Little Mathematics Library

G. E. SHILOV

CALCULUS OF RATIONAL FUNCTIONS

Mir Publishers · Moscow

ИЗДАТЕЛЬСТВО · МИР ·

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Г. Е. Шилов

МАТЕМАТИЧЕСКИЙ АНАЛИЗ В ОБЛАСТИ РАЦИОНАЛЬНЫХ ФУНКЦИЙ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

LITTLE MATHEMATICS LIBRARY

G. E. Shilov

CALCULUS OF RATIONAL FUNCTIONS

Translated from the Russian
by
V.I. Kisin

MIR PUBLISHERS
MOSCOW

First published 1976
Second printing 1982

На английском языке

Contents

Foreword

The concepts of derivative and integral are basic for the calculus. They are not elementary; in any systematic textbook on calculus the presentation of these concepts is preceded by the theory of real numbers, the theory of limits, and the theory of continuous functions. This preliminary procedure is necessary to formulate the concepts of derivative and integral in sufficiently universal form, to be applied to the widest possible class of functions. If, however, we restrict ourselves to a comparatively narrow class of rational functions and utilize the illustrative language of graphs, we can present the concepts of derivative and integral in a few pages, sufficiently accurately and at the same time pithily. And this is the purpose of the pamphlet intended for a wide circle of readers; the knowledge of secondary school students is sufficient to insure understanding of everything that will be discussed.

1. Graphs

Though we assume that the reader is conversant with graphs, we shall anyway remind the basic points.

Let us draw two mutually perpendicular straight lines, one horizontal and one vertical, and denote by O their intersection point. The horizontal line will be referred to as the *axis of abscissas* and the vertical line—as the *axis of ordinates*. The point O divides each line into two semi-axes, a positive and a negative one; the right-hand semi-axis of abscissas and the upper semi-axis of ordinates are called positive, while the left-hand semi-axis of abscissas and the lower semi-axis of ordinates are called negative. We mark the positive semi-axes by arrows. Now the position of each point M on the plane can be defined by a pair of numbers. To do this we drop perpendiculars from the point M onto each of the axes; the perpendiculars will cut on the axes the segments OA and OB (Fig. 1). The length of the segment OA, taken with the sign "+" if A is located on the positive semi-axis and with the sign "—" if it lies on the negative semi-axis, will be called the *abscissa* of the point M and will be denoted by x. Similarly, the length of the segment OB (with the same rule of signs) will be called the *ordinate* of the point M and denoted by y. The numbers, x and y, are called the *coordinates* of the point M. Each point on the plane is determined by coordinates. Points of the abscissa axis have the ordinate equal to zero, while the points of the ordinate axis have zero abscissa. The origin of coordinates O (the point of intersection of axes) has both coordinates equal to zero. Conversely, if two arbitrary numbers x and y of any signs are given, we can always plot, and this is very impor-

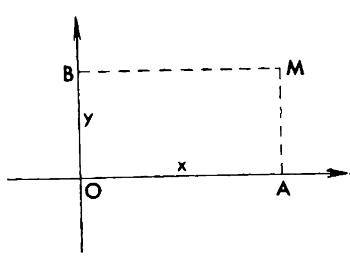

Fig. 1

tant, exactly a unique point M with the abscissa x and the ordinate y; to achieve this we have to lay off the segment $OA = x$ on the abscissa axis and to erect a perpendicular $AM = y$ (signs being taken into account); the point M will be the one sought for.

Let the rule be given which indicates the operations that should be performed over the independent variable (denoted by x) to obtain the value of the quantity of interest (denoted by y).

Each such rule *defines*, in the language used by mathematicians, *the quantity y as function of the independent variable x.* It can be said that a *given function* is just that specific *rule* by which the values of y are obtained from the values of x.

