G (fig. 10) is the centre of gravity of the cycloidal arc APD, then GK, perp. to AE (the tangent at the vertex A)* $= \frac{1}{3}$ *AB.*

Let PP′ be an element of the arc APD and let PM, P′N perp. to AB intersect the semicircle AQB in Q and Q′. Join AQ′ cutting MQ in *n*. Then ultimately PP′ is parallel and equal to *n*Q′ (Prop. IV.).

Fig. 10.



Now, representing the mass of element PP′ by its length, the moment of PP′ about AE ultimately

$$= PP'.\ AN = n\,Q'.\ AN$$
$$= MN\,.\,AQ'$$
$$(\text{since } n\,Q' : MN :: AQ' : AN)$$

and may be represented therefore by the elementary rectangle MN *q′m*, of which the side N*q′* = AQ′.

Thus the moment of the arc APD about AE may be represented by the area A *q′b* B obtained by drawing the curve A *q′b* through all the points obtained as *q′* was. But since square of N*q′* = square of AQ′ = rect. under AB, AM; A *q′b* is part of a parabola having A as vertex, AB as axis and parameter (focus at S, such that AS $= \frac{1}{4}$ AB). Therefore area ABb $= \frac{2}{3}$ AB . Bb; and moment of arc APD about AE

$$(= \text{arc APD . KG}) = \frac{2\ \text{AB . B}b}{3} = \frac{\text{arc APD . B}b}{3};$$

or $$\text{KG} = \tfrac{1}{3}\,\text{B}b = \tfrac{1}{3}\,\text{AB}.$$

Cor. 1. Moment of PP′ about AE = MN . AQ′.

Cor. 2. Still representing the mass of arc by its length, that is, taking for unit of mass the mass of one unit of length of the arc,

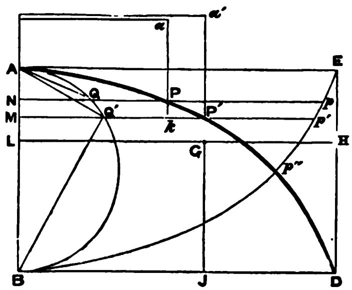
Moment of arc APD about AE $= \frac{2}{3}$ (AB)^2.

Cor. 3. Momt. of AP about AE is represented by area AMq $= \frac{2}{3}$ AM $\sqrt{\text{AB . AM}} = \frac{2}{3}$ AM$^{\frac{3}{2}}$. AB$^{\frac{1}{2}}$.

PROP. XIV.—*If G (fig. 11) is the centre of gravity of the cycloidal arc APD, then GL perp. to the axis AB = BD − $\frac{2}{3}$ AB.*

With same construction as in Prop. XII.,

FIG. 11. (AQ′ and NQ intersect in n.)



momt. of PP′ abt. AB=PN . PP′=2PN. inct. of AQ

(Prop. VI.). Draw P a, P$'a'$ parallel to AB and equal, respectively, to AQ, AQ$'$; complete the rectangles N a, M a'; and produce a P to meet MP$'$ in k. Also join BQ$'$ and let NP, MP$'$ prod. meet a cycloidal arc BE having B as vertex and E as cusp in p and p'. Then,

rect. M a' ultimately exceeds rect. N a by

rect. under PN, (P$'a'$ — P a) + rect. under a'P$'$. k P$'$.

That is,

inct. of rect. N a = PN. inct. of AQ + AQ$'$. k P$'$

= $\frac{1}{2}$ momt. of PP$'$ about AB + BQ$'$. MN

(since AQ$'$: BQ$'$ :: k P : k P$'$)

= $\frac{1}{2}$ momt. of PP$'$ about AB + momt. of $p\,p'$ about BD (Prop. XIII. Cor. 1).

Wherefore, taking all increments from A, where rect. N a has no area, to D, where N a = rect. AD, we have

2 rect. AD = momt. of arc APD about AB

+ 2 momt. of B p E about BD;

that is,

$$\text{arc APD. GL} = 2\ \text{AB . BD} - 2\,\text{arc B}\,p\,\text{E}\,.\,\frac{\text{DE}}{3};$$

or $$2\text{AB . GL} = 2\text{AB . BD} - \tfrac{4}{3}\ \text{AB . DE};$$

$$\therefore \text{GL} = \text{BD} - \tfrac{8}{3}\ \text{AB}.$$

Cor. Draw GH perp. to DE. Then GL + $\frac{8}{3}$ AB = BD = GL + GH. Therefore GH = $\frac{8}{3}$ AB.

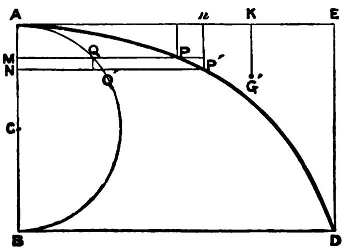
Prop. XV.—*If G (fig. 11) is the centre of gravity of the cycloidal arc APD, and GH, GJ be drawn perp. to DE and BD, JH is a square, whose sides are each equal to $\frac{2}{3}$ AB.*

From Prop. XIII. EH $= \frac{1}{3}$ AB; $\therefore$ DH $= \frac{2}{3}$ AB. From Prop. XIV. Cor., GH $= \frac{2}{3}$ AB. Therefore, the rectangle JG is a square having each of its sides $= \frac{2}{3}$ AB.

Prop. XVI.—*If G′ (fig. 12) is the centre of gravity of the area APDE, then G′K perp. to AE $= \frac{1}{4}$ AB.*

Take PP′ an element of the arc APD; draw P′ *n* perp. to AE, and PQM, P′Q′N perp. to AB, inter-

Fig. 12.



secting AQB in Q and Q′. Complete rectangles P*n*, QN. Then from Prop. IX. Cor. 1,

$$\text{rect. } Pn = \text{rect. } QN.$$

Now momt. of element P *n* about AE, ultimately

$= \frac{1}{2}$ P′ *n* . rect. P *n*

$= \frac{1}{2}$ AN . rect. NQ

$= \frac{1}{2}$ momt. of NQ about AE.

Taking all such elements, we have

Momt. of area APDE about AE = $\frac{1}{2}$ momt. of area AQB about AE.

That is, $G'K$. area APDE = $\frac{1}{2}$ AC . area AQB.

But, area APDE = area AQB;

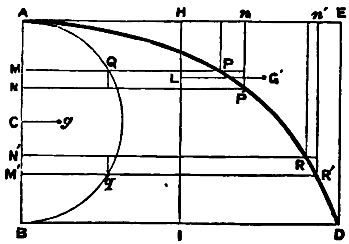
$\therefore G'K = \frac{1}{2} AC = \frac{1}{4} AB.$

PROP. XVII.—*If G′ (fig. 13) is the centre of gravity of the area APDE, HI parallel to AB through H the bisection of AE, and G′L perp. to HI, then*

$$G'L : AB :: AB : 3BI, \text{ or } G'L = \frac{4}{3\pi} . AB.$$

Take elements MN and M′N′ equal to each other and equidistant from A and B respectively; draw

FIG. 13.



MQP, NP′, N′R, and M′qR′ parallel to BD, meeting APD in P, P′, R and R′ (Q and q being points on circle AQB). Draw P′n and R′n' perp. to AE, and complete the elementary rectangles Pn, Rn', QN and qN′. These four rectangles are equal. Now, sum of moments of P n, R n' about HI

$$= \text{H}\,n\,.\,\text{rect. P}\,n + \text{H}\,n'\ \text{rect. R}\,n'$$
$$= (\text{H}\,n + \text{H}\,n')\ \text{rect. QN}$$
$$= 2\text{QM}\,.\,\text{rect. QN (Prop. III. Cor. 3)}$$
$$= 2\ \text{moment of rect. QN about AB.}$$

[This relation holds whether P n and R n' lie on the same side as in fig. 13 or on opposite sides of HI; for in the latter case, the moments being in opposite directions, their difference is the effective moment, and instead of (H n' + H n) rect. QN, we get (H n' − H n) rect. QN; but when n' and n are on opposite sides of HI, H n' − H n = 2QM. Prop. III. Cor. 3.]

Wherefore taking all the elements such as MN, M′N′, from A and B to the centre C, we get

Momt. of area APDE about HI = 2 momt. of semicircle AQB about AB;

that is, LG′. area APDE = 2 C g . area AQB (g being the centre of gravity of the semicircle AQB and C g perp. to AB). And since area APDE = area AQB, LG′ = 2C g.

But we know that

$$\text{C}\,g : \text{AB} :: \text{AB} : 3\ \text{arc AQB}^* \ (= 3\text{BD});$$
$$\left(\text{or C}\,g = \frac{2\text{AB}}{3\pi}\right):$$

wherefore LG′ : AB : 2 AB : 3 BD :: AB : 3BI;

$$\left(\text{or LG}' = \frac{4\text{AB}}{3\pi}\right)$$

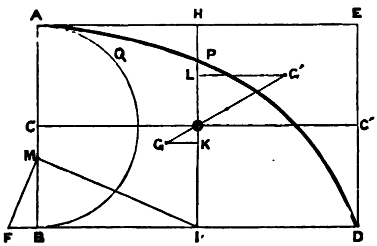
* If the reader is unfamiliar with this property, he may establish it thus:—First show that projection of any element of semicircle on tangent at the middle point of the arc has a moment about

Prop. XVIII.—*If G and G′ (fig. 14) are the centres of gravity of the areas APDB and APDE respectively, O the centre of gravity of the rectangle BE (that is the point in which HI, drawn as in last proposition, and CC′, the line of centres, bisect each other), and GK, G′L are drawn perp. to HI, then*

$$OK=\tfrac{1}{12}AB=\tfrac{1}{6}AC;\ \text{and}\ GK=\tfrac{1}{3}LG'=\frac{4}{9\pi}\,.\,AB=\frac{8AC}{9\pi}.$$

Since O is the centre of gravity of the rectangle BE, that is, of the area APDB + the area APDE, the

Fig. 14.



moments of APDB and APDE about COC′ are equal; that is,

diameter equal to the moment of the element; therefore moment of semicircular arc, or π rad. × dist. of C.G from diameter = diameter × rad.; that is distance of C.G from diam. = *diameter* ÷ π. Now a semicircular area may be supposed divided into an infinite number of equal small triangles having centre for apex, and each triangle may be supposed collected at its C.G. at a distance from centre = $\frac{2}{3}$ rad. Hence C.G. of semicircular area lies at a dist. from diameter = $\frac{2\ \text{diameter}}{3\pi}$. That is to say $Cg : 4r :: 1 : 3\pi :: r : 3\pi r$, or $Cg : 2r :: 2r : 3$ arc of semicircle.

3 area APDE . OK = area APDE . OL ;

or $$OK = \tfrac{1}{3}OL = \tfrac{1}{12}AB = \tfrac{1}{6}AC.$$

Similarly,

$$GK = \tfrac{1}{3}LG' = \frac{4}{9\pi} . AB = \frac{8AC}{9\pi}.$$

Cor. 1. Since LG′ : AB :: AB : 3 BI (Prop. XVII.),

$$GK : AB :: AB : 9\ BI.$$

Cor. 2. G, O, and G′ lie in a straight line, and OG′ = 3OG.

Cor. 3. Since moment of area AQBD about BD = (moment of ABDP − moment of AQB) about BD $= (\tfrac{5}{6} . 3AC - AC) \frac{\pi AC^2}{2} = \frac{3AC}{4} . \pi . AC^2$; it follows that the C.G. of area AQBD lies at a distance $= \tfrac{3}{4}AC$ from BD.

Schol.—The position of G may be thus obtained:—

Take $OK = \tfrac{1}{6}AC$. Also, take $BM = \tfrac{1}{3}AB$; join MI, and let MF perp. to MI intersect DB produced in F: draw KG perp. to OI and equal to BF. Then G is the centre of gravity of the area APDB. For $OK = \tfrac{1}{12}AB$; and

$$KG\ (=FB) : BM :: BM : BI;$$

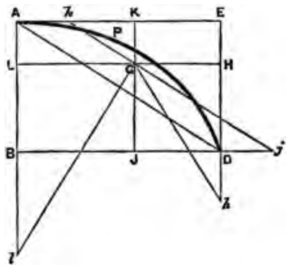
that is, $$KG : \tfrac{1}{3}AB :: \tfrac{1}{3}AB : BI :: AB : 3\ BI,$$

or $$KG : AB :: AB : 9\ BI.$$

PROP. XIX.—*If from G (fig. 15), the centre of gravity of semi-cycloidal arc APD, GL be drawn perp. to AB, and G l making with AB produced the angle G l A = the angle ADB; then the surface generated by the revolution of the arc APD about the axis AB is equal to eight times the rectangle having sides equal to AB and Ll.*

By Guldinus's First Property (see note following this Proposition), the surface generated by the revolution of APD about AB = rect. under straight lines equal to APD and circumference of circle of radius LG. But APD = 2AB, and since GL l is similar to ABD, and BD = $\frac{1}{4}$ the circumference of circle of radius AB, it follows that L l = $\frac{1}{4}$ circumference of circle of radius LG. Hence the surface produced by the revolution of APD about AB

FIG. 15.



= rect. under 2 AB and 4 L l

= 8 times the rectangle under AB and L l.

Cor. 1. In revolving round AB through half a right angle, APD generates a surface equal to rectangle under AB and L l.

Cor. 2. Since $GL = BD - \frac{2}{3}AB$ (Prop. XIV.), $Ll = (BD - \frac{2}{3}AB)\frac{\pi}{2}$; and the surface generated by revolution of APD about AB $= 4AB\,(BD - \frac{2}{3}AB)\,\pi$

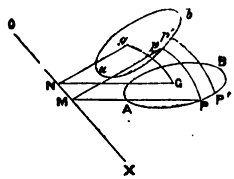
$$= 8AC\,(\pi\,.\,AC - \tfrac{4}{3}AC)\,\pi = \pi\,(\pi - \tfrac{4}{3})\,(AC)^2,$$
$$= 8\,(\pi - \tfrac{4}{3}) \text{ generating circle.}$$

NOTE.—Guldinus's properties, usually demonstrated by the integral calculus, are essentially geometrical. His First Property may be stated and established as follows :—

If a plane curve revolve through any angle α about an axis in its own plane, the curve lying entirely on one side of the axis, the area generated by the curve is equal to a rectangle having its adjacent sides equal in length to the curve and to the arc described by the centre of gravity of the curve, in revolving about the axis through the angle α.

Let APB (fig. 16) be a curve lying in the same plane as OX, and entirely on one side of OX, and let it revolve around OX through

FIG. 16.



an angle α to the position $a\,p\,b$. Then PP′, an element of the arc APB, generates a conical shred of constant breadth PP′ and of area ultimately = PP′. arc Pp = PP′. PM . α = α . moment of PP′ about OX. Taking all the elementary arcs of APB in this way, the surface generated by the arc APB = α . moment of arc APB about OX = α . GN . arc APB; (G being the centre of gravity of the arc APB, and GN perp. to OX).

Or, if length of curve APB = L, GN = $\bar{d}$, and the area of the surface generated = A, then

$$A = L\,.\,\bar{d}\,.\,\alpha$$

If the axis intersect the curve, then the two portions of the curve lying on either side of the axis must be separately dealt with.

It is easily seen that if the curve APB is not plane, or if (whether plane or not) it is not in the same plane as OX, a similar property may be established. Let the curve be carried once round OX, and let a plane through OX intersect the surface thus generated in a curve A′P′B′ (any parts of A′P′B′ through which more than one part of APB may have passed being counted twice or thrice or so many times as they may have been traversed in one circuit of APB). Let L′ be the length of A′P′B′ (thus estimated); G′ its centre of gravity (correspondingly estimating the weight of its various parts), and $\bar{d}'$ the distance of G′ from OX. Then the surface generated by the revolution of APB round OX through the angle $\alpha = L'.\bar{d}'.\alpha$ (any part of the generated surface traversed more than once by the generating curve being counted as often as it has been so traversed).

Again, if APB so move as to generate a cylindrical surface either right or oblique, and two planes through OX intersect the surface thus generated, the portion of this surface intercepted between those planes may be thus obtained :—through OX take a plane perp. to the axis of the cylindrical surface and intersecting that surface in a curve A′P′B′ of length L′ and having centre of gravity G′ at distance $\bar{d}'$ from OX; let the portion of a straight line through G′ parallel to the axis of the cylindrical surface, intercepted between the boundary planes $= h$; then the surface intercepted $= L'.\bar{d}'.h$.

The proofs of this and the preceding extensions of Guldinus's first property depend on the same principle as the proof of the property itself given above. In fact, the student who has grasped the principle of that proof will perceive the extensions to be little more than corollaries.

It may be of use to note that the two extensions require two lemmas. The first requires this lemma :—If an element of arc PP′ be projected orthogonally on a plane through OX and P into the elementary arc Pp, then PP′ and Pp in rotating through any angle round OX generate equal surfaces. This is obvious, since they generate equal elementary surfaces in rotating through an elementary angle round OX. The second extension requires this lemma :— If two planes through OX cut two parallel lines Pp, P′p' in P, P′ and p, p', the lines PP′ and pp' being elementary, then two other planes through OX near to these last cutting Pp and P′p' in R, R′ and r, r', such that PR $= p\,r$, intercept equal areas PRR′P′ and $p\,r\,r'p'$. These areas are in fact ultimately parallelograms on equal bases and between the same parallels.

PROP. XX.—*If from G (fig. 15), the centre of gravity of the semi-cycloidal arc APD, GH be drawn perp. to ED, and G h making with ED produced the angle G h H = angle ABD, then the surface generated by the revolution of the arc APD about ED as an axis is equal to eight times the rectangle under AB and H h.*

The demonstration is in all respects similar to that of Prop. XIX.

Cor. 1. In revolving through half a right angle, APD generates a surface equal to the rectangle under AB and H *h*.

Cor. 2. Since GH $= \frac{2}{3}$ AB (Prop. XIV. Cor.), H $h = \frac{\pi AB}{3}$; and the surface generated by the revolution of APD about $ED = 8 . AB . \frac{AB}{3} \pi = \frac{8\pi}{3} (AB)^2$

$= \frac{32\pi}{3} (AC)^2 = \frac{32}{3}$. generating circle.

PROP. XXI.—*If from G (fig. 15), the centre of gravity of the semi-cycloidal arc APD, GK be drawn perp. to AE, and G k parallel to AD meet AE in k, then the surface generated by the revolution of the arc APD about AE as axis = eight times the rectangle under AB and K k.*

The demonstration is similar to that of Prop. XIX.

Cor. 1. In revolving through half a right angle

APD generates a surface equal to the rectangle under AB and K k.

Cor. 2. Since GK $= \frac{1}{3}$AB (Prop. XIV.), K $k = \frac{\pi}{6}$ AB; and the surface generated by the revolution of

$$\text{APD about AE} = 8\pi \text{AB} . \frac{\text{AB}}{6} = \frac{4\pi}{3}\text{AB}^2 = \frac{16\pi}{3}(\text{AC})^2$$

$$= \frac{16}{3} . \text{ generating circle.}$$

Prop. XXIII.—*If from G (fig. 15), the centre of gravity of semi-cycloidal arc APD, GJ be drawn perp. to BD, and Gj parallel to AD to meet BD produced in j, then the surface produced by the revolution of the arc APD about BD as axis = eight times the rectangle under AB and Jj.*

The demonstration is similar to that of Prop. XIX.

Cor. 1. In revolving through half a right angle APD generates a surface equal to the rectangle under AB and J j.

Cor. 2. Since GJ $= \frac{2}{3}$AB (Prop. XV.), J $j = \frac{\pi}{3}$AB; and the surface generated by the revolution of

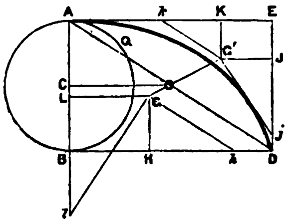
$$\text{APD about BD} = \frac{8\pi}{3}(\text{AB})^2 = \frac{32\pi}{3}(\text{AC})^2.$$

$$= \frac{32}{3} . \text{ generating circle.}$$

PROP. XXIV.—*If from G (fig. 17), the centre of gravity of the cycloidal area APDB, GL be drawn perp. to AB, and G l making with AB produced the angle G l A = angle ADB, then the volume generated by the revolution of the area APDB around the axis AB is equal to six times the volume of a cylinder having the generating circle AQB for base and height equal to L l.*

By Guldinus's Second Property (see note following this proposition) the volume generated by the revolu-

FIG. 17.



tion of surface APD around AB = volume of a right cylinder having APDB as base and height = circumference of circle of radius LG. But area APDB = $\frac{3}{2}$ generating circle; and, as in Prop. XIX., L l = $\frac{1}{4}$ circumference of circle with radius LG. Hence the volume generated by the revolution of area APD around AB is equal to ($\frac{3}{2} \times 4$ times, or) six times the volume of a cylinder having circle AQB as base and height = L l.

Cor. 1. The volume generated by the revolution of

APDB through one-third of two right angles about AB is equal to a cylinder having circle AQB as base and height $= \mathrm{L}\,l$.

Cor. 2. Since $\mathrm{LG} = \mathrm{OC} - \frac{4\mathrm{AB}}{9\pi}$ (Prop. XVIII.) $= \frac{\pi}{2} . \mathrm{AC} - \frac{8\mathrm{AC}}{9\pi}$, $\mathrm{L}\,l = \left(\frac{\pi}{4} . \mathrm{AC} - \frac{4\mathrm{AC}}{9\pi}\right)\pi$; and the surface generated by the revolution of APDB about AB $= 6\pi\,(\mathrm{AC})^2\left(\frac{\pi}{4}\mathrm{AC} - \frac{4\mathrm{AC}}{9\pi}\right)\pi = \left(\frac{3\pi^3}{2} - \frac{8\pi}{3}\right)(\mathrm{AC})^3$.

Cor. 3. Since the rectangle BE in revolving around AB generates a cylinder whose volume $= \mathrm{AB} . \pi . (\mathrm{BD})^2 = 2\mathrm{AC} . \pi\,(\pi\mathrm{AC})^2 = 2\pi^3 . (\mathrm{AC})^3$, it follows from Cor. 2 that the volume generated by APDE in revolving around AB

$$= 2\pi^3(\mathrm{AC})^3 - \left(\frac{3\pi^3}{2} - \frac{8}{3\pi}\right)(\mathrm{AC})^3 = \left(\frac{\pi^3}{2} + \frac{8}{3\pi}\right)(\mathrm{AC})^3.$$

NOTE.—Guldinus's Second Property may be thus stated and established:—

If a plane figure revolve through an angle α about an axis in its own plane (the figure lying entirely on one side of the axis), the volume of the solid generated by the figure is equal to that of a cylinder having the figure for base and its height equal to the arc described by the centre of gravity of the surface in revolving through the angle α.

Let AQB (fig. 18) be a plane figure, and let it revolve through an angle α about an axis OX in the same plane (AQB lying entirely on one side of OX) to the position of $a\,q\,b$. Then PP′, an element of the figure's area, generates a ring of constant cross section PP′ and of volume ultimately $= \mathrm{PP'} . \mathrm{P}p = \mathrm{PP'} . \mathrm{PM} . \alpha = \alpha$. moment of PP′ about OX. Taking all the elements of area of AQB in this way, the volume generated by the surface $\mathrm{AQB} = \alpha$. moment of the area AQB about $\mathrm{OX} = \alpha . \mathrm{GN}$. area AQB, G being the centre of gravity of the figure AQB, and GN perp. to OX.

Or if area of $\mathrm{AQB} = \mathrm{A}$, $\mathrm{GN} = \bar{d}$, and the volume of the solid generated $= \mathrm{V}$,

$$\mathrm{V} = \mathrm{A} . \bar{d} . \alpha.$$

PROP. XXV.—*If from G (fig. 17), the centre of gravity of the cycloidal area APDB, GH be drawn perp. to BD and G h parallel to AD to meet BD in h, then the volume generated by the revolution of the area APDB about BD as axis is equal to six times the volume of a cylinder having the generating circle AQB for base and height equal to H h.*

The demonstration is in all respects as in Prop. XXIV.

Cor. 1. The volume generated by the revolution of APDB through one-third of two right angles about

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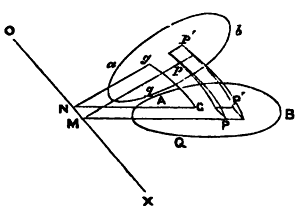
It is easily seen that if the figure AQB is not plane, or if, whether plane or not, it is not in the same plane as OX, a similar

FIG. 18.



property may be established. Let the figure AQB be carried once round OX, and let a plane through OX intersect the surface thus generated in a curve A'Q'B' (any parts of the plane figure A'Q'B' through which more than one part of AQB may have passed being counted twice or thrice, or so many times as they may have been traversed in one circuit of AQB). Let A' be the area of A'Q'B' (thus estimated), G' its centre of gravity (correspondingly estimating the weight of its various parts), and $\bar{d}'$ the distance of G' from OX. Then the volume generated by the revolution of AQB round OX through the angle $\alpha = A'.\bar{d}'.\alpha$ (any part of the volume generated which is traversed more than once by the generating curve being counted as often as it is so traversed).

BD is equal to a cylinder having the circle AQB as base and height = H h.

Cor. 2. Since $GH = \frac{5}{6}AC$ (Prop. XVIII.), $Hh = \frac{5\pi}{12}AC$; and the volume generated by the revolution of APDB about $AB = \pi.(AC)^2.\frac{5}{2}\pi AC = \frac{5}{2}\pi^2(AC)^3$.

Cor. 3. Since the rectangle BE in revolving around BD generates a cylinder whose volume = $BD.\pi(AB)^2 = \pi AC.4\pi(AC)^2 = 4\pi^2(AC)^3$, it follows from Cor. 2 that the volume generated by APDE in revolving around BD

$$= 4\pi^2(AC)^3 - \tfrac{5}{2}\pi^2(AC)^3 = \tfrac{3}{2}\pi^2(AC)^3.$$

Again, if AQB so move as to generate a cylindrical surface either right or oblique, and two planes through OX intersect the surface thus generated, the portion of the volume of this cylinder intercepted between these planes may be thus obtained:—Through OX take a plane perp. to the axis of the cylindrical surface, and intersecting that surface in a curve A′Q′B′, enclosing a figure of area A′, and having its centre of gravity G′ at a distance $\bar{a}'$ from OX; let the portion of a straight line through G′ parallel to the axis of the cylindrical surface intercepted between these bounding planes = h; then the volume intercepted = $A'.\bar{a}'.h$.

The proof of this and the preceding extension of Guldinus's second property will be found to require the two following lemmas: First, if an element of area PP′ be projected orthogonally on a plane through OX and P into the elementary area Pp', then PP′ and Pp' in rotating through any angle around OX generate equal elementary solids. This is obvious, since they generate equal elementary solids in rotating through an elementary angle around OX. Secondly, if two planes through OX cut a parallelopipedon of elementary cross section in the parallelograms PP′ and pp', Pp and $P'p'$ being two opposite edges of the parallelopipedon, then two other planes through OX near to these last, cutting Pp and $P'p'$ in R, R′, and r, r', such that $PR = pr$, intercept equal elementary solids, PRR′P′ and $prr'p'$. These solids are, in fact, ultimately parallelopipedons on equal bases and between the same parallel planes.

PROP. XXVI.—*If from G′ (fig. 17), the centre of gravity of the cycloidal area APDE, G′K be drawn perp. to AE and G′k parallel to AD to meet AE in k, then the volume generated by the revolution of the area APDE about AE as axis is equal to twice the volume of a cylinder having the generating circle AQB for base and height equal to K k.*

The demonstration is as in Prop. XXIV., except that the area APDE = a third only of the area APDB.

Cor. 1. The volume generated by the revolution of APDE through two right angles about AE = a cylinder having circle AQB as base, and height equal to K k.

Cor. 2. Since $G'K = \frac{1}{2}AC$ (Prop. XVI.), $Kk = \frac{\pi}{4}AC$; and the volume generated by the revolution of APDE about $AE = \pi(AC)^2 . \frac{\pi}{2}AC = \frac{\pi^2}{2}(AC)^3$.

Cor. 3. Since the volume generated by the revolution of rectangle BE around $AE = 4\pi^2 (AC)^3$ (see Prop. XXV. Cor. 3), it follows from Cor. 2 that the volume generated by APDB in revolving around AE $= 4\pi^2(AC)^3 - \frac{\pi^2}{2}(AC)^3 = \frac{7\pi^2}{2}(AC)^3$.

PROP. XXVII.—*If from G′ (fig. 17), the centre of gravity of the cycloidal area APDE, G′J be drawn perp. to DE and G′j parallel to AD to meet DE in j, then the volume generated by the revolution of the area APDE around DE as axis is equal to twice the volume of a cylinder having the generating circle AQB for base and height equal to Jj.*

The demonstration is as in Prop. XXIV., modified as in Prop. XXVI.

Cor. 1. The volume generated by the revolution of APDE through two right angles about AE = a cylinder having circle AQB as base, and height equal to J*j*.

Cor. 2. Since $G'J = \frac{AE}{2} - \frac{8}{3\pi}$ AC (Prop. XVII.) $= \left(\frac{\pi}{2} - \frac{8}{3\pi}\right)$ AC, $Jj = \left(\frac{\pi^2}{4} - \frac{4}{3}\right)$ AC; and the volume generated by the revolution of APDE around DE

$$= \pi (AC)^2 \left(\frac{\pi^2}{2} - \frac{8}{3}\right) AC = \left(\frac{\pi^3}{2} - \frac{8\pi}{3}\right) (AC)^3.$$

Cor. 3. Since the volume generated by the revolution of the rectangle BE around DE $= 2\pi^3$ $(AC)^3$ (Prop. XXIV. Cor. 3), it follows from Cor. 2 that the volume generated by APDB in revolving around DE $= 2\pi^3 (AC)^3 - \left(\frac{\pi^3}{2} - \frac{8\pi}{3}\right) (AC)^3 = \left(\frac{3\pi^3}{2} + \frac{8\pi}{3}\right) (AC)^3.$

SECTION II.

THE EPICYCLOID AND HYPOCYCLOID.

DEFINITIONS.

The Epicycloid is the curve (as D′AD, fig. 19, *Plate I.) traced out by a point on the circumference of a circle (as AQB) which rolls without sliding on a fixed circle (as BDB′) in the same plane, the rolling circle touching the outside of the fixed circle.*

The Hypocycloid is the curve (as D′AD, fig. 20, *Plate I.) traced out by a point on the circumference of a circle (as AQB) which rolls without sliding on a fixed circle (as BDB′) in the same plane, the rolling circle touching the inside of the fixed circle.*

What follows applies to both figures unless special reference is made to one only, and in every demonstration in this section two figures are given, one illustrating a property of the epicycloid, the other illustrating the same property of the hypocycloid, but the demonstration applying equally to either figure, unless special reference is made to one only. The student will do well to read each proof twice, using first one figure, then the other. For convenience the word 'cycloidal' throughout this section is to be understood to signify either epicycloidal or hypocycloidal according to the figure followed.

[NOTE.—It will be shown in Prop. I. of the present section that if two circles AQB and AQ′B′, touching at B, touch a fixed circle BDB′ at the extremities of a diameter BOB′, then the same curve is traced out by the point A on the circle AQB rolling in contact with the circle BDB′, as by the point A on the circle AQ′B′ rolling in contact with the same circle BDB′. We may therefore, in what follows, limit our attention to cases in which the centre O lies *outside* the rolling circle. According to the definitions given above, the curve traced out by A, fig. 19, is an epicycloid whether AQB or AQ′B′ is the rolling circle.

It may be well to mention that it has hitherto been customary to regard the curve traced out by A on AQB, fig. 19, as an *epicycloid*, and the *same* curve traced out by A on AQ′B′ as an *external hypocycloid.* Instead of defining the hypocycloid as the curve obtained when the rolling circle touches the outside of the fixed circle, it has hitherto been usual to define it as the curve obtained when either the convexity of the rolling circle touches the concavity of the fixed circle, or the concavity of the rolling circle touches the convexity of the fixed circle. There is a manifest want of symmetry in the resulting classification, seeing that while every epicycloid is thus regarded as an external hypocycloid, no hypocycloid can be regarded as an internal epicycloid. Moreover, an external hypocycloid is in reality an anomaly, for the prefix 'hypo' used in relation to a closed figure like the fixed circle implies interiorness.]

Let BDB′ (radius F) be the fixed circle, AQB (radius R) the rolling circle. If the centre of the latter circle move in the direction shown by the arrow, it is manifest that at regular intervals the tracing point will coincide with the circumference BDB′, as at D′, D, &c. (E′q′D′ and EqD being the corresponding positions of the rolling circle), while midway between two such coincidences the tracing point will be at its greatest diametral distance from D′BD as at A (AQB being the corresponding position of the rolling circle), ACB the diameter through the tracing point passing when produced through O, the diameter of the fixed circle. It is clear also from the way in which the curve is traced out that the parts AP′D and APD are similar and equal. Wherefore AB is called the *axis* of the cycloidal arc D′AD. The circular arc D′BD is the *base*, A the *vertex*, and the points D′ and D are the *cusps*. It is convenient to call the radius to the tracing point the *tracing radius*, and the diameter through the tracing point the *tracing diameter*. The tracing circle in the position AQB is called the *central generating circle*; and straight lines passing through the centres of both the fixed and rolling circles are said to be *diametral*. The arc Cc is called the *arc of centres*, and the circle of which it is part the *circle of centres*.

Let a circle E′AE be described with centre O and radius OA, and let OD′ and OD (produced if necessary) meet this circle in E and E′; then it is clear that D′d′ and Dd, the parts of the cycloidal curve on either side of D′E′ and DE, are symmetrical with regard to these lines respectively, which are therefore *secondary axes.* Also E′AE touches the curve D′AD in A.

The complete curve, either of an epicycloid or of a hypocycloid, consists of an infinite number of equal cycloidal arcs, but when the radii F and R are commensurable in length, the curve is re-entering, and may be described as consisting of a finite number of arcs.* Thus if $R = F$ the rolling circle will make one complete circuit of the fixed circle between each successive coincidence of the tracing point with the fixed circle; hence D and D′ will coincide, and there will be but one cusp. (No hypocycloid can be traced with these radii.) If $R = \frac{1}{2}F$, each base as DD′ will be equal to half the circumference of the fixed circle, and there will be but two cusps. Similarly if $R = \frac{1}{3}F$, $\frac{1}{4}F$, $\frac{1}{5}F$, &c., there will be 3, 4, 5, &c., cusps, respectively. In these cases the complete cycloidal arc will consist of a number of equal arcs, standing on equal parts of one circuit of the fixed circle's circumference. Again, if $mR = nF$, where n and m are integers prime to each other, then m circumferences of the smaller circle will be equal to n circumferences of the larger. Consequently there will be m cusps in the complete cycloidal curve, and the base of each cycloidal arc will be equal to one mth part of n circumferences of the fixed circle, that is to the $\frac{n}{m}$th part of the circumfer-

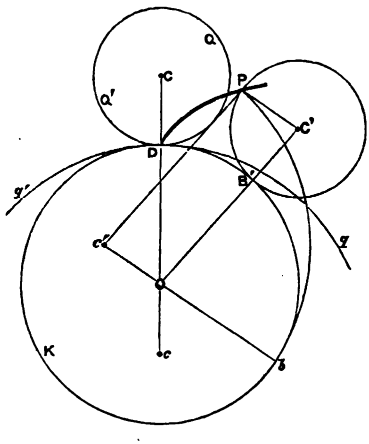
* Theoretically it consists in that case of an infinite number of arcs, occupying a finite number of positions, and consequently each arc coinciding with an infinite number of other arcs belonging to the curve.

ence of this circle. Wherefore if $n > m$, the base is greater than the circumference of the fixed circle, but if $n < m$ the base is less than this circumference. If $m =$ unity, that is $R = n F$, then the base of each cycloidal arc $= n$ times the circumference of the fixed circle.

PROPOSITIONS.

PROP. I.—*If a circle $q'Dq$ (figs.* 21 *and* 22), *having*

FIG. 21.

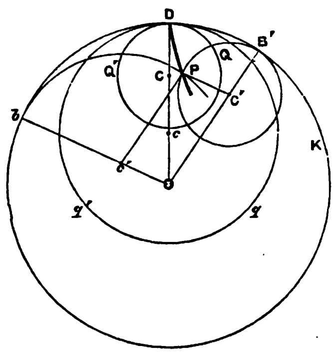


radius Dc, roll in contact with a circle KDb, having radius OD, c and O lying on the same side of D, then the point D on $q'Dq$ will trace out the same curve as the point D on a circle $Q'DQ$ having radius DC equal to $c\ O$ (measured in direction $c\ O$), rolling in contact with the circle KDb.

Let b be the point in which the rolling circle $q'Dq$

touches KD b, when the tracing point is at P, c' being the centre of q'D q (c', O, and b lying in the same straight line). Through O draw OC′, equal and parallel to c' P, meeting KD b in B′; and join PC′. Then PC′O c' is a parallelogram; PC′ $= c'$O $=$ DC; also, since OC′ $= c'$P $= c'b'$, and OB′ $=$ O b, C′B′ $=$ O c' $=$ DC. Hence a circle equal to QDQ′, touching KDb in B′ (on the same side as QDQ′), has its centre at C′

Fig. 22.



and passes through P.

Moreover, since arc P $b =$ arc D b,

$$\angle \mathrm{P}\, c'b\ (= \angle \mathrm{C'O}\, b = \angle \mathrm{PC'B'}) : \angle \mathrm{DO}\, b :: \mathrm{O}\, b : c'b,$$

$$\therefore \angle \mathrm{PC'B} : \angle \mathrm{DOB'} :: \mathrm{O}\, c' : \mathrm{O}b :: \mathrm{DC} : \mathrm{OD}.$$

Therefore arc PB′ $=$ arc DB′, and P is a point on the curve traced out by D on the circle QDQ′ rolling in contact with the circle KD b.

Schol.—It is manifest that when P arrives at the vertex of the curve the rolling circles are placed (relatively to each other) as in figs. 19 and 20.

PROP. II.—*The base of the epicycloid or hypocycloid is equal to the circumference of the generating circle.*

This, as in the case of the cycloid, needs no demonstration.

Cor. 1. Arc D′B (figs. 19 and 20) = arc BD = half the circumference of the generating circle.

Cor. 2. Arc C c : arc BD :: CO : BO.

Or for the epicycloid,

$$\text{arc } C\,c = \frac{F+R}{R}.\ \text{arc BD} = \frac{F+R}{R}\ \text{arc AQB},$$

and for the hypocycloid,

$$\text{arc } C\,c = \frac{F-R}{R}.\ \text{arc BD} = \frac{F-R}{R}.\ \text{arc AQB}.$$

Cor. 3. Area E′AEDBD′ = 2 AED′B

$$= 4 \text{ rect. under AC, C}\,c\text{*}$$

$$= 4\,\frac{CO}{BO}.\ \text{rect. under AC, BD}$$

$$= \frac{4CO}{BO}.\ \text{circle AQB}$$

for the epicycloid $= 4\,\frac{(F+R)}{R}$ circle AQB

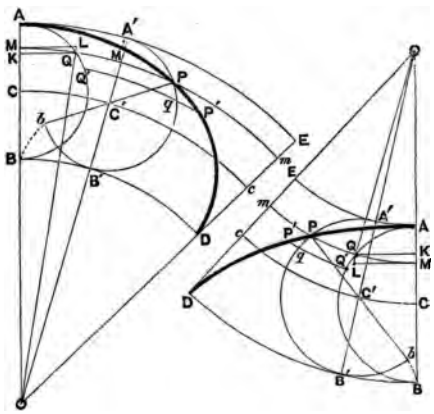
for the hypocycloid $= 4\,\frac{(F-R)}{R}$ circle AQB.

* The relation here employed is almost self-evident. It may be thus demonstrated: Divide the area AEDB into a series of elementary areas by drawing radial lines from O: each element is in the limit a trapezium whose area = rectangle under AB and half the sum of those elementary arcs of AE and BD which form (in the limit) the parallel sides of the trapezium. Therefore the area AEDB = rectangle under AB and half the sum of the arcs AE, BD = rectangle under AB and the arc C c.

PROP. III.—*If through P, a point on the epicycloidal or hypocycloidal arc APD (figs. 23 and 24), the arc PM be drawn concentric with the base BD, cutting the central generating circle in Q and meeting the axis AB in M, then arc QP : arc AQ :: OM : OB.*

Let A′PB′ be the position of the generating circle when the tracing point is at P; C′ its centre; A′C′B′O diametral, cutting PM in M′. Draw the tracing dia-

FIG. 23. FIG. 24.



meter PC′*b*. Then it is manifest that arc QM = arc M′P; arc MM′ = arc QP; and arc AQ = arc A′P. Now *b* is the point which was at B when the tracing point was at A; and since every point of the arc *b* B′ has been in rolling contact with BB′, the arc *b* B′ = the arc BB′. But arc *b* B′ = arc A′P = arc AQ; and

arc MM′ (= arc QP) : arc BB′ :: OM : OB;

∴ arc QP : arc AQ :: OM : OB.

Cor. 1. Arc $MP = \frac{OM}{OB}$. arc AQ + arc MQ.

Cor. 2.

Let arc MQ prod. meet OE (drawn as in figs. 19, 20) in *m*

then arc $Mm = \frac{OM}{OB}$ arc $BD = \frac{OM}{OB}$. arc AQB.

[But arc BQ > QK perp. to AB; $\therefore \frac{OM}{OB}$. arc BQ > ML, perp. to AB and meeting OQ produced in L (for OM : OB > OM : OK). But ML > arc MQ. *A fortiori*, then, $\frac{OM}{OB}$. arc BQ > arc MQ.] *

$\therefore$ since arc $Mm = \frac{OM}{OB}$. arc AQ + $\frac{OM}{OB}$ arc BQ,

while arc $MP = \frac{OM}{OB}$. arc AQ + arc MQ,

arc M *m* > arc MP, and P falls between OA and OE; that is, the whole arc APD lies between OA and OE.

Cor. 3. The arc P *m* = arc M *m* − arc MP

$$= \frac{OM}{OB} \text{ arc } AQ - \frac{OM}{OB} \text{ arc } AQ - \text{arc } MQ$$

$$= \frac{OM}{OB} \text{ arc } BQ - \text{arc } MQ.$$

Cor. 4. If through P′, a point near P, arc P′*p* Q′ be drawn concentric with the base BD, meeting AQB in Q′ and cutting A′PB′ in *q*, then in the limit

* The part in [] fails for hypocycloid. Substitute the following:—Let OQ produced meet arc BD in H, draw BF perp. to OH and describe ½ ⊙ BFO. Then, arc BH = arc BF (of half rad. and double ∠ at centre); but arc BF < arc BQ, ∵ chd. BF < chd. BQ (BFQ being a rt. angle) while seg. BF contains a larger angle than seg. BQ′Q. Hence arc BQ > arc BH > $\frac{OB}{OM}$. arc MQ; *i.e.* $\frac{OM}{OB}$. arc BQ > arc MQ.

(when P′ is very near to P), arc $P'Q' = \frac{OM}{OB} \cdot$ arc $A'q$;

and arc $PQ = \frac{OM}{OB} \cdot$ arc $A'P$; therefore,

$$\text{arc } P'Q' - \text{arc } PQ \; (= qP') = \frac{OM}{OB} (\text{arc } A'q - \text{arc } A'P)$$

$$= \frac{OM}{OB} \text{ arc } Pq; \quad \text{or, in the limit,}$$

$$qP' : \text{arc } Pq :: OM : OB.$$

PROP. IV.—*A, B, C, D, E, &c.* (*figs.* 25 *and* 26, *p.* 51) *representing the same points as in the preceding proposition, the area APDBQ = half the area ABDE; or area APDBQ : generating circle :: OC : OB.*

Take CL = CL′, on AB; and LK, L′K′ equal elements of AB, both towards C. Draw LQ, Kq, K′q', and L′Q′ at right angles to AB to meet AQB; and about O as centre describe arcs QP, qp, $q'p'$, and Q′P, meeting APD. Let Oq, produced if necessary, meet QP in n; draw Qk perpendicular to Kq; join Cq, and draw Cm perpendicular to Oq, produced if necessary. Then ultimately the triangles Qkq and qKC are similar, as are the triangles Qqn and qCm (for QqC being ultimately a right angle, Qqn is ultimately the complement of Cqm and therefore equal to qCm). Hence the quadrilateral Qnqk is similar to the quadrilateral qmCk, and

$$qn : Qk (= LK) :: Cm : Kq :: CO : qO$$

(triangles COm and qOK being similar). Hence

Area QPpq (ult. = rect. nq, QP) : rect. LK, QP
:: CO : qO;

but, rect. LK, QP : rect. LK, Aq :: QP : Aq
:: qO : BO (Prop. II.);

.·. *ex æq.* area QPpq : rect. LK, Aq :: CO : BO
:: Cc : BD;

similarly, area Q′P′$p'q'$: rect. L′K′, Aq' (or LK, Bq)
:: Cc : BD;

.·. QPpq + Q′P′$p'q'$: rect. LK, $\overline{\text{A}q+\text{B}q}$ (or LK, BD)
:: Cc : BD;

wherefore QPpq + Q′P′$p'q'$ = rect. LK, Cc.

.·. summing all such elements between AE and BD,
Area APDBQ = rect. under AC, Cc = ½ area ABDE.
or, area APDBQ : gen. ⊙ :: OC : OB.

Cor. 1. Since, for epicycloid, $\text{C}c = \frac{\text{F}+\text{R}}{\text{F}}$. AQB,

$$\text{area APDBQ} = \frac{\text{F}+\text{R}}{\text{F}}\,.\,\text{AC}\,.\,\text{AQB} = \frac{\text{F}+\text{R}}{\text{F}}\,.\,\text{gen.}\ \odot$$

and the area between epicycloidal arc and base

$$= \left(2\,.\,\frac{\text{F}+\text{R}}{\text{F}}+1\right)\text{gen.}\ \odot = \frac{3\,\text{F}+2\,\text{R}}{\text{F}}\ \text{gen.}\ \odot\,.$$

For the hypocycloid, area $\text{APDBQ} = \frac{\text{F}-\text{R}}{\text{F}}\,.\,\text{gen.}\ \odot$;

and the area between hypocycloidal arc and base

$$= \frac{3\,\text{F}-2\,\text{R}}{\text{F}}\,.\,\text{gen.}\ \odot\,.$$

Cor. 2. If AB is the axis of a cycloid (A the vertex) and LQ produced meet this cycloid in R, then

Area AQP : area AQR :: OC : OB.*

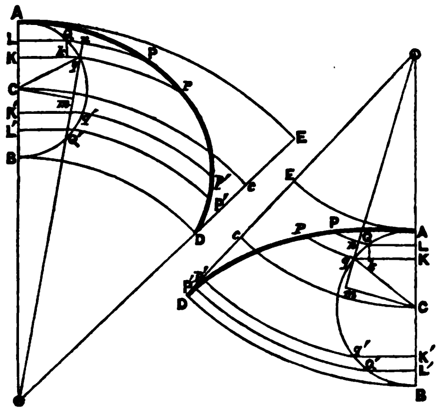
* This relation, which follows directly from the proportion on the fifth line of this page, might have been employed to establish the main proposition. I preferred, however, to give an independent proof.

Cor. 3. Epicyc. area APDE = APDBQ − AQB

$$= \left(\frac{F+R}{F} - \frac{1}{2}\right) \text{gen.} \odot = \frac{F+2R}{2F} \text{gen.} \odot .$$

Hypocycloidal area APDE $= \dfrac{F-2R}{2F}$ gen. $\odot$.

Fig. 25. Fig. 26.



Cor. 4. Area AQP + area BQ′P′D = rect. AL, C*c*;
and, area QQ′P′P = rect. under LC, C*c*.

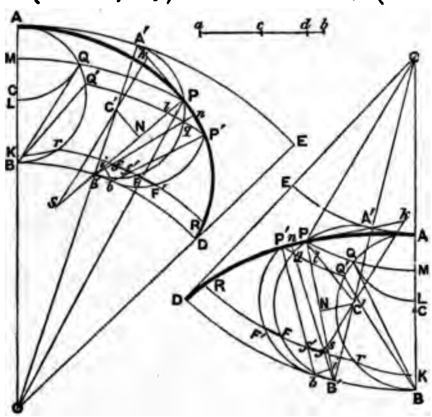
Prop. V.—*If P is a point on the epicycloidal or hypocycloidal arc APD (figs.* 27 *and* 28) *A′PB′ the generating circle when the tracing point is at P, A′CB′ diametral, then PB′ is the normal and A′P is the tangent at the point P.*

Since, when the tracing point is at P, the generating circle A′PB′ is turning round the point B′, the direction of the motion of the tracing point at P must be at right angles to PB′;—wherefore PB′ is the normal and AP is the tangent at P.

Another Demonstration. (*See p.* 8.)

Take P′ a point near to P and draw PQM, P′Q′ concentric with BD; PQM meeting AB in M and cutting AQB in Q; and P′Q′N cutting AQB and

FIG. 27. (Join PC′, AQ.) FIG. 28. (Join AQ.)