For instance, the formula

$$y = \frac{1}{1+x^2}$$

indicates that to obtain the values of y we have to square the independent variable x, add it to unity and then divide unity by the obtained result. If x takes on some numerical value x_0 then by virtue of our formula y takes on a certain value y_0. The values x_0 and y_0 define a point M_0 in the plane of the drawing. We can then replace x_0 by another number x_1 and calculate by the formula the new value y_1; the pair of numbers x_1, y_1 defines a new point M_1 on the plane. The geometric locus of all points of the plane, whose ordinates are related to abscissas by the given formula, is called the *graph* of the corresponding function.

Generally speaking, the set of graph points is infinite so that we cannot hope to plot all of them without exception by using the foregoing rule. But we shall not have to do that. In most cases a certain moderate number of points is sufficient for us to be able to realize the general shape of the graph.

The method of plotting a graph "point-by-point" consists just in plotting a certain number of graph points and in joining these points by as smooth a curve as possible.

As an example we shall consider the graph of the function

$$y = \frac{1}{1+x^2} \tag{1}$$

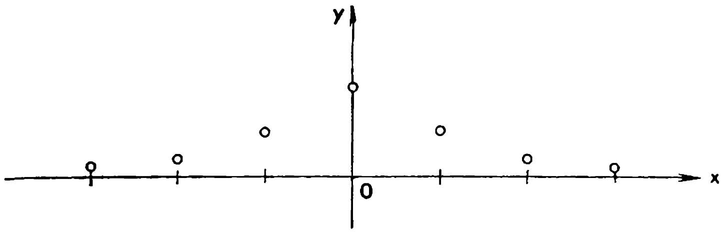

Fig. 2

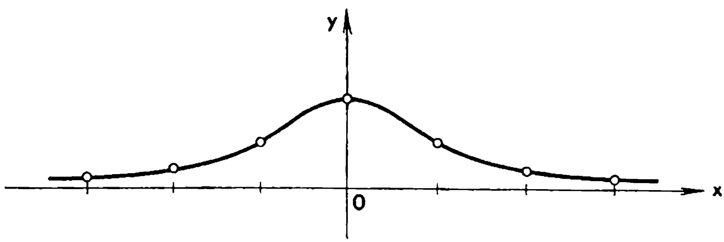

Fig. 3

Let us compile the following table

x	0	1	2	3	—1	—2	—3
y	1	1/2	1/5	1/10	1/2	1/5	1/10

The first line lists the values of $x = 0, 1, 2, 3, -1, -2, -3$. As a rule, integral values of x are more useful for calculations. The second line lists the corresponding values of y calculated by formula (1). Plotting the corresponding points on the plane (Fig. 2) and connecting them by a smooth curve, we obtain the graph (Fig. 3).

The rule of plotting a graph "point-by-point" is, as we have seen, extremely simple and requires no "science". Nevertheless, it may be for this very reason that blind adhering to this "point-by-point" rule may be fraught with serious errors.

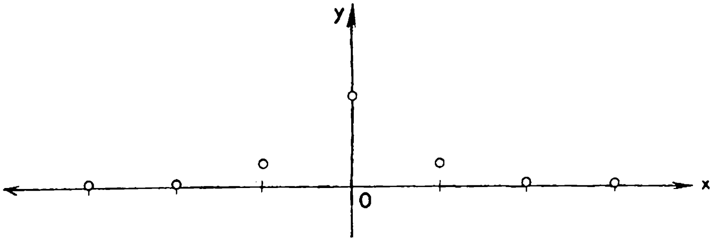

Fig. 4

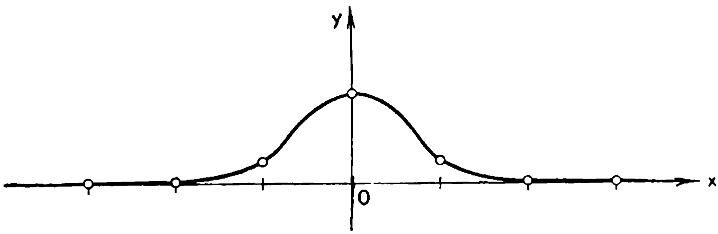

Fig. 5

Let us plot "point-by-point" the curve specified by the equation

$$y=\frac{1}{(3x^2-1)^2} \qquad (2)$$

The table of x and y values corresponding to this equation is as follows

x	0	1	2	3	−1	−2	−3
y	1	1/4	1/121	1/676	1/4	1/121	1/676

The corresponding points on the plane are plotted in Fig. 4 which is very similar to Fig. 2. Connecting the plotted points with a smooth curve we obtain the graph (Fig. 5). It may seem that we could put the pen away and feel satisfied: the art of plotting graphs has been grasped! But for the sake of a test let us calculate y for some intermediate value of x, for example, for $x = 0.5$. After performing the calculations, we obtain an unexpected result: $y = 16$. This is in

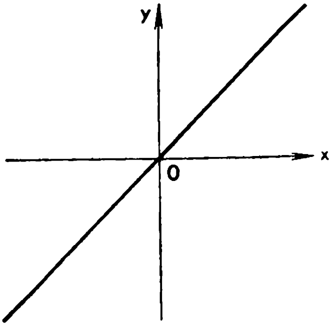

Fig. 6

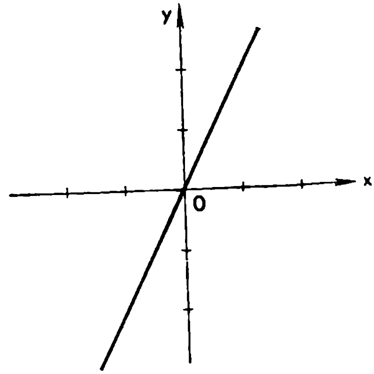

Fig. 7

striking disagreement with our graph. And we cannot guarantee that calculation of y for other intermediate values of x—and their number is infinite—will not produce even greater discrepancies. Unfortunately, the method of tracing the graph "point-by-point" proves rather unreliable.

We shall discuss below another method of graph plotting which is more reliable in the sense of safeguarding us from surprises similar to one we have encountered above. Using this method we shall be able to plot the correct graph of Eq. (2). In this method—let us term it, for instance, "by successive operations"—we have to perform directly in the graph all the operations which are written down in a given formula, viz. addition, subtraction, multiplication, division, etc.

Let us consider a few simplest examples. We shall plot a graph corresponding to the equation

$$y = x \tag{3}$$

This equation expresses that all points of the curve of interest have equal abscissas and ordinates. The locus of the points for which ordinates are equal to abscissas is the bisectrix of the angle between positive semi-axes and of the angle between negative semi-axes (Fig. 6).

The graph corresponding to the equation $y = kx$ with a coefficient k is obtained from the foregoing graph by multiplying each ordinate by the same number k. Let us set, for example, $k = 2$; each ordinate of the foregoing graph

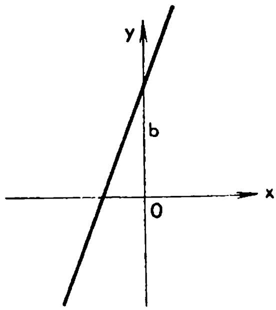

Fig. 8

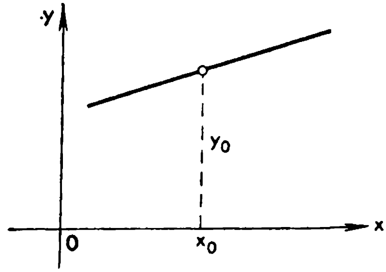

Fig. 9

must be doubled, so that as a result we obtain a straight line rising more steeply (Fig. 7). With each rightward step along the x-axis the line rises two steps up along the y-axis. By the way, this enables us to perform readily the plotting on squared paper. In the general case of the equation $y = kx$ with an arbitrary k a straight line is obtained. If $k > 0$, then with each rightward step the line will rise k steps up along the y-axis. If $k < 0$, the line will descend.

Consider the formula

$$y = kx + b \tag{4}$$

To plot the corresponding graph we have to add to each ordinate of the already known line the same number b. This will shift the straight line $y = kx$ as a whole upward in the plane by b units (for $b > 0$; if $b < 0$ the original curve will naturally be shifted downward). As a result we shall obtain a straight line parallel to the original one; it does not pass any more through the origin of coordinates but cuts on the ordinate axis the segment b (Fig. 8).