A′PB′ in Q′ and q. Join PC′, PO, and let C′s parallel to PO meet PB′ (produced in case of epicycloid) in s. Then (Prop. III. Cor. 4)

arc qP′ : arc Pq :: PO : B′O :: C′s : C′B′ (= C′P);

or the sides about the angles PqP′, PC′s are proportional; but these angles are ultimately equal, for Pq is ultimately perp. to C′P, and P′q to PO, that is to C′s. Therefore the triangles PqP′ and PC′s are ultimately similar; and the third side PP′ of one is perp. to the third side P s of the other. That is PB′ is the normal at P, and therefore PA′ perp. to PB′ is the tangent at P.

Cor. 1. If PB′ intersect Q′P′ in l, and s C′ produced meet PA′ in k, the triangle PP′l is ultimately similar to the triangle s P k.

Cor. 2. If B′q be joined and produced to meet PP′ in n, then $q\,n$ is ultimately perp. to PP′; wherefore if C′N be drawn perp. to B′P, the figure P′q P′$n\,l$ is ultimately similar to the figure PC′s N k; whence

$$PP' : Pn :: Ps : PN.$$

Schol.—As in Schol. p. 9 (obviously modified), a tangent may be drawn to APD from any point on APD or AA′E.

Prop. VI.—*With the same construction as in Prop. V.,*
Arc AP : chord AQ :: 2CO : BO.

Since $q\,n$ is ultimately perpendicular to PP′, P n is ultimately equal to the excess of chord A′q over chord A′P. Now from Cor. 2, Prop. V.,

$$PP' : Pn :: sP : NP :: 2sP : B'P$$
$$:: 2\,C'O : B'O :: 2\,CO : BO,$$

or, inct. of AP : inct. of ch. A′P (or AQ) :: 2 CO : BO. But arc AP and chord AQ begin together, wherefore

$$\text{Arc AP} : \text{chord AQ} :: 2\,\text{CO} : \text{BO}.$$

Cor. 1. Arc APD : AB :: 2 CO : BO.

Cor. 2. For the epicycloid,

$$\text{Arc APD} = \text{AB}\,.\,\frac{2\,(F+R)}{F} = \frac{4\,R\,(F+R)}{F}.$$

For the hypocycloid,

$$\text{Arc APD} = \text{AB} \cdot \frac{2(\text{F}-\text{R})}{\text{F}} = \frac{4\,\text{R}(\text{F}-\text{R})}{\text{F}}.$$

Cor. 3. $\text{PP}' : \text{P}n :: 2\,\text{CO} : \text{BO}.$

Cor. 4. $\text{PP}' : n\text{P}' :: 2\,\text{CO} : 2\,\text{CO}-\text{BO}$
$:: 2\,\text{CO} : \text{AO}.$

Cor. 5. $\text{P}n : n\text{P}' :: \text{BO} : \text{AO}.$

PROP. VII.—PROB. *To divide the arc of an epicycloid or a hypocycloid into parts which shall be in any given ratio to each other.*

Let a straight line ab (figs. 27 and 28) be divided into any parts in the points c and d : it is required to divide the arc APD in the same ratio.

Divide AB in L and K, so that

$$\text{AL} : \text{LK} : \text{KB} :: ac : cd : db;$$

with centre A and radii AL and AK, describe circular arcs LQ, K r, cutting the semicircle AQB in Q and r; through which points draw the arcs QP, r P, concentric with BD. Then

Arc AP : chord AQ (= AL) :: 2CO : BO.

Similarly Arc AR : AK :: 2CO : BO;

Therefore Arc PR : LK :: 2 CO : BO.

Similarly Arc RD : KB :: 2 CO : BO,

therefore

Arc AP : arc PR : arc RD :: AL : LK : KB
:: $ac : cd : db$;

or, the arc APD is divided into the points P and R in the required manner.

Similarly may the arc APD be divided into four, five, or any number of parts, bearing to each other any given ratios.

PROP. VIII.—*With same construction as in Prop. V.,*

Area ABB′P (figs. 27 *and* 28) : *sectorial area A′B′P* :: *area B′PD* : *segm. PFB′* :: 2 *CO* + *BO* : *BO.*

Let b be the point of contact of tracing and fixed circles, when tracing point is at P′; join b P′, BQ, and BQ′; and draw $b\,i$ perpendicular to Ps. Then triangle b B′i is similar to B′C′N, therefore to PC′N, and therefore (Prop. V., Cor. 2) to P q n; and B′ b=P q: therefore P q n and b B′i are equal in all respects; and P n=b i. Now elementary area PP′bB′ is ultimately equal to trapezium P i b P,

=half rect. under P i and (PP′ + b i)

= half rect. under PB′ and (PP′ + P n) ultimately

and elementary area QBQ′ is ultimately equal to triangle PB′ q

= half rect. under PB′ and P n, ultimately.

∴ area PP′ b B′ : area QBQ′ :: PP′ + P n : P n

:: 2 CO + BO : BO (Cor. 3, Prop. VI.).

Thus the increment of area ABB′P, or the decrement of area B′PD, bears to the increment of area A′B′P, or the decrement of area PFB′, the constant ratio

(2 CO + BO) : BO. But the areas ABB′P and B′PD commence together, and the areas A′B′P, PFB′ end together, as P passes from A to D; hence

Area ABB′P : sectorial area A′B′P
:: area B′PD : segment PFB′ :: 2 CO + BO : BO.

Cor. 1. $Pn = bi$;

and PP′ : bi :: PP′ : Pn :: 2 CO : BO.

Cor. 2. Area B′FPD : seg. B′FP :: 2 CO : BO. This can be proved independently, in the same manner as the corresponding relation for the cycloid, Cor. 1, Prop. VIII., Cycloid.*

SCHOL.—The above affords a new demonstration of the property proved in Prop. IV. Cor. 2 also, if independently established, gives another proof of the area.

* The proof may be effected in two ways, both analogous to the proof for cycloid,—viz., either by making the sides of elements such as ff' and FF′ concentric with BD, or by making them perpendicular to A′B′. In the former case we find the decrement of space PFB′D = P′q B′b, that is (ultimately) = P′ n B′b, and the rest of the proof is like the above. In the latter case we find the decrement of PFB′D = a rect under C′c' (c' centre of b F′P′) and projection of B′q on A′B′; and decrement PFB′ = triangle PB′b = ½ rect. under B′b and projection of B′q on A′B′; therefore

decrement of PFB′D : decrement of PFB′ :: 2C′c' : B′b;
that is, area PFB′D : area PFB′ :: 2C′c' : B′b :: 2CO : BO.

PROP. IX.—*If P (figs.* 28 *and* 29) *be a point on the epicycloidal or hypocycloidal arc APD, and OP, OA, OD be joined, and PM be drawn perp. to A′B′, the diametral of the generating circle A′PB′ through P, then*

Area APO : rect. OC (arc A′P+PM) :: OA : 2 BO.

The area APO = sector OBB′ + △OB′P ± area ABB′P (taking the upper sign for the epicycloid, and the lower sign for the hypocycloid, throughout);

FIG. 29.

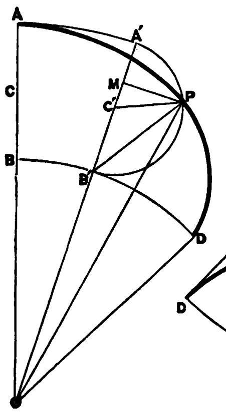
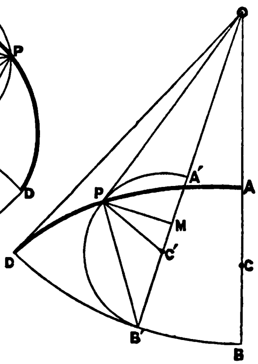


FIG. 30.



therefore,

$$2 \text{ area APO} = \text{OB . arc BB}' + \text{OB}' \text{. PM}$$
$$\pm \frac{2\,\text{CO} + \text{BO}}{\text{BO}} (2 \text{ area A}'\text{B}'\text{P});$$
$$= \text{OB arc A}'\text{P} + \text{OB . PM}$$
$$\pm \frac{2\,\text{CO}+\text{BO}}{\text{BO}} . (\text{AC . arc A}'\text{P} + \text{AC . PM});$$
$$= (\text{OB} \pm \text{AC}) \text{ arc A}'\text{P} + (\text{OB} \pm \text{AC}) \text{ PM}$$

$$\pm \frac{2\,CO.AC}{BO} \text{ arc } A'P \pm \frac{2\,CO.AC}{BO} . PM;$$

$$= \left(CO \pm \frac{2\,CO.AC}{BO}\right) \text{arc } A'P$$

$$\pm \left(CO \pm \frac{2\,CO.AC}{BO}\right) PM;$$

$$= CO\left(\frac{BO \pm 2\,AC}{BO}\right)(\text{arc } A'P + PM);$$

$$= \frac{OA}{BO} . CO . (\text{arc } A'P + PM); \qquad \text{therefore,}$$

area APO : rect. OC (arc A′P + PM) :: AO : 2 BO.

Cor. 1. Area APDO : rect. OC, BD :: AO : 2 BO.

Cor. 2. Area DPO : rect. OC (arc B′P − PM) :: AO : 2 BO.

Cor. 3. APDO : sect. OBD :: AO . CO : $(BO)^2$.

Cor. 4. APDO : sect. OC *c* (figs. 25 and 26) :: sect. OA *a* : APDO :: AO : CO.

Note.—The above demonstration might have been readily made geometrical in form as it is in substance; but it would have been more cumbrous and not so easily followed. The student should, however, note the following independent demonstration (which occurred to me after the above had been corrected for press):—

In figs. 27, 28, p. 52, let OP intersect P′*l* in *h*; draw PH perp. to *s h* and PM′ perp. to A′B′. Then the ultimate increment of area APO = ½ rect. OP, *h* P′; while the corresponding increment of rect. OC (arc A′P + PM′) = rect. OC, inct. of (arc A′P + PM′). Therefore, former inct. : latter inct. :: ½ OP . *h* P′ : OC, inct. of (arc A′P + PM′).

Now,	*h* P′ : P*q* :: *s* H : C′P
and	P*q* : inct. (arc A′P + PM′) :: C′P : B′M′
∴ *ex æq.*,	*h* P′ : inct. (arc A′P + PM′) :: *s* H : B′M′
But	OP : OB′ :: *s* C′ : C′B′
∴	OP . *h* P′ : OB′ . inct. (arc A′P + PM′) :: *s* H . *s* C′ : B′M′ . C′B′
	:: *s* P . *s* N : B′P . B′N (since C′, P′, H, P, N, lie on a ⊙).

Wherefore, increasing OB′ in 2nd term to OC, and B′P in 4th to *s* P (or both in the same ratio, since triangles *s* B′C, PB′O are similar),

$$\begin{aligned} OP \,.\, hP' : OC \,.\, \text{inct. (arc } A'P + PM') &:: sP \,.\, sN : sP \,.\, B'N \\ &:: sN : B'N :: C'B' + \tfrac{1}{2}B'O : \tfrac{1}{2}B'O \\ &:: AO : BO; \end{aligned}$$

or, inct. area APO : inct. rect. OC (arc A′P + PM′) :: AO : 2 BO

$\therefore$ Area APO : rect. OC (arc A′P + PM′) :: AO : 2 BO.

Cors. 1, 2, 3, and 4, follow as before.

SCHOL.—We have here an independent demonstration of the area of the epicycloid and hypocycloid, since

Area APDO = area ABD ± area APDB.

PROP. X.—*With the same construction as in former Propositions (figs. 31 and 32),*

Area APA′ : segment A′ h P :: AO : BO.

Let *a* P′B be the position of the tracing circle when tracing point is at P′ near to P; *a c b* O diametral. Draw *q* P′ concentric with BD and AE, join A′ *q*, *a* P′, A′P′; also producing A′*a* to T and *q* P to *t*, draw P′T and A′ *t* perp. to A′T and P *t* respectively.

Then A′PP′*a*, the increment of AA′P = ½ rect. under A′*a*, P′T ultimately; and A′P *q*, the increment of segment A′*h* P = ½ rect. under P *q*, A′*t* ultimately. But ultimately the right-angled triangles A′*t q* and P′T *a* are equal in all respects (since A′*q* = *a* P′, and angle A′*q t* = angle at circumference on segment A′*q* = angle at circumference on segment *a* P′ = angle P′*a t*) therefore P′*t* = A′T, and

increment of AA′P : increment of segment A′*h* P

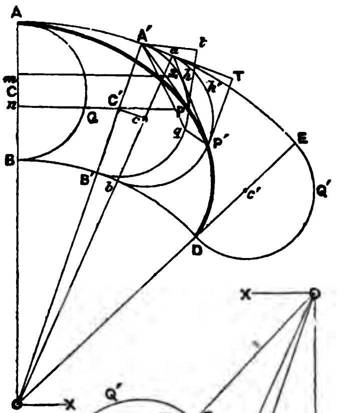
:: A′*a* : P *q* (= B′*b*) :: AO : BO;

or since these areas begin together,

area AA′P : segment A′*h*P :: AO : BO.

Cor. Area AA′*h*P : seg. A′*h*P :: 2 CO : BO (:: AO + BO : BO). This may readily be established independently—by showing that ultimately area A′*ah*′P′P*h* : △A′P*q* :: 2 C′*c* : B′*b* :: 2 CO : BO.*

Fig. 31.



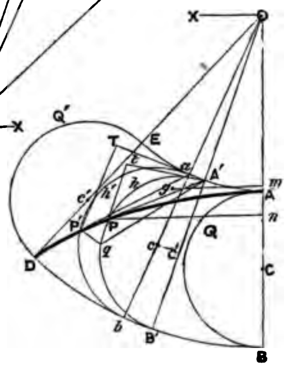


Fig. 32.

Schol.—Since it follows that

area APDE : ½ gen. ⊙ :: AO : BO,

* A line from *h*′, perp. to A′B′, to meet A′*h*P = C′*c*; and a line from P, perp. to A′B′, to meet A′*q* = P*q* = B′*b*.

we have here another demonstration of the area of APDE. Further, since

$\frac{1}{2}$ gen. ⊙ : area ABDE :: $\frac{1}{2}$ CB, BD : 2 . CB, arc C *c′*
:: BD : 4 C *c′* :: BO : 4 CO,

it follows, *ex æquali*, that

area APDE : area ABDE :: AO : 4 CO.

Yet again, from the corollary we see that

Area APDQ′E : $\frac{1}{2}$ generating circle :: 2 CO : BO
:: $\frac{1}{2}$ area ABDE : $\frac{1}{2}$ generating circle,
∴ area APDQ′E = $\frac{1}{2}$ area ABDE,

which is the relation established in Prop. IV. If established independently, as explained above, this leads to another demonstration of the area.

Note.—Arc APD divides the area AQBDQ′E into two equal areas.

Prop. XI.—*If PB′o (figs. 33, 34) is the radius of curvature at P, and PB′ the normal, then*

Po : PB′ :: 2 CO : AO.

With so much of the construction of figs. 27, 28 as is shown in figs. 33, 34, produce P′ *b* to meet PB′ produced in *o′*, then *o* is the limiting position of *o′* as P′ moves up to P. Now since PP′ is ultimately parallel to *b i*, therefore ultimately

P *o′* : B′*o′* :: PP′ : *b i* :: 2 CO : BO

(Prop. VIII., Cor. 1), wherefore

P *o′* : PB′ :: 2 CO :: 2 CO − BO :: 2 CO : CO + AC,

or ultimately P *o* : PB′ :: 2 CO : AO.

Cor. 1. For the epicycloid,

$$\text{radius of curvature} = \frac{2\,(F + R)}{F + 2\,R} \cdot \text{normal};$$

and for the hypocycloid,

$$\text{radius of curvature} = \frac{2\,(F - R)}{F - 2\,R} \cdot \text{normal}.$$

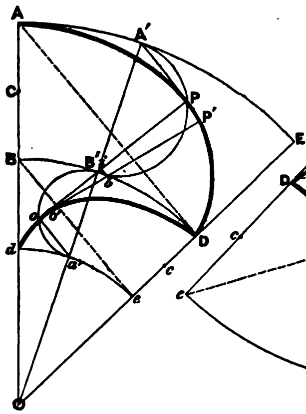


Fig. 33.

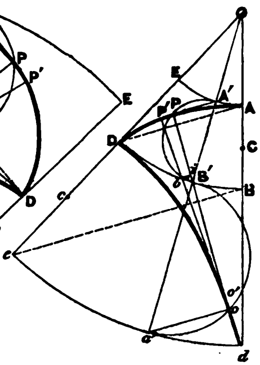


Fig. 34.

Cor. 2. PB′ : B′*o* :: 2 CO – BO : BO :: AO : BO
:: F + 2 R : F for the epicycloid;
:: F – 2 R : F for the hypocycloid.

Schol.—We see from Cor. 1 that when F = 2 R the radius of curvature of the hypocycloid is infinite, or the hypocycloid degenerates into a straight line. See further the Appendix to this section, pp. 66 to 68.

PROP. XII.—*The evolute of the epicycloid or hypocycloid APD (figs. 33 and 34) is a similar epicycloid or hypocycloid, d o D, having its vertex at D, and its cusp d so placed on OA (produced if necessary), that*

$$dB : BA :: OB : OA;$$

or, which is the same thing, $Od : OB :: OB : OA$.

Join OD and describe the arc $d a' e$ with O as centre and radius O d. Produce A′B′ to O, cutting (fig. 33) or meeting (fig. 34) $d e$ in a', and join $o a'$. Then

$$B'A' : a'B' :: BA : dB :: AO : BO :: PB' : B'o$$
(Prop. XI., Cor. 2);

that is, the sides about the equal angles $oB'a'$, PB′A′ are proportionals; therefore the triangles $o'B'a'$, PB′A′ are similar, and the angle $a'oB'$ (= the angle B′PA′) is a right angle. Hence a circle described on $B'a'$ as diameter will pass through o. Again the angles A′B′P and $a'B'o$ at the circumferences of the circles A′B′P and $a'B'o$ being equal,

$$\text{arc } o a' : \text{arc } PA' \; (=\text{arc } BB') :: a'B' : B'A' :: OA : OB$$
$$:: OB : Od :: \text{arc } d a' : \text{arc } BB'.$$

Therefore, arc $o a'$ = arc $d a'$, and o is a point on an epicycloid (fig. 33) or hypocycloid (fig. 34) having $d e$ for base, its cusp at d and $B'o a'$ as tracing circle. Since $de : BD :: od : OB :: Bd : AB$

$$:: \text{arc } B'oa' : \text{arc } A'PB' \; (=BD);$$

therefore $de = \text{arc } B'oa'$;

so that eD is the axis and D the vertex of the epicycloid or hypocycloid $d o$ D.

Cor. If c is the bisection of e D,

$$o\text{P} : o\text{B}' :: 2\text{CO} : \text{AO} :: 2c\text{O} : \text{DO};$$

therefore (Prop. VI.),

$$o\text{P} = \text{arc } o\text{D}.$$

If, then, a string coinciding with the arc doD and fastened at d, be unwrapped from this arc, its extremity will always lie on the arc APB, which may thus be traced out as the involute of the arc doD.

Schol.—A convenient construction for finding the base, &c., of the evolute doD is indicated by the dotted lines in the figures: thus, join AD, then Be parallel to AD gives Oe (on OE, produced if necessary), the radius of the base ed.

Prop. XIII.—*If doD (figs. 33, 34) be the evolute of the epicycloid or hypocycloid APD, and $oB'P$, the radius of curvature at any point P on APD, cut the base BD in B′, then*

area $APB'B$: area $dBB'o$
:: rect. under $AO\,(AO + 2BO)$: square on BO.

If P′o' be a contiguous radius of curvature cutting BD in b, and bi is drawn perp. to oB′P, then in the limit

$$o\text{P} : oi :: 2\text{CO} : \text{BO};$$

therefore

$$\text{ult. area P}o\text{P}' : \text{ult. area } oib :: 4(\text{CO})^2 : (\text{BO})^2,$$

whence, ultimately

$$\text{area PB}'b\text{P} : \text{area } o\text{B}'b :: 4(\text{CO})^2 - (\text{BO})^2 : (\text{BO})^2$$
$$:: \text{rect. } (2\text{CO} - \text{BO})\,(2\text{CO} + \text{BO}) : \text{sq. on BO}$$
$$:: \text{rect. AO}\,(\text{AO} + 2\text{BO}) : \text{sq. on BO}.$$

But the areas $PB'bP'$ and $oB'b$ are the elementary increments of the areas $APB'B$ and $dBB'o$, which begin together. Therefore,

area $APB'B$: area $dBB'o$

:: rect. under $OA\,(AO + 2BO)$: sq. on BO.

Cor. 1. Area $APDB$: area $doDB$

:: area $PB'D$: area $oB'D$

:: rect. under $OA\,(AO + 2BO)$: sq. on BO.

Cor. 2. Since

area $doDB$: area $APDE$ :: $(BO)^2 : (AO)^2$,

it follows (*ex æq.*) that

area $APDB$: area $APDE$ :: $AO\,(AO+2BO) : (AO)^2$

:: $AO + 2BO : AO$

:: $(3F + 2R) : (F + 2R)$ for the epicycloid

:: $(3F - 2R) : (F - 2R)$ for the hypocycloid.

Schol.—It follows from Cor. 2 that

Area $APDE$: area $ABDE$ :: $AO : 2(AO + BO)$

:: $AO : 4CO$,

which is one of the relations established in the scholium on Prop. X. Hence we have in Prop. XIII. another method of demonstrating the area of the epicycloid and the hypocycloid.

Appendix to Second Section.

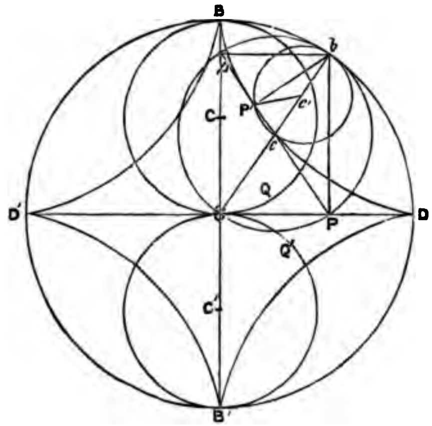
There are many forms, both of the epicycloid and of the hypocycloid, which possess interesting properties. For the most part the general properties established in the preceding section will suffice to enable the student to deduce the properties of special forms of these curves. For this reason, and also because of the requirements of space, I shall only touch briefly here on a few points in connection with the forms assumed by epicycloids and hypocycloids for certain values ot the radii of the fixed and rolling circles. I do not make set propositions of these points, but present them in such sequence as appears most convenient and suitable.

THE STRAIGHT HYPOCYCLOID.

The hypocycloid becomes a straight line when the diameter of the rolling circle is equal to the radius of the fixed circle.

This in reality has been already demonstrated, because we have seen in the scholium to Prop. XI. that the radius of curvature of the hypocycloid becomes infinite when $F = 2R$. Also the relation is involved in the demonstration of Prop. I. For when the two rolling circles (figs. 21 and 22) are equal, each having its diameter equal to the radius of the fixed circle, the curve traced out by each must be a straight line. Thus,—let BOB′ (fig. 35) be the diameter of the fixed circle, and its halves BO, OB′, the diameters of the two equal rolling circles; then by what is shown in Prop. I. of this section the point O on BQO will trace out the same curve as the point O on B′Q′O, but since the circles BQO and B′Q′O are equal, this curve, regarded

Fig. 35.



as traced out by O on BQO, must bear the same relation in all respects to the axis OB that the same curve regarded as traced out by O on B′Q′O bears to the axis OB′, and the only line which can possibly fulfil this condition is the diameter D′OD at right angles to BOB′. This then must be the path traced out by the point O in each case.

Let us proceed, however, to an independent demonstration.

When the circle OQB has rolled to the position O*p b* (O *c b* its diameter), let *p* be the point which had been at B, so that drawing the diameter *p c* P, P is the position of the tracing point. Then the arc *p b* is equal to the arc B *b*; and therefore, since F = 2R, the angle BO *b* is equal to half the angle *b c p*, that is to the angle *b*P*p*: but BO*p* and O *b* P are alternate angles; wherefore *b* P is parallel to BO; and OP, which (OP*b* being a semicircle) is perpendicular to *b* P, is perpendicular to BO. P therefore lies on the diameter D′OD at right angles to BOB′; which was to be shown.

Cor. The point *p* lies on OB (the angles *c* O *p* and *c* OB being each equal to half the angle *b c p*).

USEFUL GENERAL PROPOSITION.

The following property is worth noticing. It is true of course for the cycloid also.

A diameter of the generating circle of an epicycloid or hypocycloid constantly touches the epicycloid or hypocycloid which would be generated by a circle of half the diameter, alternate cusps of this epicycloid or hypocycloid falling on successive cusps of the former.

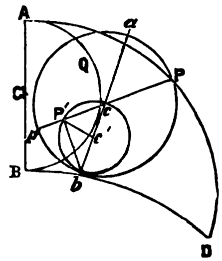
It will suffice to demonstrate the property for the epicycloid.

Let AQB (fig. 36) be the generating circle of an epicycloid when the tracing point is at A, the vertex of the epicycloid. When the circle has rolled to position *a* P *b*, let *p c* P be the position of the diameter which had originally been in position ACB. Draw *b* P′ perpendicular to *p* P, and on *c b* describe the semicircle *c*P′*b*, having *c*′ as its centre and passing through

P′ because *c* P′*b* is a right angle. Then because the angle P′*c*′*b* = twice the angle P′*c b*, and *c*′*b* = half *c b*, the arc P′ *b* = arc *p b* = arc B *b*. Wherefore P′ is a point on an epicycloid traced out by the rolling of *c* P′*b* on BD, B being a cusp. D is the next cusp, because the base of the smaller epicyloid being equal to the circumference of generating circle *c* P′*b* = circumference of semicircle AQB = BD. Also *p* P′*c* P is the tangent at P′ by what has been already shown respecting the tangent to an epicycloidal arc.

The student will find it a useful exercise to prove the property established in Prop. I. of the present

Fig. 36. (Draw in epicycloid on base BD, touching *cp* in P′.)

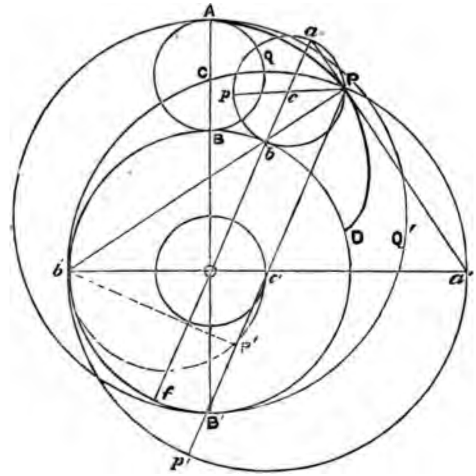


section in the manner illustrated by figs. 37 and 38, where APB is the arc traced out by point A on each of the circles AQB, AQ′B′. The construction and proof for the epicycloid (fig. 37) run as follows:—ABOB′ being a common diameter of all three circles at the beginning of the rolling motion, let P be the position of the tracing point of the smaller rolling circle when its centre is at *c*. Draw the diametral line *a c b* O*f*, and the diameter P *c p*. Join P *b* and produce to meet the circle BDB′ in b', produce b'O to c', taking Oc' = R, so that $b'c'$ = F + R, and join Pc'; then since

$$b\text{P} : b\,b' :: a\,b : b\,f :: \text{R} : \text{F} :: \text{O}\,c' : \text{O}\,b'$$

Pc' is parallel to Ob, and the triangle $b'c'$P, like triangle b'Ob, is isosceles ($c'b' = c'$P). With centre c' and radius c'P or $c'b'$ (= F + R) describe the circle

FIG. 37.



b'Pa'; produce Pc' to meet this circle in p'. Now,

$$\text{arc B}\,b = \text{arc}\ b\,p;$$

$$\therefore \text{angle}\ p\,c\,b : \text{angle BO}\,b :: \text{F} : \text{R};$$

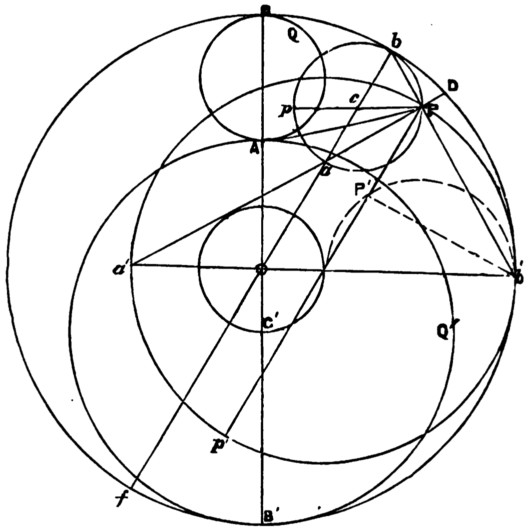
but $\quad$ angle $p\,c\,b$ = 2 angle $c\,b$ P = 2 angle O$\,b\,b'$
$\qquad$ = angle b'Of

$\therefore$ angle b'Of (= angle $b'c'p'$) : angle BOb :: F : R;

and $\angle\ b'c'p' : \angle\ b'\text{OB}' :: \text{F} : \text{F}+\text{R} :: \text{B}'\text{O} : b'c'$.

Whence it follows that arc $b'p'$ = arc $b'B'$; and P is, therefore, a point on the curve traced out by A (on the circle AQ′B′), rolling so that its inside touches the outside of the fixed circle BDB′, ABOB′ being originally diametral. The same curve APB is traced out, then, by the point A on each of the circles AQB and AQ′B′.

FIG. 38.



Cor. If we produce b'O to meet the circle b'P a' in a', and join P a', then a P and P a' are in the same straight line.

The construction and proof for the hypocycloid (fig. 38) are similar, writing only − R for + R.

The curve enveloped by a diameter of the generating circle of an epicycloid produced by the rolling of a circle larger than the fixed circle, and touching this circle internally, will be an epicycloid if the radius of the rolling circle exceeds the diameter of the fixed circle; but if the rolling circle has a radius less than the diameter of the fixed circle, the curve enveloped by a diameter of the rolling circle will be a hypocycloid. The proof for both cases is easily derived from the demonstration in pp. 68, 69, the dotted line and circle of fig. 37 showing the nature of the construction.

The curve enveloped by a diameter of the generating circle of a hypocycloid is shown by reasoning similar to that in pp. 68, 69, to be the hypocycloid traced out by a generating circle of half the diameter, alternate cusps of the smaller hypocycloid agreeing with successive cusps of the larger. The dotted line and circle in fig. 38 indicate the requisite construction when the rolling circle has a diameter greater than F.

THE FOUR-POINTED HYPOCYCLOID.

It follows from the property indicated in the preceding paragraph that the diameter OB of the rolling circle BQO (fig. 35) constantly touches a hypocycloid having four cusps, at B, D, B′, and D′. As the extremities p and P of the diameter lie always on BB′ and D′D respectively, we have in this result the solution of the problem '*to determine the envelope of a finite straight line pcP, whose extremities slide along the fixed straight lines* BOB′ *and* DOD′ *at right angles to each other.*' The direct proof is simple, however. Thus let p P be the straight line in any position. Complete the rectangle $O\,p\,b\,P$, whose diagonals $O\,b$ and $p\,P$ are equal and bisect each other in c. With centre O and radius $O\,b$, describe the circle $B\,b\,DB'$, and draw $b\,P'$ perpendicular to $p\,P$. Then a circle on $c\,b$, as diameter, passes through P'. Let c' be the centre of this circle; then $c'b = \frac{1}{4}O\,b$: but $\angle\ b\,c'P' = 2\ \angle\ b\,c\,P' = 4\ \angle\ b\,OB$; therefore arc $b\,P' =$ arc $p\,B$. Hence P' is a point on the hypocycloid traced out by circle $b\,P'\,c$ rolling on the inside of the circle BDB′, the cusps lying at B, D, B′, and D′.*

THE CARDIOID.

The cardioid, or epicycloid traced by a point on the circumference of a circle rolling on an equal circle, has some interesting properties. Here, however, space cannot be found for more than a few words about the chief characteristics which distinguish this curve.

Let AQB (fig. 39) be the rolling circle, $B\,b\,S$ the fixed circle, A the tracing point when at the vertex, so that ACBOS is diametral. Now let $a\,P\,b$ be another

* The four-pointed hypocycloid BDB′D′ is interesting in many respects. It bears the same relation to the evolute of the ellipse that the circle bears to the ellipse. Its equation may readily be obtained. Thus, let DOD′ be axis of x, BOB′ axis of y, and x, y co-ordinates of P′; put $BO\,b = \theta$; $OB = a$; then,

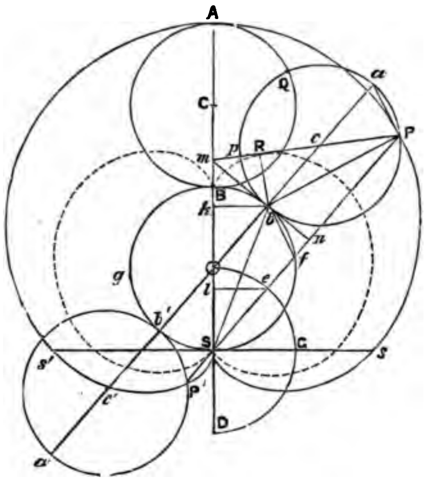
$$x = p\,P' \sin\theta = p\,b\ \sin^2\theta = a\sin^3\theta;$$
$$y = PP' \cos\theta = b\,p\ \cos^2\theta = a\cos^3\theta;$$
$$\therefore x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}, \text{ the required equation.}$$

The equation to the evolute of the ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$ is

$$\left(\frac{x}{a'}\right)^{\frac{3}{2}} + \left(\frac{y}{b'}\right)^{\frac{3}{2}} = 1\text{; where } a' = a - \frac{b^2}{a},\text{ and } b' = \frac{a^2}{b} - b.$$

position of the rolling circle, $a\,c\,b\,O$ diametral. Draw the common tangent $b\,m$, meeting ABS in m; draw also $m\,p\,c\,P$ through c, the centre of circle $a\,P\,b$; join PS, cutting $m\,b$ in n; $b\,k$ perpendicular to AS; and join $b\,P$, $b\,S$. Then, since $c\,b = b\,O$, and $b\,m$ is perp. to $c\,O$, triangle $c\,b\,m =$ triangle $O\,b\,m$ in all respects; and arc $b\,p =$ arc $b\,B$. Wherefore, P is the position of the tracing point; $P\,a$ is the tangent to the cardioid

FIG. 39. (Produce $P\,b$ to meet $\odot\,b\,BS$ in g; join $p\,b$, $b'\,P'$.)



at P, $P\,b$ is the normal. The curve will manifestly have the shape indicated in the figure, the only cusp being at S, and the tracing point returning to A after tracing the other half SP′A. AS divides the curve symmetrically.

Note first that $P\,n = n\,S$; or the cardioid is similar to the curve obtained by drawing perpendiculars from

S (as S n) to tangents at all points of a circle B b S. We might then obtain the cardioid P′APS, by drawing a circle on AS as diameter, and from S letting perpendiculars fall on tangents to this circle. This property is expressed by saying that the *pedal of a circle with respect to a fixed point on its circumference is a cardioid.*

Secondly, $\angle\ n\text{P}b = \text{alt.}\ \angle\ \text{P}bc = \angle\ b\text{P}m = \angle\ b\text{S}m$; hence $\text{S}n = \text{S}k$. So that if we draw any line S n from S, and from b, in which the bisector of BS n meets the circle on SB as diameter, draw $b\,n$ perpendicular to S n, the locus of n is a cardioid. [The larger cardioid, P′APS, would be similarly described by producing S n and S b, and from point in which Sb meets circle on AS as diameter, letting fall perpendicular on S n (meeting Sn in P).]

Or, thirdly, we may obtain a cardioid by taking any finite line as SB, drawing B b square to bisector of any angle BS n, and from b drawing $b\,n$ square to S n: the locus of n will be a cardioid.

Fourthly, draw circle OGD about S as centre cutting S n in e, and draw $e\,l$ perp. to SB; then $\text{S}n = \text{S}k = \text{SO} + \text{O}k = \text{SD} + \text{S}l$ (because S e is parallel and equal to O b) $= \text{D}l$. Thus the cardioid may be obtained by drawing radii as S e to a fixed circle OGD, and on S e, produced if necessary, taking n so that $\text{S}n = \text{D}l$. This is the usual definition of the cardioid.

Fifthly, let P n S cut circle B b S in f. Then producing P b to meet circle b BS in g, we have $b\text{P} = bg$, and rectangle $\text{P}b\,.\,\text{P}g\ (= 2\text{P}b^2) =$ rectangle Pf, PS

= 2 rect. Pf . Pn. Hence Pb^2 = Pn . Pf, and Pbf is a right angle. Wherefore pbf is a straight line, and (Pb bisecting angle pPf) Pf = Pp = SB. Hence the cardioid P′APS may be obtained by drawing straight lines as Sf to circumference of circle BfS, and taking on Sf produced Pf = BS. (The cardioid is therefore a *limaçon.*)*

Cor. If we draw s'Ss tangent to circle BfS at S, and take Ss = Ss' = BS, then s, s' are points on the cardioid. We see that $s's$ = SA; and it is easily seen that if P′SP is a straight line through S, PP′ = SA. For, according to the definition just obtained, we should have P′ on a point on the curve if fSP′ = BS = fP; therefore P′SP = SA. It may be well, however, to show how this can be directly proved when the cardioid is regarded as an epicycloid. For this purpose we have only to notice that if on abO produced we set centre of generating circle as at c', then b'P′, the arc of the generating circle to tracing point P′, must equal b'S, wherefore P′S is parallel to c'O, and in same straight line with PS. But since PSP′ is parallel to cc' joining centres of equal circles aPb, b'P′a', aP is parallel to b'P′, and therefore PP′ = $b'a$ = 2ba = SA. This property gives a method of tracing out the cardioid mechanically. For if there be a circular groove as BfS, and we take a ruler of length SA (twice diameter of groove), having a vertical pencil point at each extremity

* The *limaçon* is the curve obtained by drawing *radii vectores* to a circle from a point on its circumference, and producing and reducing all of them by a constant length.

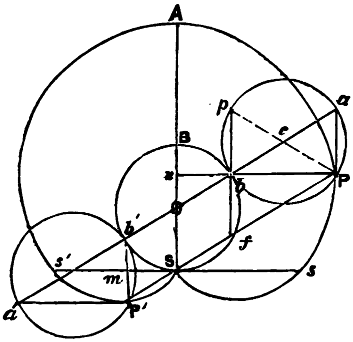
and a point at its middle point moving in the groove, while the rod itself always passes through S (either through a small ring there or by having a projecting point at S and a groove along the rod), the pencils at the extremities of the rod will trace out the cardioid. While one pencil moves over AP*s* the other will move over SP′*s*′, and while the former passes on from *s* to S, the latter passes on from *s*′ to A, completing the tracing of the curve.

The evolute of the cardioid A *s* S *s*′ is a cardioid S *r* O, having its vertex at S, cusp at *d*, on OB, such that $O\,d = \frac{1}{3}\,OB$, and linear dimensions equal to one-third those of the cardioid A *s s*′.

S, the cusp of the cardioid, is also called the *focus*. Since P *b* is the normal at P and angle SP *b* = angle *b* P *m*, we perceive that if S be a point of light, and the arc of the cardioid reflect the rays, P *m* will be the course of the ray reflected from P. Hence the *caustic* or envelope of the reflected rays will be the curve constantly touched by the diameter P *p* in the tracing out of the cardioid. This curve, as shown at pp. 68, 69, will be the epicycloid traced out by a circle whose diameter = CB, and which has S as one of its cusps. The other cusp will be at B, and the curve will have the position shown by the dotted curve BRS and its companion lobe in fig. 39.

Let us now determine how far the cardioid ranges in distance from the diameter AS, and beyond *s s*′. We note that (i.) when P (fig. 39) is at the greatest possible distance from AS, the tangent P *a* must be parallel to AS; and (ii.) when P′ is at its greatest distance from *s′* S *s*, the tangent at P′ must be parallel to *s s′*, and therefore P′*b′*, the normal, must be parallel to SA. Wherefore, since P′*b′* has been shown to be parallel to P *a*, we see that when P is at its greatest distance from SA, P′ is at its greatest distance from *s s′*. Now, when P *a* is parallel to AS, so also is *p b f*, and as the arc *b f* = arc B *b*, the position of *b f* is at once assigned: for if a chord *b f* (fig. 40) is parallel to

FIG. 40.



BS, arc B *b* = arc S*f*, and since arc *b f* = arc B *b* = S*f*, we have B *b* = $\frac{1}{3}$rd the semi-circumference B *b* S, and the angle BS*f* = two-thirds of a right angle.

∴ S*f* = SO = SP′; and SP = 3SO. Also,

$$\text{P}\,n = 3\,b\,n = \frac{3\ \sqrt{3}}{2}\,\text{SO}; \text{ and P}'n = \frac{\text{SO}}{2}.$$

$$\text{O}n = \frac{\text{SO}}{2}; \text{ and S}m = \frac{\sqrt{3}}{2}\,\text{SO}.$$

It follows from the parallelism of the tangent P *a* and the normal P′*b′*, that when the cardioid is being described by the continuous motion above indicated, one end of the rod is always moving in a direction at right angles to that of the other end of the rod. Thus the tangents and normals at P and P′ (fig. 39) intersect on the circle which has PP′ for its diameter. The normals also intersect on the circle B *g* *b*′ (at *g*), and the tangents on the circle having centre O and radius OA.

Cor. The curve cuts *s* *s*′ at equal angles, each equal to half a right angle.

THE BICUSPID EPICYCLOID.

The epicycloid with two cusps (the dotted curve of fig. 39, which, from its shape, we may call the *nephroid*) presents also many interesting relations. I merely indicate, however, in a few words the chief points to be noticed at the outset of an inquiry into the relations of the bicuspid epicycloid.

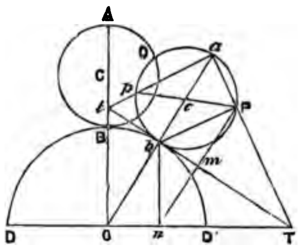
Let P (fig. 41) be a point on the epicycloidal arc traced by the rolling of AQB on the circle DBD′, whose radius BO = AB.*

Let *a* P *b* be position of rolling circle through P. Draw common tangent *b* *t*, meeting OA in *t*; and join *t* *a*, cutting *a* P *b* in *p*. Then, since O *b* = *b* *a*, angle *t* O *b* = angle *t* *a* O, and arc *p* *b* = arc B *b*; wherefore *p* *c* P is a diameter of circle AP *b*. Angle *c* *a* P = compt. of *c* *a* *p* = compt. of *t* O *b* = angle *b* OD′. Hence TO *a* is isosceles, and *t* *b* T is a straight line. Draw

* The curve has been omitted from fig. 41. The student should trace it in pencil from the cusp D through A and P (touching PT) to D′—forming a branch like either half of the dotted curve of fig. 39.

b n perpendicular to OT, and join *n* P, *b* P, *b p*; then triangle *p b* P = triangle O *n b* in all respects, *b* P = *b n*, and P *m* = *m n*. Wherefore the bicuspid epicycloid may be described thus: draw from any point *b* on

Fig. 41. (Join *b p*.)

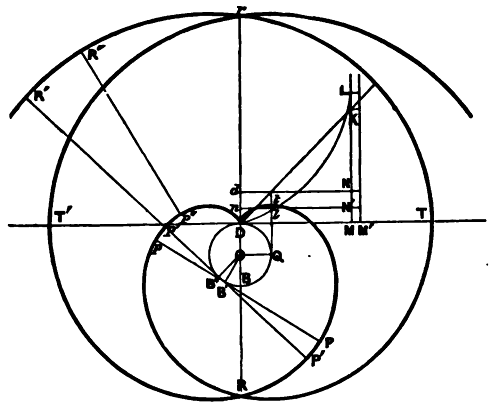


circle DBD′, *b n* perpendicular to fixed diameter, DOD′, and *n m* perpendicular to tangent at *b*; then if *n m* is produced to P so that *m* P = *m n* the locus of P is a bicuspid epicycloid.

THE INVOLUTE OF THE CIRCLE REGARDED AS AN EPICYCLOID.

The curve traced by a point on a straight line which rolls on a circle in the same plane may be regarded as an epicycloid whose generating circle has an infinite radius. The curve is the involute of the circle. Thus, let DQB (fig. 42) be a circle, T′DT a tangent at D, and let this tangent roll without sliding over the circle DQB (DOB a diameter), the point D tracing out the curve DP. Then when the tangent has the position PB′*p*, having rolled over the arc DQB′ once only, B′P having been in contact with every point of the arc B′QD is equal in length to this arc. Therefore the point P lies on that involute of the circle DQB′ which commences at the point D. But T′DT may be regarded as part of a circle of infinite radius touching the circle DQB′ in D, and the arc DPR therefore as an epicycloid. In fact this arc is the extreme case of the epicycloid when the radius of the rolling circle is indefinitely enlarged, precisely as the right cycloid is the extreme case when the radius of the fixed circle is indefinitely enlarged. The part of the curve near to DQB manifestly has the shape shown in the figure, D being the cusp. The branches of the curve extend without limit outwards. It is obvious that if the line B′P be produced to meet the next whorl of DPR (not the curve D p R), the portion of this line intercepted between P and that whorl will be equal to the circumference of the circle DQB. Again, if PB′ produced meet the branch D p R in p, PB′p is also equal to the circumference of DQB′; for B′P = arc B′QD, and B′p = arc B′B″D. The straight line r DR, perp. to T′DT, passes through all the points of intersection of the two branches, for the curve must necessarily be symmetrical on either side of OD from the way in which it is traced out. Q t, the tangent parallel to OD, and equal to the quadrant QD, determines the greatest range of the branch D t P above DT, for the curve is perp. to Q t at t; also, if Q t be produced both ways indefinitely, its intersections with the prolongation of D t P above DT determine the greatest range of each successive whorl of that branch above DT, while its intersections with the branch DpR below DT determine the greatest range of each whorl of that branch below DT. Similarly of the tangent to DQB parallel to Qt, and of the tangents perp. to DOB. Many other relations of a similar kind exist which the student will have no difficulty in discovering for himself. Both branches manifestly approach more and

Fig. 42.



more nearly to the circular form as their distance from the centre increases; for from the manner of generation the normals to the curve touch the circle DQB, and for branches at an indefinitely great distance the dimensions of DQB are relatively evanescent, wherefore the normal at any remote point of the curve is inclined at an evanescent angle to the line joining that point with O. *Or*, a whorl of the spiral may be regarded as changing its distance from the fixed point O during one complete circuit by a distance, as $p'R'$, $p''R''$, &c. (these lines being diametral), equal to the circumference of DQB, and this distance vanishes compared with the radius vector of the spiral in its remote parts, so that the radii vectores of a single whorl, though differing by a finite quantity and therefore not absolutely equal, are yet in a ratio of equality; and in that sense the whorl corresponds with the definition of a circle.

The circle DQB is the evolute of the curve RpDPR, &c.; but we have seen (second section, Prop. XII.) that the evolute of an epicycloid is a similar epicycloid: hence we must regard the circle DQB as consisting of an infinite number of infinitely close whorls, similar to the remote whorls of the curve RpDPR.

The rectification and quadrature of the epicycloid in the preceding section manifestly fail for the involute of the circle regarded as an epicycloid. But it is easy, as follows, to compare the length of any arc DtP with the corresponding arc DQB′ of the fixed circle, and the area DtPB′Q with the area of the sector DQB′O.

ARC OF THE INVOLUTE OF THE CIRCLE.

Let PP (fig. 42) be an elementary arc, PB′, P′B″ the corresponding positions of the tracing tangent, then since OB′ is perp. to B′P and OB″ to B″P′, the angle B′OB″ = the angle PB″P′, in the limit. Hence

Arc PP′ : arc B′B″ :: B′P : OB′ :: arc DQB′ : DO.

Now in Dr take Dd = OD; and in DT take DM =

arc DQB′, and MM′ = arc B′B″. Complete the rectangles DdNM, NM′. Also draw MK = DM, perp. to DM, and complete the rectangle KM′. Then if we represent the arc B′B″ by the area NM′, the arc PP′ will be represented by the area KM′, for

$$\text{KM} : \text{NM}' :: \text{P}'\text{P} : \text{B}'\text{B}''.$$

But since KM = DM, K lies on a straight line, DK, bisecting the angle rDT; and every element of arc as PP′ has a corresponding representative element of area, as KM′, in the space KDM. Therefore the length of the arc DtP is represented ultimately by the area DMK; *or*

$$\begin{aligned}\text{Arc D}t\text{P} : \text{arc DQB}' &:: \text{area DMK} : \text{area } d\text{M} \\ &:: \tfrac{1}{2}\,\text{DM}\,.\,\text{KM} : \text{DM}\,.\,\text{OD} \\ &:: \tfrac{1}{2}\,\text{DM} : \text{OD}\ (\text{since DM} = \text{KM}) \\ &:: \tfrac{1}{2}\,\text{arc DQB}' : \text{OD} \\ &:: \text{arc DQB}' : \text{BD}.\end{aligned}$$

That is, the arc DtP is a third proportional to BD and the arc DQB′.

This is the relation required. It may conveniently be replaced by the following:—

Cor. Rect. under arc DtP and BD = square on B′P,

or, $$\text{Arc D}t\text{P} = \frac{(\text{B}'\text{P})^2}{\text{BD}}.$$

AREA BETWEEN CIRCLE, ITS INVOLUTE, AND THE NORMAL TO INVOLUTE.

Take Dn = ½ OD and complete the rectangle nM. Draw ML perp. to DM, cutting nN′ parallel to DM in N′, and take L so that

$$ML : MN' (= Dn) :: (PB')^2 (= DM^2) : (OB')^2.$$

Complete the rectangle LM′. Then by construction

$$\text{Area } N'M' = \text{triangle } OB'B'' \text{ ultimately};$$

and ultimately

$$\triangle B'PP' : \triangle OB'B'' :: (PB')^2 : (OB')^2$$
$$:: \text{rect. } LM' : \text{rect. } NM'.$$

Therefore $\quad \text{Rect. } LM' = \text{triangle } B'PP'.$

Now from the construction L is a point on a parabola D*l*L, having D as vertex and *n* as focus, or BD as parameter. Hence, every elementary triangle as B′PP′ has a corresponding representative elementary rectangle LM′. Therefore

$$\text{Area } DtPB'Q = \text{parabolic area } DlLM$$
$$= \tfrac{1}{3} \text{ rect. under } DM \, . \, LM.$$

Now $\quad DM = \text{arc } DQB';$

and by property of parabola,

$$\therefore LM \, . \, BD = (DM)^2 = (PB')^2;$$

or $\quad$ LM is a third proportional to BD and PB′,

and therefore, as shown in last page,

$$LM = \text{arc } DtP,$$

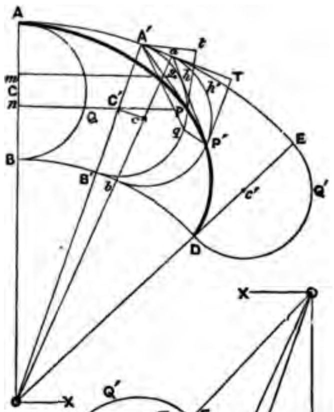
$\therefore$ area $DtPB'Q = \tfrac{1}{3}$. rect. under arcs DQB′ and D*t*P.

$$\text{Cor. Area } DtPB'Q = \tfrac{1}{3} \cdot \frac{(B'P)^3}{OD}.$$

CENTRE OF GRAVITY OF EPICYCLOIDAL AND HYPOCYCLOIDAL ARCS AND AREAS.

There is no simple geometrical method for determining the position of the centre of gravity of an epicycloidal or hypocycloidal arc or area; and therefore, strictly speaking, these problems do not belong to my subject. But it may be as well to indicate the analytical method of solving them, which has not hitherto, so far as I know, been discussed in any mathematical treatise. I shall consider the case of the epicycloid only. The solution for the hypocycloid is similar, and the result only differs in the sign of R, the radius of the rolling circle.

FIG. 43.



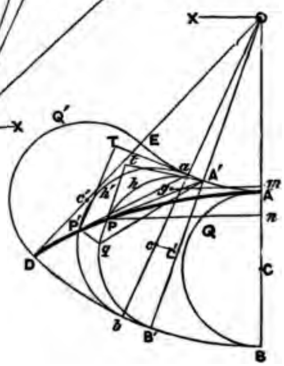


FIG. 44.

First, then, to determine the ordinates $\bar{x}$, $\bar{y}$ of the centre of gravity of the arc APD, fig. 43 (fig. 44 for the hypocycloid), O being taken as origin, OX perp. to OA as axis of x, and OA as axis of Y.

Let $\angle$ A′C′P $=\theta$; $\angle$ PC′$q = d\,\theta$. Then,

$$\text{arc PP}' = \frac{2\,R\,(F+R)}{F}\cos.\frac{\theta}{2}.\,d\,\theta.$$

Also, if P n is perp. to OA, then ultimately,

moment of arc PP′ about OA = P n . PP′

$$= \left\{(F+2\,R)\sin\frac{R}{F}\theta + 2\,R\sin\frac{\theta}{2}\cos\frac{(F+2\,R)}{2\,F}\right\}$$

$$\times\,\frac{2\,R(F+R)}{F}\cos\frac{\theta}{2}\,d\theta$$

$$= \frac{2\,R(F+R)}{F}\,m_x\,d\,\theta, \text{ say;}$$

and similarly,

moment of arc PP′ about OX = O n . PP′

$$= \left\{(F+2\,R)\cos\frac{R}{F}\theta + 2\,R\sin\frac{\theta}{2}\sin\frac{F+2\,R}{F}\right\}$$

$$\times\,\frac{2\,R(F+R)}{F}.\cos\frac{\theta}{2}\,d\,\theta$$

$$= \frac{2\,R(F+R)}{F}\,m_y\,d\,\theta, \text{ say.}$$

We have to integrate these two expressions between the limits $\theta=0$, and $\theta=\pi$, to obtain the moments of the arc APD around the axes OA and OX.

$$\text{Now}\quad \int m_x\,d\,\theta = \int\left[\frac{F+2\,R}{2}\sin\frac{F+2\,R}{2\,F}\theta\right.$$

$$\left.-\,\frac{F+R}{2}\sin\frac{F-2\,R}{2\,F}\theta + \frac{R}{2}\sin\frac{3\,F+2\,R}{2\,F}\theta\right]d\,\theta,$$

$$\therefore \int_{\pi}^{0} m_x \, d\theta = 2\,F \sin^2 \frac{F+2\,R}{4\,F}\pi$$

$$- \frac{2\,F(F+R)}{F-2\,R} \sin^2 \frac{F-2\,R}{4\,F}\pi$$

$$+ \frac{2\,FR}{3\,F+2\,R} \sin^2 \frac{3F+2R}{4\,F}\pi = M_x, \text{ say.}$$

Similarly $$\int_{\pi}^{0} m_y \, d\theta = F \sin \frac{F+2\,R}{2\,F}\pi$$

$$+ \frac{F\,(F+R)}{F-2\,R} \sin \frac{F-2\,R}{2\,F}\pi$$

$$+ \frac{FR}{3\,F+2\,R} \sin \frac{3\,F+2\,R}{2\,F}\pi = M_y, \text{ say.}$$

$$\therefore \bar{x} = \frac{2\,R\,(F+R)}{F} \cdot \frac{M_x}{\text{arc APD}} = M_x;$$

and similarly $\bar{y} = M_y$.

To determine $\overline{X}$ and $\overline{Y}$, the coordinates of the centre of gravity of the area APDE, we have,—

Area of element $A'P'a = \frac{F+2\,R}{F} R^2 \sin^2 \frac{\theta}{2}\, d\theta$;

and if g be the C. G. of this element, ultimately a triangle, $A'g = \frac{1}{3} A'P' = \frac{2\,R}{3} \sin \frac{\theta}{2}$, ultimately.

Also if gm is perp. to OA,

moment of element $A'P'a$ about $OA = gm$. area $A'P'a$,

$$= \left\{ (F+2\,R) \sin \frac{R}{F}\theta + \frac{2\,R}{3} \sin \frac{\theta}{2} \cos \frac{F+2\,R}{2\,F}\theta \right\}$$

$$\times \frac{F+2R}{F} \cdot R^2 \sin^2 \frac{\theta}{2}\, d\theta$$

$$= \frac{(F+2\,R)}{F} R^2 . \, a_x \, d\theta, \text{ say;}$$

and similarly,

moment of element A′P′a about OX = Om . area A′P′a,

$$= \left\{ (F+2R)\cos\frac{R}{F}\theta + \frac{2R}{3}\sin\frac{\theta}{2}\sin\frac{F+2R}{2F}\theta \right\}$$

$$\times \frac{F+2R}{F}. R^2 . \sin^2\frac{\theta}{2}\, d\theta$$

$$= \frac{(F+2R)R^2}{F} a_y\, d\theta, \text{ say.}$$

Now

$$\int a_x d\theta = \int \left[\frac{2F-3R}{2}\sin\frac{R}{F}\theta - \frac{F+R}{2}\sin\frac{F+R}{F}\theta\right.$$

$$\left. + \frac{3F+5R}{6}\sin\frac{F-R}{F}\theta - \frac{R}{6}\sin\frac{2F+R}{F}\theta\right] d\theta;$$

$$\therefore \int_\pi^0 a_x\, d\theta = (2F-3R)\frac{F}{R}\sin^2\frac{R}{2F}\pi - F\sin^2\frac{F+R}{2F}\pi$$

$$+ \frac{(3F+5R)F}{3(F-R)}\sin^2\frac{F-R}{2F}\pi$$

$$- \frac{FR}{3(2F+R)}\sin^2\frac{2F+R}{2F}\pi = A_x, \text{ say.}$$

Similarly

$$\int_\pi^0 a_y\, d\theta = (2F-3R)\frac{F}{2R}\sin\frac{R}{F}\pi - \frac{F}{2}\sin\frac{F+R}{F}\pi$$

$$+ \frac{(3F+5R)F}{6(F-R)}\sin\frac{F-R}{F}\pi$$

$$- \frac{FR}{6(2F+R)}\sin\frac{2F+R}{F}\pi = A_y, \text{ say.}$$

$$\therefore \overline{X} = \frac{(F+2R)}{F}R^2 . \frac{A_x}{\text{area APDE}} = \frac{2A_x}{\pi},$$

$$\text{since area APDE} = \frac{F+2R}{2F}\pi R^2;$$

and similarly $\overline{Y} = \dfrac{2A_y}{\pi}$.

It is easy to obtain in a similar manner $\overline{X}'$ and $\overline{Y}'$, the coordinates of the centre of gravity of the area APDB, though the expressions are rather more cumbrous. We take such elementary areas as PP′b B′ in fig. 27 (fig. 28 for hypocycloid), and find,

$$\text{Moment of element about OA} = \left[(3\,F+2R)\sin\frac{R}{F}\theta + \frac{5\,F+4\,R}{F}\cdot\cos R\,\frac{\theta}{2}\,\sin\frac{F+2\,R}{2\,F}\,\theta\right]R^2\cos^2\frac{\theta}{2}\,d\theta.$$

$$\text{Moment of element about OX} = \left[(3\,F+2\,R)\,\cos\frac{R}{F}\theta + \frac{5\,F+4\,R}{R}\,R\,\cos\frac{\theta}{2}\cos\frac{F+2\,R}{2\,F}\,\theta\right]R^2\cos\frac{\theta}{2}\,d\,\theta.$$

These expressions can be easily integrated. It will, however, be more convenient to proceed as follows: Moment of area ABDE about OA

$$= \tfrac{2}{3}\left[(F+2\,R)^3 - F^3\right]\sin^2\frac{R}{2F}\pi = B_x^{\,3}, \text{ say.}$$

Moment of area ABDE about OX

$$= \tfrac{1}{3}\left[(F+2\,R)^3 - F^3\right]\sin\frac{R\,\pi}{F} = B_y^{\,3}, \text{ say.}$$

$$\text{Moment of APDB about OA} = B^3_x - \frac{F+2\,R}{F}\,R^2\,A_x.$$

$$\text{Moment of APDB about OX} = B^3_y - \frac{F+2\,R}{F}\,R^2\,A_y,$$

$$\therefore\ \overline{X}' = \left(B^3_x - \frac{F+2R}{F}\cdot R^2 A_x\right) \div \frac{3\,F+2\,R}{2\,F}\,\pi\,R^2.$$

$$\overline{Y}' = \left(B^3{}_y - \frac{F+2\,R}{F}\,R^2.\ A_y\right) \div \frac{3\,F+2\,R}{2\,F}\,\pi\,R^2.$$

SCHOL.—It should be noted that these solutions might be presented geometrically, if it were worth while; but only at great length and with complicated diagrams. The student will observe that all the reasoning in each demonstration, up to the point where the integral calculus is employed, is manifestly capable of being presented geometrically, the ratios dealt with (including the trigonometrical ones) being those of lines to lines, areas to areas, or solids to solids (in dealing with moments of areas). Again, the only relations derived from the integral calculus, are these—

$$\int_{a}^{0} \sin a\theta \, d\theta = \frac{1}{a}(1 - \cos \alpha) = 2 \sin^2 \frac{\alpha}{2}$$

$$\int_{a}^{0} \cos a\theta \, d\theta = \frac{1}{a} \sin \alpha.$$

These (which are in effect one) are both capable of easy geometrical demonstration, and are in fact demonstrated further on in the quadrature of the 'companion to the cycloid.'

The student not familiar with the integral calculus, will find no difficulty in proving by trigonometrical series,* that the sum of the series whose general term is $\frac{a}{n} \sin \frac{ra}{n}$ (r taking all integral values from 0 to n), is $2 \sin^2 \frac{a}{2}$ when n is indefinitely increased; and that the sum of the series whose general term is $\frac{a}{n} \cos \frac{ra}{n}$, is $\sin a$. These summations (or such as these) suffice for summing the elements dealt with in the above demonstration.

* See the chapter on the Summation of Trigonometrical Series in Todhunter's 'Plane Trigonometry.'

SECTION III.

TROCHOIDS.

NOTE.—*Any curve traced by a point, within or without the circumference of a circle, which rolls without sliding upon a straight line or circle in the same plane, is a trochoid; but the term is usually limited to the right trochoid, and will be so employed throughout this section.*

DEFINITIONS.

The *right trochoid* is the curve traced out by a point either within or without the circumference of a circle, which rolls without sliding upon a fixed straight line in the same plane.

If the tracing point is within the circle, the trochoid is called a *prolate* or *inflected cycloid*. The shape of such a trochoid is shown in fig. 45, Plate I.

If the tracing point is outside the circle, the trochoid is called a *curtate* or *looped cycloid*. The shape of such a trochoid is shown in fig. 46, Plate I.

An *epitrochoid* is the curve traced out by a point either within or without the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the rolling circle touching the outside of the fixed circle.

A *hypotrochoid* is the curve traced out by a point either within or without the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the rolling circle touching the inside of the fixed circle.

It may readily be shown that every epitrochoid can be traced out in two ways—viz., either by a point within or without a circle which rolls in external contact with a fixed circle, or by a point without or within a circle which rolls in internal contact with a fixed circle of smaller radius. Also every hypotrochoid can be traced out either by a point within or without a circle which rolls in internal contact with a fixed circle of radius larger than rolling circle's diameter, or by a point without or within a circle which rolls in internal contact with a larger fixed circle, but of radius not larger than rolling circle's diameter. Instead, however, of giving a demonstration of these relations, after the manner of Prop. I., Section II., I leave the point for more general demonstration in Section V.

In what follows, reference is made to right trochoids, unless special mention is made of epitrochoids and hypotrochoids. Either fig. 45 or fig. 46 may be followed. The reader is recommended to read the following remarks twice over—once with each figure, and to adopt the same plan with the demonstration of each of the following propositions.

Let AQB (radius R) be the rolling circle, KL the fixed straight line. Let the distance of the tracing point from the centre be *r*, so that the tracing point lies on the circumference of the circle *a q b*, of radius *r*, and concentric with AQB. This circle, *a q b*, is called the *tracing circle*. Let D′D be the fixed straight line, touching the circle AQB in B. Let the centre of the rolling circle move along a line *c′* C *c*, parallel to D′D through C, the centre of AQB, in the direction shown by the arrow. Draw *e′ e* and *d′ d* parallel to *c′ C′ c*, and touching the tracing circle *a q b*. Then it is manifest that at regular intervals the tracing point will fall upon the straight lines *e′ e* and *d′ d*. When at *a* on the straight line *e′ e*, the tracing point is turning around the centre of the rolling circle in the direction in which this centre is advancing, and is at its greatest distance from the fixed straight line. When at *d′* and *d*, the tracing point is turning round the centre of the rolling circle in the opposite direction, and is at its greatest distance from *c′ c* on the side towards which lies the fixed straight line KL. The curve will manifestly be symmetrical on either side of the diameter *a* C *b*, perp. to KL. Therefore *a b* is called the *axis* of the trochoidal curve: *d′ d* is the *base*; and *a* the *vertex*. The radius C *a*, drawn to the tracing point, may conveniently be called the *tracing radius*. D′AD is called the *generating base*. The rolling circle AQB is called the *generating circle*, and when in the position AQB, is called the *central generating circle*. The circle *a q b* is called the *tracing circle*, and when in the position *a q b*, is called the *central tracing circle.* The complete trochoid consists of an infinite number of equal trochoidal arcs, but it is often convenient to speak of a single trochoidal arc, $d' a\, d$, as the trochoid.

It is clear that if D′c'E′, DcE, be drawn perp. to the fixed straight line through d' and d, and intersecting $e' a\, e$ in e' and e, respectively, the parts of the trochoid on either side of $d' e'$ and $d e$ are symmetrical with respect to these lines. Therefore $d' e'$ and $d e$ may conveniently be called *secondary axes.*

The straight lines $e' a\, e$ and $d'\, b\, d$ are tangents to the trochoid at a, and at d' and d, respectively.

PROPOSITIONS.

Prop. I.—*The base of the trochoid is equal to the circumference of the generating circle* (*figs.* 45, 46).

For $d'\, b\, d$ = D′BD ⩦ circumference of the circle AQB.

Cor. 1. $d'\, b = b'\, d$ = half the circumference of the generating circle.

Cor. 2. Area $e\, d\, d'\, e'$ = 2 rect. $a\, d$ = 4 rect. Cd $= 4\,\frac{r}{R}$ rect. CD $= 4\,\frac{r}{R}$ circle AQB.

Cor. 3. The base $d'\, b\, d$: circumference of the tracing circle $a\, q\, b$ :: circumference AQB : circumference $a\, q\, b$ :: R : r.

Cor. 4. Area $e\, d\, d'\, e'$ = 4 rect. under Cb, $b\, d$ = 4 rect. under C b, $\frac{R}{r}$. arc $a\, q\, b$ = 4 $\frac{R}{r}$. circle $a q b$.

PROP. II.—*If through p, a point on the trochoidal arc a p d (figs. 47, 48), the straight line p q M be drawn parallel to the base b d, cutting the central tracing circle in q, and meeting the axis A B in M; then,*

$$q\,p = \frac{R}{r} \text{ arc } a\,q.$$

Let A′PB′, $a'\,p\,b'$ be the position of the generating and tracing circles when the tracing point is at p,

FIG. 47.

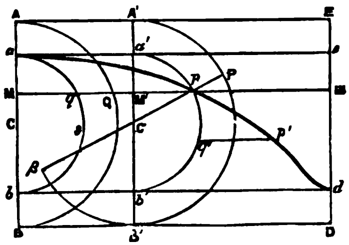
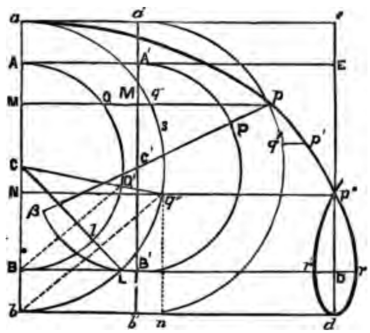


FIG. 48.



C′ their common centre, A′C′B′ diametral cutting p M in M′. Draw the diameter P p C′ β. Then it is manifest that $M q = M' p$; $MM' = q p$; and arc $a q =$ arc $a' p$. Now β is the point which was at B when the tracing point was at a, and since every point of the arc $\beta B'$ has been in rolling contact with BB′, the arc $\beta B' = BB'$.

But arc $\beta B' =$ arc $A'P = \frac{R}{r}$ arc $a' p = \frac{R}{r}$ arc $a q$;

and $BB' = MM' = q p$; wherefore $q p = \frac{R}{r}$ arc $a q$.

Cor. 1. $M p = \frac{R}{r}$ arc $a q + M q$.

Cor. 2. Since $b d = AQB = \frac{R}{r}\, a q b$

$$= \frac{R}{r}(\text{arc } a q + \text{arc } q b),$$

it follows that in the case of the prolate cycloid, where $R > r$, and therefore $\frac{R}{r}$. arc $q b$ necessarily $> M q$, $b d > M p$, and the whole arc $a p d$ lies on the same side of $d e$ as $a b$.

But in the case of the curtate cycloid (fig. 48), where $R < r$, there must be a point q'' on $a q b$ where

$$\frac{R}{r} \text{ arc } b q'' = N q'' \text{ (drawn perp. to AB)},$$

and if p'' be the point in which $N q''$ produced meets the trochoid, then will p'' fall on $e d$, for

$$p'' N = \frac{R}{r} \text{ arc } a q'' + N q''$$

$$= \frac{R}{r}(\text{arc } a q'' + \text{arc } b q'') = b d.$$

The part of the trochoid lying between p'' and d manifestly falls on the side of $e\,d$ remote from $a\,b$; and as the complete curve is symmetrical with respect to $e\,d$, it follows that the curtate cycloid has a loop of the form $p''\,r\,d\,r'$. It is also clear that the point p'' lies between D and e, since if L be the point in which BD cuts the arc $a\,q\,b$, and CL cuts AQB in l, the arc Bl is less than BL. The point p'' may lie nearer to e than E does, however, and the arc $d\,r'\,p''$ may intersect $a\,b$. It is easily seen either from the mode of generation or from Cor. 1, that if the ratio r : R be small, the curve may cut $e\,d$ a great number of times before the tracing circle has been carried entirely past $e\,d$.

Observe that if Cq'' cuts AQ′B in point Q′

$$\text{arc BQ}' = \text{N}\,q''.$$

Cor. 3. Let Mp produced (if necessary, in the case of curtate cycloid) meet $e\,d$ in m; then

$$\begin{aligned} p\,m &= \text{M}\,m - p\,\text{M} \\ &= \frac{\text{R}}{r}\,\text{arc}\,a\,q\,b - \frac{\text{R}}{r}\,\text{arc}\,a\,q - \text{M}\,q \\ &= \frac{\text{R}}{r}\,\text{arc}\,b\,q - \text{M}\,q. \end{aligned}$$

For points of the arc $p''\,r\,d$ (fig. 48) this relation still holds, regarding lines drawn perp. to $e\,d$ from the right as negative.

Cor. 4. Arc $a'\,p = \frac{r}{\text{R}}\,.\,b\,b'$, and

$$\text{arc } b'\,p = \frac{r}{\text{R}}\,.\,b'\,d.$$

Cor. 5. If from p' on $p\,d$, $p'\,q'$ be drawn parallel to $b\,d$ to meet $a'\,p\,b'$ in q',

$$q'\,p' : \text{arc}\,p\,q' :: \text{R} : r.$$

The proof of this is similar to that of Prop. II., sec. 1, cor. 5.

SCHOL.—The reader will find no difficulty in making the necessary modifications for the epitrochoid and hypotrochoid, deducing properties bearing to those established above the same relation which those established in Prop. III., section 2, bear to the properties established in Prop. II., section 1.

PROP. III.—*The area $d'\,a\,d$ (figs. 45, 46) between the trochoid and its base : area of the generating circle* $:: (b\,C + b\,A) : b\,C :: 2\,R + r : r$.

This may be proved in either of two ways corresponding in all respects with the two proofs of Prop. III., section 1. In the first proof, we show that elementary rectangles $q\,p, q'\,p'$ (figs. 49, 50) are equal to elementary rectangle $L\,l$; whence areas $a\,q\,p, q'b'dp'$, together, are equal to rectangle $L\,l$; and the area $a\,q\,b\,d\,p$ to the rectangle $C\,e = \frac{R}{r}$ circle $a\,q\,b$. Whence

$$\text{area } d'\,a\,d \text{ (figs. 45, 46)} = \odot\ a\,q\,b + \frac{2\,R}{r} \odot\ a\,q\,b,$$

or

$$\text{area } d'\,a\,d : \odot\ a\,q\,b :: 2\,R + r : r :: b\,c + b\,A : b\,C.$$

In the second proof, having drawn the inverted trochoid $a\,p''\,d$, with $a\,e$ as half base, and $d\,e$ as axis, we show that the elementary rectangles $p''p$ and $q''q$ are equal, whence

$$\text{area } q''a\,q = \text{area } p''a\,p; \text{ and area } a\,p''d\,p = \text{circle } a\,q\,b.$$

The equal areas $a\,p''d\,b$ and $a\,p\,d\,e$ are, therefore, each

$$= \tfrac{1}{2}\,(\text{rect. } b\,e - \text{circle } a\,b\,q)$$
$$= \left(\frac{R}{r} - \tfrac{1}{2}\right) \text{circle } a\,b\,q;$$

therefore

$$\text{the area } a\,p\,d\,b = \left(\frac{R}{r} + \tfrac{1}{2}\right) \text{circle } a\,b\,q;$$

and $$d'\,a\,d = \left(\frac{2\,R + r}{r}\right) \text{circle } a\,b\,q,$$

as before.

Fig. 49.

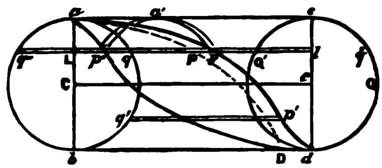
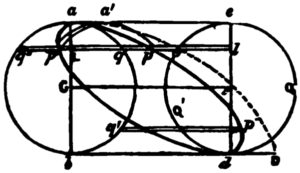


Fig. 50.



SCHOL.—The reader will find no difficulty in dealing in like manner, so far as first proof is concerned, with the area between the epitrochoid or hypotrochoid and the base. The demonstration bears precisely the same relation to that of Prop. IV., section 2, which the above first proof bears to the first proof of Prop. III., section 1. We thus show that the area between the generating semicircle $a\,q\,b$, the arc base $b\,d$ (radius F) and the trochoidal arc $a\,p\,d$: generating circle $a\,q\,b$ :: 2 CO (b C + b A) : b O . b C, that is, in the ratio compounded of the ratios 2 CO : b O and (b C + b A) : b C.

In all cases,—for cycloid, epicycloid, hypocycloid, trochoid, epitrochoid, and hypotrochoid,—
area $a\,q\,b\,d\,p$ in the trochoidal figures $= \frac{1}{2}$ area $a\,b\,d\,e$.

Prop. IV.—*If the cycloid, a PD, and the trochoid, a p d (figs. 49 and 50), have a common axis a b,*

area a q b d p : area a q b DP :: R : r.

From Prop. II., section 1,

$$q\,\mathrm{P} = \text{arc } a\,q\,;$$

but from Prop. II., of the present section,

$$q\,p = \frac{\mathrm{R}}{r} \text{ arc } a\,q$$

$$\therefore \qquad q\,p : q\,\mathrm{P} :: \mathrm{R} : r,$$

and elem. rectangle $p\,q$: elem. rectangle q P :: R : r;

whence area $a\,q\,p$: area $a\,q$ P

:: area $a\,q\,b\,d\,p$: area $a\,q\,b$ DP :: R : r.

Cor. Area $a\,q\,p$: area $a\,q$ P :: R : r.

Schol.—A similar property can be readily established for epicycloids and epitrochoids, or for hypocycloids and hypotrochoids, having a common axis. In this case, $q\,p$, q P, and $b\,d$, are concentric arcs, and in place of elementary rectangles we have elementary areas like Qp, q' P′ of figs. 26 and 27; but the ratios are the same, and we therefore still find

$$\text{area } a\,q\,p : \text{area } a\,q\,\text{P} :: \text{area } a\,q\,b\,d\,p : \text{area } a\,q\,b\,\text{DP} :: \text{R} : r.$$

Prop. V.—*If p (figs. 51, 52) is a point in a trochoidal arc, a′ p b′, the tracing circle when the tracing point is at p, a′ C′ b′ diametral, meeting the generating base in B′, then B′ p is the normal at p; and if T a′ t is the tangent to the tracing circle at a′, T p, t p, tangents to the trochoid and tracing circle respectively at p, then*

$$Tt : a't :: R : r.$$

Fig. 51.

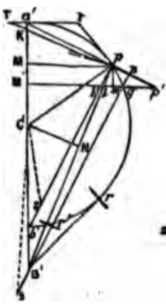
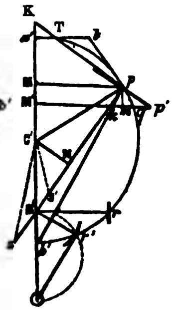


Fig. 52.



Since, when the tracing point is at p, the generating circle is turning around the point B′, the direction of the tracing point's motion at p must be at right angles to B′p, which is, therefore, the normal at p. The tangent p T at p is therefore perp. to p B′. Also, since $p\,a'$ is perp. to $p\,b'$, and $p\,t$ to $C'p$, triangle $p\,T\,a'$ is similar to $p\,B'b'$, and $p\,a't$ to $p\,b'C'$; therefore

$$T\,t : a't :: B'C' : C'\,b :: R : r.$$

Another Demonstration.

From p and p' (near p), on the trochoidal arc, draw $p\,M$, $p'M'$ perp. to $a'b'$, $p'M'$ cutting $a'p\,b'$ in q. Then

$$q\,p' : p\,q :: C'B' \cdot C'b' \ (=C'p),$$

Prop. II., cor. 5, and since ultimately the sides $q\,p'$, $q\,p$ are perp. to the sides $C'\,B'$, $C'p$,

$$\text{angle } p\,q\,P' = \text{angle } p\,C'B'.$$

Hence the triangle $p\,q\,p'$ is ultimately similar to the triangle $p\,C'B'$, and $p\,p'$, the third side of one, is ultimately perp. to $p\,B'$, the third side of the other. Wherefore $p\,B'$ is the normal at p. And, as in the preceding proof, $T\,t : a't :: C'B' : C'b' :: R : r$.

Cor. 1. Triangle $p\,q\,p'$ being similar to triangle $p\,C'B'$,

$$p\,p' : p\,q :: B'p : C'p.$$

Cor. 2. If $p\,m$ be perp. to $p'M'$, and $p\,T$ cut or meet $a'b'$ in K, then $p\,p'\,m$ is in the limit similar to triangles $p\,B'M$, $K\,p\,M$, $KB'\,p$.

Cor. 3. If $B'p$ cut $p'\,M'$ in l, the triangles $l\,p\,m$ and $l\,p'p$ are similar to the four triangles named in cor. 1. Also, $l\,p\,q$ is similar to $K\,p\,C'$, and $q\,p\,p'$ to $C'\,p\,B'$. Wherefore

$$l\,q : q\,p' :: KC' : C'B' :: p\,N : NB'.$$

Cor. 4. If $p\,b'$ cut p' M′ in k, $k\,p\,p'$ is similar to $a'\,p$ B′, and $k\,p\,q$ to $a'\,p$ C′. Wherefore

$$p\,q = q\,k, \text{ and } k\,q : q\,p' :: p\,q : q\,p' :: r : \mathrm{R}.$$

Cor. 5. If in the case of the prolate cycloid, illustrated in fig. 51, the tracing point is at r, where the tangent from B′ meets the tracing circle $a\,q\,b'$, then the normal B′r has its greatest inclination to a'B′, and its least inclination to the base. It is manifest, therefore, that r is a point of inflection. At the point of the prolate cycloid corresponding to r', in which B′ p cuts the tracing circle, the tangent is parallel to the tangent at p.

Cor. 6. If in the case of the curtate cycloid, illustrated in fig. 52, the tracing point is at r on the generating base, the normal B′r coincides with the generating base. Therefore the curtate cycloid cuts the generating base at right angles.

Cor. 7. B′ q produced to meet $p\,p'$ in n is ultimately perp. to $p\,p'$, and if C′N is drawn perp. to p B′, $p\,q\,n$ is similar to p C′N, and $p'q\,n$ to B′C′N; and $p\,p' : p\,n :: p$ B′ : p N.

Schol.—It is easy to prove that p B′ is the normal in the case of epitrochoid or hypotrochoid. We have only to draw C′s parallel to the line joining p with the centre of the fixed circle, to meet p B′,* and proceed as in Prop. V., section 2. (In both figs. C′s is drawn for the case of the epitrochoid; C′s', for the case of

* The reader will note that, in fig. 51, C′s' does not extend far enough. It should be produced to meet p B′.

the hypotrochoid). If, in the former case, the straight line joining p with O, the centre of the fixed circle, be perp. to $B'p$, which can only happen when $r > R$ (fig. 51), the tangent at p passes through O. This determines the position of the tangent from the centre to the curtate epicycloid corresponding to the direction of the stationary point in the looped epitrochoid, regarded as a planetary curve. It is well to note the construction for determining this point. Produce $C'b'$ (fig. 51) to O, the centre of the fixed circle, and on $B'O$ describe a semicircle cutting $a'p\,b'$ in r'; then $B'r'$ is perp. to $r'O$, and therefore a circle described about O as centre, with radius $O\,r'$, intersects the curtate epicycloid in the point where the tangent passes through O. This relation is demonstrated and dealt with under Prop. X.

Cor. 8. In the case of epitrochoids and hypotrochoids the triangle $p\,q\,p'$ is similar—not to $p\,CB'$—but to $p\,C's$ (the s accented throughout for hypotrochoid);

$$p\,p' : p\,q :: p\,s : p\,C',$$

$$\text{and } p\,p' : n\,p :: p\,s : PN.$$

Since then $N\,p$ and $n\,p$ are the same for the epitrochoid or hypotrochoid as for the right trochoid, with the same generating and tracing circles (and, of course, the same angle, $p\,C'a'$, between tracing radius and diametral), while

$$p\,B' : B's :: F : R,$$

and therefore $p\,B' : p\,s :: B'O : C'O$ (see figs. 28 and 29), it follows that $p\,p'$, regarded as an arc of an epitrochoid or hypotrochoid, bears to $p\,p'$, regarded as an arc of a trochoid ($p\,q$ being the same for both), the ratio $s\,p : p\,B'$, or $C'O : B'O$, or $F \pm R : R$ (the upper sign for epitrochoid, the lower for hypotrochoid).

The student will find it a useful exercise to complete the construction indicated in the scholium, noting that the figs. 51 and 52 are correct for the cases there considered, as well as for the case considered in the text, except only that the lines $p\,M$ and $p'q\,M'$ must be concentric with the generating base through B'—that is, must have for centre the point O mentioned in the scholium.

PROP. VI.—*From a point p (figs. 53 and 54), on the trochoid a p d, above the line of centres c c' C, let q p be drawn parallel to c C to meet the central tracing circle a c'b in q, and q n, p m, perp. to c C; then, if the rectangle a c n f be completed,*

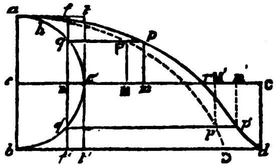
$$\text{area } a\,h\,q\,p + \text{rect. } p\,n : \text{rect. } c\,f :: R : r.$$

And if from p' on a p d below c C, p'q' parallel to c C meet a c'b in q'; q'n', p'm' are drawn perp. to c C; and rect. n c b f' is completed, then

$$\text{area } a\,h\,c'q'p' - \text{rect. } p'n : \text{rect. } c\,f' :: R : r.$$

Let a PD be a semi-cycloid having $a\,b$ as axis; then it is easily seen that every element of either area $a\,h\,q\,p + p\,n$ or $a\,h\,q'p' - p'n$ parallel to c C, bears to the corresponding element for the case of cycloid a PD, the ratio $R : r$; and therefore the sum of all such elements of either area in case of trochoid : sum of all such elements of either area in case of cycloid (*i.e.*,

FIG. 53.



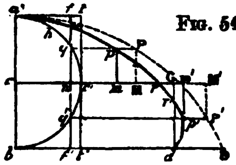


FIG. 54.

cf or cf', as shown in Prop. V. sec. 1) :: R : r. That is,

$$\left.\begin{array}{l}\text{area } a\,h\,q\,p + \text{rect. } q\,m : \text{rect. } c\,f \\ \text{area } a\,h\,q'p' - \text{rect. } q'm' : \text{rect. } c'f'\end{array}\right\} :: \mathrm{R} : r.$$

Cor. Area $a\,c'b\,d\,r = \text{rect. } c\,d = \frac{\mathrm{R}}{r}\,.\,\text{circle } a\,q\,b$

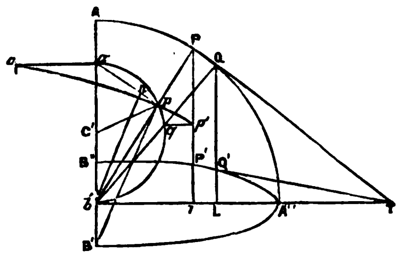
(Prop. I., cor. 4). Thus we have here another demonstration of the area of trochoid.

PROP. VII.—*Let a (fig. 55) be the vertex of the trochoidal arc $a\,p$, $a'p\,b'$ the tracing circle through p, $a'C'b'$ diametral, $A'C'B'$ the corresponding diameter of generating circle. Describe the quadrant $A'PA''$ having b' as centre and $b'A'$ as radius; produce $b'p$ to meet $A'PA''$ in P; and draw Pl perp. to $b'A''$. Then, if $b'B'' = b'B'$, and $B''P'A''$, an elliptic quadrant having $b'B''$ and $b'A''$ as semi-axes, intersect Pl in P',*

arc $a\,p$ = twice the elliptic arc $B''P'$.

Let p'* be a point on the trochoid near p, and let $p'q$ parallel to the base meet $a'p\,b'$ in q. Produce $b'q$

FIG. 55.



to meet A'PA'' in Q; draw QL perp. to b'A'', cutting B''Q'A'' in Q'. Join $a'p$, B'p, and draw $b'n$ parallel to B'p (dividing $a'p$ in n, so that $a'n : np :: a'b' : b'B' ::$ A'B'' · B''b'). Join C'p, PQ, and P'Q'. The secants PQ, P'Q' being ultimately tangents at

* p' does not lie on P l.

P and P′, meet ultimately when produced on b'A″; let them thus meet in T.

Then $P'Q' : PQ :: P'T : PT :: b'n : b'a'$ (since triangle $a'b'p$ is similar to PT l, and $a'p$ and P l are similarly divided in n and P′ respectively) $:: B'p : B'a'$.

Also, $PQ : pq :: A'b' (= a'B') : a'b'$

(because Pb'Q is an angle at centre of quadrant A′PA″ and at circumference of semicircle $a'p\,b'$).* Wherefore, *ex æquali*,

$P'Q : pq :: B'p : a'b'$. But

$pp' : pq :: B'p : C'p$ (Prop. V., cor. 1) $:: 2\,B'p : a'b'$;

therefore, $pp' = 2\,P'Q'$.

But pp' and 2P′Q′ are increments of arc ap and arc B″P′ respectively, which arcs begin together.

Therefore, arc $ap = 2$ arc B″P′.

Cor. The arcs $a\,p\,d$ (figs. 45 and 46) = elliptic arc B″A″B′, and arc $d'a\,d$ = circumference of an ellipse having semi-axes b A, b B, that is, $R + r$ and $R \sim r$.

PROP. VIII.—*If $a'p\,b'$ (figs. 56 and 57) is the position of the tracing circle through p, $a'b'$ diametral, $a\,b$ the axis, and $p\,b'$ be joined, then*

$$\left.\begin{array}{l} \textit{area } apb'b : \textit{sect. area } abq \ (\textit{or } a'b'ph) \\ \textit{area } pb'd : \textit{segment } bsq \ (\textit{or } b'fp) \end{array}\right\} :: 2R + r : r.$$

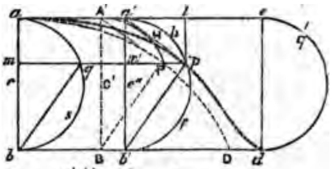
Let a PD be a cycloid, having $a\,b$ as axis, and let Pp be parallel to $b\,d$; then area $a\,q\,b$ B′P = 2 sec-

* $\frac{PQ}{A'b'}$ = circ. meas. of $p\,b'q = \frac{1}{2}$ circ. meas. of $p\,C'q = \frac{1}{2}\frac{p\,q}{C'a} = \frac{p\,q}{a'b'}$.

torial area A′B′P. But every element of the area $a\,q\,b\,b'p$ parallel to base $b\,d$ (as in Prop. III.) : corresponding element in case of cycloid :: R : r. Wherefore area $a\,q\,b\,b'p$: sectorial area $a\,b\,q$ :: 2 R : r, and area $a\,c\,b\,b'$: sectorial area $a\,b\,q$ :: 2 R + r : r. Similarly

area $p\,b'd$: segment $b\,s\,q$ :: 2 R + r : r.

FIG. 56.



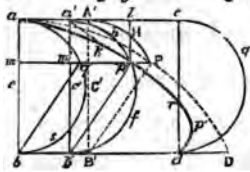


FIG. 57.

Cor. 1. Area $p\,f\,b'd$: segment $p\,f\,b'$ :: 2 R : r.

Cor. 2. Area $a\,q\,b\,b'p$: sectorial area $a\,b\,q$ :: 2 R : r.

Cor. 3. If $p\,q$ produced meet $a\,b$ in m,

$$\text{area } q\,s\,b\,d\,p = \text{rect. } b\,m,\ q\,p + \frac{2\,\mathrm{R}}{r}\,\text{segment } b\,s\,q.$$

SCHOL.—Two independent methods of demonstrating the area of trochoids can be derived from the above proposition, as in the case of cycloid. For, carrying p to d, we have area $a\,p\,d\,b$: $\frac{1}{2}$ circle $a\,q\,b$ :: 2 R + r : r, as in Prop. III.

The proof may be extended to epitrochoids and hypotrochoids, and the following proportion established :—

Area $a\,b\,b'\,p$: sectorial area $a'\,b\,p$

:: area $b'p\,d$: seg. $b\,s\,q$

:: $(2\text{ CO}+\text{BO})(2\text{ R}+r)$: BO . r, where BO is the radius of the base, and CO is the radius of the arc of centres, or

$$:: (3\text{ F}\pm 2\text{ R})\,(2\text{ R}+r) : \text{F} \,.\, r$$

(where F is the radius of fixed circle), the upper sign for epitrochoid, the lower for hypotrochoid.

PROP. IX.—*To determine the area of the loop of the curtate cycloid $a\,p\,d$, fig.* 48.

By cor. 3, Prop. VIII., area $q''p''r\,d\,b$, fig. 48,

$$(= \text{rect. N } d+\tfrac{1}{2}\text{ loop } r'r - \text{area N } b\,q'')$$

$$= \text{rect. } b\text{ N, } q''p'' + \frac{2\text{ R}}{r}\text{ seg. } q''\text{L } b;$$

$$\therefore \tfrac{1}{2}\text{ loop } r'r = \text{area N } b\,q'' - \text{rect. under } b\text{ N, N } q'' + \frac{2\text{ R}}{r}\text{ seg. } q''\text{L } b$$

$$= \frac{2\text{ R}+r}{r}\text{ seg. } q''\text{L } b - \text{triangle } b\text{ N } q'';$$

$$\therefore \text{loop } r\,r' = \frac{4\text{ R}+2\,r}{r}\text{ seg. } q''\text{L } b - \text{rect. N } n.$$

PROP. X.—*With the same construction as in Proposition VIII., area $a\,p\,h\,a'$: segment $a'h\,p$ :: $2\,R : r$.*

Since area $a\,q\,p$: area AQP :: R : r :: area $a\,q\,p\,h\,a'$: $a\,q$ PHA′ (PHA′ being the arc of tracing ⊙ A′PB′, for cycloid, not wholly shown in the figure);

it follows that area $a\,p\,h\,a'$: area a PHA' :: R : r. But area a PHA' = 2 segment A'HP or 2 segt. $a'h\,p$; ∴ area $a\,p\,h\,a'$: segt. $a'\,h\,p$ :: 2 R : r.

Cor. 1. Area $a\,p\,d\,q'e$: $\frac{1}{2}$ circle $e\,q'd$ :: 2 R : r.

$$\text{Since } a\,p\,d\,q'e = a\,p\,d\,e + \tfrac{1}{2} \text{ circle } a\,q\,b,$$

and rect. $b\,e$: $\frac{1}{2}$ circle $e\,q'd$ :: 4 R : r, it follows that

$$\text{rect. } b\,e : \text{area } a\,p\,d\,e + \tfrac{1}{2} \text{ circle } a\,q\,b :: 2 : 1$$

as in schol. to Prop. III., so that we have here a new demonstration of the area.

Cor. 2. In the case of the prolate cycloid, fig. 57, in which $p\,a'$ does not intersect the arc $a\,p$,

$$\text{area } a\,p\,a' : \text{segment } a'h\,p :: 2\text{ R} - r : r.$$

Cor. 3. Proceeding to d, area $a\,p\,d\,e$: $\frac{1}{2}$ circle $e\,q'd$ :: 2 R $-\,r$: r, in case of prolate cycloid.

Cor. 4. In the case of the curtate cycloid, fig. 56, $p\,a'$ cuts the curve in some point k, between p and a'. Here then

$$\text{area } a\,k\,a' - \text{area } k\,p : \text{segment } a'h\,p :: 2\text{ R} - r : r,$$

or passing to d,

$$\text{area } a\,r\,e - \text{semi-loop } r\,p'a : \tfrac{1}{2} \text{ circle } e\,q'd :: 2\text{ R} - r : r.$$

Schol.—Another independent demonstration of the area of trochoids is worthy of notice. Let us suppose that the circle $a\,q\,b$, figs. 49 and 50, slides uniformly between $a\,e$ and $b\,d$ to the position e Q d ($e\,d$ diametral). Let $p''a'p$ be the position of the upper segment when the circle passes through $p''p$ ($=q''q$, so that the circle reaches p'' and p simultaneously), and let a closely adjacent segment, as in the figure, give the elementary areas $a'p$ and $a'p''$. These are ultimately in a ratio of equality, but they are the respective increments of the areas $a\,p\,a'$, $a\,p''a'$ (or as actually drawn in the figure, they are the elementary increments next before the attainment of these areas $a\,p\,a'$, $a\,p''a'$), and these areas begin together. Hence

$$\text{area } a\,p\,a' = \text{area } a\,p''a'\,;$$

and carrying the moving circle to its final position,

$$\text{area } a\,p\,d\,\mathrm{Q}\,e = \text{area } a\,p''d\,\mathrm{Q}'e = \text{area } a\,p\,d\,b\,q',$$

whence the result of Prop. III. follows at once.

PROP. XI.—*Let* $p\,o$ (*figs.* 58–62) *be the radius of curvature at* p, *on the trochoid;* $a'p\,b'$ *the tracing circle through* p. *Then, if* $a'\,C'b'$ *meet the generating base in* B', *and* $C'N$ *be drawn perp. to* $p\,B'$,

$$p\,o : p\,B' :: p\,B' : p\,N.$$

With so much of the construction of Prop. V. as is indicated in fig. 58 (illustrating the prolate cycloid), let p'L be the normal at p' (near p). Then

$$q\,p' = \frac{\mathrm{R}}{r}\text{ arc } p\,q \text{ (Prop. II., Cor. 5)} = \mathrm{B}'\,\mathrm{L}.$$

Join $q\,\mathrm{B}'$. Now p'L, being parallel to $q\,\mathrm{B}'$, is not parallel to $p\,\mathrm{B}'$, unless the point q falls on $p\,\mathrm{B}'$; that is, unless the tangent to the circle $a'q\,b'$ passes through B′, the case illustrated by fig. 60. In this case the radius of curvature is infinite, or p is a point of inflection. In all other cases, p B′ and p'L meet when produced,—towards B′L, when $p'q$ has to be produced to meet p B′ (in l), and towards $p\,p'$ when p B′ intersects

Fig. 58.

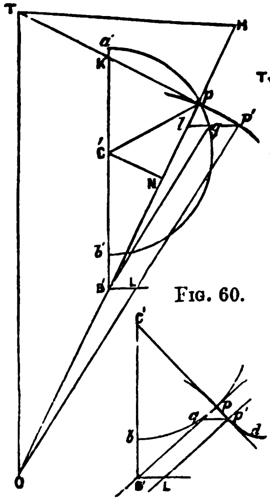


Fig. 59.

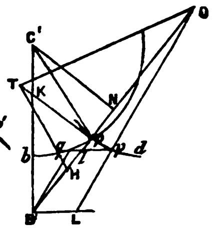


Fig. 61.

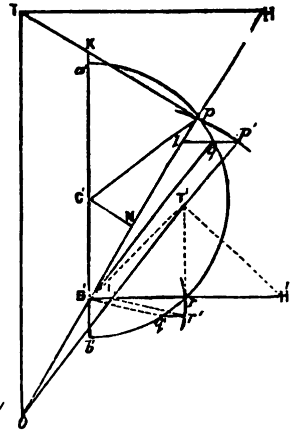
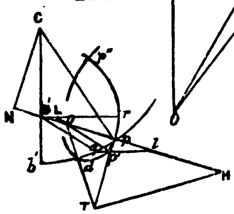


Fig. 60.

Fig. 62.



$p'q$ (in l) between p' and q, fig. 59. Let them meet in o. Then in the limit

$$lo : l\mathrm{B}' :: lp' : lq :: p\,\mathrm{B}' : p\,\mathrm{N} \text{ (Prop. V., cor. 3).}$$

That is, ultimately,

$$o\,p : p\,\mathrm{B}' :: p\,\mathrm{B}' : p\,\mathrm{N}.$$

Cor. Rect. under $o\,p, p$ N = square on p B′.