The number k is called the *slope* of a straight line $y = kx + b$; we already mentioned that this number k shows by what number of steps the straight line moves upward per each rightward step. In other words, k is the tangent of the angle between the direction of the x-axis and the straight line $y = kx + b$.

The equation

$$y = k(x - x_0) + y_0 \tag{4'}$$

corresponds to the straight line with the slope k; it passes through the point (x_0, y_0) (Fig. 9), since setting $x = x_0$ gives $y = y_0$.

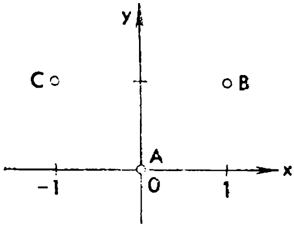

Fig. 10

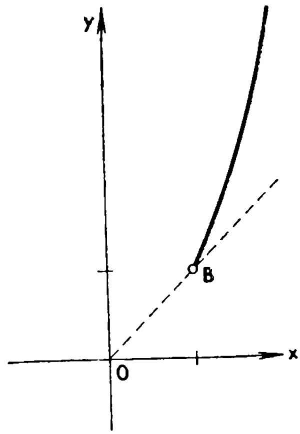

Fig. 11

Thus, the graph of any first-degree polynomial in x is a straight line which is plotted according to the aforesaid rules. Let us pass over to the second-degree polynomials.

Consider the formula

$$y = x^2 \tag{5}$$

It can be presented in the form

$$y = y_1^2, \text{ where } y_1 = x$$

In other words, the required graph will be obtained if each ordinate of the already known line $y = x$ is squared. Let us find out what this should produce.

Since $0^2 = 0$, $1^2 = 1$, $(-1)^2 = 1$, we obtain three reference points A, B, C (Fig. 10). If $x > 1$, then $x^2 > x$; therefore to the right of the point B the curve will be above the bisectrix of the quadrant angle (Fig. 11). If $0 < x < 1$, then $0 < x^2 < x$; therefore the curve between the points A and B will be under the bisectrix. Moreover, we state that, as it approaches the point A, the curve will enter an angle bounded above by the line $y = kx$ (however small k) and below by the x-axis; indeed, the inequality $x^2 < kx$ is satisfied for all $x < k$. This fact means that the sought-for curve is *tangent* to the abscissa axis at the point O (Fig. 12). Let us move now leftward along the x-axis from the point O. We know that the numbers $-a$ and $+a$ when squared give

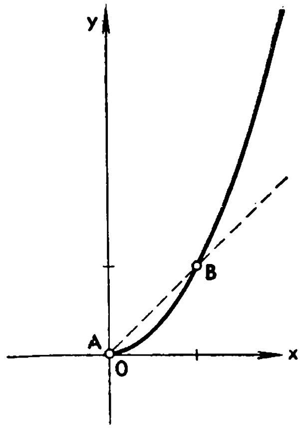

Fig. 12

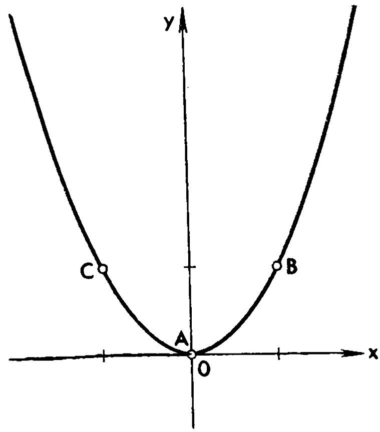

Fig. 13

the same result a^2. The ordinate of our curve will therefore be the same both for $x = +a$ and for $x = -a$. In geometrical terms this means that the graph of the curve in the left-hand semi-plane can be obtained by reflection relative to the ordinate axis of the curve already plotted in the right-hand semi-plane. We obtain the curve which is called the *parabola* (Fig. 13)

Now, following the same procedure, we sketch a more complicated curve

$$y = ax^2 \tag{6}$$

and a still more complicated one

$$y = ax^2 + b \tag{7}$$

The first of these curves is obtained by multiplying all ordinates of parabola (5)—we shall refer to it as a *reference parabola*—by a number a.