Schol.—The following construction is indicated for determining the centre of curvature. On B′p, produced if p is beyond N, otherwise not, take p H = p N, nd on the tangent p KT at p take p T = p B′; then

To perp. to HT will meet pB′ produced in o, the centre of curvature at p. For

$$op, p\text{H} = (p\ \text{T})^2,$$

that is, $$op, p\text{N} = (p\text{B}')^2.$$

The student will find no difficulty in dealing with the corresponding demonstration for the curtate cycloid. Fig. 61 gives the construction for one general case, p above the base; and for the case of a point on the generating base where B′ becomes the centre of curvature (for the latter case r and r' are put for p and p', while the letters H, T, and N are accented). Fig. 62 gives the construction for a general case, p below the base.

For the vertex, N coincides with C′, pN $= a'\text{C}' = r$, and $p\text{B}' = a'\text{B}' = \text{R} + r$. Therefore,

$$\text{radius of curvature at } a = \frac{(\text{R} + r)^2}{r},$$

both for prolate and curtate cycloids.

For the point d, N also coincides with C′, $p\text{N} = r$ in absolute length, and must be regarded as negative in case of prolate cycloid, because N falls outside pB′ beyond p, whereas in case of curtate cycloid N falls on the same side of p as B′, though beyond B′. Also $\pm p\text{B}' = (\text{R} - r)$. Therefore, rad. of curvature at d $= \frac{(\text{R} - r)^2}{r}$, negative for prolate cycloid, and positive for curtate cycloid.

But it is to be noticed that in considering the curvature in the case of the curtate cycloid as constantly positive, regard is had to the intrinsic nature of the curve. If the curvature is considered with reference to the base, there is a change of sign at the moment when N passes the point B′, or where the curve cuts the generating base—viz., at r.

At this point r,

$$\text{radius of curvature} = \frac{(r\,B')^2}{r\,B'} = r\,B'\,; \qquad \text{or}$$

$$\text{square on rad.} = (r\,B')^2 = (C'\,r)^2 - (CB')^2 = r^2 - R^2.$$

PROP. XII.—*Let $p\,o$ (figs. 63, 64) be the radius of curvature at the point p of an epitrochoid or hypotrochoid; $a'p\,b'$ the tracing circle through p; and $a'b'O$ diametral, cutting generating base in B'. Draw $C'N$ perp. to $p\,B'$; and $C's$ parallel to $p\,O$ meeting $p\,B'$ (produced if necessary) in s. Then*

$$p\,o : p\,B' :: p\,s : p\,s - NB'.$$

[Two illustrative cases only are dealt with (one of a prolate epicycloid, one of a prolate hypocycloid). The student will find no difficulty in modifying the demonstration and figure for other cases.]

Let p' be a point near p; p'L the normal at p'; $p'q$ concentric with generating base B′L, meeting $a'p\,b'$ in q. Draw $q\,n$ perp. to $p\,p'$; $q\,i$ in direction perp. to $a'b'$ to meet $p\,p'$ in i, and L h perp. to B′p. Then, as in case of right trochoid, $\quad q\,i = \frac{R}{r}\,\text{arc}\,p\,q = B'L,$

and triangle B′L h is equal in all respects to triangle $q\,i\,n$. Also triangles $p\,q\,n, p\,q\,i, p\,q\,p'$ are similar to triangles p C′N, p C′B′, p C′s. (See Prop. V., Cors. and Schol.) Now L h is parallel to $p'p$; wherefore,

$$p\,o : h\,o :: p\,p' : h\,L\ (= n\,i) :: p\,s : NB',$$

or ultimately $p\,o : p\,B' :: p\,s \quad (p\,s - NB')$.

Cor. Since $p\,s : C'O :: p\,B' : B'O$, we see that

$$p\,o : C'O :: (p\,B')^2 : (p\,s - NB')\,B'O$$
$$:: (p\,B')^2 : p\,B'.\,C'O - NB'.\,B'O.$$

See p. 166. At vertex, and at pt. on base, rad. of curvature $= \frac{(R+r)^2(F+R)}{R^2+r\,(F+R)}$, and $= \frac{(R-r)^2(F+R)}{R^2-r\,(F+R)}$, respectively, R being regarded as negative for hypocycloid.

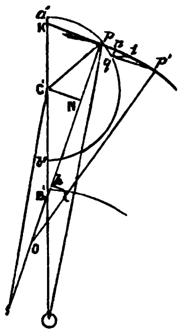


Fig. 63.

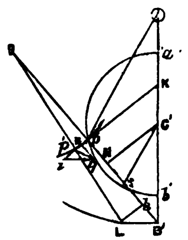


Fig. 64.

Schol.—A construction similar to that for the radius of curvature at points on right trochoids can readily be obtained. Thus produce $B'p$ to H (as in fig. 58), taking $p\,H = p\,s - NB'$; on the tangent p K take p T, a mean proportional between $p\,B'$ and $p\,s$; then T o perp. to TH will intersect $p\,B'$ produced, in o, the centre of curvature at p. For by the construction

$$p\,o\,(p\,s - NB') = (p\,T)^2 = p\,B'\,.\,p\,s$$
$$\therefore p\,o : p\,B' :: p\,s : (p\,s - NB').$$

At a point of inflection the radius of curvature becomes infinite. Now $p\,B'$ is always finite, and since $ps : pB' :: C'O : B'O$, ps is also necessarily finite. Wherefore, the radius of curvature can only become infinite by the vanishing of $ps - NB'$, that is,

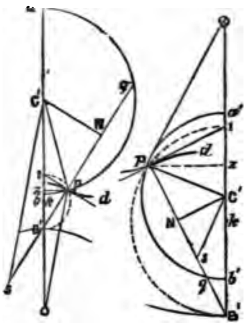
when $$NB' = ps, \text{ or } Np = B's,$$

or p must have such a position as is shown in figs. 65 and 66, for the epitrochoid and hypotrochoid respectively. Wherefore,

$$NB' : pB' :: ps : pB' :: C'O : B'O :: F \pm R : F$$

(upper sign for epitrochoid, lower for hypotrochoid),

Fig. 65. Fig. 66.



or, drawing pI parallel to NC'—that is, perp. to $B'N$ —to meet $C'O$ in I,

$$C'B' : B'I :: C'O : B'O :: F \pm R : F.$$

Wherefore, the construction for determining points of inflection is as follows:—Take I in $C'O$ (figs. 65 and 66), so that

$$C'B' : B'I' :: C'O' : B'O :: F \pm R : F$$

$$\text{or } B'I = \frac{C'B' . B'O}{C'O} = \frac{R . F}{F \pm R}.$$

Then if the circle on IB′ as diameter cuts the tracing circle, as at p, a circle about centre O with radius Op cuts the epitrochoid or hypotrochoid in its points of inflection. If the circle on IB′ as diameter does not cut the tracing circle, there are no points of inflection.

Cor. $C'B' : C'I :: C'O : C'B'$,

and $(C'B')^2 = C'I \cdot C'O$; that is, $C'I = \frac{R^2}{F \pm R}$.

If, in case of epitrochoid, I falls at b',—that is, if

$$C'B' : B'b' :: C'O : B'O :: F + R : F,$$

the radius is infinite at the point d; but there is no change of curvature: two points of inflection coincide, and the curvature has the same sign on both sides of the double point of inflection. In this case,

$$C'b' : C'B' :: C'B' : C'O :: R : F + R$$
$$\text{or } r : R :: R : F + R.$$

This indicates the relation between r, R, and F, when in the case of epitrochoid the curve just fails, at d, of becoming concave towards the centre.

If, in case of hypotrochoid, I falls at a', that is, if

$$C'B' : B'a' :: C'O : B'O :: F - R : F,$$

the radius is infinite at the vertex a. Two points of inflection coincide, the curvature having the same sign on both sides of the double point of inflection. In this case

$$C'a' : C'B' :: C'B' : C'O :: R : F - R$$
$$\text{or } r : R :: R : F - R.$$

This indicates the relation between r, R, and F, when, in the case of the hypotrochoid, the curve just fails at *a* of becoming concave towards the centre.

PROP. XIII.—*If p (figs.* 65 *and* 66) *be a point of inflection of an epitrochoid or hypotrochoid, a′ q p the corresponding position of the generating circle; a′ C′ O diametral, meeting the generating base in B′; p z perp. to B′ C′; and k the centre of semicircle B′p I; then will*

$$\text{rect. } C'B'.C'I \pm \text{sq. on } C'p = 2 \text{ rect. } C'k,\ C'z$$

(the upper sign for epitrochoid, the lower for hypotrochoid).

We have

$$(C'p)^2 = (C'z)^2 + (pz)^2 = (C'z)^2 + (kI)^2 - (kz)^2,$$

and for epitrochoid

$$C'B' . C'I = (C'k)^2 - (kI)^2$$

$$\therefore C'B' . C'I + (C'p)^2 = (C'z)^2 + (C'k)^2 - (kz)^2$$
$$= 2\, C'k . C'z.$$

For hypotrochoid

$$C'B' . C'I = (kI)^2 - (C'k)^2$$

$$\therefore C'B' . C'I - (C'p)^2 = (kz)^2 - (C'z)^2 - (C'k)^2$$
$$= 2\, C'k . C'z.$$

SCHOL.—This prop. may also be treated in the manner adopted for the next—*i.e.*, starting from the relation $(Ip)^2 + (B'p)^2 = (IB')^2$, and taking triangles $IC'p$ and $B'C'p$.

PROP. XIV.—*Let p (figs. 67, 68) be the point of the loop of an epitrochoid or hypotrochoid where the tangent to the curve passes through the centre of the fixed circle; $a'p\,b'$ the corresponding position of the tracing circle; and $a'\,C'B'$ diametral, meeting the generating circle in A' and B'; then, if $p\,K$ is drawn perp. to OC'',*

Rect. OA', $C'K$ = sq. on $C'b'$ + rect. $O\,C'$, $C'B'$,

for epitrochoid, and

= rect. $OC'.C'B'$ − sq. on $C'b'$,

for hypotrochoid.

Since $p\,B'$ is the normal at p, $B'p\,O$ is a right angle, and sq. on $B'p$ + sq. on $p\,O$ = sq. on $B'O$.

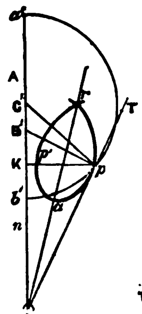


FIG. 67.

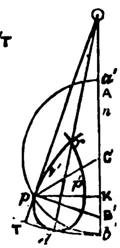


FIG. 68.

$$\text{Now } (B'p)^2 = (C'p)^2 + (C'B')^2 - 2\,C'B'\,.\,C'K$$

$$\text{and } (O\,p)^2 = (C'p)^2 + (C'O)^2 \mp 2\,C'O\,.\,C'K$$

(lower sign for hypotrochoid)

$$\therefore (B'p)^2 + (O\,p)^2 = 2\,(C'p)^2 + (C'B')^2 + (C'O)^2 - 2\,(C'B' \pm C'O)\,C'K;$$

that is, $$(B'O)^2 = 2\,(C'p)^2 + (C'B')^2 + (C'O)^2 \mp 2\,OA'\,.\,C'K.$$

Or, for epitrochoid,

$2\,OA'.\,C'K = 2\,(C'b')^2 + (C'O)^2 + (C'B')^2 - (B'O)^2$;
i.e. (Euc. II., 7) $OA'.\,C'K = (C'b')^2 + OC'.\,C'B'$.

For hypocycloid,

$2OA'.\,C'K' = (B'O)^2 - 2(C'b')^2 - (C'O)^2 - (C'B')^2$;
i.e. (Euc. II., 4) $OA'.\,C'K' = OC'.\,C'B' - (C'b')^2$.

SCHOL.—This prop. may also be treated in the manner adopted for the preceding, bisecting K O in n, and noting that rect. $OC'.\,C'B' = \pm\,[(C'n)^2 - (n\,B')^2]$, upper sign for epitrochoid, lower for hypotrochoid.

Observe that C'K (regarded as positive or negative, according as K lies on C'O, or C' on KO)

$$= \frac{r^2 \pm (F \pm R)\,R}{F \pm 2\,R} \text{ or } = \frac{r^2 + R^2 \pm FR}{F \pm 2\,R};$$

the upper sign for epicycloid, the lower for hypocycloid.

This is the relation existing at a stationary point in an epicycloidal planetary orbit.

PROP. XV.—*If G (figs.* 47 *and* 48) *is the centre of gravity of the trochoidal area d′a d,*

$$b\,G : 3\,R + 2\,r :: r : 2\,(2\,R + r).$$

Since every elementary rectangle of the part of area $d'a\,d$ outside circle $a\,q\,b$, taken parallel to base : corresponding element of part of cycloid having $a\,b$ as axis lying outside same circle $a\,q\,b$: : R : r, it follows that the distance of CG of former areas from $b\,d$ (along

b C, evidently) = distance of CG of latter areas from b (along b C) $= \frac{3}{4} b$ C (Prop. XVIII., sec. 1st, cor. 3).

$\therefore$ Mom. of $d'a\,d$ about $b\,d$

$$= 2\,\frac{R}{r}\,\text{circle}\;a\,q\,b\,.\,\frac{3r}{4} + \text{circle}\;a\,q\,b\,.\,r$$

$$= \frac{3\,R + 2\,r}{2}\,.\,\text{circle}\;a\,q\,b$$

and $$\text{area } d'a\,d = \frac{2\,R + r}{r}\,\text{circle}\;a\,q\,b$$

$$\therefore\; bG\,.\,\frac{2\,R + r}{r}\,\text{circle}\;a\,q\,b = \frac{3\,R + 2\,r}{2}\,\text{circle}\;a\,q\,b$$

$$b\,G\,.\,\frac{2\,R + r}{r} = \frac{3\,R + 2\,r}{2}$$

or $b\,G : 3\,R + 2\,r :: r : 2\,(2\,R + r)$.

Cor. $b\,G = \dfrac{3\,R + 2\,r}{2\,R + r}\,.\,\dfrac{r}{2}$

Prop. XVI.—*The volume generated by the revolution of a trochoid about its base is equal to that of a cylinder having the circle $a\,q\,b$ for base and height equal to the circumference of a circle of radius* $\frac{3}{2}$ R + r; *that is, this volume* $= r^2(3R + 2\,r)\,\pi^2$.

By Guldinus' 2nd prop., vol. = (area $d'a\,d$) $2\,\pi\,b$ G

$$= \odot\;a\,q\,b\,\frac{2\,R + r}{r}\,\pi\,\frac{3\,R + 2\,r}{2\,R + r}\,r = \odot\;a\,q\,b\,(3\,R + 2\,r)\,\pi$$

= vol. of cylinder having circle $a\,q\,b$ for base, and height equal to circumference of a circle of radius $\frac{3}{2}$ R + r; or, vol. $= r^2\,(3\,R + 2\,r)\,\pi^2$.

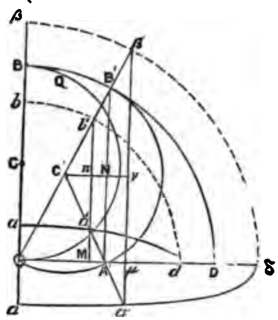
Appendix to Section III.

ELLIPTICAL HYPOTROCHOIDS.

The hypotrochoid becomes an ellipse when the diameter of the rolling circle is equal to the radius of the fixed circle.

Let BB′D (fig. 69) be the fixed circle, BQO the rolling circle, when tracing point a is on the radius

Fig. 69. (Note that two lower a's are Greek.)



BCO. We have already seen (p. 68) that when the circle has rolled to position B′A′O, the tracing radius has its extremity A′ on OD perp. to OB, and B′A′ is perp. to OD (OC′B′ being diametral). Take C′a' on C′A′, equal to C a, then a' is the tracing point. Taking C b = C a, describe arc $b\,b'd$ about O as centre, cutting OB′ in b'. Then C′b' = C′a', and ∴ $b'a'$ is parallel to B′A′ and perp. to OD, which let it meet in M, and draw C′N perp. to B′A′, bisecting $b'a'$ in n. Then

$$a'M : a'n :: a'A' : a'C' :: a\,O : a\,C$$

$$\therefore\ a'M : b'M\,(= a'M + 2\,a'n) :: a\,O : O\,b.$$

Wherefore a' is a point on an ellipse having O a as semi-minor axis, and $b\,b'd$ as auxiliary circle,—*i.e.*, having O d and O a (or R + r and R − r) as semi-axes.

If $r >$ R, or the tracing point is in CO produced, as at a, it may be shown in like manner that when the tracing radius has any other position C′A′a', the tracing point a' lies on an ellipse having O δ (D δ = O a) and O a as semi-axes, that is, having semi-axes equal to r + R and r − R, respectively.

Schol.—An ellipse with given semi-axes, a and b, can be traced out equally by taking the radius of the fixed circle equal to $\frac{1}{2}(a+b)$ or $\frac{1}{2}(a-b)$. In the former case, the tracing radius $= \frac{1}{2}(a+b) - b = \frac{1}{2}(a-b)$; in the latter the tracing radius $= \frac{1}{2}(a-b) + b = \frac{1}{2}(a+b)$.

THE TRISECTRIX.

When the radius of the rolling circle of an epitrochoid is equal to that of the fixed circle, and $r = 2$ R, the curve is called the *trisectrix*. The property of trisecting angles from which it derives its name may be thus established.

Let BDB′ (fig. 70), centre O, be the fixed circle; EQD, centre C, the rolling circle (ECDO diametral), when the tracing radius is in the position CDO, or (since CD = DO = R = $\frac{1}{2}r$) the tracing point is at O. When the rolling circle is in position B′QA′, A′C′B′O diametral, let C′Pp be the tracing radius, cutting B′QA′ in P. Then arc PB′ = arc B′D; ∴ angle OC′p = angle C′OC; and since C′p = OC, the triangles OC′p and C′OC are equal in all respects. Thus,

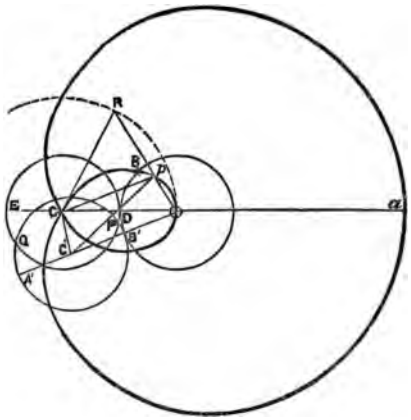
$$\text{angle } p\,OC' = \text{angle } CC'O$$

and

$$\text{angle } COC' = \text{angle } p\,C'O;$$

$$\begin{aligned} \therefore \text{angle } pOC &= \text{angle } pC'C = \text{angle } OC'C - p\,C'O \\ &= \text{right angle} - \tfrac{1}{2}\text{ angle } COC' - \text{angle } p\,C'O \\ &= \text{right angle} - \tfrac{3}{2}\text{ angle } p\,C'O \\ &= \text{right angle} - \tfrac{3}{2}\text{ angle } OC\,p. \end{aligned}$$

Fig. 70.



Wherefore, if Op produced meet in R a circle described about C as centre, through O,

$$\begin{aligned} \text{angle } ROC + \text{angle } CRO &= 2\text{ angle } p\,OC \\ &= 2\text{ right angles} - 3\text{ angle } OC\,p; \end{aligned}$$

but angle ROC + angle CRO

$$= 2 \text{ right angles} - \text{angle RCO};$$

$$\therefore \text{angle RCO} = 3 \text{ angle OC}p.$$

Hence the trisectrix affords the following construction for trisecting any given angle RCO. With centre C and radius CO describe arc OR, cutting CR in R. Join OR, cutting the loop OBC in p; then angle RCO = 3 angle p CO, or C p trisects the angle RCO.

SCHOL.—Both the tricuspid epicycloid and the tricuspid hypocycloid are *trisectrices*. See Exs. 91, 92.

THE SPIRAL OF ARCHIMEDES REGARDED AS AN EPITROCHOID.

The curve traced out by a point retaining a fixed position with respect to a straight line which rolls without sliding on a circle, in the same plane as line and point, may be regarded as an epitrochoid, whose generating circle has an infinite radius.

Supposing the tracing point on R r, fig. 71, T′DT the rolling straight line, it will easily be seen that if this point is near D, the curve will resemble DPR, only instead of a cusp near D there will be simply strong curvature convex towards O, and two points of inflexion, one on each side of R r. When the point is remote from D, the curve will be concave towards O throughout. It is easily seen from the formula at page 119 (or it can be readily proved independently*) that

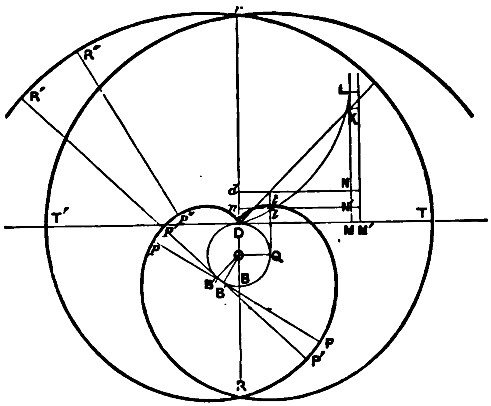
* For the independent geometrical proof, it is only necessary to show that the tracing point recedes from R r initially at the

if the tracing point lies at d such that $\mathrm{D}\,d = \mathrm{DO}$, the radius of curvature will be infinite at d, the two points of inflexion coinciding there, for from the proportion

$$r : \mathrm{R} :: \mathrm{R} : \mathrm{F}+\mathrm{R},$$

we have $$\mathrm{R}-r : \mathrm{R} :: \mathrm{F} : \mathrm{F}+\mathrm{R}.$$

FIG. 71.



Wherefore, since the ratio $\mathrm{R} : \mathrm{F}+\mathrm{R}$ is one of equality when R is infinite,

$$\mathrm{R}-r = \mathrm{F}\text{; that is, } d\,\mathrm{D} = \mathrm{DO}.$$

When the tracing point is on DR there will be a loop. We need not consider the various curves traced out according to the varying position of the point d, either

same rate at which the point of contact between the generating line and the fixed circle recedes from R r; which is obvious, since D d as it moves with the rolling tangent is constantly parallel to the radius from O to the point of contact just named, and in its initial motion the point D moves in direction D r.

on D r or on DR. There is, however, one case which is historically interesting, and may therefore be considered here, though briefly.

When the tracing point is at O, the curve traced out becomes the spiral of Archimedes, a curve so called because, though invented by Conon, it was first investigated by his friend Archimedes. It was defined as the curve traversed by a point moving uniformly along a straight line, which revolves uniformly around a centre. So traced it is only perfect as a spiral when the moving point is supposed first to approach the centre from an infinite distance, and after reaching the centre to recede along the prolongation of its former course to an infinite distance. Regarded as a trochoid, the complete spiral (or rather the part near the centre) will be traced out by supposing TDT′ to roll first in one direction from the position where the tracing point is at D, and afterwards in the other direction.

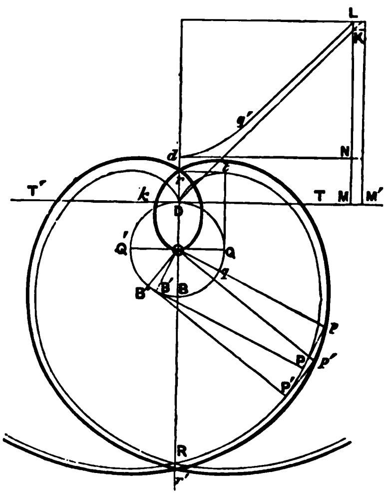
The identity of this epitrochoid with the spiral of Archimedes is easily demonstrated. Thus, let p (fig. 72) be a point on the curve, B′P the corresponding position of the rolling tangent, P p being the position of the line which had been coincident with OD, so that Pp is perp. to B′P, and B′P equal in length to the arc DQB′. Then, since OB′ is perp. to B′P and equal to Pp, Op = B′P. And Op is parallel to B′P, the rolling line, whose direction has changed through an angle measured by the arc DQB′, which is equal to B′P or Op. Hence the distance of p from O is proportional to the angle through which

Op has revolved from its initial direction OQ′ (parallel to DT′). Therefore p is a point on a spiral of Archimedes.

AREA OF THE SPIRAL OF ARCHIMEDES.

The area of the curve is thus determined:—Let $p\,p'$ be neighbouring positions of the tracing point; B′Pp, B″P′p' corresponding positions of the rolling tangent

FIG. 72.



with its perp. Then O p is equal and parallel to B′P; O p' to B″P′. Wherefore, in the limit, area p O p' = area PB″P′. Hence, increment of area O $k\,r\,p$ = increment of area D t PB′Q; and these areas begin together: they are therefore equal. But PB′ and P′B″ are normals to D t P, the involute of the circle DQB;

therefore, area $\mathrm{D}\,t\,\mathrm{PB'Q} = \frac{1}{3}\frac{(\mathrm{B'P})^3}{\mathrm{OD}}$; (see p. 85)

that is, area $\mathrm{O}\,r\,p = \frac{1}{3}\frac{(\mathrm{O}p)^3}{\mathrm{OD}}$.

ARC OF THE SPIRAL OF ARCHIMEDES.

The arc of this spiral may be thus determined. Drawing DK (fig. 72), as in fig. 71, and representing element of arc PP′ by an element of area KM′ (KM = DM = B′P), let LM be so taken that element of area LM′ represents the increment of arc pp'. Now the tangent at p is perp. to B′p, so that in the limit (angle $p\,\mathrm{O}\,p'$ being equal to angle PB′P′),

$$p\,p' : \mathrm{PP'} :: \mathrm{B'}p : \mathrm{B'P};$$

$$\therefore (p\,p')^2 : (\mathrm{PP'})^2 :: (\mathrm{B'}p)^2 : (\mathrm{B'P})^2$$
$$:: (\mathrm{PO})^2+(\mathrm{OB'})^2 : (\mathrm{B'P})^2$$

or $$(\mathrm{LM})^2 : (\mathrm{KM})^2 :: (\mathrm{DM})^2+(\mathrm{OD})^2 : (\mathrm{DM})^2$$
$$:: (\mathrm{KM})^2+(\mathrm{OD})^2 : (\mathrm{KM})^2$$
$$\therefore (\mathrm{LM}^2 = (\mathrm{KM})^2+(\mathrm{DO})^2$$

or $$(\mathrm{LM})^2-(\mathrm{KM})^2 = (\mathrm{OD})^2$$

Wherefore L is a point on rectangular hyperbola $d\,q'$L, having D$\,d$ = OD as semi-axis, D as centre, and DK as an asymptote; and

arc Orp : arc DtP :: hyperb. area DdLM : △DKM.

$$:: \text{rect. DL} + \text{sq. on OD}\left(\log_e\frac{\mathrm{DM}+\mathrm{ML}}{\mathrm{DO}}\right) : \text{sq. on DM}.$$

or, since arc $\mathrm{D}\,t\,\mathrm{P} = \frac{(\mathrm{B'P})^2}{2\,\mathrm{OD}}$ (p. 84) $= \frac{\mathrm{DM}^2}{2\,\mathrm{OD}}$;

$$\text{Arc O}\,r\,p = \frac{\text{rect. DL}}{2\,\mathrm{OD}} + \left(\log_e\frac{\mathrm{DM}+\mathrm{ML}}{\mathrm{DO}}\right)$$

COR. The loop cuts the axial line BO d in a point r, such that O r = Q t (the tangent drawn to DQB, parallel to OD, meeting involute D t R in t) = arc DQ.

SCHOL.—The curve, as it recedes from O, approaches more and more closely to the involute of the circle Q′DQ, the curves being asymptotic. All that has been said about the figure of the involute of the circle at a great distance from O (pp. 82, 83), applies therefore to the spiral of Archimedes.

We have seen that the epicycloid, traced by the point O, fig. 72, carried along with T′DT, as it rolls on the fixed circle Q′DQ, is a spiral of Archimedes. *To prove the converse of this,*—

Let a point start from O in direction OQ′, travelling uniformly with velocity v along radius OQ′, while this radius turns uniformly with angular velocity ω around O in direction Q′DQ. After a time t, let the point be at p; then O p = $v\,t$ and

$$\text{angle Q}'\text{O}\,q \text{ (greater than 2 rt. angles)} = \omega\, t.$$

Now if, with radius OQ′ = F, we describe a circle Q′DQB about O as centre, intersecting O p in q, then arc Q′D q = F $\omega\, t$; and if F be such that F $\omega = v$ (in other words, if F be such that motion in a circle of radius F, with angular velocity ω round the centre, gives linear velocity v), then arc Q′D q (= F $\omega\, t$) = $v\, t$ = O p. Wherefore, drawing OB′ perp. to O p, and completing the rectangle OB′P p,

$$\text{B}'\text{P} = \text{O}\,p = \text{arc Q}'\text{D}\,q = \text{arc B}'\text{QD};$$

and $$\text{P}\,p = \text{OB}' = \text{F}.$$

∴ P is the position of the point D on tangent DT after rolling round arc DQB′ to tangent at B′, and P p is the position then taken up by DO. *Hence as T′DT rolls on the circle Q′DQ, the point O regarded as rigidly attached to T′OT, the tangent to circle Q′DQ of radius F, at D, will trace out a spiral of Archimedes in which the linear velocity of the moving point along the revolving radius is equal to F . angular velocity of the latter.*

PROP.—*The axis of a planet's shadow in space is a spiral of Archimedes.*

The spiral of Archimedes is interesting as the path along which the centre of a planet's shadow (the earth's for example) may be regarded as constantly travelling outwards with the velocity of light.

This is easily seen if we suppose the earth and its shadow momentarily reduced to rest, and, with the sun as pole, imagine a radius vector carried from an initial position coinciding with the earth and retrograding through the various portions of the shadow. Let V be the velocity of the earth in her orbit, D her distance from the sun, and therefore $\frac{V}{D}$ her angular velocity about the sun. Also let L be the velocity of light. Then if our radius vector, carried back through an angle θ, corresponding to the earth's motion in time t, is equal to r, we have $\frac{V}{D} t = \theta$, or $t = \frac{D}{V} \theta$; and $r = L t = \frac{L.D}{V} \theta$. Wherefore, since the radius vector varies as the vectorial angle, the corresponding point of the shadow's axis (which was at the earth at time t before the epoch we are considering) lies on a spiral of Archimedes. We have in fact L, the velocity of light, for the velocity along the radius vector (v in the preceding demonstration), when the angular velocity about the sun is taken equal to the earth's angular velocity in her orbit, or $\frac{V}{D}$ (corresponding to ω in preceding demonstration). The radius F of the fixed circle by which this tremendous spiral could be traced out, would therefore be such that $F\frac{V}{D} = L$, or $F = \frac{L}{V}D$ = the radius of the earth's orbit increased in the ratio in which the velocity of light exceeds the velocity of the earth in her orbit. Thus

$$F = 92{,}000{,}000 \text{ miles} \times \frac{187{,}000}{18.4} \text{ (roughly)}$$
$$= 5{,}000{,}000 \times 187{,}000 \text{ miles}$$
$$= 935{,}000{,}000{,}000 \text{ miles.}$$

[It is convenient to remember that the sun's distance is nearly equal to five million times the mean distance traversed by the earth in one second.]

NOTE.—The student will find further information respecting spiral epitrochoids in the examples on pp. 254–256. The solution of these examples presents no difficulty.

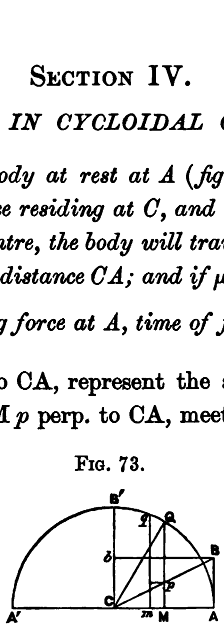
Section IV.

MOTION IN CYCLOIDAL CURVES.

Lemma.—*When a body at rest at A (fig. 73) is acted on by an attractive force residing at C, and varying as the distance from the centre, the body will travel to C in the same time whatever the distance CA; and if μ . CA is the measure of the accelerating force at A, time of fall to A* $= \dfrac{\pi}{2\sqrt{\mu}}$.

Let AB, perp. to CA, represent the accelerating force at A; join CB, then Mp perp. to CA, meeting CB in p, repre-

Fig. 73.



sents the accelerating force at M; (vel.^{2} at M) is represented

by $2 \,.\, \mathrm{M}p\,\mathrm{BA}^* = 2\,\mathrm{CAB}\,\dfrac{(\mathrm{CA})^2-(\mathrm{CM})^2}{\mathrm{CA}^2} = \text{rect. } b\,\mathrm{A}\,.\left(\dfrac{\mathrm{QM}}{\mathrm{CQ}}\right)^2$,

(AQA′ being a circle about C as centre). That is,

* Any elementary rectangle pm represents Mm . accelerating force at M; or since the force may be considered uniform throughout the space Mm, pM represents half the increase of the square of the velocity (by well-known relation in case of uniform force). Hence, area pA represents $\frac{1}{2}$ (vel.^{2} at M − vel.^{2} at A) = $\frac{1}{2}$ (vel.^{2} at M).

$$\text{Vel. at M is represented by } \frac{QM}{CQ} \cdot \text{rect. } b\,A\,;$$

or, $$\text{Vel. at M} = \frac{QM}{CQ} \cdot V,$$

where V is the velocity with which a particle would reach C after traversing distance AC under the force at A continued constant.

But if $Q\,q$ is a small element of arc at Q and $q\,m$ perp. to CA, then, ultimately,

$$m\,M = \frac{QM}{CQ} \cdot Q\,q.$$

Therefore time of traversing $m\,M = \dfrac{m\,M}{\text{vel. at M}} = \dfrac{Q\,q}{V}$; or,

$$\text{incrt. of time from beginning} = \frac{1}{V} \cdot \text{the incrt. of arc AQ.}$$

Hence, $$\text{time of fall from A to M} = \frac{\text{arc AQ}}{V}.$$

But if $\mu \,.\, CA$ is the measure of the accelerating effect of the force at A, $V^2 = 2\,\mu\,CA \,.\, \dfrac{CA}{2} = \mu\,(CA)^2$

$$\text{or } V = \sqrt{\mu} \,.\, CA\,;$$

Thus, vel. at $M = \sqrt{\mu} \,.\, QM$; and time of fall from A to M

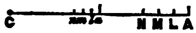
$$= \frac{\text{arc . AQ}}{\sqrt{\mu} \,.\, CA} = \frac{1}{\sqrt{\mu}} \,.\, \text{circular measure of } \angle\, CQA.$$

Thus, time of fall to $C = \dfrac{1}{\sqrt{\mu}}$. circ. meas. of rt. angle$= \dfrac{\pi}{2\sqrt{\mu}}$; and is therefore independent of the original distance CA.

Schol.—The general relation of this lemma may be regarded as obvious, seeing that a force varying as the distance from the centre is in this case a force varying as the distance remaining to be traversed; and this relation holding from the beginning, it follows that whether such distance be large or small, it will be traversed in the same time. The general relation may be considered, in this aspect, as follows :—

Let C, fig. 74, be the centre of force, and let one particle start from A, another from *a*, in the same straight line CA. Divide CA and C *a* each into the same number of equal elements, and let *l*, *m*, *n*, and L, M, N be the points of division nearest to *a* and A, respectively. Then the force on the particles starting from A and *a* may be regarded as severally uniform while these particles traverse the spaces AL, *a* respectively; hence these spaces being proportional to AC, *a* C, that is to the uniform forces under which they are traversed, will be traversed in equal times; and velocities proportional to the forces, that is to ML and *l m* respectively, will be generated in those times. Again, since the forces acting on the particles at L and *l* are proportional to the spaces LM, *l m*, and the velocities with which the particles begin to traverse these spaces also proportional to LM, *l m*, it follows that the times in LM, *l m*, will be equal; and the total velocity acquired at the end of those times will still be proportional to ML and *m l*, or to MN and *m n*, the spaces next to be traversed. And so on continually. Hence the particles will arrive at C simultaneously; and the velocities with which they reach C will be proportional to AC and *a c*.

Fig. 74.



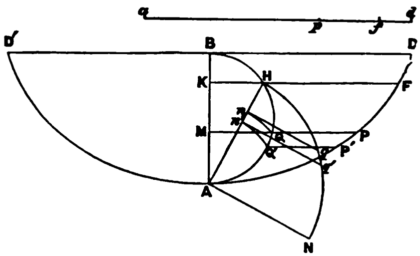
It is manifest, also, that if the particles during their progress to C be resisted in a degree constantly proportional to the velocity, the times of reaching C will still be equal.

PROPOSITIONS.

PROP. I.—*If A (fig. 75) be the vertex, A B the axis of an inverted cycloid DPA, a particle let fall from a point F on the arc APD (supposed perfectly smooth) will reach A in the same time wherever F may be.*

Let P be a point on the arc AF; draw PM perp. to AB cutting the generating circle in Q and join AQ. Represent

FIG. 75. (Join AQ.)



the accelerating force of gravity by g. Then since the tangent at P is parallel to AQ,

$$\text{Acc}^{g}.\text{ force at P along PP}' : g :: \text{AM} : \text{AQ} :: \text{AQ} : \text{AB};$$

or, the accelerating force at P in direction of motion

$$= g \cdot \frac{\text{AQ}}{\text{AB}} = g \cdot \frac{\text{arc AP}}{2\,\text{AB}}$$

Hence if the straight line $a\,d =$ arc AD, and we take af=arc AF, and $a\,p$=arc AP, the acceleration of the particle at P is the same as that of a particle moving from f to a under the action of a force varying as the distance from a, and equal at p to $g \cdot \frac{ap}{2\text{AB}}$, or at d to $g \cdot \frac{ad}{2\text{AB}} = g$. The time of fall, then, (by lemma, p. 135) is independent of the position of F.

Since in this case the accelerating force at D $= g = \frac{g}{4R}$. arc APD, the μ of lemma $= \frac{g}{2R}$, and time of fall from any point of arc APD to A $= \sqrt{\frac{4R}{g}} \cdot \frac{\pi}{2} = \pi \sqrt{\frac{R}{g}}$.

The time of oscillation from rest to rest on either side of A $= 2\pi \sqrt{\frac{R}{g}}$.

SCHOL.—This proposition is easily established independently. Thus take an elementary arc PP′; draw ordinates FHK, PQM, and P′Q′ (Q,Q′, on BQA); arcs Qn, Q′n' about A as centre, to AH; and $n q$, $n'q'$ perp. to AH, meeting quadrantal arc HqN on AH in q, q'. Then, (vel.)^2 along PP $= 2\,g$. KM

$$= 2g\,(\text{AK}-\text{AM}) = \frac{2g}{\text{AB}}\,(\text{AH}^2 - \text{AQ}^2) = \frac{2g}{\text{AB}} \cdot (n\,q)^2; \text{ or,}$$

$$\therefore \text{ vel. at P} = \sqrt{\frac{2g}{\text{AB}}}\,(n\,q); \ \& \ \text{PP}' = 2\,(\text{AQ}-\text{AQ}') = 2\,n\,n';$$

$$\text{time along PP}' = 2\,n\,n' \div \sqrt{\frac{2g}{\text{AB}}}\,(nq) = \sqrt{\frac{2\text{AB}}{g}} \cdot \frac{qq'}{\text{A}\,q};$$

$$= \sqrt{\frac{2\text{AB}}{g}} \text{. circ. meas. of } q\,\text{A}\,q';$$

and time along FPA

$$= \sqrt{\frac{2\text{AB}}{g}} \cdot \frac{\pi}{2} = \pi \sqrt{\frac{\text{R}}{g}}, \text{ as before.}$$

PROP. II.—*A particle will pass in the same time to A along a smooth epicycloidal arc APD (A the vertex, APB O diametral,) under the action of a repulsive force at O varying directly as the distance, from whatever point on APD the particle starts.*

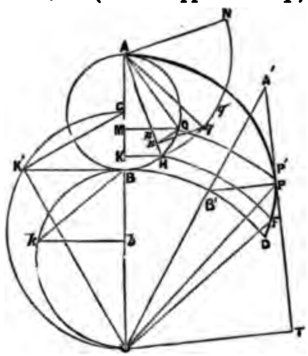
Let the particle start from F. At P on the arc FA, draw the tangent A′PT, and the normal PB′; then OB′A′ cuts the generating circle through P diametrally in B′A′, (B′ on the base BD); and OT perp. to A′T is parallel to B P. Describe arc PQ about O as centre, to meet central generating circle AQB, and join OP, AQ.

Then if the measure of accelerating force at A $= \mu$. OA, accelerating force at P $= \mu$. OP; and the accelerating force in direction PP′ $= \mu \, \text{OP} \cdot \frac{\text{PT}}{\text{OP}} = \mu \, \text{PT}$

$$= \mu \, \text{A P} \cdot \frac{\text{OB}'}{\text{B A}'} = \mu \cdot \frac{2\text{AQ} \cdot \text{OC}}{\text{OB}} \cdot \frac{(\text{OB})^2}{2\text{BA} \cdot \text{OC}}$$

$$= \frac{\mu \cdot \text{F}^2}{4\text{R}(\text{F}+\text{R})} \cdot \text{Arc APD (Prop. VI., sec. 2).}$$

Fig. 76. (Accent upper *n* and *q*.)



Therefore (applying lemma, p. 135, as in the case of cycloid) the time in which particle reaches A

$$= \frac{\sqrt{4\text{R}(\text{F}+\text{R})}}{\text{F}\sqrt{\mu}} \cdot \frac{\pi}{2} = \frac{\pi}{\text{F}} \cdot \sqrt{\frac{\text{R}(\text{F}+\text{R})}{\mu}}.$$

The time of oscillation from rest to rest on either side of A is twice this.

Schol.—This proposition may be proved independently of the lemma, by a demonstration similar to that used for the cycloid. The figure indicates the construction. We begin by showing that $(\text{Vel.})^2$ at $M = \mu\,[(OP)^2 - (OF)^2]$

$$= \mu\,[(OM)^2 + (MQ)^2 - (OK)^2 - (KH)^2]$$

$$= \mu\,[MK\,(OM+OK) + AM\,(MK+KB) - KB(MK+AM)]$$

$$= \mu \cdot 2MK\,(OA+OB)$$

$$= \mu\,[(AH)^2 - (AQ)^2)]\,\frac{F+R}{R}$$

$$= \mu\,(n\,q)^2\,\frac{F+R}{R}.$$

The rest follows directly, as in case of cycloid.

Prop. III.—*A particle will pass in the same time to the vertex of a smooth hypocycloidal arc under the action of an attractive force at the centre varying directly as the distance, from whatever point on the arc the particle starts.*

The construction and demonstration are in all respects similar to those in the case of epicycloid, Prop. II.

$$\text{Time of motion from F to A} = \frac{\pi}{F}\sqrt{\frac{R\,(F-R)}{\mu}};$$

and $$(\text{Vel.})^2 \text{ at } P = \mu\,(n\,q)^2\left(\frac{F-R}{R}\right).$$

Schol.—The time of oscillation in the epicycloid under force above considered : time of an oscillation in cycloid under gravity (the radii of generating circles being equal) $:: \sqrt{g\,(F+R)} : F\sqrt{\mu}\cdot$

This follows directly from the values above determined for the times of motion to A.

That the times of oscillation may be equal, we must have

$$(F+R)\,g = \mu\,F^2;\ \text{or}\ \mu = \frac{F+R}{F^2}\,g.$$

Since this gives $\mu\,F = \dfrac{F+R}{F}\,g$, it follows that the accelerating force at A in the epicycloid must exceed the force of gravity in the ratio OC : OB, in order that the oscillations may be performed in the same time as in a cycloid of equal generating circle, under gravity. The force in the epicycloid will equal gravity at a distance from $O = \frac{F^2}{F+R} = OK'$, obtained as in fig. 76 by drawing BK′ perp. to OC to meet semicircle on OC as diameter in K′.

If we take $\mu F = g$, a cycloid in which the oscillations under gravity will be the same as the oscillations in the epicycloid must have a generating circle whose radius $= \frac{(F+R)R}{F} = \frac{OC \cdot CB}{OB} = \frac{(CK')^2}{OB} = Bb$, obtained by drawing semicircle BkO, taking $Bk = CK'$, and drawing kb perp. to BO.

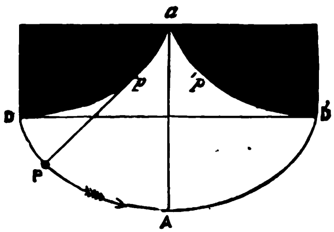
Corresponding considerations and constructions apply in the case of hypocycloid.

It is manifest (see scholium to lemma) that if the particle in its passage along the epicycloidal, hypocycloidal, or cycloidal arc, be resisted in a degree constantly proportional to the velocity, the periods of oscillation will still be isochronous; the arc of oscillation, however, will no longer be symmetrical on either side of the axis, but will continually be reduced, each complete arc of oscillation being less than the arc last described.

A weight may be caused to oscillate in the arc of an inverted cycloid in the manner indicated in fig. 77. Here a A is a string swinging between two cycloidal cheeks $a\,p$ D, $a\,p'$ D′, a being a cusp, and DD′, the common tangent at the vertices D, D′, being horizontal. The length of the string a A being equal to twice the axis of $a\,p$ D, or to the arc $a\,p$ D, the weight swings in the cycloidal arc DAD′ (Prop. XI. section 1). Such a pendulum would vibrate isochronously, if there were no friction and the string were weightless; but in practice the cycloidal pendulum does not vibrate with perfect isochronism.

An approach to isochronism is secured in the case of an ordinary pendulum by having the arc of vibration small compared with the length of the pendulum. In this case the small circular arc described by the bob may be regarded as coincident with a small portion of the cycloidal arc DAD′ (fig. 75) near to A, and the isochronism thence inferred. But

FIG. 77.



in reality the approach to isochronism in the case of a long pendulum oscillating in a small arc, is best proved as a direct consequence of the relation established in the lemma.

Thus, let ACA′ (fig. 73) be the arc of oscillation of a pendulum, whose length l is so great, compared with AA′, that ACA′ may be regarded as straight. Then the accelerating force in the direction of the bob's motion when at M

$= g$. sin. deflection from the vertical $= g \cdot \frac{\text{CM}}{l}$ very nearly,

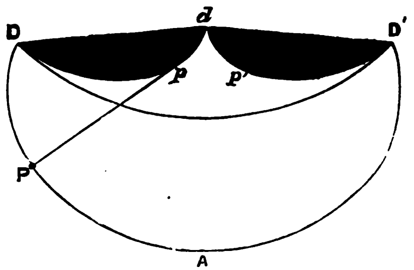
or varies as CM. Hence the time of oscillation is very nearly constant, whatever the range on either side of C, so only that the arc of oscillation continues very small compared with l.

The accelerating force towards C at M being $\frac{g}{l}$. CM, the time of an oscillation from rest to rest is $\pi\sqrt{\frac{l}{g}}$;

and the Vel. at $M = QM\sqrt{\frac{g}{l}} = \sqrt{\frac{g}{l}(CA^2 - CM^2)}$.

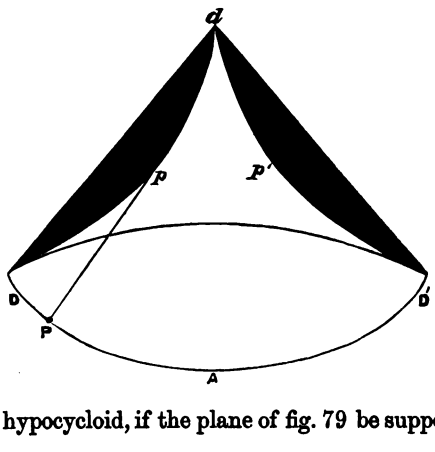
A pendulum may be made to swing in an epicycloidal arc in the way shown in fig. 78, or in a hypocycloidal arc in

FIG. 78.



the way shown in fig. 79 (Prop. XII. sect. 2); but of course the oscillations will not be isochronous under gravity. In the

FIG. 79.



case of the hypocycloid, if the plane of fig. 79 be supposed horizontal, P a smooth ring running on the arc DAD′, and this ring be connected with the centre of the fixed circle by an exceedingly elastic string, very much stretched, the oscillations of the ring will be very nearly isochronous. For the tension of a stretched elastic string is proportional to the extension, and if when the ring is at A the string is stretched to many times its original length, the extension when the ring is at different parts of the arc DAD′ is very nearly proportional to the extended length. Suppose, for instance, that when at A the string were extended to 100 times its original length, then the extension would only be less than the actual length by one 100th part.

If the circular arc DD′ represent part of a great circle of the earth's surface, DAD′ a hypocycloidal tunnelling having DD′ as base, then, since the attraction at points below the surface of the earth varies directly as the distance from the centre, a body would oscillate in DAD in equal periods. It would not, however, be possible to construct such a tunnelling, or to make its surface perfectly smooth.

PROP. IV.—*The path of quickest descent from D to any point F not vertically below D, is a cycloidal arc through F, having its cusp at D and its axis vertical.*

The following is a modification of Bernouilli's original demonstration.

The path of descent will necessarily be in the vertical plane through D and F. Let it be DPF, and let PP′ P″ be a small portion of this path, represented on a much enlarged scale in fig. 80*a*.

Let *p* be a point on a horizontal line through P′, and close to P. Then since DPF is the path of quickest descent, the time of descent down the arc PP′P″ is a minimum, and from the nature of maxima and minima it follows that the change in the time of fall resulting from altering the arc PP′P″ into the arc Pp′P″ is evanescent, compared with the total time of fall down PP′P″. If this time were increased in an appreciable ratio by passing from P′ to a point p on one side, it would be appreciably diminished by passing from P′ to a point on the other side of P′, which is contrary to the supposition that DPF is the arc of quickest descent. Now regarding PP′ and P′P″ as straight lines, draw $p'l$ perp. to PP′ and P′m perp. to P″p', so that ultimately Pl = Pp', and

FIG. 80.

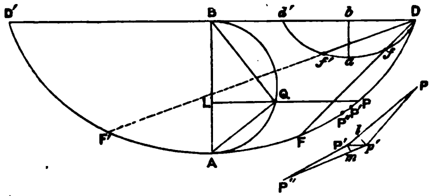


FIG. 80*a*.

P″m=P″P′. Therefore, if we suppose PP′ and Pp' traversed with the uniform velocity V, then $\frac{P'l}{V}$ represents the excess of time in PP′ over time in Pp'; and if we suppose P′P″ and p'P″ traversed with the uniform velocity V′, then $\frac{p'm}{V'}$ represents the defect of time in P′P″ from time in p'P″. Therefore since time along PP′P″ = time along Pp'P″, we must have $\frac{P'l}{V} = \frac{p'm}{V'}$, or $\frac{V}{V'} = \frac{P'l}{p'm}$ $= \frac{\cos PP'p'}{\cos P'p'P''}$. That is, the velocity at different points along the arc of descent varies as the cosine of the angle at which the arc is inclined to the horizon at these points. But this is a property of motion in an inverted cycloid. For if DPFAD′ is a cycloidal arc having D and D as cusps, AB as axis, and AB vertical, and PL is drawn perp. to AB, cutting central generating circle in Q, then

$$(\text{Vel.})^2 \text{ at } P = 2\,g\,.\,BL = 2\,g\,\frac{(BQ)^2}{AB} = 4\,g\,R\,.\left(\frac{BQ}{AB}\right)^2$$

$$\text{i.e. Vel. at } P = 2\,\sqrt{g\,R}\,.\,\cos\,ABQ = 2\,\sqrt{g\,R}\,.\,\cos AQL,$$

the required relation, since AQ is parallel to the tangent at P.

Hence DPF is part of a cycloid having its cusp at D and its axis vertical.

To describe the required arc, draw any cycloid D*f d′* having D as cusp, its base D*d′* horizontal, and cutting DF in *f*; then D′ so taken that

$$DD' : D\,d' :: DF : D\!f$$

is the base of the required cycloid through F. The axis BA, bisecting DD′ at right angles, bears to *b a*, the axis of D *a d′*, the ratio DF : D*f*.

Schol.—The arc is not necessarily one of descent throughout. If F′ be the point to be reached, and the angle of inclination of D*f′* to the horizon is less than the angle *b* D *a*, the path from D to F′ will include the vertex A, and the particle will be ascending from A to F′.

The cycloid DAD′ is the path of quickest motion from D to D′ at the same horizontal level as D.

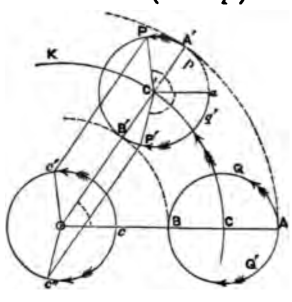
SECTION V.

EPICYCLICS.

DEF.—*If a point travels uniformly round the circumference of a circle, whose centre travels uniformly round the circumference of a fixed circle in the same plane, the curve traced out by the moving point is called an epicyclic.*

Let AQB (fig. 81) be the circle round which the tracing point travels, CC′K the circle in which the centre C of the moving circle AQB is carried, O the centre of the fixed circle CC′K. Then the circle CC′K is called the *deferent*, AQB the

FIG. 81. (Join C′p.)



epicycle, O the *centre*, C the *mean point*, P the *tracing point.* At the beginning of the motion let the tracing point be at A in OC produced, or at its greatest possible distance from O. When the centre is at C′ let the tracing point be at P. Draw the epicyclic radius C′*a* parallel to CA, and let OC produced meet the epicycle in A′; also let OA and OA′ cut the epicycle respectively in B and B′. Then C′*a* is the position to which CA has been carried by the motion of the epicycle, and *a* A′P is the arc over which the tracing point has travelled, in the same time. The angle PC′*a* is called the *epicyclic angle*, and the angle C′OC the *deferential angle*. Both motions being uniform, the deferential angle bears a constant ratio to the epicyclic angle. Call this ratio 1 : *n*; so that 1 : *n* is the ratio of the angular velocities of mean point round centre, and of tracing point round mean point. If we represent the radius of the deferent by D, and the radius of the epicycle by E, the linear velocities of the motions just mentioned are in the ratio D : *n* E.

The deferential motion may be conveniently supposed to take place in all cases in the same direction around O,—that indicated by the arrow on CC′. Such motion is called *direct*. Angular motion in the reverse direction is called *retrograde*. When the motion of the tracing point round the mean point is direct, *n* is positive; we may for convenience say in this case that *the epicycle is direct*, or that the curve is a *direct epicyclic*. When the motion of the tracing point round the mean point is retrograde (as, for instance, if the tracing point had moved over arc *a q*′ P′ while mean point moved over arc CC′), *n* is negative, and we say *the epicycle is retrograde*, or that the curve is a *retrograde epicyclic*.

The straight line joining the centre and the tracing point in any position is called the *radius vector*. A point such as A, where the tracing point is at its greatest distance (D + E) from O, is called an *apocentre*. A point where the tracing point is at its least distance (D − E) from the centre is called a *pericentre*. Taking an apocentre as A for starting point, OA is called the *initial line*, and the angle between the radius vector and the initial line is called the ***vectorial angle.*** This angle is estimated always in the same direction as the deferential angle: so that if at the beginning the motion of the tracing point round O was retrograde, the vectorial angle would at first be negative.

Whatever value n may have, save 1 (in which case the tracing point will manifestly move in the circle AA′), the tracing point will pass alternately from apocentre on the circle AA′ to pericentre on the circle BB′, thence to apocentre on the circle AA′, and so on continually. The angle between an apocentral radius vector and the next pericentral radius vector is called the *angle of descent.* It is manifestly equal to the angle between a pericentral radius vector and the next apocentral radius vector, called the *angle of ascent.*

PROPOSITIONS.

PROP. I.—*The angle of descent : two right angles* :: $n \sim 1 : 1$.

When n is positive and greater than 1, the epicyclic angle PC′ a (fig. 81) exceeds the deferential angle C′OC, or A′C′a, by PC′A′, or angle PC′A′$=(n-1)$ deferential angle. But, at the first pericentre, angle PC′A′$=2$ right angles, and the deferential angle is the angle of descent. Hence,

$$2 \text{ right angles} = (n-1) \text{ angle of descent,}$$

or the angle of descent : two right angles :: $n-1 : 1$.

When n is positive and less than 1, A′C′ a exceeds the epicyclic angle p C′a by p C′A′, or angle p C′A′ $= (1-n)$ deferential angle; and proceeding as in the last case, we find

the angle of descent : two right angles :: $1-n : 1$.

When n is negative, we have the epicyclic angle a C′P′ + angle A′C′a = angle P′C′A′, or (taking the absolute value of n without regard to sign) angle P′C′A′ = $(n+1)$ deferential angle. Wherefore (proceeding as before),

the angle of descent : two right angles :: $(n + 1)$: 1.

But n being negative, the sum of the absolute values of 1 and n is the difference of their algebraic values, or $n \sim 1$. Hence for all three cases,

angle of descent : two right angles :: $n \sim 1$: 1.

SCHOL.—The angle of descent is always positive. See note, p. 185.

PROP. II.—*The epicycle traced with deferential and epicyclic radii D and E, respectively, and epicyclic vel. : deferential vel. :: n : 1, can also be traced with deferential and epicyclic radii E and D respectively, and epicyclic vel. : deferential vel. :: 1 : n.*

In fig. 81, complete the parallelogram PC′Oc'. Then Oc' = CP = E and c'P = OC′ = D. Moreover ∠ c'OC = ∠ PC′a, and c'P is parallel to OC′. Wherefore we see that while the epicyclic curve is traced out by the motion already described, the point c' travels in a circle of radius E about O as centre, with the same velocity as P round C; while P travels uniformly in a circle of radius E round c, and with the same velocity as C′ round O.

Therefore the same epicyclic curve is traced out with deferent and epicycle of radii D, E, respectively, having angular velocities as n : 1, or by deferent and epicycle of radii E, D, respectively, having angular velocities as 1 : n.

SCHOL.—Thus the deferential and epicyclic radii, D and E, can always be so taken that D is not less than E. When D = E, the curve can still be regarded as traced in either of two ways, viz., with epicyclic vel. to deferential vel. :: n : 1 or :: 1 : n. In this case all the pericentres fall at the centre.

PROP. III.—*Every epitrochoid is a direct epicyclic; and every hypotrochoid is a retrograde epicyclic.*

Let O be the centre of a fixed circle BB′D (fig. 82) on which rolls the circle AQB; and let the tracing point be at r on CA.* Let the circle AQB roll uniformly to the position A′Q′B′, C′p P being the position of the generating radius, p the tracing point. Draw C′Q′ parallel to OC. Then the centre C of the rolling circle has travelled uniformly in circle CC′ about O as centre. Again $\angle$ Q′C′p = $\angle$ Q′C′A′ + $\angle$ A′C′p=COC′$\left(1+\frac{F}{R}\right)$ (since arc A′P = arc B′B). Wherefore p is a point on an epicyclic arc, whose deferent and epicycle have radii OC and C r, or (R + F) and r respectively, and whose epicyclic angle : deferential angle :: R+F : R. *Or*, by preceding proposition, we may have r and R + F for radii of deferent and epicycle respectively, having R : R+F for ratio of epicyclic and deferential angles.

In this case n is greater than 1 and positive.

Next, fig. 83, let the circle AQB roll around instead of on the circle BB′D. Then the above proof holds in all respects, save that the angle Q′C′p now = $\angle$ Q′C′A′ − $\angle$ A′C′p, and radius OC = R − F instead of R + F. Thus in this case, the epitrochoid gives an epicyclic curve having for deferential and epicyclic radii (R−F) and r, respectively, and deferential angle : epicyclic angle :: R−F : R; *or* else, deferential and epicyclic radii r and (R−F) and ratio of deferential and epicyclic angles as R : R−F.

In this case n is less than 1 and positive.

Next let O be the centre of a fixed circle BB′D, inside which, figs. 84 and 85, rolls the circle AQB; and let the

* Or at r', on CA produced, in which case read p' for p throughout the demonstration, for all four cases.

tracing point be at r. Then following the words of proof for the case of epitrochoid with modifications corresponding to the two figs. 84 and 85, the student will have no difficulty in showing that the hypotrochoid, in the case illustrated by each of these figures, may either have deferential and epicyclic radii (F − R) and r, and deferential angle : epicyclic angle :: F − R : R; *or* epicyclic and deferential radii r

FIG. 82.

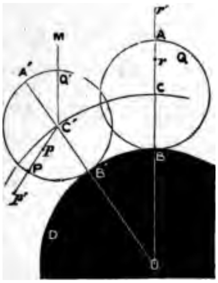
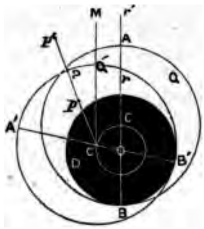


FIG. 83.



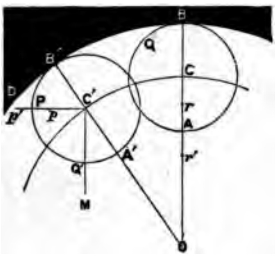


FIG. 84.

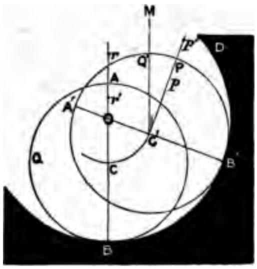


FIG. 85.

and (F − R), and deferential velocity : epicyclic velocity :: R : F − R.

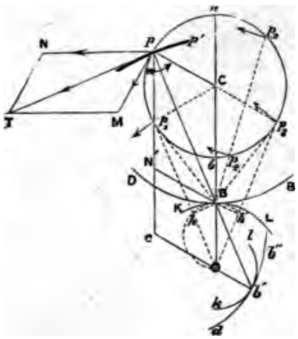
Since P has moved round C′ in a direction contrary to that in which C′ has moved round O, n is negative in both cases. If F − R > R or F > 2 R, n is > 1; this is the case illustrated by fig. 84. If F − R < R or F < 2 R, the case illustrated by fig. 85, n is < 1.

SCHOL.—We may find in this proposition another reason for regarding the curve traced out by a point on, or within, or without a circle which rolls outside a fixed circle, but is touched by that circle internally, as an epitrochoid, not as a hypotrochoid, for this definition leads again (while the other does not) to a symmetrical classification, giving epitrochoids as direct epicyclic curves, and hypotrochoids as retrograde epicyclic curves.

PROP. IV.—*Every direct epicyclic is an epitrochoid; and every retrograde epicyclic is a hypotrochoid.*

Let p be a point on an epicyclic curve $p\,p'$, OC (= D) the radius of deferent, Cp (= E) the radius of epicycle;

FIG. 86.



n positive and > 1. Then the motion of p may be resolved into two, one perp. to CO, the other perp. to Cp. Represent these by the straight lines pN, pM, taking $p\text{M} = p\text{C}$ and therefore $p\text{N} = \frac{\text{CO}}{n}$; then the diameter pT of the parallelogram NpMT represents the motion of p in direction and magnitude. Complete the parallelogram pCOc; take PN$' = p$N; and draw N$'$B parallel to cO to meet OC in B.

Suppose the parallelogram NM turned (in its own plane) round the point p through one right angle in the direction shown by the curved arrow, making p M coincide with p C and the parall. NM with parall. N′C. Then p B, the diameter of the parallelogram N′C, is the normal at p.

Now, by the preceding proposition, if a circle DBB′, having centre at C and radius CB, roll on the fixed circle KBL having centre at O and radius OB, the epitrochoid traced out by p, at distance C p from C, will be the epicyclic having C p as radius of epicycle, CO as radius of deferent, and epicyclic ang. vel. : deferential ang. vel. :: CO : CB :: n : 1. It will therefore be the epicyclic $p\,p'$.

Fig. 87.

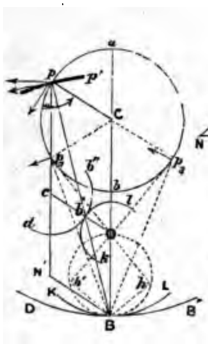
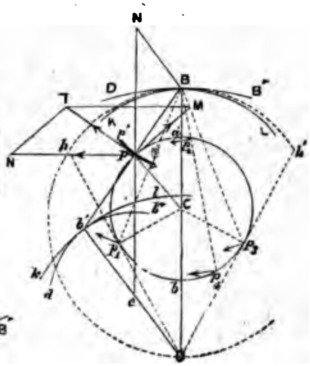


Fig. 88.



Thus the epicyclic $p\,p'$ is an epitrochoid having

$$F = BO = D\left(1 - \frac{1}{n}\right);\ R = CB = \frac{D}{n};\ \text{and}\ r = E.$$

We get precisely the same construction for the position of the normal p B by interchanging the radii and the angular velocities of deferent and epicycle, that is, taking O c as radius of deferent and $c\,p$ as radius of epicycle. Let p B and c O, produced (if necessary) intersect in b'. Then

$b'O : b'c :: OB : cp :: n-1 : n$; and by the preceding proposition, if a circle $db'b''$, with centre at c and radius cb', roll outside but in internal contact with the circle $kb'l$ having centre at O and radius Ob', the epitrochoid traced out by p at distance cp from c will be the epicyclic having cp as radius of epicycle, cO as radius of deferent, and epicyclic ang. vel. : deferential ang. vel. :: $cO : cb' :: 1 : n$. It will therefore be the epicyclic pp'. Therefore pp' is an epitrochoid having

$$F = b'O = D\,(n-1);\ R = cb' = D\,.\,n;\ \text{and } r = E.$$

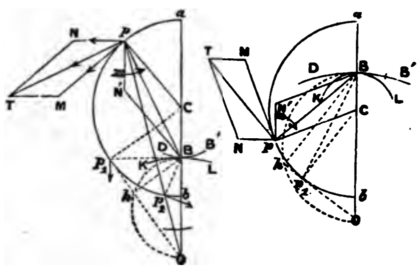
It will be found that the demonstration applies equally to the case of the direct epicyclic where $n < 1$, illustrated in fig. 87, only that N′ lies on pc produced. The two corresponding epitrochoids have

$$(1)\ F = BO = D\left(1 - \frac{1}{n}\right);\ R = CB = \frac{D}{n};\ \text{and } r = E.$$

$$(2)\ F = b'O = D\,(1-n);\ R = cb = Dn;\ \text{and } r = E.$$

Moreover, it will be found that the demonstration applies with slight (and obvious) alterations to the case of

Fig. 89. Fig. 90.



the retrograde epicyclic illustrated in fig. 88. (In the case illustrated, $n > 1$: it is not necessary to illustrate separately the case in which $n < 1$). We obtain for the two corresponding hypotrochoids,—

(1) $F = BO = D\left(1 + \frac{1}{n}\right)$; $R = CB = \frac{D}{n}$; and $r = E$.

(2) $F = bO = D\ (1 + n)$; $R = cb' = Dn$; and $r = E$.

SCHOL.—A number of cases resulting from varieties in the position of p are illustrated by the dotted constructions, and in figs. 89 and 90 (cases in which there is retrogression about O, b lying between O and B). The reader will have no difficulty either in understanding these, or in illustrating many other cases resulting from variations in the values of D, E, and n.

PROP. V.—*The normal at any point p of an epitrochoid or hypotrochoid passes through the point of contact B of the fixed circle with the rolling circle when the tracing point is at p.*

The demonstration of the preceding proposition includes the proof of this general proposition. The motion of p being at the instant precisely the same as though the circle B were rolling on the tangent to the fixed circle at B, it follows that if Np (= CB) represent the linear velocity of p in direction perp. to CO due to the advance of centre C of rolling circle DBB', pM = pC represents on the same scale the linear velocity of p in direction perp. to Cp ; wherefore pT, the diameter of the parallelogram NM, represents the resultant linear velocity of p ; and as in the demonstration of preceding proposition, if the parallelogram NM be rotated round p in its own plane, through a right angle, in the direction indicated by the curved arrow, pT is brought to coincidence with pB, which is therefore the normal at p.

PROP. VI.—*To determine the apocentral and pericentral velocities in epicyclic curves.*

From Prop. IV. fig. 86, we see that if the linear velocity of p around C is represented by p C, that is, by E, the linear velocity of p is represented by p T, perp. to p B, in direction, and by p T in magnitude, where CB $(=\frac{D}{n})$ represents the linear velocity of C about O.

Hence the velocity at an apocentre is represented on the same scale by B a, and the velocity at a pericentre by O b, a and b being the points in which OC, produced if necessary, meets the circle $p p_1 p_3$, a the remoter. That is, the linear velocity at apocentre $=\frac{D}{n}+E$. On the same scale the linear velocity of the mean centre $=\frac{D}{n}$; and

lin. vel. at apocen. : lin. vel. of mean cen. : lin. vel. at pericen.

$$:: \frac{D}{n}+E : \frac{D}{n} : \frac{D}{n}-E$$

$$:: D+nE : D : D-nE;$$

n being positive in case of direct epicyclic and negative in case of retrograde epicyclic.

Thus in the case of the direct epicyclic the motion at an apocentre is always direct; while the motion at a pericentre is direct, retrograde, or negative, according as D < or > n E, or as CB, fig 86, $\left(=\frac{D}{n}\right)$ > or < C b. In the case of the retrograde epicyclic the motion at an apocentre is direct or retrograde according as D > or < n E, or as CB $\left(=\frac{D}{n}\right)$ > or < C a, fig. 88; while the motion at a pericentre is always direct.

SCHOL.—If $D=nE$, there is a cusp at pericen. or apocen.

Prop. VII.—*To determine the position of the points, if any, where the motion of the radius vector becomes retrograde.*

It is manifest that if, as in the cases illustrated by figs. 86, 87, and 88, the point B lies outside the circle $p\, p_1\, p_3$, or D > n E, the motion, direct both at apocentres and pericentres, is direct throughout. For the motion to be retrograde in part of the epicyclic, then, we require that D be < n E, or CB < C a. Since the direction at p is perp. to B p, the motion will be directly towards or from centre if B p is at right angles to O p, for then O p will be the tangent at p. We have then the relations presented in fig. 91 for direct epicyclic, and in fig. 92 for retrograde epicyclic.

Fig. 91.

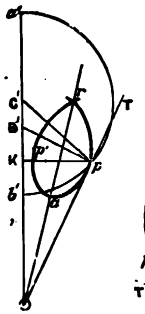
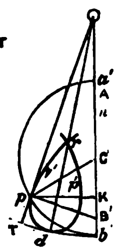


Fig. 92.



O p is the distance from O at which the epicyclic becomes retrograde (for all smaller distances in case of direct epicyclic, and for all greater distances in case of retrograde epicyclic). Manifestly the distance O p is determined by describing a semicircle on OB intersecting $a'p\,b'$ in p. Now the angle $p\,C'a' = (n-1)$ deferential angle (measured from apocentral initial radius vector), say $\angle\ p\,C'a' = (n-1)\,\phi$, and we might proceed by the epicyclic method of treatment to determine ϕ geometrically. We have, however, already the means of doing this, in the result of Prop. XIV., Sect. III. Thus, draw p K perp. to C O; then

$$\text{Cos } p\,\text{C}\,a = -\frac{\text{CK}}{\text{C}p} = -\frac{\text{CK}\,.\,\text{OA}'}{\text{C}p\,.\,\text{OA}} = -\frac{(\text{C}a)^2 + \text{OC}\,.\,\text{CB}}{\text{C}p\left(\text{OC} + \frac{\text{D}}{n}\right)}$$

$$\left(\text{C A of Prop. XIV. sec. } 3 = \frac{\text{D}}{n}\right)$$

$$\text{Cos }(n-1)\,\phi = -\frac{\text{E}^2 + \text{D}\,.\,\frac{\text{D}}{n}}{\text{E}\left(\text{D} + \frac{\text{D}}{n}\right)} = -\frac{\text{D}^2 + n\,\text{E}^2}{(1+n)\,\text{DE}};$$

n being negative in case of retrograde epicyclic.

Cor. If ϕ_1 be the value of ϕ determined from this equation, the motion is retrograde from $\phi = \phi_1$ to $\phi = \frac{360°}{n-1} - \phi_1$.

Schol.—The angle ϕ_1 is of course the angle which OC makes with the initial line, and does not directly indicate the arc of retrogradation, which is twice the angle $p\,\text{O}\,d$. This, however, may be readily deduced in any given case. For $\tan p\,\text{O}\,b = \frac{p\,\text{K}}{\text{KO}} = \frac{\text{E}\sin(n-1)\phi_1}{\text{D} + \text{E}\cos(n-1)\phi_1}$ is known, and therefore, $p\,\text{O}\,d = \phi_1 + p\,\text{O}\,b' - \frac{180°}{n-1}$ is also known.

It can easily be shown that

$$\sin(n-1)\phi_1 = \frac{\sqrt{(\text{D}^2-\text{E}^2)(n^2\text{E}^2-\text{D}^2)}}{(1+n)\,\text{DE}}$$

and $$\tan p\,\text{O}\,b = \frac{1}{n}\sqrt{\frac{n^2\,\text{E}^2-\text{D}^2}{\text{D}^2-\text{E}^2}}$$

Prop. VIII.—*To determine the tangential, transverse, and radial velocities, and the angular velocity around the centre at any point of an epicyclic curve.*

Let p_1 (figs. 86, 87, 88) be the position of the point on the epicycle $a\,p_1 b$. Join O p_1 and draw B h perp. to O p_1. Then when C p_1 (= E) represents the linear velocity in the epicycle, O p_1 represents the linear vel. at p_1 in magnitude, but is at right angles to the direction of motion at p_1. Hence $p_1\,h$ represents the linear velocity perp. to the radius vector, and B h represents the linear velocity in the direction of the radius vector, the direction of the motion in either case being determined by conceiving p_1 C turned around p_1, carrying with it p_1 B and $p_1\,h$, in the plane of the figure, through a right angle, to coincidence with the direction of p_1's motion in the circle $a\ p\ b$. This includes all cases geometrically, and the student will have no difficulty in effecting the construction and deducing the proper directions for the tangential, transverse, and radial velocities, for any given values of D, E, and n, and for any given position of the moving point. The angular velocities are determined by the same construction. Thus in the case illustrated by fig. 86 :

The tangential velocity of p_1 is represented by p_1 B in magnitude and is in advancing direction shown by arrow at p_1.

The transverse velocity of p_1 is represented by $p_1\,h$ in magnitude, and in direction by B h.

The radial velocity of p_1 is represented by B h in magnitude, and in direction by $p_1\,h$.

The angular velocity of p_1 about O : uniform angular velocity of p_1 about C $:: \frac{p_1 h}{\mathrm{O} p_1} : \frac{p_1 \mathrm{C}}{p_1 \mathrm{C}} :: p_1 h : \mathrm{O} p_1$.

And similarly for all other cases.

It is more convenient, however, where so many cases arise, to obtain the analytical expressions for these quantities; for we know that by rightly considering the signs of the values used and obtained, the same expression will be correct for all possible cases. Let then the angle $p_1\,C\,a$ (fig. 86) $=(n-1)\,\phi$; that is, let the deferential angle $=\phi$; let the linear velocity of the mean point (C) be V, wherefore the linear velocity of the moving point in the epicycle $=n\,V\,.\,\frac{E}{D}$. This is what we have represented linearly by $p_1\,C$ in figs. 86, 87, and 88, so that since $p_1\,C = E$, we have to affect all the above linear representations of velocity with the co-efficient $\frac{n\,V}{D}$:

Therefore, the tangential vel.

$$=\frac{n\,V}{D}\,.\,p_1\,B=\frac{n\,V}{D}\sqrt{(P_1C)^2+(CB)^2+2p_1\,C\,.\,CB\cos p\,C\,a}.$$

$$\frac{n\,V}{D}\sqrt{E^2+\frac{D^2}{n^2}+\frac{2\,DE}{n}\cos{(n-1)\,\phi}}$$

$$=\frac{V}{D}\sqrt{D^2+n^2\,E^2+2\,n\,DE\cos{(n-1)}\phi}.$$

The transverse vel. $=\frac{n\,V}{D}\,.\,p_1\,B\,.\cos B\,p_1\,O$:

now, $\cos B p_1\,O=\frac{(B\,p_1)^2+(p_1\,O^2)-(BO)^2}{2\,p_1\,B\,.\,p_1\,O}$; $\therefore p_1\,B\,.\cos B p_1\,O$

$$=\frac{E^2+\frac{D^2}{n^2}+2\frac{DE}{n}\cos{(n-1)}\phi+E^2+D^2+2\,DE\cos{(n-1)}\phi-\left(D-\frac{D}{n}\right)^2}{2\,p_1\,O}$$

and transverse vel. (direct)

$$=\frac{V}{D}\,.\,\frac{D^2+n\,E^2+(n+1)DE\cos{(n-1)}\phi}{\sqrt{D^2+E^2+2\,DE\cos{(n-1)}\,\phi}}.$$

The radial vel. $=\frac{V}{D}\,p_1B\,.\,\sin Bp_1O$:

now $$\frac{\sin Bp_1O}{\sin p_1OB} = \frac{BO}{p_1B};$$

therefore,

$$p_1B \sin Bp_1O = BO \sin p_1OB = \left(D - \frac{D}{n}\right) \cdot \frac{E \sin(n-1)\phi}{p_1O};$$

and radial vel. (towards centre)

$$=(n-1)V \cdot \frac{E \sin(n-1)\phi}{\sqrt{D^2+E^2+2DE\cos(n-1)\phi}}$$

The angular velocity about O

$$= \frac{\text{transv. vel.}}{\text{rad. vect.}} = \frac{V}{D} \cdot \frac{D^2+nE^2+(n+1)DE\cos(n-1)\phi}{D^2+E^2+2DE\cos(n-1)\phi}$$

The transverse vel. and the angular vel. about O vanish, if

$$D^2 + nE^2 + (n+1)DE\cos(n-1)\phi = 0,$$

the condition already obtained.

If v is the velocity in epicycle, $v = V\frac{nE}{D}$, or $V = v\frac{D}{nE}$ which value substituted for V in the above formulæ gives formulæ enabling us to compare the various velocities with the velocity in the epicycle.

Schol.—We see from the geometrical construction that the radial velocity has its maximum value towards or from the centre, when the moving point is at p_1 or p_3 (figs. 86, &c.), where a tangent from O meets the circle $a\,p_1\,b$; for then Bh or Bh has its greatest value. This also may be thus seen: —Since the deferential motion gives no radial velocity, the radial velocity will have a maximum value when the epicyclic motion is directly towards or from the fixed centre,—that is, at the points where a tangent from the fixed centre to the epicycle meets this circle.

Cor. The angular vel. at apocentre > = or < angular vel. at pericentre, according as

$$\frac{aB}{aO} \gtreqless \frac{bB}{bO}, \text{ or as } \frac{aB}{bB} \gtreqless \frac{aO}{bO}$$

PROP. IX.—*To determine when epicyclic loops touch.*

For this we must have $\angle\ p\,O\,d$ (figs. 93, 94) $=$ angle of descent; that is, see Schol. to Prop. VII.,

FIG. 93.

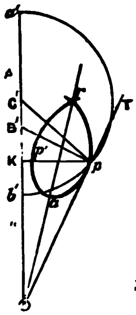
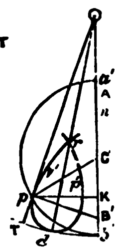


FIG. 94.



$$\phi_1 + p\,O\,b' - \frac{180^\circ}{n-1} = \frac{180^\circ}{n-1}; \qquad \text{or}$$

$$\frac{1}{n-1}\cos^{-1}\left[-\frac{D^2+n\,E^2}{(1+n)\,DE}\right]+\tan^{-1}\left[\frac{1}{n}\sqrt{\frac{n^2E^2-D^2}{D^2-E^2}}\right]=\frac{360^\circ}{n-1}$$

PROP. X.—*To determine the position of points of inflexion.*

If p, figs. 95, 96, as in Prop. XIII., sec. 3, is a point of inflexion, we have as in that proposition

$$2\,C'k\,.\,C'z = C'B'\,.\,C\,I \pm (C'p)^2$$

(lower sign for retrograde epicyclic)

$$\text{or } (C'B \pm C'I)\,C'z = C'B'\,.\,C'I \pm (C'p)^2$$

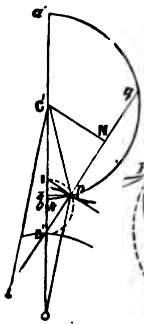
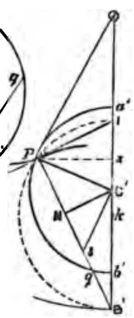
$$\therefore \frac{C'\,z}{C'p} = -\cos\,(n-1)\,\phi = \frac{C'B'\,.\,C'I \pm (C'p)^2}{(C'B' \pm C'I)\,C'p}$$

Now by Cor. to Prop. XII., Sect. III., $C'I = \left(\frac{D}{n}\right)^2 + D = \frac{D}{n^2}$

to be regarded as negative for retrograde epicyclic. Hence

$$\text{Cos}\,(n-1)\,\phi = -\frac{\frac{D}{n}.\frac{D}{n^2}+E^2}{\left(\frac{D}{n}+\frac{D}{n^2}\right)E} = -\frac{D^2+n^3\,E^2}{n\,(1+n)\,DE}$$

FIG. 95. FIG. 96.

n being negative in case of retrograde epicyclic.

Cor. If ϕ_2 be the value of ϕ determined from this equation, there is a point of contrary flexure when $\phi = \phi_2$ and another when $\phi = \dfrac{360°}{n-1} - \phi_2$.

SCHOL.—The angular range round O of the arc between the points of flexure can be determined, as in case of arc of retrogradation, see scholium to Prop. VII. We have

$$\tan p\,O\,b' \text{ (figs. 95 and 96)} = \frac{pz}{zO} = \frac{E\sin(n-1)\,\phi_2}{D+E\cos(n-1)\,\phi_2};$$

wherefore, if d be the pericentre

$$p\,O\,d = \frac{180°}{n-1} - \phi_2 - p\,O\,b\text{, is also known.}$$

It is easily shown that

$$\sin\,(n-1)\,\phi_2 = \frac{\sqrt{(n^2\,E^2-D^2)(D^2-n^4\,E^2)}}{n\,(1+n)\,DE}$$

and $\tan p\,O\,b' = \frac{\sqrt{(n^2 E^2 - D^2)(D^2 - n^4 E^2)}}{D^2(n^2 + n - 1) - n^3 E^2}.$

For the critical case where the points of inflexion coincide,

we have, from Cor. 1, $\cos (n-1)\phi_2 = -1$;

that is $D^2 + n^3 E^2 = n(1+n) DE$

(the same condition, both for direct and retrograde epicyclic, due account being taken of the sign of n);

or $n(n^2 E - D) E = (n^2 E - D) D$

or $(n E - D)(n^2 E - D) = 0,$

which is satisfied, (i), if $n = \frac{D}{E}$, the condition (Schol. p. 158) for a cusp (at pericentre in case of direct epicyclic, and at apocentre in case of retrograde epicyclic), and (ii), if $n^2 = \frac{D}{E}$, corresponding to the case when this curve becomes straight at pericentre both for direct and retrograde epicyclic. Compare scholium to Prop. XII., Section III., from which the relation between n^2, D, and E, can be directly obtained.

PROP. XI.—*To determine the radius of curvature, ρ, at a point on epicyclic where deferential angle $= \phi$.*

From Cor. p. 117, noting value of $p\,B'$ (as in p. 162); that $C'O = n \,.\, C'B'$; and that $B\,N = C'B \cos p\,B'C'$; while $p\,B' \cos p\,B\,C = B'C' + p\,C' \cos (n-1)\phi$, it is easily shown that

$$\rho = \frac{[D^2 + n^2 E^2 + 2\,n\,DE \cos (n-1)\phi]^{\frac{3}{2}}}{D^2 + n^3 E^2 + n(n+1)\,DE \cos (n-1)\phi}$$

at apocentre, $\rho = \frac{(D + n\,E)^2}{D + n^2 E}$; at pericentre $\rho = \frac{(D - n\,E)^2}{D - n^2 E}$.

Appendix to Section V.

RIGHT TROCHOIDS REGARDED AS EPICYCLICS.

It is often convenient to regard right trochoids as epicyclics. The radius of the deferent is in their case infinite, the centre of the epicycle moving in a straight line. It is necessary to substitute linear for angular velocities, the value of n becoming infinite when the deferent becomes a straight line. It is manifest that if the centre of the rolling circle of a right trochoid moves with velocity v in the line of centres, the tracing point moves with velocity $\frac{r}{R} v$ around the tracing circle; and conversely, it is manifest that if a point moves with velocity $m v$ round the circumference of a circle of radius E, whose centre moves with velocity v in a straight line in its own plane, the point will trace out a right trochoid, having a tracing circle of radius E and a generating circle of radius m E. We may put $v = 1$, in which case m represents the velocity of the tracing point round the circumference of the moving circle $\left(m = \frac{r}{R}\right)$. It is obvious also that if $m > 1$ there is a loop; if $m = 1$, a cusp; if $m < 1$ the curve is inflected. These cases correspond to those of right trochoids in which $r > R$, $r = R$, and $r < R$.

Since right trochoids may be regarded as special cases of epicyclic curves, it is not necessary to discuss them further in their epicyclic character. It will be found easy to deduce any required relation for right trochoids from the relations above established for epicyclics, combined with the considerations noted in the preceding paragraph. A single illustration will suffice to show how this may be effected.

Suppose we wish to determine when the tracing point ceases to advance in the looped trochoid. We have, from Prop. VII., in case of epicyclic,

$$\cos (n-1)\,\phi_1 = -\frac{D^2 + n\,E^2}{(1+n)\,DE}$$

and if m represents the ratio of linear velocities in epicycle and deferent, $n = m\frac{D}{E}$. Also $n\,\phi$ is the angle swept out in epicycle, and when D becomes infinite is the same as $(n-1)\phi$, so that the angle ϕ_1 (the angle a CL of fig. 48) is determined by the equation

$$\cos \phi_1 = -\frac{D^2 + m\,DE}{(E + mD)\,D} = -\frac{1}{m} \text{ when D is infinite.}$$

The student will, however, find it a useful exercise to go independently through the various propositions relating to epicyclics, for the case in which the deferent is a straight line. The relations involved are simpler than those dealt with in the present section. It is to be noticed that m may always be regarded as positive, the same curve being obtained for a negative value of m as for the same positive value, if r remains unaltered.

SPIRAL EPICYCLICS.

When the radii of epicycle and deferent are both infinite but (D—E) finite, the epicyclic becomes one of the system of spirals of which the involute of the circle and the spiral of Archimedes are special cases. We must of course suppose the curve traced out on either side of the pericentre, since the remoter parts of the curve pass off on each side to infinity. Instead, however, of imagining a deferent of infinite radius carrying an epicycle also of infinite radius, it is more convenient, in independent researches into these spirals by epicyclic methods, to consider a deferent radius as revolving uniformly round a fixed point, this radius bearing at its extremity a straight line perp. to it in the plane of its own motion, along which line a point moves with uniform velocity. Let the length of the revolving radius $= d$, velocity of its extremity 1, and velocity of moving point m. Then if $m = 1$, the curve is the involute of the circle traced out by the end of the revolving radius; if $m >$ or < 1, the curve is one of the system of spirals bearing the same relation to the involute of the circle which the curtate and prolate epicycloid respectively bear to the right epicycloid. If $d = 0$, the infinite straight line revolves about a point in its own centre; and the curve traced out by the moving point is the spiral of Archimedes, whatever the uniform angular velocity of the revolving line, and whatever the uniform velocity of the tracing point along the line. See also examples 131–133.

PLANETARY AND LUNAR EPICYCLES.

The ancient astronomers discovered that the paths in which the planets travel with reference to the earth are approximately epicyclic. It is easily shown that this follows from the fact that the planets, as well as our earth, travel in nearly circular paths about the sun as centre.

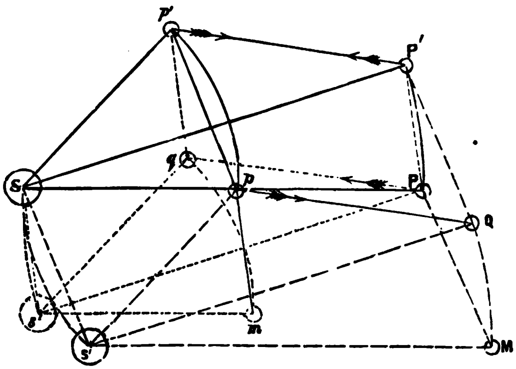
The general property is as follows :—

PROP. I.—*Regarding the planets as travelling uniformly in circles about the sun as centre, and in the same plane, the path of any planet P (fig. 97) with reference to any other planet, p, regarded as at rest, is the same as the path of p with reference to P regarded as at rest, the corresponding radii vectores lying in opposite directions; and each such path is a direct epicyclic.*

Let S be the sun, p and P two planets (p being the inferior planet, and P the superior), in conjunction on the line

S p P. Let the planet p move to p', while P moves to P'. Draw p Q and P q parallel and equal to p' P'. Then, with reference to the planet p, regarded as at rest, the planet P has moved as if from P to Q; while considered with reference to P, regarded as at rest, the planet p has moved as if from p to q: and since p Q is equal and parallel to P q, the path of the outer planet with reference to the inner, regarded

Fig. 97.



as at rest, is the same as the path of the inner planet with reference to the outer regarded as at rest,—each path being, however, turned round through 180° with regard to the other.

Join $p'q$, P'P, $p'p$, and P'Q. Draw S s' parallel to $p'q$, and SS' parallel to P'Q, and join $s'q$, s'P, S'p, and S'Q. Also draw $s'm$ and S M parallel to SP, and complete the parallelograms PMS'S, and $p\,m\,s'$S.

Then, by construction, the figures S'p', p'Q, S'P', Sq, qP', and s'P', are parallelograms. Wherefore $p\,\text{S}'=p'\text{S}=\text{S}p$; and $\angle\, \text{S}p\text{S}' = \angle\, p\text{S}p'$; S'M = SP = SP' = S'Q and $\angle$ MS'Q = $\angle$ PSP'; so that the relative motion of the outer planet from P to Q around p may be regarded as effected by the uniform motion of S to S' in a circle about p as centre (corresponding to the real motion of p to p' around S as centre), accompanied by the uniform motion of P (which, if at rest, would have been carried to M), in a circle around the moving S as centre to Q,—that is, through the arc M Q = P P′. Hence the motion of P with reference to p is that of a direct epicyclic having D = S p, E = S P, and

$$n = \frac{\text{Ang. vel. of P round S}}{\text{Ang. vel. of } p \text{ round S}}.$$

Similarly the relative motion of the inner planet from p to q, around P, may be regarded as effected by the uniform motion of S to s' around P as centre (corresponding to the real motion of P to P′ around S as centre), accompanied by the uniform motion of p (which, if at rest, would have been carried to m) in a circle around the moving S as centre to q,—that is, through the arc $m\,q = p\,p$. Hence the motion of p with reference to P is that of a direct epicyclic having D = SP, E = S p, and

$$n = \frac{\text{Ang. vel. of } p \text{ round S}}{\text{Ang. vel. of P round S}}.$$

SCHOL.—If the distances of the planets p and P from the sun are r and R respectively, the epicyclic of either planet about the other has D = R, E = r, and

$$n = \left(\frac{R}{r}\right)^{\frac{3}{2}};$$

for the angular velocities of planets round the sun vary inversely as the periods—that is, as the sesquiplicate power of the mean distance.

Since $$\left(\frac{R}{r}\right)^{\frac{3}{2}} > \frac{R}{r}, \text{ or } n > \frac{D}{E},$$

the motion of one planet with reference to another is always retrograde when the planets are nearest to each other; therefore every planetary epicyclic is looped.

The arc of retrogradation of one planet with reference to the other may be obtained as explained in scholium to Prop. VII. of this section. The duration of the retrogradation follows directly from the formula for determining $\cos\,(n-1)\,\phi_1$ as in that proposition; for ϕ_1 is the angle swept out by the superior planet around the sun between the time of inferior conjunction and first station. This formula, with the values above given for D, E, and n, becomes

$$\cos \frac{R^{\frac{3}{2}} - r^{\frac{3}{2}}}{r^{\frac{3}{2}}} \phi_1 = \frac{R^2 + \left(\frac{R}{r}\right)^{\frac{1}{2}} r^2}{R\,r + \left(\frac{R}{r}\right)^{\frac{1}{2}} R\,r} \,.$$

or, putting P, p, for the respective periods of the planets,

$$\cos \frac{P-p}{p} \phi_1 = -\frac{R^2 r^{\frac{1}{2}} + R^{\frac{1}{2}} r^2}{R\,r^{\frac{1}{2}} + R^{\frac{1}{2}}\,r} = -\frac{R\,r^{\frac{1}{2}} + R^{\frac{1}{2}}\,r}{R^{\frac{1}{2}} + r^{\frac{1}{2}}}$$

$$= -\frac{R^{\frac{1}{2}}\,r^{\frac{1}{2}}}{R - R^{\frac{1}{2}}\,r^{\frac{1}{2}} + r} = \frac{\sqrt{R\,r}}{\sqrt{R\,r} - (R+r)}\text{; and}$$

$$\sin \frac{P-p}{p} \phi_1 = \sqrt{1 - \cos\,(n-1)\,\phi_1}\,\sqrt{1 + \cos\,(n-1)\,\phi_1}$$

$$= \frac{\sqrt{(R+r)\,(R - 2\,R^{\frac{1}{2}}\,r^{\frac{1}{2}} + r)}}{R - R^{\frac{1}{2}}\,r^{\frac{1}{2}} + r}$$

$$= \frac{(R^{\frac{1}{2}} - r^{\frac{1}{2}})\,\sqrt{R+r}}{R - R^{\frac{1}{2}}\,r^{\frac{1}{2}} + r} \,.$$

Wherefore $\tan\,p\,O\,b'$ (see fig. 91, and schol. p. 160)

$$= \frac{r\,(R^{\frac{1}{2}} - r^{\frac{1}{2}})\,\sqrt{R+r}}{R\,(R - R^{\frac{1}{2}}\,r^{\frac{1}{2}} + r) - R^{\frac{1}{2}}\,r^{\frac{1}{2}}}$$

$$= \frac{r\,(R^{\frac{1}{2}} - r^{\frac{1}{2}})\,\sqrt{R+r}}{R(R+r) - R^{\frac{1}{2}}\,r^{\frac{1}{2}}\,(R+r)} = \frac{r}{R^{\frac{1}{2}}\,\sqrt{R+r}}$$

The arc of retrogradation,—

$$= 2\,\phi_1 + 2\,p\,O\,b' - 360^\circ \left(\frac{p}{P-p}\right),\text{—}$$

can be readily determined. Thus, the arc of retrogradation

$$= 2 \angle p \mathrm{O} b' - \frac{p}{\mathrm{P} - p}\left(360^\circ - \cos^{-1} \frac{\sqrt{\mathrm{R}r}}{\sqrt{\mathrm{R}r} - \mathrm{R} - r}\right)$$

$$= 2 \tan^{-1} \frac{r}{\mathrm{R}^{\frac{1}{2}} \sqrt{\mathrm{R} + r}}$$

$$- \frac{p}{\mathrm{P} - p}\left(180^\circ + \cos^{-1} \frac{\sqrt{\mathrm{R}r}}{\mathrm{R} - \sqrt{\mathrm{R}r} + r}\right) \qquad (1)$$

This formula gives the arc of retrogradation. The angle between pericentral and stationary *radii vectores* is half the arc of retrogradation.

Thus the epicyclic path of a superior planet (period P) with respect to an inferior planet (period p), or of latter planet with respect to former, will have—

Apocentral distance $= \mathrm{R} + r$;
Pericentral distance $= \mathrm{R} - r$;

Angle of descent $= \frac{p}{\mathrm{P} - p} \cdot 180^\circ$.

The arc of retrogradation is determined by formula (1) above. All the tables of planetary elements give R, r, P and p. If one of the planets is the earth, the calculation is simplified, because the tables of elements give the distances of other planets with the earth's mean distance as unity.

If a satellite be regarded as travelling uniformly in a circle around its primary, while the primary travels uniformly in a circle in the same plane around the sun, the path of the satellite is an epicyclic about the sun as fixed centre.

All the satellites travel in the same direction round their primaries as the primaries round the sun, except the satellites of Uranus, whose inclination is so great that their motion does not approach the epicyclic character. The direction of the motion of Neptune's satellite, sometimes given in tables of astronomical elements as retrograde, cannot yet be regarded as determined. The inclination of Saturn's satellites, seven of which travel nearly in the same plane as the rings, is considerable; but these bodies may be regarded as having paths of an epicyclic character. Our own moon's path is but little inclined to the ecliptic, and the paths of Jupiter's moons are still nearer the plane of their planet's motion. The discussion of the actual motions of these bodies belongs rather to astronomy than to our present subject. We need consider here only some general relations.*

PROP. II.—*To determine under what conditions a satellite, travelling in a direct epicycle about the sun, will have its motion (referred to the sun) looped, cusped, or direct throughout, or partly convex towards the sun, or just failing of becoming convex at perihelion, or partly concave towards the sun.*

Let M be the sun's mass, m the primary's, R the distance of primary from the sun, r the distance of satellite from primary; also (though these values are only for convenience) let P be the primary's period, p the satellite's, and assume that m is so small compared with M, and the satellite's mass so small compared with m, that both the ratios $(M + m) : M$, and (m + satellite's mass) : m may be regarded throughout this inquiry as equal to unity.