If $a > 1$ the curve will be similar to (5) but will rise more steeply (Fig. 14).

If $0 < a < 1$ the curve will be less steep (Fig. 15), and when $a < 0$ its branches will turn downward (Fig. 16). Curve (7) will be obtained from curve (6) by shifting it upward by a segment b if $b > 0$ (Fig. 17). If $b < 0$, we have to shift the curve downward (Fig. 18). All these curves are also called parabolas.

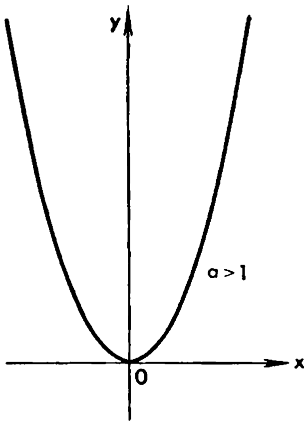

Fig. 14

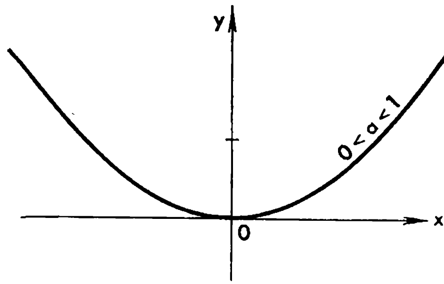

Fig. 15

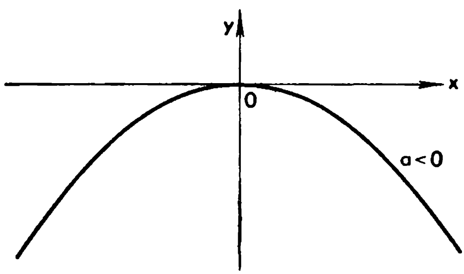

Fig. 16

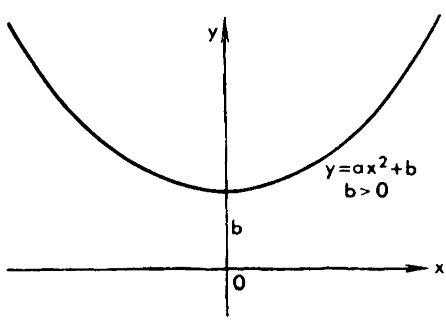

Fig. 17

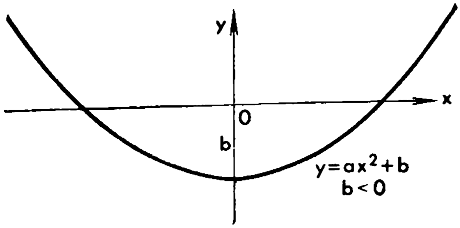

Fig. 18

Consider now a more complicated example of plotting graphs by means of multiplication. Let the problem be to plot the graph of the equation

$$y = x(x-1)(x-2)(x-3) \qquad (8)$$

Here we have the product of four multipliers. Let us plot each of them separately: all of them are straight lines parallel to the bisectrix of the quadrant angle and cutting the segments on the ordinate axis (see Fig. 19)

$$0, \ -1, \ -2, \ -3$$

At the points 0, 1, 2, 3 of the x-axis the sought-for curve will have the ordinate 0 since the product is equal to zero if at least one of the factors is equal to zero. At other points the product will differ from zero and its sign can easily be found by considering the signs of the co-factors. Thus, all factors are positive to the right of the point 3; hence, the