We have first to obtain the means of comparing the velocities in the primary and secondary orbits under any

* In a work on the 'Principles of Astronomy,' which I am at present writing, the nature of the planetary and lunar epicycles will be found fully treated of.

given conditions. The most convenient way of doing this is perhaps as follows:—Let V, v, be the respective velocities of bodies moving in circles around the sun, and round the primary, at the same distance, R; and let v be the velocity of the satellite at distance r. Then we know that

$$\frac{V^2}{R} : \frac{v^2}{R} :: M : m,$$

or $$V : v' :: \sqrt{M} : \sqrt{m},$$

and $$v' : v :: \sqrt{r} : \sqrt{R}.$$

$$\therefore V : v :: \sqrt{Mr} : \sqrt{mR},$$

and $$\frac{V}{R} : \frac{v}{r} :: \sqrt{Mr^3} : \sqrt{mR^3}.$$

This is the ratio of the angular velocities of primary and satellite in their respective orbits. It gives us

$$n : 1 \; (:: P : p) :: \sqrt{mR^3} : \sqrt{Mr^3}.$$

The path of the satellite will therefore be looped, cusped, or direct throughout, according as

$$\sqrt{\frac{mR^3}{Mr^3}} \begin{matrix} > \\ = \\ < \end{matrix} \frac{R}{r}$$

or as $$mR \begin{matrix} > \\ = \\ < \end{matrix} Mr; \text{ or } \frac{m}{M} \begin{matrix} > \\ = \\ < \end{matrix} \frac{r}{R}.$$

And the path of the satellite will be partly convex towards the sun, or just fail of becoming convex at perihelion, or be partly concave towards the sun, according as

$$\frac{mR^3}{Mr^3} \begin{matrix} > \\ = \\ < \end{matrix} \frac{R}{r},$$

or as $$mR^2 \begin{matrix} > \\ = \\ < \end{matrix} Mr^2; \text{ or } \frac{m}{M} \begin{matrix} > \\ = \\ < \end{matrix} \frac{r^2}{R^2}; \text{ or } \sqrt{\frac{m}{M}} \begin{matrix} > \\ = \\ > \end{matrix} \frac{r}{R}$$

The student will find no difficulty in obtaining formulæ for the range of the arc of retrogradation, if any, or of the arc of convexity towards the sun, if any, following the course pursued at pp. 172, 173 (using in the latter case the formula of p. 165), remembering that in this case D = R and E = r and $n = \frac{P}{p}$, as in the case of planetary motion, but that in reducing the formula he must employ the relation

$$n = \sqrt{\frac{m\,R^3}{M\,r^3}}.$$

I have not thought it necessary to occupy space here with the reduction of these formulæ, because they are of no special use. The path of our own moon has no points of retrogradation or of flexure, and the position of such points on the paths of Jupiter's moons, or Saturn's, is not a matter of much moment.

We may pause a moment, however, to inquire into the limits of distance at which, in the case of these planets and our earth, convexity towards the sun, or retrogradation, would occur.

In the case of our earth, $\frac{M}{m} = 322{,}700 = (568)^2$ about; and R = 92,000,000. Therefore a moon would travel in a cusped epicycle, or come exactly to rest at perihelion, if (the earth's whole mass being supposed collected at her centre) the moon's distance from the earth's centre were $\frac{92{,}000{,}000}{322{,}700}$ miles, or about 285 miles. That a moon should travel in a path convex to the sun in perihelion, the distance should not exceed $\frac{92{,}000{,}000}{568}$, or about 162,000 miles. Thus the moon's actual distance being 238,828 miles, her path is entirely concave towards the sun.

In the case of Jupiter, $\frac{M}{m} = 1{,}046 = (32\frac{1}{3})^2$ about; and

R = 478,660,000 miles. Therefore a moon would travel in a cusped epicycle, or come exactly to rest in perihelion, if its distance from Jupiter's centre were $\frac{478,660,000}{1,046}$, or about 457,600 miles. Thus the two inner moons, whose distances are 259,300 and 412,000 miles, have loops of retrogradation; whereas the two outermost, whose distances are 658,000 and 1,155,800 miles, have paths wholly direct. But all the moons travel on paths convex towards the sun for a considerable arc on either side of perihelion; since for the path of a Jovian moon to just escape convexity towards the sun at perihelion, its distance from Jupiter should be $\frac{478,660,000}{32\frac{1}{3}}$ miles, or about 14,804,000 miles; which far exceeds the distance even of the outermost moon.

In the case of Saturn $\frac{M}{m} = 3,510 = (59)^2$ about, and R = 877,570,000 miles. Hence a moon would travel in a cusped epicycle if its distance from Saturn were $\frac{877,570,000}{3,510}$ or about 250,700 miles. This is rather less than the distance of his fourth satellite, Dione, 253,442 miles; and, owing to the eccentricity of Saturn's orbit, it must at times happen that Dione comes almost exactly to rest for an instant at a cusp in epicyclic perihelion, or only has a motion perpendicular for the moment to the path of Saturn. The three satellites nearer to Saturn travelling at distances of 124,500, of 159,700, and of 197,855 miles, have loops of retrogradation, as have all the satellites composing the ring system. The other satellites, having distances of 353,647, of 620,543, of 992,280, and of 2,384,253 miles respectively, have no loops; but their paths are convex towards the sun for a considerable arc on either side of epicyclic perihelion; since, for a satellite's path just to escape convexity towards the sun, the satellite's distance should be $\frac{877,570,000}{59}$ miles, or about 14,874,000 miles.

PROP. III.—*Regarding the planets as moving uniformly in circles round the sun in the invariable plane, the projections of the paths of the planets in space upon a fixed plane parallel to the invariable plane of the solar system are right trochoids.*

This follows directly from the fact that the sun is advancing in a right line (appreciably, so far as ordinary time-measures are concerned), with a velocity comparable with the orbital velocities of the planets. His course being inclined to the invariable plane, the actual path of each planet is a skew helix, as shown in the last chapter of my treatise on the sun.

PROP. IV.—*To determine the tangential, transverse, and radial velocities (linear) of a planet in its orbit relatively to another planet, and its angular velocity about this planet.*

Let R be the distance, P the period, V the velocity of the planet which is regarded as the centre of motion; r the distance, p the period, v the velocity of the other planet.

Then, in the formulæ for the tangential transverse, and radial velocities in epicyclics, we have to put

$$D = R;\ E = r;\ \text{and } n = \left(\frac{R}{r}\right)^{\frac{3}{2}} = \frac{P}{p};$$

but it will be convenient to retain n, remembering its value. We may also conveniently write $\frac{r}{R} = \rho$, so that $n = \rho^{-\frac{3}{2}}$

Moreover, with the units of distance and time in which R, r, P, and p are expressed,

$$V = \frac{2\pi R}{P}.$$

Also ϕ is the angle swept out around the sun by the planet of reference since the last conjunction of the sun and the other planet, the conjunction being superior in the case of an inferior planet.*

Thus the tangential velocity is equal to

$$\frac{V}{R}\sqrt{R^2 + \left(\frac{R}{r}\right)^3 r^2 + 2\left(\frac{r}{R}\right)^{\frac{1}{2}} \cos\frac{P-p}{p}\phi}$$
$$= V\sqrt{1 + \rho^{-1} + 2\rho^{\frac{1}{2}}\cos(n-1)\phi}\,.$$

The formula can obviously assume many forms, but perhaps this, which enables us at once to compare the tangential velocity with V, the velocity of the planet of reference in its orbit, is the most convenient.

The transverse velocity (direct)

$$= \frac{\frac{V}{R}\left\{R^2 + \left(\frac{R}{r}\right)^{\frac{3}{2}} r^2 + \frac{R^{\frac{3}{2}} + r^{\frac{3}{2}}}{r^{\frac{3}{2}}} R\,r\cos(n-1)\phi\right\}}{\sqrt{R^2 + r^2 + 2\,R\,r\cos(n-1)\rho}}$$

$$= V\frac{1 + \rho^{\frac{1}{2}} + (\rho^{-\frac{1}{2}} + \rho)\cos(n-1)\phi}{\sqrt{1 + \rho^2 + 2\rho\cos(n-1)\phi}}.$$

The radial velocity (towards centre)

$$= (\rho^{-\frac{3}{2}} - 1)\,V.\ \frac{r\sin(n-1)\theta}{\sqrt{R^2 + r^2 + 2\,R\,r\cos(n-1)\phi}}$$

$$= V\frac{(\rho^{-\frac{1}{2}} - \rho)\sin(n-1)\phi}{\sqrt{1 + \rho^2 + 2\rho\cos(n-1)\phi}}.$$

* The conjunction must be such that the sun is between the two planets. It is a convenient aid to the memory, in distinguishing between the superior and inferior conjunctions of inferior planets, to notice that inferior conjunction is that kind of conjunction with the sun which only inferior planets *can* enter into.

The angular velocity of the planet about the planet of reference

$$= \frac{\mathrm{V}}{\mathrm{R}} \cdot \frac{\rho^{-\frac{1}{2}} r^2 + \mathrm{R}^2 + (\rho^{-\frac{1}{2}} + 1)\, \mathrm{R} r \cos (n-1) \phi}{\mathrm{R}^2 + r^2 + 2 r \cos (n-1) \phi}$$

$$= \omega \cdot \frac{\rho^{\frac{1}{2}} + 1 + (\rho^{-\frac{1}{2}} + \rho) \cos (n-1) \phi}{1 + \rho^2 + 2\rho \cos (n-1) \phi},$$

putting $\frac{\mathrm{V}}{\mathrm{R}} = \omega =$ angular velocity of the planet of reference in its orbit.

Cor. 1. In conjunction (superior if moving planet is inferior) $\phi = 0$;

$\therefore$ Angular velocity in superior conjunction

$$= \omega \frac{\rho^{\frac{1}{2}} + 1 + \rho^{-\frac{1}{2}} + \rho}{1 + \rho^2 + 2\rho}$$

$$= \omega \frac{(1+\rho) \times (1 + \rho^{\frac{1}{2}})}{(1+\rho)^2}$$

$$= \omega \left(\frac{1 + \rho^{-\frac{1}{2}}}{1 + \rho} \right).$$

Cor. 2. Similarly since in opposition if the moving planet is superior, or in inferior conjunction if the moving planet is inferior, $(n-1)\, \phi = 180°$, angular velocity of a planet in opposition or inferior conjunction

$$= \omega \frac{\rho^{\frac{1}{2}} + 1 - \rho^{-\frac{1}{2}} - \rho}{1 + \rho^2 - 2\rho}$$

$$= \omega \frac{(1-\rho) - \rho^{-\frac{1}{2}} (1-\rho)}{(1-\rho)^2} = \omega \left(\frac{1 - \rho^{-\frac{1}{2}}}{1 - \rho} \right)$$

$$= \frac{-\omega}{\sqrt{\rho}} \cdot \frac{1 - \rho^{\frac{1}{2}}}{1 - \rho} = - \frac{\omega}{\sqrt{\rho} + \rho}.$$

Schol.—All the above formulæ are susceptible of many modifications depending on the relations subsisting between the periods, distances, real velocities, and angular velocities of the planets in their orbits. From Kepler's third law all such modifications may be directly deduced.

PROP. V.—*A planet transits the sun's disc at such a rate that the sun's diameter S would be traversed in time t; assuming circular orbits and uniform motion, determine the planet's distance from the sun.**

Let the planet's distance $= \rho$, earth's distance being unity, and let ω be the earth's angular vel. about the sun = sun's angular vel. about earth. Then, if t' be the time in which the sun in his annual course moves through a distance equal to his own apparent diameter, $\omega t' = S$, and the planet's angular velocity about the earth when in inferior conjunction

$$= -\frac{\omega}{\sqrt{\rho} + \rho}.$$

Wherefore, the planet's retrograde gain on the sun (which advances with angular velocity ω)

$$= \frac{\omega}{\sqrt{\rho} + \rho} + \omega,$$

$$= \omega\left(\frac{1 + \sqrt{\rho} + \rho}{\sqrt{\rho} + \rho}\right) = \frac{S}{t} = \frac{\omega t'}{t},$$

or
$$\rho + \sqrt{\rho} = \frac{t}{t' - t};$$

a quadratic giving

$$\sqrt{\rho} = -\tfrac{1}{2} \pm \frac{\sqrt{3t + t'}}{2} = \tfrac{1}{2}\left(\pm\frac{\sqrt{3t + t'}}{t' - t} - 1\right),$$

or
$$\rho = \tfrac{1}{2}\left(\frac{t + t}{t' - t} \pm \frac{\sqrt{3t + t'}}{t' - t}\right).$$

The lower sign must be taken, the upper giving a value of ρ greater than unity.

Cor. Let us take the supposed case of Vulcan, whose

* This was the problem Lescarbault had to deal with in the case of the supposed intra-Mercurial planet Vulcan. He failed for want of such formulæ as are here given.

rate of transit was such that the sun's diameter would have been traversed in rather more than four hours. Since in March (the time of the supposed discovery) the sun traversed by his annual motion a space equal to his own apparent diameter in rather more than 12 hours, we may say that (with as near an approximation as an observation of this kind—inexact at the best—merits) $t' = 3t$. Thus

$$\rho = \tfrac{1}{2}(2 - \sqrt{3})$$
$$= \tfrac{1}{2}(2 - 1{\cdot}732) = \tfrac{1}{2}(0{\cdot}268) = 0{\cdot}134.$$

This is very near the estimated value of the imagined planet's distance.

FORMS OF EPICYCLIC CURVES.

The relations discussed in the propositions of this section enable us to determine the shape and general features of epitrochoids or direct epicyclics and of hypotrochoids or retrograde epicyclics, for various values of D, E, and n. I propose to consider these features, but briefly only, because in reality their consideration belongs rather to the analytical than to the geometrical discussion of our subject.

In the first place, since we obtain the same curve by interchanging deferent and epicycle, and at the same time interchanging the relative angular velocities of the motions in these circles, we shall obtain all possible varieties of epicyclic curves by taking D as not less than E, so long as we give to n all possible values from positive to negative infinity.

The whole curve lies, in every case, between circles of radii D+E and D−E, the apocentres falling on the former circle, the pericentres on the latter. When D = E, the whole curve lies within the apocentral circle; and all the pericentres lie at the fixed centre.

PLATE II.

Fig. 98.

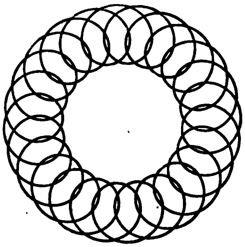


Fig. 99.

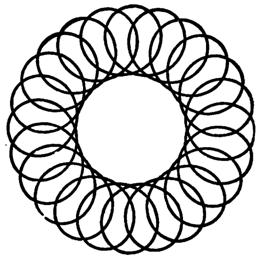


Fig. 100.

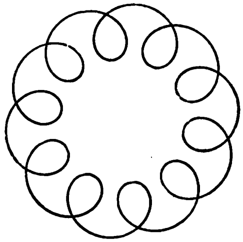


Fig. 101.

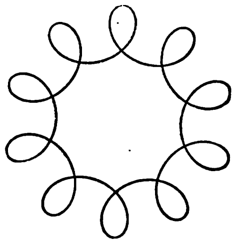


Fig. 102.

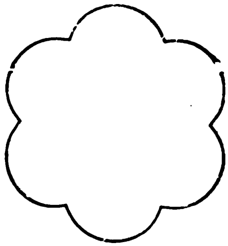


Fig. 103.

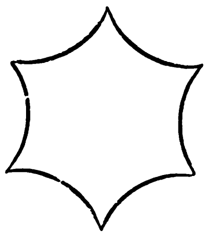


Fig. 104.

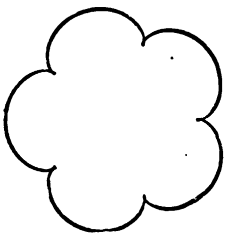


Fig. 105.

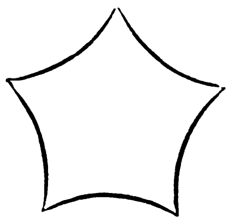


PLATE III.

Fig. 106.

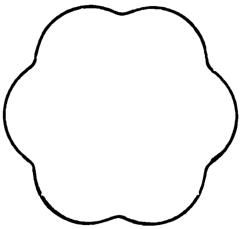


Fig. 107.

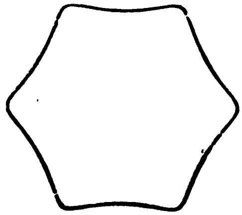


Fig. 108.

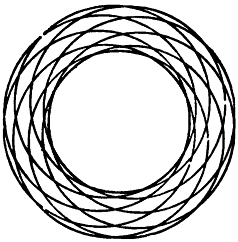


Fig. 109.

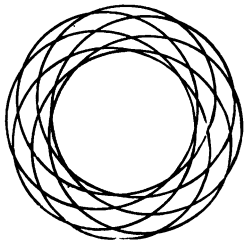


Fig. 110.

Fig. 111.

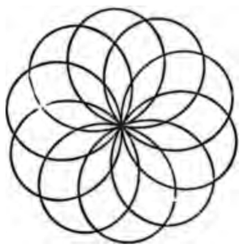


Fig. 112.

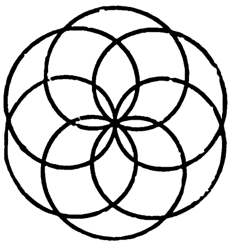
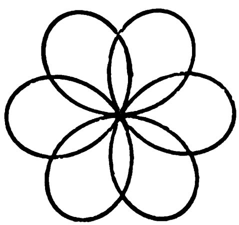


Fig. 113.



If n be infinite, whether positive or negative, we may consider the deferential velocity zero, and that of the epicyclic finite, giving for the curve the direct epicycle itself if n is positive, and the retrograde epicycle itself if n is negative.

When n is very great, we obtain such a curve as is shown in fig. 98, Plate II. (p. 184) if n is positive, and such a curve as in fig. 99, if n is negative.

As n diminishes the angle of descent increases, the loops separate and we obtain such forms as are shown in figs. 100 and 101, for n positive or negative respectively.

With the further reduction of n, the loops become smaller, the point of intersection approaching the pericentre when n is positive, the apocentre when n is negative, until finally, when $n = \frac{D}{E}$, the loops disappear and we have pericentral cusps as in figs. 102 and 104, or apocentral cusps as in figs. 103 and 105, according as n is positive or negative. In the former case the curve is the epicycloid, in the latter the hypocycloid.

As n diminishes from $\frac{D}{E}$ towards unity the cusps disappear and we have points of inflexion on either side of the pericentres if n is positive, or of the apocentres if n is negative, as shown respectively in figs. 106 and 107, Plate III.

As n further diminishes the points of inflexion draw further apart for a while in case of direct epicyclic, and afterwards approach until $n^2 = \frac{D}{E}$, when they coincide again at the pericentres, the curve being entirely concave towards the centre for all smaller values of n. In the case of the retrograde epicyclic, the points of inflexion draw apart on either side of the apocentres, and continue so to do till they meet points of inflexion advancing from next apocentres on either side; so that in this case, as in that of direct epicyclic, we have when $n^2 = \frac{D}{E}$ two points of inflexion coinciding at the pericentres. These two cases are illustrated in figs. 114 and 115. The former is a direct epicyclic; $n = 5$; and D : E : : 25 : 1; (apocentral dist. : pericentral dist. : : D+E : D−E : : 13 : 12. The latter is a retrograde epicyclic; $n = -3$; and D : E : : 9 : 1; (apocentral dist. : pericentral dist. : : D+E : D−E : : 5 : 4). Compare figs. 118, 121, 154, 158.

As n continues to decrease from the value $\sqrt{\frac{D}{E}}$ the angle of descent continually increases if n is positive and we have curves of the form shown in fig. 108.

Fig. 114.

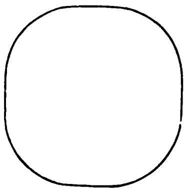
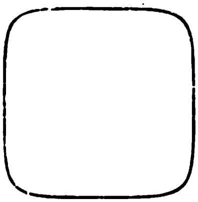


Fig. 115.



In diminishing from the value $\sqrt{\frac{D}{E}}$, n passes through the value unity. When $n = +1$ the curve is a circle having the fixed point as centre, and having for radius whatever distance the tracing point may have from that centre initially; the radius vector therefore always lies in value between D + E and D − E.

As n continuing positive diminishes in absolute value from 1 to 0, the angle of descent which had become infinite diminishes, remaining positive.* The curve continues concave

* De Morgan says, 'becomes very great and negative.' This is correct on his assumption that the angle of descent is to be re-

towards the centre, resembling the appearance it had had before n reached the value unity. As n approaches the value 0, however, the angle of descent becomes less and less, until when $n=0$ it becomes 180°, the curve being now a circle having radius D and centre at distance E from the fixed centre. Thus, if the tracing point is initially at A, fig. 81, p. 148, the centre is at c, but if the tracing point is initially at P, the centre is at c', (O c being parallel to C P).

As n diminishes in absolute value from $-\sqrt{\frac{D}{E}}$ to -1, the angle of descent increases till it is equal to 90°, the curve, always concave towards the fixed centre, forming a series of arcs more and more approaching the elliptical form, as in fig. 109, till when $n = -1$ the curve is the elliptical hypocycloid, see p. 124. We see that the equality of the diameters of the fixed and rolling circles is equivalent to the condition $n = -1$ for retrograde epicyclic. The semi-axes are $(D+E)$ and $(D-E)$.

Lastly as n, still negative, diminishes from -1 towards 0, the curve at first resembles in appearance that obtained before n reached the value -1, but the angle of descent gradually increases, until at length, when $n = 0$, it is 180° and the curve becomes the circle already described.

garded as positive when the radius of the epicycle gains in direction on the radius of the deferent, and negative when the radius of the deferent gains in direction on the radius of the epicycle. There is no occasion, however, to make this assumption, which is altogether arbitrary. If we consider the actual motion of the tracing point coming alternately at apocentre and at pericentre upon the deferential radius, *which constantly advances* whatever the value of n positive or negative (except $+1$ only), we must consider the angle of descent as always positive. We arrive at the same conclusion also if we consider that the radius vector advances on the whole between apocentre and following pericentre, for all epicyclics, direct or retrograde.

The varieties of form assumed by epicyclics according to the varying values of n, D, and E, are practically infinite. It will be noticed that in all the illustrative figures, n is a commensurable number, so that the curve re-enters into itself. Of course, no complete figure of an epicycle in which n is not a commensurable number could be drawn.

Certain special cases may here be touched on briefly.

When D = E, the direct epicyclic assumes such forms as are shown in figs. 110, 112, the retrograde epicyclic such forms as are shown in figs. 111 and 113. The distinction between the two classes of epicyclics in these cases is recognised by noting that the angle of descent, which must be positive, can only be made so by tracing the curves in figs. 110 and 112 the direct way, and by tracing those in figs. 111 and 113 the reverse way.

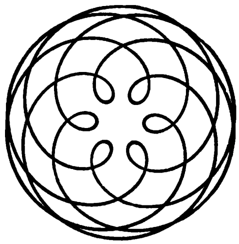
A distinction must be noted between direct and retrograde epicyclics, when D is nearly equal to E, and n approaches the value $\frac{D}{E}$, which is nearly equal to unity. For the direct epicyclic, the angle of descent, $180° \div (n-1)$, becomes very great, and we have a curve which passes from apocentre to pericentre through a number of revolutions, before beginning to ascend again by as many revolutions to the next pericentre.* On the other hand, in the case of the retrograde epicyclic, when D is very nearly equal to E, the angle of descent $180° \div (n+1)$ approaches in value to 90°, or the angle between successive apocentres approaches in value to two right angles, so that the curve has such a form as is shown farther on in fig. 119.

We have followed the effects of changes in the value of

* Prof. De Morgan strangely enough takes figs. 116 and 117 as illustrating this case. But in both these figs. $n = \frac{11}{5}$; in fig. 117, $D = \frac{11}{5}$ E. In neither is E very nearly equal to D.

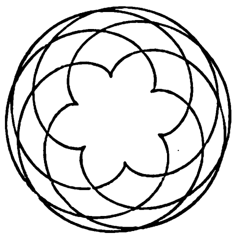
n, where D and E are supposed to remain unchanged throughout. The number of apocentres and pericentres depends, as we have already seen, on the value of n. It will be a useful exercise for the student to examine the effect of varying the value of E, keeping D and n constant, or (which amounts

FIG. 116.



really to the same thing) to examine the effect of varying the value of $\frac{E}{D}$, keeping n constant. Since the angle of descent is equal to $180° \div (n - 1)$ if n is positive, and to $180° \div$

FIG. 117.

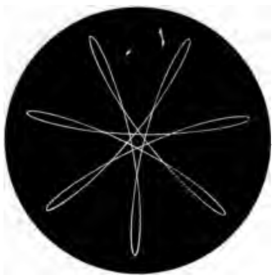


$(n+1)$ if n is negative, changing the value of $\frac{E}{D}$ will not give all the curves having any given number m of apocentres or pericentres (for each revolution of the deferent). For this purpose it is necessary to assume first $n = (m + 1)$, giving all the direct epicyclics having m apocentres and m pericentres, and secondly $n = -(m-1)$ giving all the retrograde epicyclics having m apocentres and m pericentres, for each revolution of the deferent. (Of course, m is not necessarily a whole number.)

FIG. 118.

Suppose we take $n = \frac{11}{5}$, so that the angle of descent $(= 180° \div \frac{6}{5})$ is equal to $\frac{5}{6}$ths of two right angles. Then if $E > \frac{5}{11} D$ we have such a curve as is shown in fig. 116. As E diminishes until $E = \frac{5}{11} D$, the loops turn into cusps as

FIG. 119.



shown in fig. 117; as E diminishes still further until $E = \frac{25}{121} D$ (that is $n^2 = \frac{D}{E}$), the curve assumes the orthoidal form shown in fig. 118. Again, take $n = -\frac{4}{3}$. Then when E is nearly equal to D the curve has such a form as is shown in fig. 119, merging into the cuspidate form as in fig. 120, when $E = \frac{3}{4} D$; and into the orthoidal (or straightened) form, as in fig. 121, when $E = \frac{9}{16} D$ (or

FIG. 120.

$n^2 = \frac{D}{E}$). For further illustrations see p. 256.

If we compare fig. 98 with fig. 122, we perceive that in the former the loop between two successive whorls overlaps

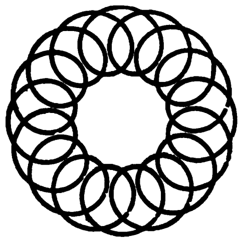
FIG. 121.

two preceding loops, while in the latter each loop overlaps but one preceding loop. A number of varieties arise in this way. The determination of the condition under which any given preceding loop may be just touched is not difficult;

but in no case does the condition lead to a formula giving n directly in terms of D and E. The simplest of these cases is dealt with in Prop. IX. of this section. (See fig. 160, p. 256.)

Figs. 123 and 124 illustrate eight-looped epicyclics direct and retrograde. By noting the different proportions between

FIG. 122.



their respective loops, and by comparing fig. 123 with fig. 100, a ten-looped direct epicyclic, and fig. 124 with fig. 101, a ten-looped retrograde epicyclic, the student will recognise the effect of varying conditions on the figures of epicyclics. (In

FIG. 123.

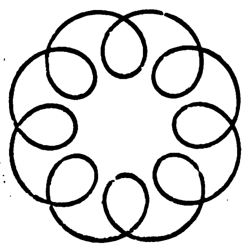
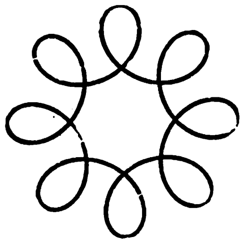


fig. 100, $n = 11$; in fig. 101, $n = -9$; in fig. 123, $n = 9$, and in fig. 124, $n = -7$).

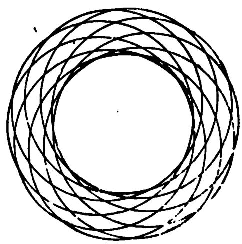
It is a useful exercise to take a series of epicyclics and determine the value of D, E, and n, from the figure of the curve. Suppose, for instance, the curve shown in fig. 125, is given for examination. This closely resembles fig. 108 in appearance; but in reality fig. 125 is a retrograde, whereas fig. 108 is a direct epicyclic. The character of the curve in this respect is determined by tracing it directly from any apocentre and noting that the next apocentre falls behind

FIG. 124.



the one from which we started. The values of D and E are determined at once from the dimensions of the ring within which the curve lies,—its outer radius being D + E, its inner D − E. The value of n is conveniently determined

FIG. 125.



by noting the angle between two neighbouring apocentres (indicated best by the intersections of the curve next within the apocentres, for from the symmetry of the curve all intersections lie of necessity either on apocentral radii vectores or on these produced). This angle = one-tenth of 360°, so that the angle of descent is $\frac{9}{10}$ths of 180°; or $n + 1 = \frac{10}{9}$. Thus in absolute value $n = \frac{1}{9}$, but n is negative.

In like manner we find that in fig. 126, $n = -\frac{1}{5}$.

In each of the figs. 127, 128, and 129, $n = 2$, since there is only one apocentre. In fig. 127, the trisectrix,

FIG. 126.

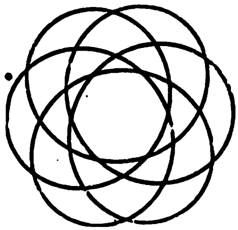
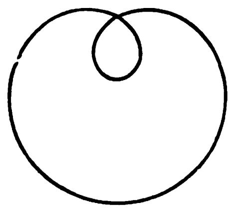


FIG. 127.



D = E; in fig. 128, the cardioid, D = 2 E; in fig. 129, D = 3 E.

Figs. 130 and 131, Plate IV., illustrate some of the pleasing combinations of curves which may be obtained by the use of the geometric chuck, the instrument with which all the curves of the present part of this section have been drawn. In

FIG. 128.

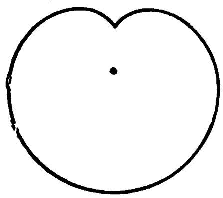


FIG. 129.

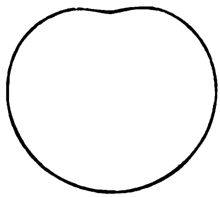


fig. 130 we have two direct epicyclics, (D − E) of the outer being equal to (D + E) of the inner. It will be found that for the outer $n = 7$, while for the inner $n = 15$. In fig. 131 we have four direct epicycles, having (D + E) constant, but ratio D : E different in each. It will be found that there

PLATE IV.

Fig. 130.

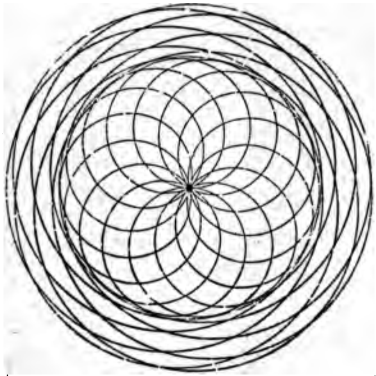
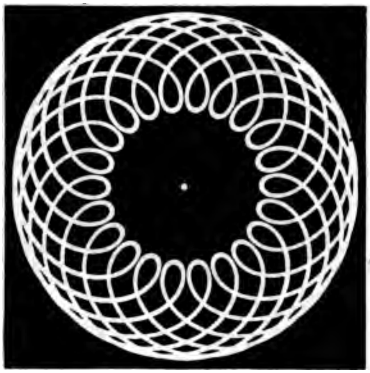


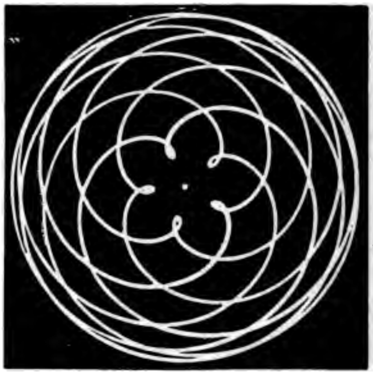
Fig. 131.

Fig. 134. MERCURY.



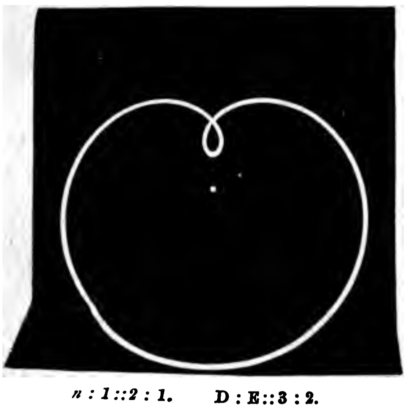
n : 1 :: 29 : 7. D : E :: 5 : 2.

Fig. 135. VENUS.



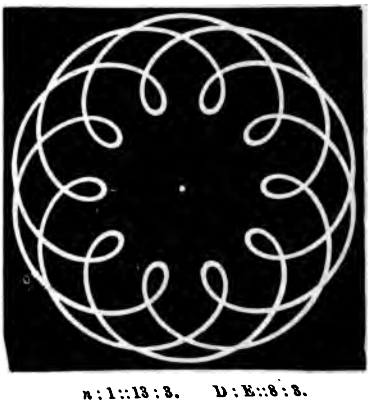
n : 1 :: 13 : 8. D : E :: 10 : 7.

Fig. 136. MARS.



n : 1 :: 2 : 1. D : E :: 3 : 2.

Fig. 137. JUNO.



n : 1 :: 13 : 3. D : E :: 8 : 3.

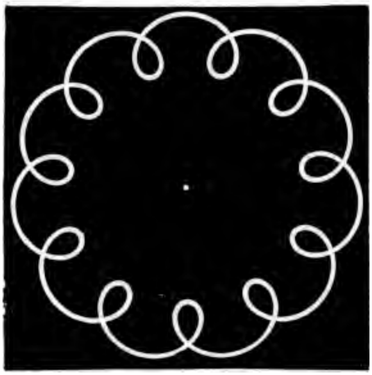
APPROXIMATE FORMS OF

PLATE V.

FIG. 132.

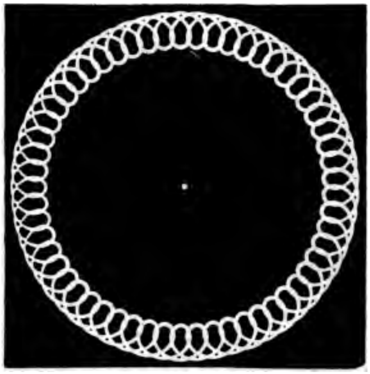
FIG. 133.

FIG. 138. JUPITER.



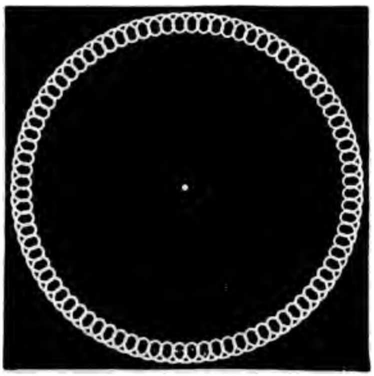
n : 1 :: 12 : 1. D : E :: 5 : 1.

FIG. 139. SATURN.



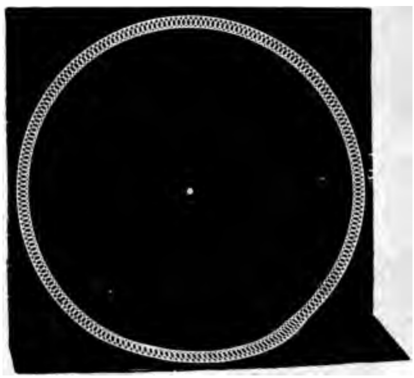
n : 1 :: 59 : 2. D : E :: 19 : 2.

FIG. 140. URANUS.



n : 1 :: 85 : 1. D : E :: 19 : 1.

FIG. 141. NEPTUNE.



n : 1 :: 217 : 1. D : E :: 36 : 1.

THE PLANETARY EPICYCLICS.

are $5\frac{1}{3}$ apocentres in each circuit; whence $(n - 1) = \frac{3}{16} . 360 = 67\frac{1}{2}$, and $n = 68\frac{1}{2}$. The inner part of the figure is a retrograde epicyclic having $5\frac{1}{3}$ apocentral distances in each circuit; whence in absolute value $(n + 1) = 67\frac{1}{2}$, and $n = - 66\frac{1}{2}$.

Figs. 132, 133, Plate V., are further examples for the student.

The remaining eight figures of Plates IV. and V., for which I am indebted to Mr. Perigal, present the approximate figures of the epicyclics traversed by the planets, with reference to the earth regarded as fixed. Of course the real curves of the planetary orbits with reference to the earth do not return into themselves as these do, the value of n not being in any case represented by a commensurable ratio. Moreover, the orbits of the earth and planets around the sun are not in reality circles described with uniform velocity, but ellipses around the sun as a focus of each and described according to the law of areas called Kepler's second law. Therefore figs. 134—141 must be regarded only as representative types of the various epicyclics to which the planetary geocentric paths approximate more or less closely. In the case of Mars, I may remark that either of the ratios 15 : 8 or 32 : 17 would have given a more satisfactory approximation to the planet's epicyclic path around the earth. It so chances that I have taken occasion during the opposition-approach of Mars in 1877 to draw the true geocentric path of Mars around the earth for the last forty years and for the next fifty, taking into account the eccentricity and ellipticity of the paths, and the varying motion of the earth and Mars in their real orbits around the sun. The resulting curve, though presenting the epicyclic character, yet falls far short of any of the curves of Plates IV.

PLATE IV.

Fig. 130.

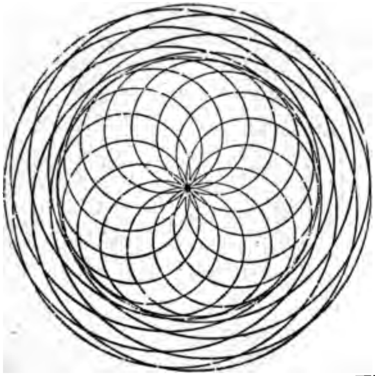
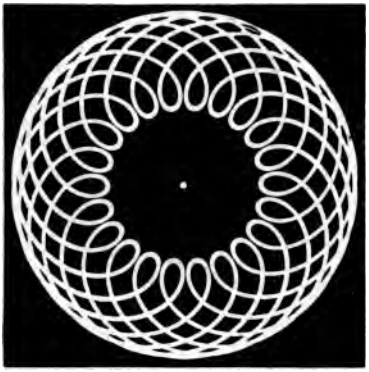


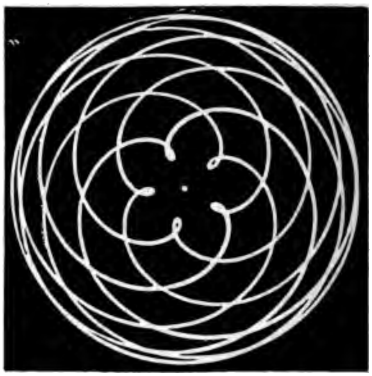
Fig. 131.

Fig. 134. MERCURY.



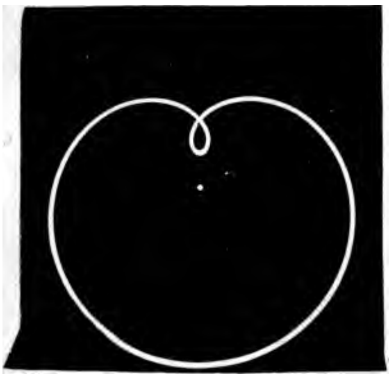
$n : 1 :: 29 : 7$. D : E :: 5 : 2.

Fig. 135. VENUS.



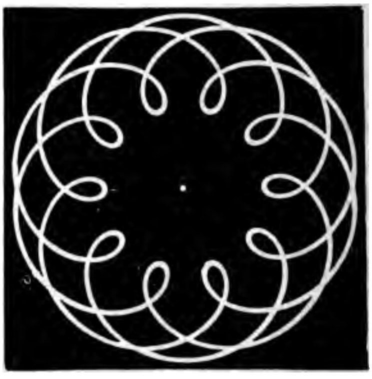
$n : 1 :: 13 : 8$. D : E :: 10 : 7.

Fig. 136. MARS.



$n : 1 :: 2 : 1$. D : E :: 3 : 2.

Fig. 137. JUNO.



$n : 1 :: 13 : 3$. D : E :: 8 : 3.

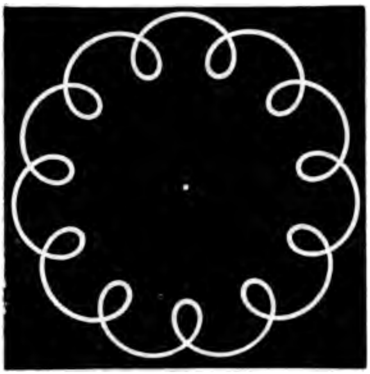
APPROXIMATE FORMS OF

PLATE V.

FIG. 132.

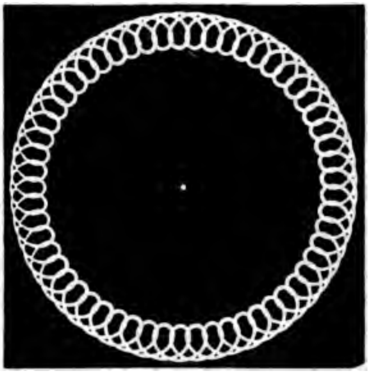
FIG. 133.

FIG. 138. JUPITER.



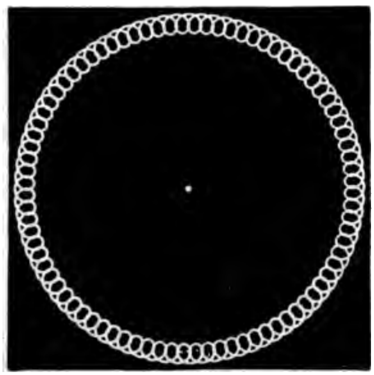
n : 1 :: 12 : 1. D : E :: 5 : 1.

FIG. 139. SATURN.



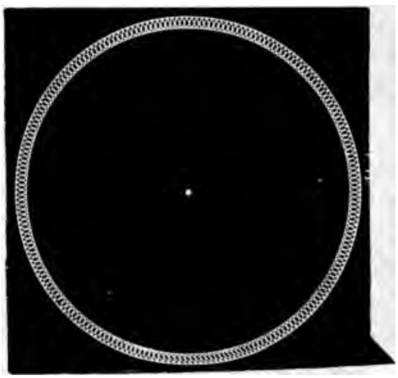
n : 1 :: 59 : 2. D : E :: 19 : 2.

FIG. 140. URANUS.



n : 1 :: 85 : 1. D : E :: 19 : 1.

FIG. 141. NEPTUNE.



n : 1 :: 217 : 1. D : E :: 36 : 1.

THE PLANETARY EPICYCLICS.

are $5\frac{1}{3}$ apocentres in each circuit; whence $(n - 1) = \frac{3}{16} . 360 = 67\frac{1}{2}$, and $n = 68\frac{1}{2}$. The inner part of the figure is a retrograde epicyclic having $5\frac{1}{3}$ apocentral distances in each circuit; whence in absolute value $(n + 1) = 67\frac{1}{2}$, and $n = - 66\frac{1}{2}$.

Figs. 132, 133, Plate V., are further examples for the student.

The remaining eight figures of Plates IV. and V., for which I am indebted to Mr. Perigal, present the approximate figures of the epicyclics traversed by the planets, with reference to the earth regarded as fixed. Of course the real curves of the planetary orbits with reference to the earth do not return into themselves as these do, the value of n not being in any case represented by a commensurable ratio. Moreover, the orbits of the earth and planets around the sun are not in reality circles described with uniform velocity, but ellipses around the sun as a focus of each and described according to the law of areas called Kepler's second law. Therefore figs. 134—141 must be regarded only as representative types of the various epicyclics to which the planetary geocentric paths approximate more or less closely. In the case of Mars, I may remark that either of the ratios 15 : 8 or 32 : 17 would have given a more satisfactory approximation to the planet's epicyclic path around the earth. It so chances that I have taken occasion during the opposition-approach of Mars in 1877 to draw the true geocentric path of Mars around the earth for the last forty years and for the next fifty, taking into account the eccentricity and ellipticity of the paths, and the varying motion of the earth and Mars in their real orbits around the sun. The resulting curve, though presenting the epicyclic character, yet falls far short of any of the curves of Plates IV.

and V. in symmetry of appearance. The loops are markedly unequal, a relation corresponding of course to the observed inequality of the arcs of retrogradation traversed by Mars at different oppositions.

NOTE.—Mr. H. Perigal, to whom I am indebted for all the illustrations of this part of the present work (except figs. 118–121, 132, 133, and 154–161, engraved by Mr. L. W. Boord, with a similar instrument), gives the following account of the geometric chuck:—

'The geometric chuck, a modification of Suardi's geometric pen, was constructed by J. H. Ibbetson, more than half a century ago, as an adjunct to the amateur's turning-lathe. It is admirably adapted for the purposes of ornamental turning; but is still more valuable as a means of investigating the curves produced by compound circular motion. In its simplest form it generates bicircloid curves, so called from their being the resultants of two circular movements. This is effected by a stop-wheel at the back of the instrument giving motion to a chuck in front, which rotates on its centre, while that centre is carried round with the rest of the instrument and the train of wheels which imparts the required ratio of angular velocity to the two movements. A sliding piece gives the radial adjustment, which determines the phases of the curve dependent upon the radial-ratio.

'By the simple geometric chuck a double motion is given to a plane on which the resultant curve is delineated by a fixed point; but it may act as a geometric pen when it is made to carry the tracing point with a double circular motion, so as to delineate the curve on a fixed plane surface. The curves thus produced being reciprocals, all the curves generated by the geometric chuck may be produced by the geometric pen, and *vice versâ*, by making the angular velocity of the one reciprocal to that of the other. For instance, the ellipse may be generated by the geometric chuck with velocity-ratio = 1 : 2' (see, however, remarks following this extract), 'and by the geometric pen with velocity-ratio = 2 : 1, the movements of both being inverse, that is, in contrary directions.

'The accompanying curves were turned in the lathe with the geometric chuck (by myself, many years ago), of sufficient depth to enable casts to be taken from them in type metal, so as to print the curves as black lines on a white ground. These curves are therefore veritable autotypes of motion.'

Mr. Perigal has invented, also, an ingenious instrument, called the kinescope (sold by Messrs. R. & J. Beck, of Cornhill), by which all forms of epicyclics can be ocularly illustrated. A bright bead is set revolving with great rapidity about a centre, itself revolving rapidly about a fixed centre, and by simple adjustment, any velocity-ratio can be given to the two motions, and thus any epicyclic traced out. The motions are so rapid that, owing to the persistence of luminous images on the retina, the whole curve is visible as if formed of bright wire.

He has also turned hundreds of epicyclics (or bicircloids, as he prefers to call them) with the geometric chuck. There is one point to be noticed, however, in his published figures of these curves. The velocity-ratio mentioned beside the figures is not the ratio $n : 1$ of this section, but $(n-1) : 1$, *i.e.*, he signifies by the velocity-ratio, *not* the ratio of the actual angular velocity of the tracing radius in the epicycle to the angular velocity of the deferent radius, but the ratio of the angular gain of the tracing radius *from the deferent* to the angular velocity of the deferent. This may be called the mechanical ratio, as distinguished from the mathematical ratio; for a mechanician would naturally regard the radius C′A′ of the epicycle PA′P′ (fig. 81) as at rest, and therefore measure the motion of the tracing radius C′P′ from C′A′, whereas in the mathematical way of viewing the motions, C′*a* is regarded as the radius at rest, and the motion of C′P is therefore measured from C′*a*. The point is not one of any importance, because no question of facts turns upon it; but it is necessary to note it, as the student who has become accustomed to regard the velocity-ratios as they are dealt with in the present section (and usually in mathematical treatises on epicyclic motion), might otherwise be perplexed by the numerical values appended to Mr. Perigal's diagrams. These values, be it noticed, are those actually required in using the geometric chuck or the kinescope; for in all adjustments the epicycle is in mechanical connection with the deferent.

FORMS OF RIGHT TROCHOIDS.

Right trochoids may be regarded as epicyclics having the radius of deferent infinite, the centre of the epicycle travelling in a straight line. A good idea of the form of trochoids may be obtained by regarding them as pictures of screw-shaped wires (like fine corkscrews), viewed in particular directions. This may be shown as follows:—

If a point move uniformly round a circle whose centre advances uniformly in a straight line perpendicular to the plane of the circle, the point will describe a right helix, the convolutions of which will lie closer together, relatively to the span of each, as the motion of the point in the circle is more rapid relatively to the motion of the circle's centre. Now if any plane figure be projected on a plane at right angles to its own, by parallel lines inclined half a right angle to each plane (or perpendicular to one of the two planes bisecting the plane angle between them), the projection of the figure is manifestly similar and equal to the figure itself. Therefore if the circle and the point tracing out the helix just described be projected on a plane parallel to the axis of the helix, by lines making with this plane and the plane of the circle an angle equal to half a right angle, the circle will be projected into a circle whose centre advances uniformly in the plane of projection in a right line. The projection of the tracing point will be a point travelling uniformly round this circle; and therefore the projection of the helix will be a right trochoid. We may say then that every helix viewed at an angle of 45° to its axis is seen as a trochoid,—or rather that portion of the helix which is so viewed from a distant point appears as a trochoid. When the tracing point of a helix moves at the same rate as the centre of the circle, the helix viewed at an angle of 45° to its axis appears as a right cycloid. Thus a helicoid or corkscrew wire having a slant of 45° and viewed from a great distance at the same slant (so that the line of sight coincides with the direction of the helix where touched, at one side, by a plane through the remote point of view), appears as a cycloid.

The helix is projected into other curves if the line of sight is inclined to the axis at an angle less or greater than 45°. In this case the projected curve is that generated by a point travelling round an ellipse in such a way that the eccentric angle increases uniformly while the centre of the ellipse advances uniformly,—in the direction of the minor axis if the angle of inclination exceeds half a right angle, and of the major axis if the angle of inclination is less than half a right angle.

A set of such curves, obtained from a helix of inclination 45°, are shown in fig. 144, plate VI., A b_{10} T′ being a semicycloid, and A b_9 T, A b_8 T′, &c., other projections of the same portion of the helix by lines inclined to the plane of projection at an angle exceeding a right angle, A b T′ being the orthogonal projection of this portion of the helix.

Such curves, and varieties of them resulting when the helix is skewed (the centre of the circle advancing in a direction not perpendicular to the plane of the circle), possess interesting properties; but they do not belong to our subject, not being trochoidal. Moreover, for their thorough investigation much more space would be required than can here be spared. But one of these curves, the orthogonal projection A b T′ (fig. 144, Plate VI.) of a helix of inclination 45°, must be briefly mentioned here, because associated historically as well as geometrically with the right cycloid.

THE COMPANION TO THE CYCLOID.

This curve, called also 'Roberval's Curve of Sines,' may be obtained as follows:—

Let AB (fig. 142) be a fixed diameter of a circle AQB, and through any point Q on AQB draw MQp perp. to ACB and equal to the arc AQ; the locus of this point p is the companion to the cycloid APD having AB as axis.

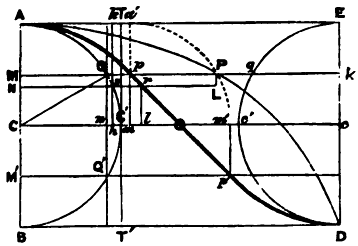
If COc, the line of centres of semicycloid APD, be bisected in O, the curve passes through O, because CO = quadrant AQC′.

Drawing $p\,m$, Q, n, perp. to COc, we have mO =

$CO - Cm = AC' - AQ =$ arc QC'; $pm : Om :: Qn :$ arc QC' $::$ sin $QCC' :$ circ. meas. of QCC'. Hence the part $A\,p\,O$ of the companion to the cycloid is a curve of sines.

Produce $Q\,n$ to meet $AC'B$ in Q', draw $M\,Q'\,p'$ paralle

Fig. 142.



to BD to meet the curve $A\,p\,D$ in p' and AB in M', and draw $p'm'$ perp. to $CO\,c$. Then

$$C\,m' = M'p' = AC\,Q', \text{ and } OC = AC'$$

$$\therefore\ O\,m' = \text{arc } C'Q' = \text{arc } C'Q = O\,m\,;$$

And $$p'\,m' = n\,Q' = n\,Q = p\,m.$$

Therefore the part $O\,p'D$ of the curve bears precisely the same relation to the line $O\,c$, which the part $A\,p\,O$ bears to OC. Thus the entire curve is a curve of sines.

Area $A\,p\,OC$ = area $O\,p'\,D\,c$; wherefore, adding CODB,
area AODB = rect. CD = $\frac{1}{2}$ rect. BE = circle AQB.

It is also obvious that the same curve $D\,p'\,O\,p\,A$ will be obtained by taking $E\,c'\,D$ as the generating semicircle, and drawing $m'\,q'\,p' =$ arc $q'\,D$, $m\,q\,p =$ arc $q\,q'\,D$; so that the figure $ED\,p\,\,O\,p\,A$ is in all respects equal to the figure $BA\,p\,O\,p'\,D$.

Since MQP = arc AQ + MQ; and M p = arc AQ,

$$MQ = p\,P;$$

so that an elementary rectangle QN = elementary rectangle p L of same breadth; whence it follows that area A p D P = semicircle AQB: for we may regard p L and NQ as elementary rectangles of these areas respectively, and the equality of every such pair of elements involves the equality of the areas. Since

area AODB=circle AQB; and area A p DP=$\frac{1}{2}$ circle AQB;

$\therefore$ Area APDB = $\frac{3}{2}$ circle AQB;

and 2 area APDB = 3 circle AQB:

this is Roberval's demonstration of the area of the cycloid.

Draw $s\,r$ parallel and near to Qp, and $k\,s\,h$, C T, $r\,l$ perp. to OC; then

$$l\,\mathrm{C} = \mathrm{A}\,s;\ m\,\mathrm{C} = \mathrm{AQ};\ \therefore m\,l = \mathrm{Q}\,s;\ \text{and}$$

$$m\,l : n\,h :: \mathrm{Q}\,s : n\,h :: \mathrm{CQ}\ (=h\,k) : \mathrm{Q}\,n\ (\text{ult.} = r\,l)$$

$\therefore$ rect. $m\,l\,.\,r\,l$=rect. $n\,h\,.\,h\,k$; that is, rect. $r\,m$=rect. $n\,k$; or inct. of area A $p\,m$ C=inct. of rect. A n. But these areas begin together. Hence area A $p\,m$ C = rect. A n; also

Area AOC = rect. CT; and area $p\,m$ O = rect. n T.

Representing angles by their circular measure:—

$$p\,m = r\sin\frac{\mathrm{QC}'}{r} = r\sin\frac{\mathrm{O}\,m}{r};\ \text{and rect. } n\mathrm{T} = r^2\left(1 - \cos\frac{\mathrm{O}\,m}{r}\right);$$

therefore, the proof that area $p\,m$ O = rect. n T, may be regarded as a geometrical demonstration of the relation

$$\int_x^0 \sin x\,d\,x = 1 - \cos x;$$

and similarly, since $p\,m = r\cos\frac{\mathrm{AQ}}{r} = r\cos\frac{\mathrm{C}\,m}{r}$, the proof that area A p m C = rect. A n may be regarded as a geometrical demonstration of the relation

$$\int_x^0 \cos x \, d x = \sin x.$$

It will easily be seen that for points on O p'D,

Area AO p' M' − rect. M' m' = rect. A n, or B n,

leading again to the relation

area AODB = rect. B c.

SECTION VI.

EQUATIONS TO CYCLOIDAL CURVES.

Although, properly speaking, the discussion of the equations to cycloidal curves belongs to the analytical treatment of our subject, it may be well, for convenience of reference, to indicate here the equations to trochoids (including the cycloid), epicyclics, and the system of spirals which may be regarded as epitrochoidal (*see* p. 127, *et seq.*). For the sake of convenience and brevity I follow the epicyclic method of considering all these curves.

Let the centre of a circle $a\,q\,b$ (figs. 45, 46, Plate I.), of radius e, travel with velocity 1 along a straight line C c in its own plane, while a point travels with velocity m round the circumference of the circle. Take the straight line C c for axis of x, C a for axis of y, and let the point start from a, in direction $a\,q\,b$. When it has described an angle $m\phi$ about C, the centre has advanced a distance $e\,\phi$ along C c, and therefore, if x and y are the coordinates of the tracing point,

$$x = e\,\phi + e \sin m\,\phi, \qquad y = e \cos m\,\phi. \qquad (1)$$

If we remove the origin to b, the centre of the base, taking $b\,d$ as axis of x and $b\,a$ as axis of y, the equations are,

$$x = e\phi + e \sin m\,\phi, \qquad y = e + e \cos m\,\phi. \qquad (2)$$

If we remove the origin to a, the vertex, taking $a\,e$ as axis of x and $a\,b$ as axis of y, the equations are

$$x = e\,\phi + e \sin m\,\phi, \qquad y = e - e \cos m\,\phi. \qquad (3)$$

If we remove the origin to c', taking c' C as axis of x, and $c'\,d'$ as axis of y, the tracing point starting from d in the same direction as before, the equations are

$$x = e\,\phi - e\sin\phi, \qquad y = e\cos m\,\phi. \qquad (4)$$

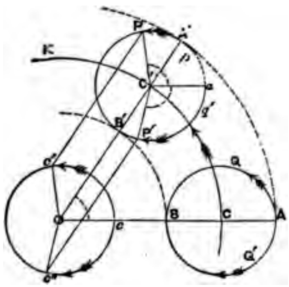
If in this case we remove the origin to e', taking $e'e$ as axis of x and $e'\,d'$ as axis of y, the equations are

$$x = e\,\phi - e\sin\phi, \qquad y = e + e\cos m\,\phi. \qquad (5)$$

And lastly, if we remove the origin to d', taking $d'd$ as axis of x and $d'\,e'$ as axis of y, we have the equations

$$x = e\,\phi - e\sin\phi, \qquad y = e - e\cos m\,\phi. \qquad (6)$$

FIG. 143. (Join C$'p$.)



If $m = 1$, these equations represent the right cycloid; if $m < 1$, they represent the prolate cycloid; and if $m > 1$, they represent the curtate cycloid.

For epicyclics, take O (fig. 143), the centre of fixed circle as origin, OA through an apocentre A as axis of x, and a perp. to OA through O as axis of y. Put OC, radius of deferent $= d$; CA, radius of epicycle $= e$ (using italics as more convenient in equations than capitals); $\angle$ COC$' = \phi$, and angle a C P $= n\,\phi$. Then, if x and y are the co-ordinates of P

$$x = d\cos\phi + e\cos n\,\phi, \qquad y = d\sin\phi + e\sin n\,\phi. \qquad (7)$$

If OC, instead of passing through an apocentre when produced, intersects the curve in a pericentre at B, the equations are

$$x = d\cos\phi - e\cos n\phi, \qquad y = d\sin\phi - e\sin n\phi. \quad (8)$$

For a retrograde epicyclic, angle $a\,C'P' = n\phi$, and the equations (A being an apocentre) are

$$x = d\cos\phi + e\cos n\phi, \qquad y = d\sin\phi - e\sin n\phi. \quad (9)$$

If B is a pericentre of retrograde epicyclic, the equations are

$$x = d\cos\phi - e\cos n\phi, \qquad y = d\sin\phi + e\sin n\phi. \quad (10)$$

But all these equations are derivable from form (7);—(8) by rotating the axis through the angle of descent, $\frac{\pi}{n-1}$; and (9) and (10) from (7) and (8) respectively by changing the sign of n. So that equations (7) may be used as the equations for the epicyclic in rectangular coordinates, without loss of generality.

When, in (7) and (10), $n = \frac{d}{e}$, the equations are those of the epicycloid and hypocycloid respectively, when an axis coincides with the axis x; if, in equations (8) and (9), $n = \frac{d}{e}$, the equations are those of the epicycloid and hypocycloid, respectively, when a cusp falls on axis of x. It will be remembered that if F is radius of fixed circle and R radius of rolling circle, $d = R + F$, and $e = R$; R being regarded as negative in case of hypocycloid.

From (7) we get

$$x^2 + y^2 = r^2 = d^2 + e^2 + 2\,d\,e\cos(n-1)\phi, \qquad (11)$$

$$\text{and } \tan\theta = \frac{d\cos\phi + e\cos n\phi}{d\sin\phi + e\sin n\phi}; \qquad (12)$$

which are the polar equations to the curve, O being the pole and OA, though an apocentre, the initial line. [Equation (11) is obviously derivable at once from the triangle OC′P.]

For the epicyclic spirals, suppose OC′, fig. 143 $= f$, and that a tangent at C to circle CK, carrying with it the perp. BCA, rolls over the arc CK, uniformly, till it is in contact at C, the angle C′OC being ϕ. Then if AC $= g$, and x and y are the rectangular coordinates of the point to which A has been carried, it is obvious (since CA in its new position is parallel to OC) that (taking projections on axes of x and y)

$$x=(f+g)\cos\phi+f\phi\sin\phi;\quad y=(f+g)\sin\phi-f\phi\cos\phi; \quad (13)$$

the equations to the epicyclic spiral traced by A. The spiral traced by B obviously has for its equations

$$x=(f-g)\cos\phi+f\phi\sin\phi;\quad y=(f-g)\sin\phi-f\phi\cos\phi. \quad (14)$$

From (13) we get

$$x^2+y^2=r^2=(f+g)^2+f^2\phi^2; \qquad \text{or}$$

$$f\phi=\sqrt{r^2-(f^2+g^2)};\ \tan\theta=\frac{(f+g)\sin\phi-f\phi\cos\phi}{(f+g)\cos\phi+f\phi\sin\phi} \quad (15)$$

the polar equations to these spirals. See also Ex. 133, p. 253.

If $g=0$, or the tracing point is on the tangent, equations (13) become

$$x=f\cos\phi+f\phi\sin\phi,\qquad y=f\sin\phi-f\phi\cos\phi; \quad (16)$$

the equations to the involute of a circle. The polar equation to this curve is (from 15),

$$\tan\theta=\frac{f\tan\dfrac{\sqrt{r^2-f^2}}{f}-\sqrt{r^2-f^2}}{f\tan\dfrac{\sqrt{r^2-f^2}}{f}+\sqrt{r^2-f^2}}. \quad (17)$$

If $g=-f$, equations (13) become

$$x=f\phi\sin\phi;\quad y=-f\phi\cos\phi;$$

giving $\qquad x^2+y^2=f^2\phi^2;\quad \text{or } r=f\phi;$

and $\qquad \tan\theta=-\cot\phi;\quad \text{or } \theta=\phi-\frac{\pi}{2};$

whence $$r = f\theta + f\frac{\pi}{2}; \qquad (18)$$

the polar equation to the spiral of Archimedes, with OD, fig. 72, p. 130, as initial line. If OQ be taken as initial line, the equation is

$$r = f\theta. \qquad (19)$$

All the pairs of equations in rectangular coordinates can readily, by eliminating ϕ, be reduced to a single equation between x and y. Thus (1) becomes

$$x = \frac{e}{m}\cos^{-1}\left(\frac{y}{e}\right) + \sqrt{e^2 - y^2}; \qquad (20)$$

the general equation to the right trochoid.

From equation (11)

$$\phi = \frac{1}{n-1}\cos^{-1}\frac{x^2 + y^2 - d^2 - e^2}{2de};$$

which combined with either of equations (7) gives the general equation to the epicyclic in rectangular coordinates. To obtain this general equation in a symmetrical form, note that from (7)

$$y\cos\phi - x\sin\phi = e\sin(n-1)\phi. \qquad (21)$$

However, in nearly all analytical investigations of the properties of these curves, it is more convenient to use the pair of equations (1) for trochoids, (7) for epicyclics, and (13) for epicyclic spirals, or the polar equations (11) and (12) for epicyclics, and (15) for epicyclic spirals.

The only use I propose to make, here, of the equations to these curves, is to obtain the general equations to the evolutes of trochoids, epicyclics, and epicyclic spirals. These general equations, though they may be deduced from relations established geometrically in the text, are more conveniently dealt with analytically.

We have in equations (1), (7), and (13), x and y expressed as functions of a third variable ϕ; wherefore

$$\rho = \frac{\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{d\phi^2}\frac{dx}{d\phi} - \frac{d^2x}{d\phi^2}\frac{dy}{d\phi}};$$

and the equation to the evolute is derived from the two equations

$$\xi = x - \frac{\frac{dy}{d\phi}\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}}{\frac{d^2y}{d\phi^2}\frac{dx}{d\phi} - \frac{d^2x}{d\phi^2}\frac{dy}{d\phi}};$$

$$\eta = y + \frac{\frac{dx}{d\phi}\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}}{\frac{d^2y}{d\phi^2}\frac{dx}{d\phi} - \frac{d^2x}{d\phi^2}\frac{dy}{d\phi}};$$

where ξ and η are coordinates of the point in the evolute corresponding to the point x, y, on the curve.

In the case of trochoids, we obtain from (1)

$$\frac{dx}{d\phi} = e + m\,e\cos m\phi; \quad \frac{dy}{d\phi} = -\,m\,e\sin m\phi;$$

$$\therefore \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = e^2(1 + 2\,m\cos m\phi + m^2).$$

Also, $$\frac{d^2y}{d\phi^2} = -m^2 e\cos m\phi; \quad \frac{d^2x}{d\phi^2} = -\,m\,e\sin m\phi;$$

$$\therefore \frac{d^2y}{d\phi^2}\frac{dx}{d\phi} - \frac{d^2x}{d\phi^2}\frac{dy}{d\phi} = -\,e^2 m^2(\cos m\phi + m);$$

wherefore $$\rho = -\,e\frac{(1 + 2\,m\cos m\phi + m^2)^{\frac{3}{2}}}{m^2(\cos m\phi + m)};$$

and if we put $-\dfrac{(1 + 2\,m\cos m\phi + m^2)}{m^2(\cos m\phi + m)} = k$, the equations to the evolute are

$$\begin{aligned} \xi &= e\phi + e(1 + km)\sin m\phi, \\ \eta &= ek + e(1 + km)\cos m\phi. \end{aligned} \qquad (22)$$

If we put $\frac{\phi}{1+k\,m}=\phi'$ and $m\,(1+k\,m)=m'$, these equations may be written

$$\xi = e\,(1 + k\,m)\,\phi' + e\,(1 + k\,m)\sin m'\varphi',$$

and $$\eta = e\,k + e\,(1 + k\,m)\cos m'\varphi'\,;$$

from which we see that the evolute of the trochoid may be regarded as traced by an epicycle of variable radius $e\,(1 + k\,m)$, in which the tracing point moves with velocity bearing the variable ratio m' to the velocity of the epicycle's centre, while the deferent straight line shifts parallel to the axis of x so that its distance from this axis is constantly equal to $e\,k$ on the negative side of the axis of y.

If $m = 1$ (or curve (1) becomes the cycloid), $k = -2$, and equations (22) become

$$\xi = e\,\phi - e\sin\phi\,;\quad \eta = -2\,e - e\cos m\,\phi\,; \tag{23}$$

showing that the evolute is an equal and similar cycloid, with parallel base, removed a distance $2\,e$, or one diameter of the tracing circle, from the base of the involute cycloid towards the negative side of the axis of y (that is *from* the concavity of the involute), and having vertices coincident with the cusps of the involute cycloid.

From equations (7) we obtain

$$\frac{d\,x}{d\,\phi} = -d\sin\phi - n\,e\sin n\,\phi\,;\quad \frac{d\,y}{d\,\phi} = d\cos\phi + n\,e\cos n\,\phi\,;$$

$$\therefore \left(\frac{d\,x}{d\,\phi}\right)^2 + \left(\frac{d\,y}{d\,\phi}\right)^2 = d^2 + n^2\,e^2 + 2\,n\,d\,e\cos(n-1)\,\phi,$$

$$\frac{d^2\,y}{d\,\varphi^2} = -\,d\sin\phi - n^2\,e\sin n\,\phi,$$

$$\frac{d^2\,x}{d\,\varphi^2} = -\,d\cos\phi - n^2\,e\cos n\,\phi\,;$$

$$\therefore \frac{d^2 y}{d\,\varphi^2}\cdot\frac{d\,x}{d\,\phi} - \frac{d^2\,x}{d\,\varphi^2}\cdot\frac{d\,y}{d\,\varphi} = d^2 + n^3 e^2 + (n^2 + n)\,d\,e\cos(n-1)\,\phi$$

$$\text{wherefore, } \rho = \frac{\{d^2 + n^2 e^2 + 2\,n\,d\,e \cos{(n-1)}\,\phi\}^{\frac{3}{2}}}{d^2 + n^3 e^2 + (n^2 + n)\,d\,e \cos{(n-1)}\phi};$$

$$\text{and if we put} \quad \frac{d^2 + n^2 e^2 + 2\,n\,d\,e \cos{(n-1)}\,\phi}{d^2 + n^3 e^2 + (n^2+n)\,d\,e \cos{(n-1)}\,\phi} = k,$$

we obtain for the equations to the evolute

$$\xi = d \cos\phi + e \cos n\,\phi - k\,(d \cos\phi + n\,e \cos n\,\phi),$$

$$\text{and} \quad \eta = d \sin\phi + e \sin n\,\phi - k\,(d \sin\phi + n\,e \sin n\,\phi);$$

$$\left.\begin{array}{ll} \text{or} & \xi = d\,(1-k) \cos\phi + e\,(1-n\,k) \cos n\,\phi \\ \text{and} & \eta = d\,(1-k) \sin\phi + e\,(1-n\,k) \sin n\,\phi \end{array}\right\}; \quad (24)$$

whence we see that the evolute may be regarded as traced by an epicycle of variable radius $e\,(1-n\,k)$ carried on a deferent also of variable radius $d\,(1-k)$.

It is easily seen (see p. 117, and figs. 63, 64), that

$$k = \frac{\text{C}'\text{B}'}{\text{C}'\text{O}'}\left(\frac{p\,s}{p\,s - \text{NB}'}\right).$$

When $d = n\,e$, so that the involute epicyclic is the epicycloid or the hypocycloid (according as n is positive or negative), k reduces to $\frac{2}{1+n}$, and the equations of the evolute become

$$\left.\begin{array}{l} \xi = \dfrac{n-1}{n+1} \,.\, d \cos\phi - \dfrac{n-1}{n+1}\; e \cos n\,\phi \\ \eta = \dfrac{n-1}{n+1} \,.\, d \sin\phi - \dfrac{n-1}{n+1}\; e \sin n\,\phi \end{array}\right\}; \quad (25)$$

which (we see from 8) are the equations of an epicycloid or hypocycloid (according as n is positive or negative), whose deferential and epicyclic radii (and in fact whose linear proportions) bear to those of the involute the ratio $(n-1)$: $(n+1)$, and whose vertices touch the cusps of the involute epicycloid or hypocycloid. If n is positive the ratio $(n-1)$: $(n+1)$ is the same as $(d-e)$: $(d+e)$, or F : (F + 2 R), as in Section II. If n is negative the ratio $(n-1)$: $(n+1)$ is the same as $(d+e)$: $(d-e)$, or F : (F − 2 R), as in Section II.

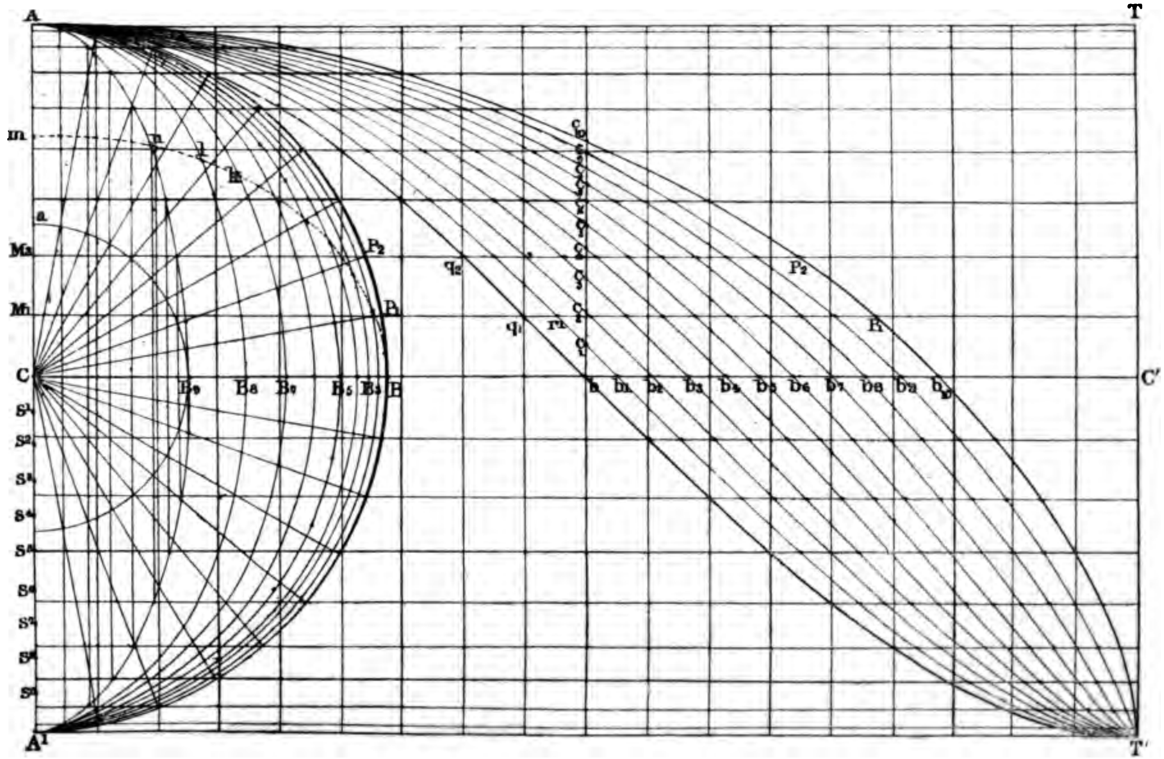


Fig. 144. CONSTRUCTION FOR MEASURING MOTION IN ELLIPTIC ORBITS UNDER GRAVITY.

SECTION VII.

GRAPHICAL USE OF CYCLOIDAL CURVES.

GRAPHICAL USE OF THE CYCLOID AND ITS COMPANION TO DETERMINE THE MOTION OF PLANETS AND COMETS.

[*From the Monthly Notices of the Astronomical Society for April* 1873.]

The student of astronomy often has occasion to determine approximately the motion of bodies, as double stars, comets, meteor systems, and so on,—in orbits of considerable eccentricity. The following graphical method for solving such problems in a simple yet accurate manner is, so far as I know, a new one.* By its means a diagram such as fig. 144, Plate VI., having, once for all, been carefully inked in on good drawing card, the motion of a body in an orbit of any eccentricity can be determined by a pencilled construction of great simplicity, which can be completed (including the construction of the ellipse) in a second or two.

Let APA′, fig. 145, be an elliptical orbit of which ACA′ is the major axis, C the centre, S being the centre of force, so that A is the aphelion, and A′ the perihelion. Let H be

* New as a method of construction, though the principle on which it depends is of course not new. The curve A p T′ (fig. 145), for instance, is an orthogonal projection of a particular prolate cycloid which, as Newton long since showed, if accurately drawn, gives the means of determining the motion in the ellipse APA′. But, as he remarks, this prolate cycloid cannot readily be drawn; whereas the curve A p T′ can be very readily drawn.

half the periodic time, and T the time in which the body moves from A to P.

On AA′ describe the auxiliary semicircle A b A′.

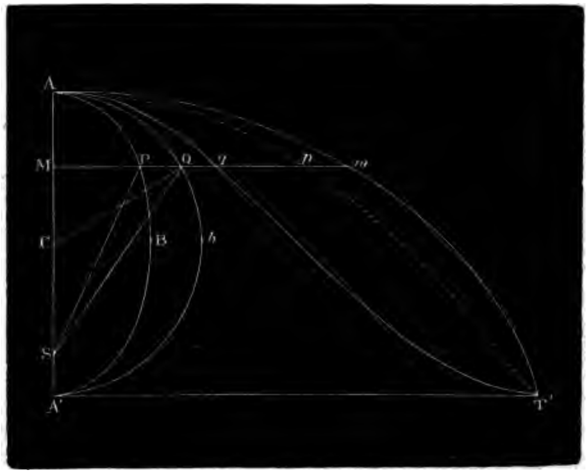
Then

$$\begin{aligned} T : H &:: \text{area ASP} : \text{area ABA}' \\ &:: (ACQ + SCQ) : \text{area A } b \text{ A}' \\ &:: AC \,.\, AQ + CS \,.\, QM : AC \,.\, AQA' \\ &:: AQ + \frac{CS}{AC} \,.\, QM : AQA' \end{aligned}$$

Now if A m T′ be a cycloid having AA′ as its diameter, then

$$\text{Ordinate } M\,m = AQ + QM.$$

Fig. 145.



And if we take M q = AQ, we have q a point on A q T′, the companion to the cycloid. The line $q\,m$ is then equal to Q M; and if we take a point p on m Q such that

$$q\,p = \frac{SC}{AC} \,.\, q\,m = \frac{SC}{AC} \,.\, QM$$

we have

$$M p = AQ + \frac{SC}{AC} \cdot QM \text{ ; and } A'T' = AQA' \text{ ;}$$

wherefore

$$T : H :: M p : A'T'$$

Thus we may represent the time in traversing the arc AP by the ordinate M p to a curve A p T′, obtained by dividing all such lines as $q\,m$ (joining the cycloid and its companion, and parallel to A′T′) so that $q\,p$: $q\,m$ as SC : AC.

Accordingly, if we construct such a diagram as is shown in fig. 144, plate VI., in which AT′ is a semi-cycloidal arc and A b T′ its companion, while intermediate curves are drawn dividing all such lines as b b_{10} into ten or any other convenient number of equal parts, the curves through the successive points b, b_1, b_2, &c., to b_{10}, give us the time-ordinates for bodies moving in ellipses having A and A as apses, and their centres of force respectively at C, S_1, S_2, S_3, . . . S_9, and A′.

In the plate the semi-ellipses corresponding to these positions of the centre of force are drawn in, and it will be manifest that any ellipse intermediate to those shown can be pencilled in at once, with sufficient accuracy. Ellipses within AB_9A' have their focus of force between S_9 and A′, and are exceptionally eccentric.* It is easy to construct such an ellipse, however, in the manner indicated for the semi-ellipse AB_9A'. For the radial lines and the parallels to AT through their extremities are supposed to be inked in; and (taking the case of ellipse AB_9A') we have only to draw the semicircle a B_9 a′, and parallels to AA′ through the points where the radial lines intersect this semicircle, to obtain by

* It is manifest that when the centre of force is at A′ we have the case of a body projected directly from a centre of force, and the time-curve becomes the cycloid A b_{10}T′. Thus the above lines give a geometrical demonstration of the relation established analytically in the paper which follows.

the intersections of these parallels with the parallels to AT a sufficient number of points on the semi-ellipse.

The illustrative diagram has been specially constructed for the use of those who may have occasion to employ the method, and will be found sufficiently accurate for all ordinary purposes. Before proceeding, however, to show how the method is applied in special cases, I shall describe how such a diagram should be constructed :—

First the semicircle ABA′ must be drawn, and the lines AT, A′T′ perp. to AA′. Then CA′ must be divided into ten equal parts (and when the figure is large, a plotting scale for hundredths, &c., should be drawn). Next A′T and AT must be each taken equal to 3·1416 where CA′ is the unit. Join TT′. Now AT and A′T′ represent, as time-ordinates, the half-period of any body moving in an ellipse having AA′ as major axis. Each must now be divided into the same number of equal parts, and it is convenient to have eighteen such parts. (So that in the illustrative case of our Earth, three divisions represent a month.) Next the semicircle ABA′ must be divided into eighteen equal parts. Through the points of division on the semicircle, parallels to AT and A′T′ are to be drawn,* and the points of division along AT and A′T′ are to be joined by parallels to AA′ and TT′. Then the curve A b T′, the 'companion to the cycloid,' runs through the points of intersection of the first parallel to AT and the first to AA′, the second parallel to AT and the second to AA′, the third parallel to these lines, the fourth, and so on. We have now only to take $b\,b_{10}$ equal to CB; $q_1\,p_1$ equal to $M_1\,P_1$; $q_2\,p_2$ equal to $M_2\,P_2$; and so on, to obtain the required points on the cycloid $A\,b_{10}\,T'$; and the equidivision

* Practically it is convenient to draw another semicircle on TT′, divide its circumference into eighteen parts, and join the corresponding points of division on the two semicircles.

of all such lines as $b\,b_{10}$, $q_1\,p_1$, $q_2\,p_2$ (into ten parts in the illustrative diagram) gives us the required points on the intermediate curves.

Next let us take some instances of the application of the diagram.

I. Suppose we wish to divide a semi-ellipse of given eccentricity into any given number of parts traversed in equal times, and let the eccentricity be $\frac{1}{2}$, and 18 the given number of parts * :—

Then S_5 is the centre of force; AB_5A' the semi-ellipse; and $A\,b_5\,T'$ the time-curve. The dots along $A\,b_5T'$ give the intersection of the time-curve with the time-ordinates parallel to AA'; and therefore parallels to AT, though these dots (not drawn in the figure, to avoid confusion) indicate by their intersection with the semi-ellipse AB_5A_1 the points of division required.

II. Suppose we wish to know how far the November meteors travel from perihelion in the course of one quarter of their period, that is, one half the time from perihelion to aphelion :—

The curve AB_9A_1 is almost exactly of the same eccentricity as the orbit of the November meteors. To avoid additional lines and curves, let us take it as exactly right. Then $A\,b_9\,T'$ is the time-curve. For the quarter period from the perihelion (or aphelion), we take of course the middle vertical line, which intersects $A\,b_9T$ in c_9. This point by a coincidence is almost exactly on a parallel to AT, and this parallel meets the semi-ellipse $AB_9\,A'$ in n, the required point on the orbit. In other words, the journey of the November meteors from A to n occupies the same time as their journey from n to A', S_9 being the position of the

* This selection is made solely to avoid the addition of lines and curves not necessary to the completeness of the diagram.

Sun, and the Earth's distance from the Sun approximately equal to $A'S_0$.

III. Suppose we require, in like manner, the quarter-period positions in different orbits, all having AA as major axis, but their centres of force variously placed along CA. We get any number of points, n, l, k, precisely as n was obtained; m, of course, is on the parallel through C_{10}; and we obtain, in fine, the curve m n l k B, which resembles, but is not, an elliptic quadrant.

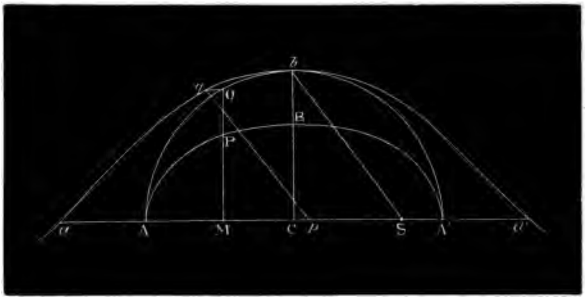
IV. Suppose we require to know in what time the half orbit from aphelion or perihelion is described in orbits of different eccentricity. The required information is manifestly indicated by the intersection of CC′ with the time-curves, in b, b_1, b_2, &c. Thus in the circle, AB is described in the time represented by C b; in the semi-ellipse AB_3 A, AB_3 is described in the time represented by C b_3, and B_3A' in the time represented by b_3C'; and so on for the other semi-ellipses.

V. Suppose we require to determine approximately the 'equation of the centre' for a body when at any given point of its orbit of known eccentricity. Take the case of Mars, whose eccentricity being nearly $\frac{1}{10}$, his path is fairly represented by the ellipse next within ABA′, and his time-curve by A b_1 T′. Then the equation of the centre, when Mars is at his mean distance, is represented by b b_1; when Mars is at P_1 (not on the circle, but on the curve just within), the equation of his centre is represented by q_1 r_1; and so on.

Many other uses and interpretations of the time-curves will suggest themselves readily to those who are likely to use the diagram.

After the above method had been briefly described, Professor Adams, who was in the chair, mentioned a method (devised by himself many years since) by which the same results can be obtained from the 'companion to the cycloid' or 'curve of sines.' Professor Adams's method may be thus exhibited:—Let $a\,b\,a'$ be the y-positive half of one wave of the 'curve of sines,' bC its diameter; AbA′, a semicircle with radius b C. Let ABA′, fig. 146, be a half-ellipse having its focus at S. Then the time in any arc AP of this ellipse may be thus determined. Join b S, produce the ordinate PM to Q on circle ABA′, draw Q q parallel to $a\,a'$, and $q\,p$ parallel to b S; then $a\,p$ represents the time in traversing AP, where $a\,a'$ represents the half period. And *vice versâ*, if we require

FIG. 146.



the position of the moving body after any time from the apse, say aphelion, then take $a\,p$ to represent the time, where $a\,a'$ is the half period, ACA′ the major axis, S the centre of force; join S b, draw $p\,q$ parallel to S b, q Q parallel to AA′, and QP perpendicular to AA′ gives P the point required.

It will be manifest that in principle my method is identical with this, for in my figure the time is represented by M p, where M q (fig. 145) is equal to the arc AQ, and $q\,p$ is equal to QM reduced in the ratio of CS to CA. Now $a\,p$ in fig. 146 is the projection of $a\,q$ and $q\,p$; and the projection of $a\,q$ is equal to the arc AQ (see p. 200), while the projection of $q p$ is equal to QM reduced in the proportion of CS to AC.

Although Professor Adams's construction has the advantage of requiring but a single curve, yet for the particular purpose described my construction is more convenient. We see from the fig. 146 that to give the relation between the times and positions in the case of the ellipse $A p A'$, we require a series of parallels to b C, $a a'$ and b S; and the parallels to b S only serve for this one case. Therefore we could not construct a reference figure for many cases, without having *many series of parallels* and a very confusing result. In my construction we have, instead, *many curves*, but a result which is not confusing because each curve is distinct from the rest.

Graphical Use of the Cycloid to Measure the Motion of Matter projected from the Sun.

[*From the Monthly Notices of the Astronomical Society for December* 1871.]

Whatever opinion we may form as to the way in which the matter of certain solar prominences is propelled from beneath the photosphere, there can be little question that such propulsion really takes place. It seems clear indeed that some prominences, more especially those seen in the Sun's polar and equatorial regions, are formed—or rather make their appearance—in the upper regions of the solar atmosphere, and even assume the appearance of eruption-prominences by an extension *downwards*, somewhat as a waterspout simulates the appearance of an uprushing column of water though really formed by a descending movement. But it is certain that other prominences are really phenomena of eruption.

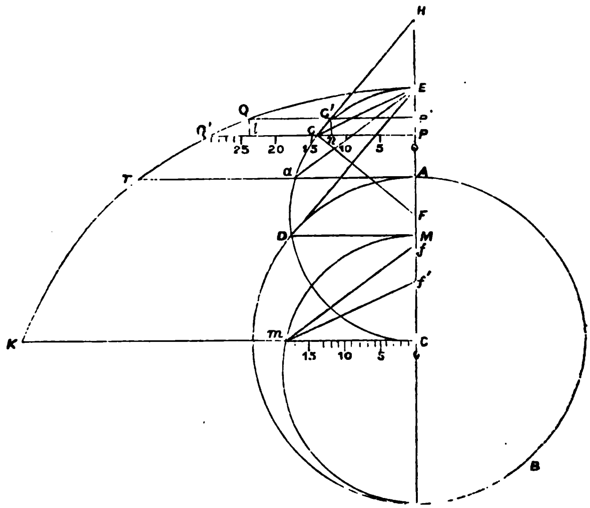
In the case of any matter thus erupted, we shall clearly obtain an inferior limit for the value of the initial velocity of outrush, if we assume that the apparent height reached by the matter is the real limit of its upward motion (that is, that there is no foreshortening), and that the solar atmosphere exercises no appreciable influence in retarding the motion. The latter supposition is, however, wholly untenable under the circumstances, while the former must in nearly all cases be erroneous; and I only make these suppositions in order to simplify the subject, noting that their effect is to reduce the estimated velocity of outrush to its lowest limiting value.

We are to deal then, for the present, with the case of matter flung vertically upwards from the sun's surface and subject only to the influence of solar gravity; I propose to consider the time of flight between certain observed levels, not the mere vertical distance attained by the erupted matter; and (as I wish to deal with cases where a great distance from the sun has been attained) it will be necessary to take into account the different actions of the solar gravity at different distances. Zöllner, in dealing with prominences of moderate height, has regarded the solar gravity as constant; but this is evidently not admissible when we come to deal with matter hurled to a height of 200,000 miles, since at that height solar gravity is reduced to less than one-half the value it has at the surface of the sun.

It is easy to obtain the required formula; and though it is doubtless contained in all treatises on Dynamics, it will be as well to run through the work in this place. In reducing the formula I have noticed a neat geometrical illustration (and a partial proof) which I do not remember to have seen in that form in any book. It not only presents in a striking manner the varying rate at which a body falls towards a centre attracting according to the law of nature, but it supplies a means whereby the time of flight between any given distances may be readily obtained from a simple construction.

Let C, fig. 147, be the centre of a globe ABD, of radius R, and attracting according to the law of nature; let g be the accelerating force of gravity at the surface of the globe. Then the attraction exerted at a unit of distance, if the whole mass of the globe were collected at a point, would be $g\,R^2$.

FIG. 147.



Illustrating the motion of a body descending from rest towards a globe attracting according to the law of nature.

Let a particle falling from rest at E reach the point P i time t; and let AE = H, and CP = x. Then the equation of motion is

$$\frac{d^2 x}{d\,t^2} = -\frac{g\,R^2}{x^2},$$

giving

$$\left(\frac{d\,x}{d\,t}\right)^2 = v^2 = \frac{2\,g\,R^2}{x} + C\,;$$

so that, since the particle starts from rest at a distance (R + H) from C, we have

$$0 = \frac{2\,g\,R^2}{R+H} + C.$$

For convenience write D for (R + H); then we have

$$\left(\frac{d\,x}{d\,t}\right)^2 = v^2 = 2\,g\,R^2\left(\frac{1}{x} - \frac{1}{D}\right)$$

$$= \frac{2\,g\,R^2}{D}\left(\frac{D - x}{x}\right). \qquad (1)$$

Thus

$$R\sqrt{\frac{2\,g}{D}}\cdot\frac{d\,t}{d\,x} = \frac{x}{\sqrt{D\,x - x^2}}.$$

Integrating, we have

$$R\sqrt{\frac{2\,g}{D}}\,.\,t = \sqrt{D\,x - x^2} - \frac{D}{2}\cos^{-2}\left(\frac{D - 2\,x}{D}\right) + C.$$

But when $t = 0$, $x = D$; so that $C = \frac{D\,\pi}{2}$,

hence we have

$$R\sqrt{\frac{2\,g}{D}}\,t = \sqrt{D\,x - x^2} + \frac{D}{2}\cos^{-1}\left(\frac{2\,x - D}{D}\right), \qquad (2)$$

(where D is equal to the radius of the globe added to the height from which the particle is let fall).

Equation (1) gives the velocity acquired in falling (from rest) from a height H to a distance x from the centre, and (2) gives the time of falling to that distance. The geometrical illustration to which I have referred, relates to the deduction of (2) from (1). We see from (1) that at the point P

$$v^2 = \frac{2g R^2}{D}\left(\frac{D-x}{x}\right).$$

Bisect CE in F, and describe the semicircle CDE; then if DE is a tangent to the circle DAB, and if DM is drawn perpendicular to CE,

$$CM = \frac{(CD)^2}{CE} = \frac{R^2}{D};$$

so that

$$v = \sqrt{2g \cdot CM}\sqrt{\frac{PE}{CP}}. \qquad (a)$$

But if close by G, either on the tangent GH or on the arc GE, we take G′ and draw G′P′ perpendicular to CE, and G n perpendicular to GP, we have

$$\frac{GG' + Gn}{PP} = \frac{GF + FP}{GP} = \frac{CP}{\sqrt{CP \cdot PE}}$$

$$= \sqrt{\frac{CP}{PE}}.$$

Hence, from (a),

$$\frac{v}{\sqrt{2g \cdot CM}} = \frac{PP'}{GG + Gn};$$

so that

$$\left\{\begin{matrix}\text{the vel.}\\ \text{at P}\end{matrix}\right\} : \left\{\begin{matrix}\text{velocity acquired in falling through}\\ \text{space CM, under const. accel. force } g\end{matrix}\right\}$$

$$:: \left\{\begin{matrix}\text{elem. space}\\ \text{PP}'\end{matrix}\right\} : \left\{\begin{matrix}\text{sum of elementary}\\ \text{spaces GG and G}n\end{matrix}\right\}.$$

Therefore the falling particle traverses the space PP′ in the same time that a particle travelling with the velocity acquired in falling through space CM under constant accelerating force g, would traverse the space (GG′ + G n). It follows that the time in falling from E to P is the same as would be occupied by a particle in traversing (arc EG + GP) with the velocity acquired in falling through the space CM under a constant accelerating force g. In other words,

$$t = \frac{\text{PG} + \text{arc GE}}{\sqrt{2g \cdot \text{CM}}};$$

or

$$\text{R}\sqrt{\frac{2g}{\text{D}}} \cdot t = \sqrt{\text{PE} \cdot \text{PC}} + \text{CF arc GE}$$

$$= \sqrt{(\text{D} - x)x} + \frac{\text{D}}{2}\cos^{-1}\left(\frac{2x - \text{D}}{\text{D}}\right),$$

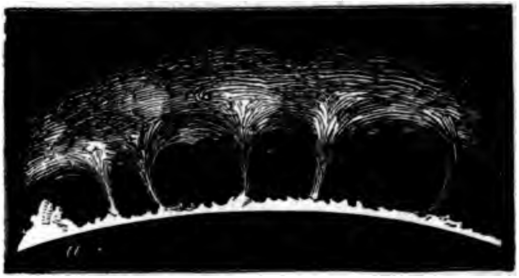
as before.

The relation here considered affords a very convenient construction for determining the time of descent in any given case. For, if PG be produced to Q so that GQ = arc GE, Q lies on a semi-cycloid KQC, having CE as diameter; and the relative time of flight from E to any point in AE is at once indicated by drawing through the point an ordinate parallel to CK. The actual time of flight in any given case can also be readily indicated. For let T be the time in which LC would be described with the velocity acquired in falling through a distance equal to LC under accelerating force g, and on LM describe the semicircle L m M; then clearly C m ($= \sqrt{\text{CL} \cdot \text{CM}}$) will be the space described in time T with the velocity acquired in falling through the space CM under accelerating force g; and we have only to divide C m into parts corresponding to the known time-interval T, and to measure off distances equal to these parts on PQ to find the time of traversing PQ with this uniform velocity, *i.e.*, the time in which the particle falls from E to P. The division in the figure illustrates such measurements in the case of the sun, the value of T being taken as 18⅔ minutes.

Moreover it is not necessary to construct a cycloid for each case. One carefully constructed cycloid will serve for all cases, the radius CA being made the geometrical variable.

As an instance of this method of construction, I will take Professor Young's remarkable observation of a solar outburst, premising that I only give the construction as an illustration, and that a proper calculation follows.

FIG. 148.



12h 55m.

FIG. 149.

1h.5m.

On September 7, 1871, Professor Young saw wisps of hydrogen carried in ten minutes from a height of 100,000 miles to a height exceeding 200,000 miles from the sun's surface. A full account of his observations is given in the second and

FIG. 150.

1h.40m

third editions of my treatise on the sun. Figs. 148, 149, 150, and 151, with the times noted, indicate the progress of the changes. I assumed in what follows that there was no fore-

FIG. 151.

1h.55m.

shortening. The height, 100,000 miles (upper part of cloud in fig. 148), was determined by estimation; but the ultimate height reached by the hydrogen wisps (that is, the elevation at which they vanished as by a gradual dissolution) results from the mean of three carefully executed and closely accordant measures. This mean was 7 49″, corresponding to a height of 210,000 miles (highest filaments in fig. 149). We may safely take 100,000 miles as the vertical range actually traversed, and 200,000 miles as the extreme limit attained. We need not inquire whether the hydrogen wisps were themselves projected from the photosphere,—most probably they were not,—but if not, yet beyond question there was propelled from the sun some matter which by its own motion caused the hydrogen to traverse the above-mentioned range in the time named, or caused the hydrogen already at those heights to glow with intense lustre. We shall be underrating the velocity of expulsion, in regarding this matter as something solid propelled through a non-resisting medium, and attaining an extreme range of 200,000 miles. What follows will show whether this supposition is admissible.

Now g for the sun, with a mile as the unit of length and a second for the unit of time, is 0·169, and R for the sun is 425,000. Thus the velocity acquired in traversing R under uniform force g,

$$= \sqrt{2g \cdot R}$$

$$= \sqrt{338 \times 425}$$

$$= 379, \text{ very nearly.}$$

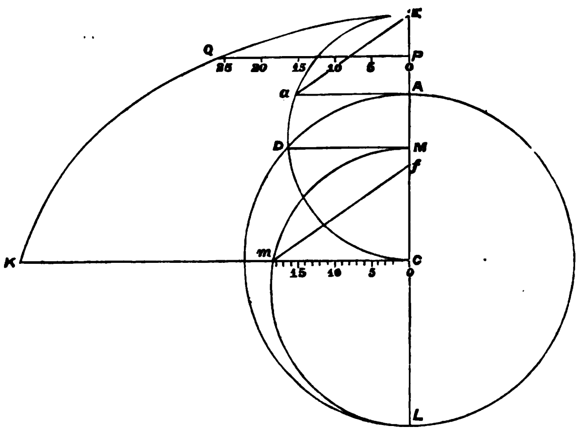
(This is also the velocity acquired under the sun's actual attraction by a body moving from an infinite distance to the sun's surface.)

And a distance 425,000 would be traversed with this velocity in $18^{m}\ 40^{s}$ (= T).

Let KQE, fig. 152, be our semi-cycloid (available for many successive constructions if these be only pencilled), and CDE half the generating circle.

Then the following is our construction:—Divide EC into $6\frac{1}{4}$ equal portions, and let EP, PA be two of these parts, so that EA represents 200,000 miles and CA 425,000 miles (the sun's radius). Describe the semicircle ADL about the centre C and draw DM perpendicular to EC; describe the half circle M *m* L. Then *m* C represents T where the ordinate PQ represents the time of falling from E to P.

Fig. 152.



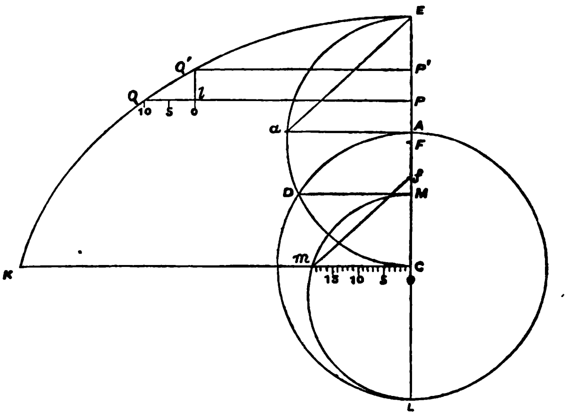
Illustrating the construction for determining time of descent of a particle from rest towards a globe attracting according to the law of nature.

$T = 18^h\ 50^m$, and PQ (carefully measured) is found to correspond to about twenty-six minutes.

Thus a body propelled upwards from A to E would traverse the distance PE in twenty-six minutes. But the hydrogen wisps watched by Professor Young traversed the distance represented by PE in ten minutes. Hence either E was not the true limit of their upward motion, or they were retarded by the resistance of the solar atmosphere. Of course if their actual flight was to any extent foreshortened, we should only the more obviously be forced to adopt one or other of these conclusions.

But now let us suppose that the former is the correct solution; and let us inquire what change in the estimated limit of the uprush will give ten minutes as the time of moving (without resistance) from a height of 100,000 to a height of 200,000 miles. Here we shall find the advantage

FIG. 153.



Illustrating the construction for determining time of descent between given levels when a body descends from rest at a given height towards a globe attracting according to the law of nature.

of the constructive method; for to test the matter by calculation would be a long process, whereas each construction can be completed in a few minutes.

Let us try 375,000 miles as the vertical range. This gives CE = 800,000 miles, and our construction assumes the appearance shown in fig. 153. We have AC=425,000 miles;

AP=PP′=100,000 miles; and Q l or (PQ−P′Q′) to represent the time of flight from P to P′.

The semicircles ADL, M m L, give us m C to represent T or $18^{\text{m}}\ 50^{\text{s}}$; and QL carefully measured is found to correspond to rather less than ten minutes. It is, however, near enough for our purpose.

It appears, then, that if we set aside the probability, or rather the certainty, that the sun's atmosphere exerts a retarding influence, we must infer that the matter projected from the sun reached a height of 375,000 miles, or thereabouts. This implies an initial velocity of about 265 miles per second.*

But it will be well to make an exact calculation,—not that any very great nicety of calculation is really required, but in order to illustrate the method to be employed in such cases, as well as to confirm the accuracy of the above constructions.

In equation (2) put $\sqrt{2\,g\,\text{R}} = 379$; R = 425,000; D = 625,000; and $x = 525{,}000$; values corresponding to Professor Young's observations. It thus becomes—

$$\sqrt{\frac{425}{625}}\,(379)\,t = \sqrt{(100{,}000)(525{,}000)} + 312{,}500\cos^{-1}\left(\frac{1050-625}{625}\right),$$

* The value is of course deduced directly from (1), p. 219; but it is worthy of notice that it can be deduced at once from fig. 153, by drawing A a parallel to KC, and $m\,f$ parallel to a E; then C f represents the required velocity, CL representing 379 miles per second. A similar construction will give the velocity at P, P′, &c. Applied to fig. 147, it gives C f to represent the velocity at A, C f' to represent the velocity at P; $m\,f$ and $m\,f'$ being parallel to a E and GE respectively. Applied to the case dealt with in fig. 152, we get C f t represent the velocity at A, where E is the limit of flight: C f i found to be rather more than $\frac{5}{8}$ of CL; so that the velocity at A is rather more than 210 miles per second.

or

$$379\sqrt{17}\,.\,t = 250{,}000\sqrt{21} + 1{,}562{,}500\cos^{-1}\left(\frac{17}{25}\right),$$

$$1562{\cdot}7\,t = 1{,}145{,}100 + 1{,}285{,}800 = 2{,}430{,}900,$$

$$t = 1{,}556^{s} = 25^{m}\,56^{s}.$$

This then is the time which would have been occupied in the flight of matter from a height of 100,000 to a height of 200,000 miles, if the latter height had been the limit of vertical propulsion in a non-resisting medium.

In order to deduce the time of flight t between the same levels, for the case where the total vertical range is 375,000 miles, we have, putting t_1 for the time of *fall* to 200,000 miles above the sun's surface, and t_2 for the time of fall to 100,000 miles, the equation,

$$\sqrt{\frac{425}{800}}(379)\,t_1 = \sqrt{(175{,}000)(625{,}000)} + (400{,}000)\cos^{-1}\left(\frac{125-80}{80}\right),$$

$$\sqrt{\frac{425}{800}}(379)\,t_2 = \sqrt{(275{,}000)(525{,}000)} + (400{,}000)\cos^{-1}\left(\frac{105-80}{80}\right),$$

giving (since $t_2 - t_1 = t'$)

$$\sqrt{\frac{425}{800}}(379)\,t' = 25{,}000\left\{\sqrt{11\times 21} - \sqrt{7\times 25}\right\} + 400{,}000\left\{\cos^{-1}\left(\frac{5}{16}\right) - \cos^{-1}\left(\frac{9}{16}\right)\right\}$$

$$276{\cdot}25\,t' = 49{,}250 + 111{,}816 = 161{,}066,$$

$$t' = 583^{s} = 9^{m}\,43^{s}.$$

This is very near to Professor Young's ten minutes. I had found that an extreme height of 400,000 miles gave $9^{m}\,24^{s}$ for the time of flight between vertical altitudes 100,000 miles and 200,000 miles. It will be found that a height of 360,000 miles gives $9^{m}\ 58^{s}$, which is sufficiently near to Professor Young's time.

Now to attain a height of 360,000 miles a projectile from the sun's surface must have an initial velocity

$$= \sqrt{2gR} \,.\, \sqrt{\frac{360,000}{785,000}} = 379 \sqrt{\frac{72}{157}}$$

$$= 257 \text{ miles per second.}$$

The eruptive velocity, then, at the sun's surface, cannot possibly have been less than this. When we consider, however, that the observed uprushing matter was vaporous, and not very greatly compressed (for otherwise the spectrum of the hydrogen would have been continuous and the spectroscope would have given no indications of the phenomenon), we cannot but believe that the resisting action of the solar atmosphere must have enormously reduced the velocity of uprush before a height of 100,000 miles was attained, as well as during the observed motion to the height of 200,000 miles. It would be safer indeed to assume that the initial velocity was a considerable multiple of the above-mentioned velocity, than only in excess of it in some moderate proportion. Those who are acquainted with the action of our own atmosphere on the flight of cannon-balls (whereby the range becomes a mere fraction of that due to the velocity of propulsion), will be ready to admit that hydrogen rushing through 100,000 miles even of a rare atmosphere, with a velocity so great as to leave a residue sufficient to carry the hydrogen 100,000 miles in the next ten minutes, must have been propelled from the sun's surface with a velocity many times exceeding 257 miles per second, the result calculated for an unresisted projectile. Nor need we wonder that the spectroscope supplies no evidence of such velocities, since if motions so rapid exist, others of all degrees of rapidity down to such comparatively moderate velocities as twenty or thirty miles per second also exist, and the spectral lines of the hydrogen so moving would be too greatly widened to be discerned.

Now the point to be specially noticed is, that supposing matter more condensed than the upflung hydrogen to be propelled from the sun during these eruptions, such matter would retain a much larger proportion of the velocity originally imparted. Setting the velocity of outrush, in the case we have been considering, at only twice the amount deduced on the hypothesis of no resistance (and it is incredible that the proportion can be so small), we have a velocity of projection of more than 500 miles per second; and if the more condensed erupted matter retained but that portion of its velocity corresponding to three-fourths of this initial velocity (which may fairly be admitted when we are supposing the hydrogen to retain the portion corresponding to so much as half of the initial velocity), then such more condensed erupted matter would pass away from the sun's rule never to return.

The question may suggest itself, however, whether the eruption witnessed by Professor Young might not have been a wholly exceptional phenomenon, and so the inference respecting the possible extrusion of matter from the sun's globe be admissible only as relating to occasions few and far between. On this point I would remark, in the first place, that an eruption very much less noteworthy would fairly authorise the inference that matter had been ejected from the sun. I can scarcely conceive that the eruptions witnessed quite frequently by Respighi, Secchi, and Young —such eruptions as suffice to carry hydrogen 80,000 or 100,000 miles from the sun's surface—can be accounted for without admitting a velocity of outrush exceeding considerably the 379 miles per second necessary for the actual *rejection* of matter from the sun. But apart from this it should be remembered that we only see those prominences which happen to lie round the rim of the sun's visible disk, and that thus many mighty eruptions must escape our notice even though we could keep a continual watch upon the whole circle of the sierra and prominences (which unfortunately is very far from being the case).

It is worthy of notice that the great outrush witnessed by Professor Young was not accompanied by any marked signs of magnetic disturbance. Five hours later, however, a magnetic storm began suddenly, which lasted for more than a day; and on the evening of September 7, there was a display of aurora borealis. Whether the occurrence of these signs of magnetic disturbance was associated with the appearance (on the visible half of the sun) of the great spot which was approaching or crossing the eastern limb at the time of Young's observation, cannot at present be determined.

I would remark, however, that so far as is yet known the disturbance of terrestrial magnetism by solar influences would appear to depend on the condition of the photosphere, and therefore to be only associated with the occurrence of great eruptions in so far as these affect the condition of the photosphere. In this case an eruption occurring close by the limb could not be expected to exercise any great influence on the earth's magnetism; and if the scene of the eruption were beyond the limb, however slightly, we could not expect any magnetic disturbance at all, though the observed phenomena of eruption might be extremely magnificent.

In this connection I venture to quote from a letter addressed to me by Sir J. Herschel in March 1871 (a few weeks only before his lamented decease). The letter bears throughout on the subject of this paper, and therefore I quote more than relates to the association between terrestrial magnetism and disturbances of the solar photosphere.

After referring to Mr. Brothers' photograph of the corona (remarking that 'the corona is certainly *extra*-atmospheric and *ultra*-lunar'), Sir John Herschel proceeds thus :—

'I can very well conceive great outbursts of vaporous matter from below the photosphere, and can admit at least the possibility of such vapour being tossed up to very great heights; but I am hardly yet exalted to such a point as to conceive a positive ejection of erupted particles with a velocity of two or three hundred miles per second. But now the great question of all arises : what *is* the *photosphere?* what are those intensely radiant *things*—scales, flakes, or whatever else they be—which really *do* give out all (or at least $\frac{99}{100}$ths of) the total light and heat of the sun? and if the prominences, &c., be eruptive, why does not the eruptive force scatter upwards and outwards this luminous matter? . . . Through the kindness of the Kew observers I have had heliographs of the two great outbursting spots which I think I mentioned to you as having been non-existent on the 9th, and in full development on the 10th, both [being] large and conspicuous, and including an area of disturbance at least 2′ (54,000 miles) across. They were both nearly absorbed, or in rapid process of absorption, on the 11th. In my own mind I had set it down as pretty certain that the outbreak must have taken place *very* suddenly at somewhere about the intervening midnight. Well, now! The magnet's declination curves at Kew have been sent me, and, lo! while they had been going on as smoothly as possible on the 6th, 7th, 8th, and 9th, and up to 11½ P.M. on the latter day (9th), suddenly a great downward jerk in the curve, forming a gap as far as 3½ A.M. on the 10th. Then comparative tranquillity till 11 A.M., and then (corresponding to the re-absorption of the spots) a furious and convulsive state of disturbance extending over the 11th and the greater part of the 12th. I wonder whether anything was shot out of those holes on that occasion! and, if so, what is going on in the inside of the sun?'

EXAMPLES.

All the examples which have no name appended to them are original, except four or five familiar ones (as 125, 126, &c.), the authors of which are not known.

1. A chord of a cycloid parallel to the base is equal in length to the perimeter of the uppermost of the two segments into which the chord divides the generating circle.

2. A′PB′ is the generating circle through P on the cycloidal arc APD; A′B′ diametral; and equal arcs P q and P q' are taken on A′PB′. Show that straight lines drawn from q and q', parallel to the base, to meet APD, are equal.

3. AQB is a semicircle on diameter AB; and from Q, QL is drawn perp. to AB, and produced to P, so that QP = arc AQ. Show that the locus of P is a cycloid having a cusp at A, and AB as secondary axis.

4. If B′P (fig. 4, p. 8), the normal at P, be produced to meet AA′ produced, in G, then $PB' . PG = A'P^2$.

5. If the tangent A′P (fig. 4, p. 8), produced, meet the tangent at D in T, show that A′T : A B :: arc PB′ : PB.

6. Show that the rectangle under PG (fig. 4, p. 8) and the diameter of curvature at P = $(\text{arc AP})^2$.

7. Show that the chord in which the tangent at P (fig. 4, p. 8) intersects the circle on B′G as diameter, is equal to the arc AP.

8. PC′p is the tracing diameter of P on the cycloidal arc

D′APD. If p P′, parallel to the base, meet the arc D′A in P, show that the tangents and normals at P and P′ form a rectangle.

9. An equilateral triangle AQC is described on AC (fig. 4, p. 8) as side; show that QP, parallel to the base of the cycloid, bisects the arc APD in P.

10. If through C, CP be drawn parallel to the base, to meet the cycloid in P, show that (arc AD)$^2 = 2$ (arc AP)^2.

11. If there are two cycloids APD and AP″D placed as in fig. 3, p. 6, and the straight line drawn from any point P in one to a point Q in the other, P and Q lying on different sides of C c, is equal to the diameter of the generating circle, show that the circle on PQ as diameter touches BD and AE.

12. When the angle BAQ (fig. 4, p. 8) is equal to two thirds of a right angle, then in the limit when P′ moves up to P,

$$PP' = 2\,MN, \text{ and } q\,P' = 2\,n\,q = 2\,l\,n.$$

13. When the angle BAQ = one-third of a right angle, then in the limit

$$PP' = q\,P' = 2\,n\,q = \tfrac{2}{3}\,l\,n.$$

14. In fig. 3, p. 6, if arc AQB intersect arc AP′ D in R, show that

$$\text{area AQRP}'' = \text{area BRD}.$$

15. APD, AP′D are two equal semi-cycloids placed as in fig. 8, p. 17; show that every generating circle A′PB′ divides the area APDP′ into three parts, which are equal each to each to the three parts into which the area of the circle A′PB′ is divided by the arcs APD, AP′D.

16. In the same case, if two generating circles P′RA′PB′ and $p'r\,a\,p\,b$ cut APD in R, P and r, p, respectively, and AP′D in P′, p', show that

$$\text{area P}'\text{R}\,r\,p' = \text{difference of areas RA}'\text{P},\ r\,a\,p.$$

17. In fig. 5, p. 10, Area RD c = area AQC′T.

18. In fig. 11, p. 22,

Area AQB p'' — area E p'' D = $\frac{1}{2}$ generating circle.

19. If in fig. 5, p. 10, RJ is drawn perp. to BD, and a quadrant AIC about T as centre, show that

$$\text{area RJD} = \text{area AQC'I}.$$

20. If CQP parallel to base BD cut the central generating circle in Q and meet the cycloid in P, show that the area AQP is equal to the triangle ABQ.

21. A semi-cycloid having BA as axis, B as vertex, cuts the semi-cycloid APD (A vertex, AB axis, and D cusp) in P, and AQB is the central generating circle, Q lying on the same side of AB as P; show that the area AQBP is equal to the square inscribed in the circle AQB.

22. The normal at any point of a cycloidal arc divides the area of a generating circle through the point, and the area of the cycloid, in the same ratio.

23. In Example 20, show that

$$(\text{arc AP})^2 = \tfrac{1}{2}(\text{arc APD})^2.$$

24. If a cycloidal arc DAD′ is divided into any two parts in P, and PB′ is the normal at P (B′ on the base), show that

$$\text{arc DP . arc PD'} = 4\,(\text{PB'})^2.$$

25. D is the cusp of a cycloid APD, C′ the centre of the tracing circle PKB′ through P. If DC′ cut the tracing circle PKB′ in K, and DP = 2 arc PK, show that DP touches the tracing circle at P.

26. If APD is a semi-cycloid, having axis AB and base BD; AP′D the quadrant of an ellipse having semi-axes AB, BD; and AP″D the arc of a parabola, having AB as axis, show that

$$\text{area APDP} : \text{area AP'DB} : \text{area AP''DB} :: 9 : 3\pi : 8.$$

27. With the same construction, the radii of curvature of the three curves at A are in the ratio $16 : 2\pi^2 : \pi^2$.

28. On the generating circle AQB the arc AQ $= \frac{1}{3}$ circumference is taken, and through Q a straight line parallel to the base is drawn, cutting the cycloid in the point P; show that the radius of curvature at P is equal to the axis AB.

29. The axis AB of a cycloid APD is divided into four equal parts in the points D, C, and E, through which straight lines are drawn parallel to the base, meeting the cycloid in the points P_1, P_2, and P_3; if the radii of curvature at A, P_1, P_2, and P_3, are respectively equal to ρ_1, ρ_2, ρ_3, and ρ_4, show that

$$\rho_1^2 : \rho_2^2 : \rho_3^2 : \rho_4^2 :: 4 : 3 : 2 : 1.$$

30. OI (fig. 14, p. 27) is produced to a point J, such that I J = 2 OK, and on OJ as base a cycloid is described; show that radius of curvature at vertex of this cycloid = LG′.

31. If a cycloid roll on the tangent at the vertex, the locus of the centre of curvature at the point of contact is a semicircle of radius 4 R.

32. If a cycloidal arc be regarded as made up of a great number of very small straight rods jointed at their extremities, and each such rod has its normal (terminated on the base of the cycloid) rigidly attached to it, show that if the arc be drawn into a straight line, the extremities of the normals will lie in a semi-ellipse, whose major axis = 8 R, and minor axis = 4 R.

33. PB′ and P′B″ are the normals at two points P, P′, close together on a cycloidal arc, and PQ parallel to the base BD′ meets the central generating circle in Q; show that if PP′ is of given length, B′B″ varies inversely as the chord BQ.

34. From different points of a cycloidal arc, whose axis is

vertical, particles are let fall down the normals through those points; show that they will reach the base simultaneously in time $2\sqrt{\frac{R}{g}}$.

If they still continue to fall along the normals produced, they will reach the evolute simultaneously in time $2\sqrt{\frac{2R}{g}}$.

35. If the distance of P on semi-cycloidal arc APD (fig. 10, p. 21) from base BD $= \frac{3}{4}$ AB, show that

3 moment of PD about AE = 14 moment of AC about AE.

36. In same case, if PM parallel to BD meet AB in M, show that

$$\text{moment of PD about AE} = \tfrac{2}{3}(AB)^{\frac{1}{2}}\left[(AB)^{\frac{3}{2}} - (AM)^{\frac{3}{2}}\right].$$

37. Show that the moment of arc AP (fig. 10, p. 21) about AB

$$= 2\,(NQ + \text{arc } AQ)\,AQ - \tfrac{4}{3}\,AB^{\frac{1}{2}}\left(AB^{\frac{3}{2}} - BM^{\frac{3}{2}}\right).$$

38. If equal rolling circles on the same fixed circle trace out an epicycloid and hypocycloid having coincident cusps, the points of contact of the rolling circles with the fixed circles coinciding throughout the motion, show that the tangents through the simultaneous positions of the tracing point intersect on the simultaneous common tangent to the three circles.

39. A tangent at a point P on an epicycloidal arc APD is parallel to AB the axis, and a circular arc PQ about O as centre intersects the central generating circle in Q; show that

$$\text{Arc } AQ : \text{arc } BQ :: F : 2R.$$

40. Two tangents P′T, PT to the same epicycloidal arc D′P′APD intersect in T at right angles, and through P′ and

P circular arcs P′Q′ and PQ are drawn around Q as centre to meet the central generating circle in Q′ and Q, neither arc cutting this circle; show that

arc Q′AQ : a semicircle :: F : F + 2 R.

41. If the rolling circle by which an epicycloid is traced out travel uniformly round the fixed circle, the angular velocity of the point of contact about centre of fixed circle being ω, show that the directions of the normal of the tangent also change uniformly with angular velocity $\frac{F + 2R}{2R}\omega$.

42. On the same assumption, the direction of the tracing radius changes uniformly with angular velocity $\frac{F + R}{R}\omega$.

43. If the rolling circle by which a hypocycloid is traced out travel uniformly round the fixed circle, the angular velocity of the point of contact about centre of fixed circle being ω, show that the direction of the normal and of the tangent also change uniformly with angular velocity $\frac{F - 2R}{2R}\omega$.

44. On the same assumption the direction of the tracing radius changes uniformly with angular velocity $\frac{F - R}{R}\omega$.

45. A is the vertex of a hypocycloidal arc APDP′, D the cusp, P′ a point on the next arc; and the tangent at P′ is parallel to the axis AB. If a circular arc P′Q around O as centre intersect the remoter half of the central generating circle in Q, show that

Arc ABQ : arc BQ :: F : 2 R.

46. Two tangents P′T, PT to the same hypocycloidal arc D′P′APD, the base D′D less than a quadrant, intersect in T at right angles; and through P′ and P circular arcs P′Q′ and

PQ are drawn around O as centre to meet (without cutting) the central generating circle in Q′ and Q; show that

$$\text{Arc } Q'AQ : \text{a semicircle} :: F : F - 2R.$$

47. AQ, QB are quadrants of the central generating circle of an epicycloid or a hypocycloid, and the circular arc QB about O as centre meets APD in P; show that

$$\text{Area } APQ : \text{triangle} ABQ :: CO : BO.$$

48. In last example, show that $(\text{arc } AP)^2 = \frac{1}{2}(\text{arc } APD)^2$.

49. At any point B′ in the base of an epicycloid DAD′ a tangent PB′P′ is drawn to the fixed circle, meeting the epicycloid in P and P; show that

$$PB' < \text{arc } DB', \text{ and } P'B' < \text{arc } D'B'.$$

50. With the same construction, show that PB′P′ has its greatest value when B′ is at B, the foot of the axis AB.

51. At P, a point on the epicycloid DAD′, a tangent PKD′ is drawn cutting the fixed circle in K and K′, and the normal PB′b' cutting the fixed circle in B′ and b' (B′ on the base DBD′); show that

$$PK \cdot PK' : (PB')^2 :: F + R : R :: (Pb')^2 : PK \cdot PK'.$$

52. With the same construction if OM be drawn perp. to PKP′, show that

$$OM : PB' : Pb' :: F + 2R : 2R : 2(F + R).$$

53. If tangent at P to epicycloid DAD′ touches the fixed circle, and PB′b' the normal at P meets the fixed circle in B′ and b' (B′ on the base DBD′), show that

$$PB'(F + 2R) = 2R^2; \text{ and } Pb'(F + 2R) = 2R(F + R).$$

54. If tangent at P to epicycloid DAD′ touches the fixed circle and cuts the rolling circle in A′, then

$$(A'P)^2 : (2R)^2 :: (F + R)(F + 3R) : (F + 2R)^2$$

55. In figs. 21 and 22 (pp. 44, 45) the points P, B′, b, lie in a straight line.

56. In figs. 21 and 22, the tangent to DP at P cuts b O c' produced in a point a such that $b\,a = 2\,b\,c'$.

57. At D the cusp of an epicycloid D′AD (fig. 19, frontispiece) a tangent D t to the fixed circle DBD′ meets D′AD in t, and from t another tangent t K is drawn meeting the fixed circle in K; show that D t is always less than the arc DBK if the radius of the rolling circle is finite.

58. ACB is the axis of an epicycloid DAD′; D, D′ its cusps; CQ, O q radii of central generating circle and fixed circle respectively, perp. to ABO and on same side of it. If C q cut Q q parallel to CO in K, and a straight line d K d' through K parallel to O q is the generating base of a prolate cycloid having AQB as central generating circle, show that the area between the epicycloid DAD′ and its base DD′ is equal to the area between the prolate cycloid d A d' and its base $d\,d'$.

59. ACB is the axis of a hypocycloid DAD′; D, D′ its cusps; CQ, O q radii of central generating circle and fixed circle perp. to BAO and on the same side of it. If C q cut Q q parallel to CO in K, and a straight line d K d' through K parallel to O q is the generating basis of a curtate cycloid having AQB as central generating circle, show that the area between the hypocycloid DAD′ and its base DD′ is equal to the area between the curtate cycloid d A d' and its base $d\,d'$.

60. The area between the cardioid and its base is equal to five times the area of the fixed circle.

61. The area between the cardioid and a circle concentric with the fixed circle, touching the cardioid at the vertex, is equal to three times the area of the fixed circle.

62. The area of a circle touching the cardioid at the vertex and concentric with the base, is divided into three equal parts by the arc of the cardioid and the axis produced to meet the circle.

63. Area A o P (fig. 39, p. 74) $= 3\,R\,(b\,k + \text{arc } B\,b)$.

64. If $\theta = \angle$ BO b (fig. 39, p. 74)

$$\text{Area PSA} = R^2\,(3\,\theta + 4 \sin\theta + \tfrac{1}{2}\sin 2\,\theta).$$

65. The area between one arc of the tricuspid epicycloid and the base is equal to $3\frac{5}{8}$ times the area of the generating circle.

66. A complete focal chord is drawn to a cardioid. Show that the lesser of the two segments into which the focus divides the chord, is equal to the portion intercepted between the fixed circle and the tracing circle through the extremity of the longer segment.

67. A circle is described on the axial focal chord as diameter, show that the segments of a complete focal chord intercepted between the curve and this circle are equal. (*Purkiss.*)

68. Lines perp. to focal radii vectores through their extremities have a circle for envelope. (*Purkiss.*)

69. From S, the focus of cardioid, a perp. SQ to a complete focal chord PSP′, is drawn, meeting the fixed circle in Q; show that SQ is a mean proportional between SP and SP′.

70. If SP be any focal radius vector of a cardioid whose vertex is A, and the bisector of the angle PSA meet the circle on SA in Q, SQ will be a mean proportional between SP and SA. (*Purkiss.*)

71. PSP′ is a complete focal chord of a cardioid; SQAQ′ a circle on SA as diameter; SQ, SQ′ bisectors of the angles

PSA, P′SA respectively; and S q perp. to PSP′ meets circle SQA in q; show that

$$SQ : Sq :: SB : SQ'.$$

72. The pedal of a cardioid with respect to the focus is also the locus of the vertex of a parabola which is confocal with the cardioid and touches the circle on SA as diameter. (*Purkiss.*)

The demonstration of this will be more easily effected by taking for the cardioid the locus of n, fig. 39 (*see* p. 75). From n draw $n\,y$ a parallel to $b\,f$, then S y, perp. to $n\,y$, gives y a point on the pedal of this cardioid with respect to S. It can readily be shown that a parabola having S as focus and y as vertex touches the circle B b S in b.

73. From a fixed point A any arc AQ is taken and bisected in Q′. If P is a point on the chord QQ′ such that QP = 2 Q′P, show that the locus of P is a cardioid.

74. If rays diverge from a point on the circumference of a circle and be reflected at the circumference, the caustic will be a cardioid. (Coddington's 'Optics,' or Parkinson's 'Optics,' Art. 72, which see.)

If S b, fig. 39, p. 74, represent path of a ray, to circle B b S, reflected ray $b\,g$ is in the line P $b\,g$, normal to the caustic APS, and therefore the envelope of the reflected rays is the evolute of the cardioid APS, or is a cardioid having its vertex at S, SO diametral and linear dimensions one third those of APS. This, however, is not a direct proof. The preceding proposition will be found to supply a direct proof. For if from A two rays proceed to neighbouring points Q, q, and thence respectively after reflection to neighbouring points Q′ and q', arc Q′ q' = 2 arc Q q; and the point of intersection of QQ′ and $q\,q'$ therefore lies on QQ′ (equal to AQ), at a point ultimately equal to one-third of the distance QQ′ from Q.

75. A series of parallel rays are incident on a reflecting semicircular mirror and in the plane of the semicircle; show that the caustic curve is one half (from vertex to vertex) of

a bicuspid epicycloid or *nephroid*. (Coddington's 'Optics,' or Parkinson's 'Optics,' Art. 71, which see.)

76. A series of rays are incident on the concave side of a reflecting cycloidal mirror to whose axis they are parallel and in whose plane they lie; show that the caustic curve consists of two equal cycloids each having one half of the base of the cycloidal mirror for base, and the axis of this larger cycloid as the tangent at their cusp of contact.

77. The linear dimensions of the evolute of the bicuspid epicycloid (or *nephroid*) are $\frac{1}{2}$ those of the curve itself.

78. The area between one arc of the nephroid and the base is equal to four times the generating circle.

79. The evolute of a nephroid is drawn, the evolute of this evolute, the evolute of this second evolute, and so on continually: show that the sum of all the areas between all the evolute nephroids, and their respective base-circles, are together equal to one-third of the area between the original nephroid and its base-circle.

80. If in the epicycloid $m\,\mathrm{R} = n\,\mathrm{F}$, show that the linear dimensions of the evolute are to those of the epicycloid as $m : m + 2n$.

81. If $m\,\mathrm{R} = n\,\mathrm{F}$, area between an arc of epicycloid and its base $= \dfrac{3m + 2n}{m}$. gen. $\odot = \dfrac{(3m + 2n)\,n^2}{m^3}$. fixed $\odot$.

82. If $\mathrm{PB}'o\,\mathrm{Q}$ is the diameter of curvature at the point P of an epicycloid, o the centre of curvature, B′ a point of the base, then

Area of epicycloid : area of gen. $\odot$ :: QB′ : B′o.

83. If the arc of an epicycloid, from cusp to cusp $= a$, and $m\,\mathrm{R} = n\,\mathrm{F}$, show that a + arc of evolute from cusp to cusp + arc of evolute's evolute from cusp to cusp, and so on *ad infinitum*,

$$= \frac{(m + 2n)\,a}{2n}.$$

84. If the area between an epicycloid and its base $= A$, and $m R = n F$, show that $A +$ area between an arc of the evolute and its base $+$ area between an arc of the evolute's evolute and its base, and so on *ad infinitum*,

$$= \frac{(m + 2n)^2 A^2}{4n(m+n)}.$$

85. If in the hypocyloid $m R = n F$, show that the linear dimensions of the evolute are to those of the hypocycloid as $m : m - 2n$.

Interpret this result when $R = \frac{F}{2}$.

86. If $m R = n F$, area between an arc of hypocycloid and its base $= \frac{3m - 2n}{m}$ gen. $\odot = \frac{(3m - 2n)n^2}{m^3}$. fixed $\odot$.

87. If $PB'oQ$ is the diameter of curvature at the point P of a hypocycloid, o the centre of curvature, B' a point on the base,

$$QB : B'o :: 3CF - 2R : F.$$

88. If the arc of a hypocycloid from cusp to cusp$=a$, and $m R = n F$, show that $a +$ arc of hypocycloid of which the given hypocycloid is the evolute $+$ arc of hypocycloid of which *this* hypocycloid is the evolute, and so on *ad infinitum*,

$$= \frac{m}{2n} a.$$

89. If the area between a hypocycloid and its base $= A$, and $m R = n F$, show that $A +$ the area between one arc of the hypocycloid of which the given hypocycloid is the evolute, and *its* base $+$ the area between one arc of the hypocycloid of which *this* hypocycloid is the evolute and *its* base, and so on *ad infinitum*,

$$= \frac{m^2 A}{4n(m-n)}.$$

90. D′AD is an arc of a tricuspid epicycloid, from cusp to cusp, ACB the axis, AQB the central generating circle, C its centre, OBCA diametral; show that an angle may be trisected by the following construction:—Let ACQ be the angle to be trisected. Join QB, QO; about O as centre describe arc QP meeting D′AD in P (on AD): join PO; make the angle OPB equal to the angle OQB, and towards the same side, PB′ meeting the base D′BD in B′; and join B′O. Then the angle BOB′ is equal to one-third of the angle ACQ.

91. D′AD is an arc of a tricuspid hypocycloid from cusp to cusp; ACB the axis; AQB the central generating circle, C its centre, OACB diametral. Show that an angle may be trisected by the following construction. Let ACQ be the angle to be trisected. Join QB, QO; about O as centre describe arc QP meeting D AD in P (on AD); join PO and make the angle OPB′ equal to the angle OQB, and towards the same side, PB′ meeting the base D BD in B′; and join BO. Then the angle BOB′ is equal to one-third of the angle ACQ.

92. D′AD is an arc of an epicycloid from cusp to cusp; ACB the axis; AQB the central generating circle, C its centre; OBCA diametral. A radius CQ is drawn to AQB; and BQ, OQ are joined. About O as centre the arc QP is drawn meeting D′AD in P (on AD); PO is joined, and the angle OPB is made equal to the angle OQB and towards the same side, PB′ meeting the base D′BD in B′. If OB′ is joined, show that

$$\text{angle BOB}' = \frac{R}{F} \cdot \text{angle ACQ},$$

so that, by means of a suitable epicycloid, an angle may be divided in any required ratio.

93. D′AD is an arc of a hypocycloid from cusp to

cusp; ACB the axis; AQP the central generating circle, C its centre; OACB diametral. From C a radius CQ is drawn to AQB; and BQ, OQ are joined. About O as centre the arc QP is drawn meeting D′AD in P (on AD); PO is joined; and the angle OPB′ is made equal to the angle OQB, and towards the same side, PB′ meeting the base D′BD in B′. If OB′ is joined, show that

$$\text{angle BOB}' = \frac{R}{F} \cdot \text{angle ACQ},$$

so that by means of a suitable hypocycloid an angle may be divided in any required ratio.

94. If PC p is the tracing diameter at P on an epicycloid or hypocycloid APD (vertex at A), o the centre of curvature at P, show that op produced meets the tangent at P in a point T such that TP is equal to the arc AP.

95. If an epicycloid roll upon the tangent at the vertex, show that the locus of the centre of curvature at the point of contact is a semi-ellipse having semi-axes

$$4\,R\frac{(F+R)}{F} \quad \text{and} \quad \frac{4\,R^2}{F}\left(\frac{F+R}{F+2R}\right).$$

96. If a hypocycloid roll upon the tangent at the vertex, show that the locus of the centre of curvature at the point of contact is a semi-ellipse having semi-axes

$$\frac{4\,R^2}{F}\left(\frac{F-R}{F-2R}\right) \quad \text{and} \quad \frac{4\,R(F-R)}{F}.$$

97. An arc DAD′ of the bicuspid epicycloid, or nephroid, has its axis AB coincident in position with A b, the axis of a cycloid whose vertex is at A; but AB $= \frac{2}{3}$ A b. If the nephroid and the cycloid roll on T′AT, the common tangent at A, in such sort that they simultaneously touch the same point on T′T, show that the centre of curvature of the nephroid at the point of contact will trace out the same curve as the foot of normal to the cycloid at the point of

contact (the foot of normal being understood to mean the intersection of the normal with the base).

98. If a quadricuspid hypocycloid (radius of fixed circle F) is orthogonally projected on a plane through two opposite cusps, in such sort that the distance 2 F between the other two cusps is projected into distance $2f$, show that the projected curve is the evolute of an ellipse having axes equal to

$$\frac{F^2 f}{F^2 - f^2} \text{ and } \frac{F f^2}{F^2 - f^2}.$$

99. Show that the arc of the projected curve in 98, from cusp to cusp,

$$= \frac{F^2 + Ff + f^2}{F + f}.$$

100. ACA′, BCB′ are the major and minor axes of an ellipse, C its centre; and a B a' B′ is a similar ellipse having BCB as major axis; if the ellipse ABA′B′ is orthogonally projected into a circle, show that the evolute of a B a'B′ will be projected into a quadricuspid hypocycloid, and determine its dimensions.

101. With the same construction, show (independently) that the portion of the projection of any normal of a B a B, intercepted between the projections of AA′ and BB′, is of constant length. (This will be found to follow readily from Propos. X. and XIV. of Drew's 'Conics,' chapter ii.)

NOTE.—*This proposition, demonstrated geometrically, combined with what is shown at pp. 72, 73, affords a geometrical demonstration of the nature of the evolute to the ellipse. See next problem.*

102. Let ACA′, BCB be the major and minor axes of an ellipse, b C b' the orthogonal projection of BCB on a plane through ACA′, so situated that $b\,b'$: BB′ :: BB : AA. From B draw BL perp. to AB to meet A′C in L; and about

C in the plane A b A, describe a circle with radius LA cutting CA, CA, C b, and C b', in K, K$'$, k, and k, respectively. Draw a four-pointed hypocycloid, having cusps at K, k', K$'$, and k. Then a plane perpendicular to the plane A b A$'b'$, through any tangent to the hypocycloid K k'K$'k$, will intersect the plane ABA$'$B$'$ in a normal to the ellipse ABA$'$B$'$, and a right hypocycloidal cylinder on K k'K$'k$ as base, will intersect ABA$'$B$'$ in the evolute of this ellipse.

103. Two straight lines intersect at right angles in a plane perpendicular to the sun's rays, one of the lines being horizontal. If the extremities of a finite straight line slide along the fixed straight lines, and the shadow of all three lines be projected on a horizontal plane, show that the envelope of the projection of the sliding line is the evolute of an ellipse. Determine the position and dimensions of this ellipse.

If the sun's altitude is α, and the length of the sliding line l, then taking for axis of x the shadow of the horizontal fixed line, the equation to the envelope is $x^2 + y^2 \sin^2\alpha = l^2$; and the equation to the involute ellipse is $x^2 \cos^4 \alpha + y^2 \sin^2 \alpha \cos^4 \alpha = l^2$.

104. At P a point on the hypocycloid DPAD$'$ the tangent KPK$'$ is drawn, meeting the fixed circle in K and K$'$, and the normal b'PB$'$ meeting the fixed circle in b' and B$'$ (B$'$ on the base DBD$'$); show that

$$KP \,.\, PK' : (PB')^2 :: F-R : R :: (Pb')^2 : KP \,.\, PK'.$$

105. With the same construction, OM is drawn perp. to KPK$'$; show that

$$OM : PB' : P\,b' :: F-2\,R : 2\,R : 2\,(F-R).$$

106. If the tangent to the cardioid at P touches the fixed circle, and cuts the rolling circle in A$'$, and the normal at P cuts the fixed circle in B$'$ and b', then

$$PB' = \tfrac{4}{3} R;\ P b' = \frac{4 R}{3};\ \text{and } A'P = \frac{4\sqrt{2}}{3} R.$$

107. In the trochoid, if R b', the normal at p, meets the generating base in B$'$, and the tangent at p meets the tangent at vertex in T, $a'b'$ being diametral to tracing circle; show that triangle TB$'p'$ is similar to triangle $a'b'p$.

108. With same construction

$$\angle TB'a' = \angle b'p B' = \angle T p a'.$$

109. In fig. 48, triangle C $b q'' = \frac{R}{r}$ sector b C q''.

110. In fig. 48, p. 96, show that

$$\text{loop } p' r d r = 2\frac{(r-2R)}{r} \text{arc } a b \text{NL} q'' + 2\frac{(r-R)}{r} \text{rect. N} n.$$

111. Show that the result obtained in the last example agrees with that obtained in Prop. IX., Section III.

112. If in Q$'$ q'', fig. 48, produced, a point X is taken such that $(CX)^2 =$ rect. a B a C, and a circular arc XY (less than semicircle) with C as centre and CX as radius cuts a b produced in Y, show that

$$\text{loop } p'' r d = 2 \text{ segment XY} - \text{rect. N} n.$$

113. In fig. 48, $p'' x$ is drawn parallel to $q''b$ to meet the base b d in y; show that

$$\text{area } y d r p'' : \text{seg. } q''\text{L} b :: a \text{B} : a \text{C}.$$

114. From B (fig. 45, frontispiece) a straight line B q q' is drawn cutting the central tracing circle in q and q', and straight lines q p and $q'p'$ parallel to the base meet the arc a d in p and p'; show that the tangent at p is parallel to the tangent at p'.

115. P and P$'$ are two points on an epitrochoid or hypotrochoid, C and C$'$ the corresponding positions of the centre of generating circle, O the fixed centre, OA, OB the apocentral and pericentral distances. If OP . OP$'$ = OA . OB,

show that the tangents at P and P′ make equal angles with OC and OC′ respectively.

116. A cycloid on base BD (fig. 45, frontispiece) has its cusps at B and D; show that it touches the prolate cycloid $a\,p\,d$ at a point of inflexion.

117. A series of prolate cycloids have the same line of centres, their axes in the same straight line, and their bases equal. Show that their envelope is a pair of arcs of a cycloid having its base equal to half the base of each prolate cycloid of the system, and the line of their axes as a secondary axis.

118. If the normals at p and q, two points on a prolate cycloid $a\,p\,q\,d$, are parallel, and meet the generating base in b' and b'' respectively, then ρ and ρ' being the radii of curvature at p and q respectively,

$$\rho : \rho' :: (p\,b')^2 : (q\,b'')^2.$$

119. If ρ is the radius of curvature at the point where a curtate cycloid cuts the generating base, and μ is a mean proportional between the radii of curvature at the vertex and at d on the base, show that $\rho^2 = \mu\,r$.

120. Show that that involute of the central generating circle of a cycloid which has its cusp at the vertex passes through the cusps of the cycloid.

121. That involute of any generating circle of a cycloid, which has its cusp at the tracing point, passes through the cusps of the cycloid.

122. The sum of the two nearest arcs of the involute of the circle, cut off by any tangent to the circle, is least when the tangent touches the circle at the farther extremity of the diameter through the cusp of the involute.

123. If the rolling straight line by which the involute of a circle of radius f is traced out has rolled over an arc a from the cusp, show that the arc traced out $= \frac{f}{2} a^2$.

124. If the rolling straight line by which a spiral of Archimedes is traced out, has rolled over an arc a from first position, when the extremity of perp. carried with it was at the centre of the fixed circle (radius f), show that

$$\text{arc traced out} = \frac{f}{2}\left\{ a\sqrt{1+a^2} + \log\left(a + \sqrt{1+a^2}\right)\right\}.$$

125. All involutes of circles are similar.

126. All spirals of Archimedes are similar.

127. If a straight line carrying a perp. of length d roll on a circle of radius f, and another straight line carrying a perp. of length D (on same side with reference to centre of fixed circle) roll on a circle of radius F, show that the curves traced out by the extremities of these perps. will be similar if

$$\text{P} : p :: \text{F} : f.$$

128. In the spiral of Archimedes the subtangent is equal to that arc of a circle whose radius is the radius vector, which is subtended by the spiral angle. (Frost's 'Newton').

The subtangent is the portion of a perp. to radius vector, through pole, intercepted between pole and tangent at extremity of radius vector. What is required to be shown in this example is that if $p'p$ (fig. 72, p. 130), produced, meet B′O produced in Z, OZ is equal to the arc corresponding to DQB′ in a circle of radius Op.

129. Establish the following construction for determining the centre of curvature at point p (fig. 72, p. 130) of a spiral of Archimedes. Draw radius OB′ to fixed circle, perp. to Op; join pB′; and draw OL perp. to pB′. Then if B′L is divided in o so that

$$\text{B}'o : o\text{L} :: \text{B}'p : \text{B}'\text{L},$$

o is the centre of curvature at p.

130. From this construction (established geometrically) show that, taking the usual polar equation to the spiral of Archimedes, viz., $r = a\,\theta$,

$$\rho = \frac{a(1+\theta^2)^{\frac{3}{2}}}{2+\theta^2}.$$

131. A straight line turns uniformly in a plane round a fixed point, while the foot of a perpendicular of length l moves uniformly along the revolving line; show that the other end of this perpendicular will trace out one of the spirals described at pp. 128, 129.

132. If the angular velocity in preceding problem is ω, the linear velocity of the foot of perpendicular v, and $l = \frac{v}{\omega}$, the perpendicular lying on the side towards which the revolving line is advancing, show that the other extremity of the perpendicular will describe the involute of the circle.

133. If DT, fig. 42, p. 82, rolls on the circle DQB of radius a, and a point initially on DO and distant b from D is carried with DT to trace out a spiral in the manner described at pp. 128, 129, show that the polar equation to the spiral, OQ being taken as initial line, and the rolling taking place in the usual positive direction, is

$$\theta = \frac{\sqrt{r^2 - (a-b)^2}}{a} + \tan^{-1} \frac{(a-b)}{\sqrt{r^2 - (a-b)^2}}.$$

134. Show that the construction given in Example 129 for determining the centre of curvature at a point on the spiral of Archimedes is applicable to all the spirals of Examples 131 and 133.

135. In the case of one of these spirals, putting the arc over which the rolling line has passed from its initial position $= \phi$, show that

$$\rho = \frac{(a^2\phi^2 + b^2)^{\frac{3}{2}}}{a^2\phi^2 + ab + b^2}.$$

136. The locus of the foot of perpendicular from a point on a cycloid upon the diametral of the generating circle through the point is the companion to the cycloid.

137. From D, the cusp of an inverted cycloid, and P, a point near D, two particles roll down the smooth arc to the vertex; show that in the limit the path of either relatively to the other is a semicircle.

138. A particle is projected with given velocity from the vertex of a cycloid whose axis is vertical, and vertex uppermost; find where it will leave the curve, and the latus rectum of its future parabolic path.—(Tait and Steele's 'Dynamics.')

139. A particle falling from rest at a point in an inverted cycloid has its velocity suddenly annihilated when it has passed over half its vertical height above the lowest point; then proceeds, again losing its velocity when half-way down from its last position of no velocity, and so on continually. Show that it will be at $\frac{1}{2^{2n}}$th of its original height above the vertex after n times the time it would have taken to fall to the vertex undisturbed.—(Tait and Steele's 'Dynamics.')

140. If a curve of any form is rolling upon another curve in the same plane, and p is a point on the curve traced by any given point carried with the rolling curve and in the same plane with it, b the point of contact of the fixed and rolling curves, show that the following relation exists between ρ_1, ρ_2, the radii of curvature of the fixed and rolling curves at b, and ρ_3 the radius of curvature of the traced curve at p (putting $pb = n$ and the angle between pb and the normal of fixed curve at $b = \theta$),

$$\{n(\rho_1 + \rho_2) - \rho_1 \rho_2 \cos\theta\}\rho_3 = n^2(\rho_1 + \rho_2).$$

141. A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends

lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. Equal quantities of fluids of specific gravity σ_1 and σ_2 are poured in at the two cusps, the quantity of each being such as would fill a length a of the tube (a being the length of the cycloid's axis, so that $4a$ is the length of the tube). If the fluids do not mix and the distance of the upper levels of the fluids from the vertex (measured along the cycloidal arc) be x_1, x_2 respectively, show that

$$4x_1(\sigma_1 + \sigma_2) = a(\sigma_1 + 3\sigma_2),$$

and
$$4x_2(\sigma_1 + \sigma_2) = a(3\sigma_2 + \sigma_3).$$

142. If in problem 141 an equal quantity of a third fluid of specific gravity σ_3 is poured in upon the free surface of the second fluid (sp. gr. σ_2), and x_1, x_2, are the respective distances of the free surfaces of the first and third fluids from the vertex (measured along the cycloidal arc), show that

$$4x_1(\sigma_1 + \sigma_2 + \sigma_3) = a(\sigma_1 + 3\sigma_2 + 5\sigma_3),$$

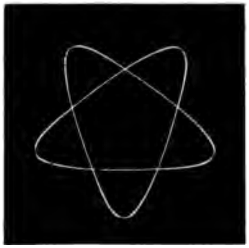
and
$$4x_2(\sigma_1 + \sigma_2 + \sigma_3) = a(5\sigma_1 + 3\sigma_2 + \sigma_3).$$

Under what condition will either the first or third fluid run over?

143. If n fluids are poured in, as in Ex. 141, the specific gravities of 1st, 2nd, 3rd, &c., to the nth, being σ_1, σ_2, σ_3, &c., to σ_n, respectively, the arcs occupied by the respective fluids being l_1, l_2, l_3, ... l_n, and no fluid overflowing; and if x is the distance of the free surface of the first fluid from the vertex (measured along the cycloidal arc), show that

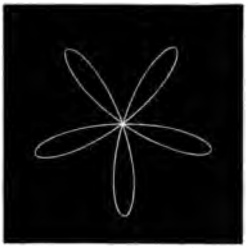
$$\begin{aligned}4x(\sigma_1 l_1 + \sigma_2 l_2 + \sigma_3 l_3 + \dots + \sigma_n l_n) = {} & \sigma_1 l_1^2 + \sigma_2(l_2^2 + 2l_1 l_2) \\ & + \sigma_3(l_3^2 + 2l_1 l_3 + 2l_2 l_3) + \dots \\ & + \sigma_n(l_n^2 + 2l_1 l_n + 2l_2 l_n + \dots + 2l_{n-1} l_n).\end{aligned}$$

Ex. 144. Fig. 154.

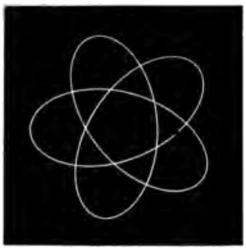


Ex. 145. Fig. 155.

Ex. 146. Fig. 156.



Ex. 147. Fig. 157.



Ex. 148. Fig. 158.

Ex. 149. Fig. 159.

Ex. 150. Fig. 160.

Ex. 151. Fig. 161.

Interpret figs. 154–161 in the way explained in pp. 191–193.

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