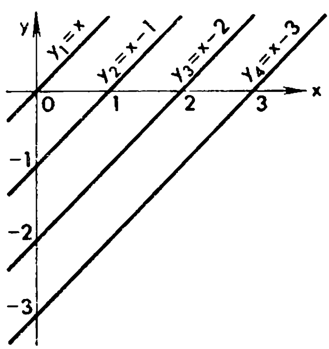

Fig. 19

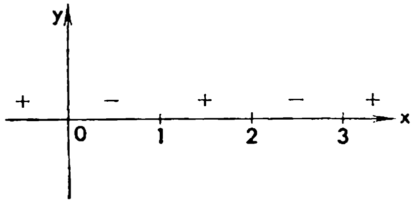

Fig. 20

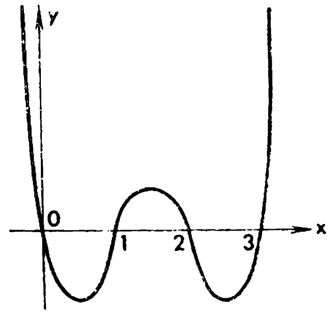

Fig. 21

product is positive too. Between the points 2 and 3 one of the multipliers is negative and therefore the product is negative. Two of the multipliers are negative between the points 1 and 2, so that the product is positive, etc. We obtain the following distribution of signs (Fig. 20). To the right of the point 3 all multipliers will increase as x increases, so that the product will increase greatly at that. To the left of the point 0 all multipliers increase their negative values and the product (which is positive) also rises steeply.

Now it is not difficult to sketch the general shape of the graph (Fig. 21).

So far we only used the operations of addition and multiplication. Now we supplement them with division. Let us plot the curve

$$y=\frac{1}{1+x^2} \tag{9}$$

To do this we shall plot separately the graph of the numerator and that of the denominator.

The graph for the numerator

$$y_1 = 1$$

is the straight line parallel to the abscissa axis at the distance 1 from it. The graph of the denominator

$$y_2 = x^2 + 1$$

is a reference parabola shifted upward by 1. The two graphs are shown in Fig. 22.

Let us divide each ordinate of the numerator by the corresponding (i.e. taken for the same x) ordinate of the deno-

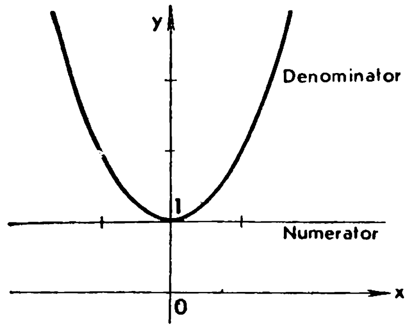

Fig. 22

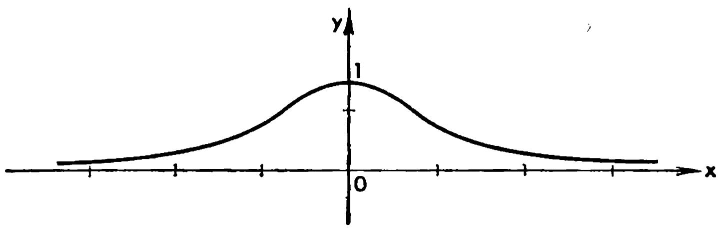

Fig. 23

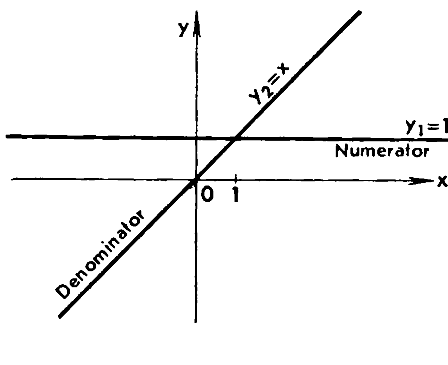

Fig. 24

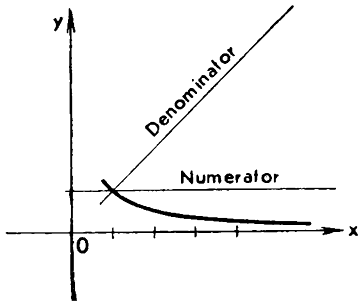

Fig. 25

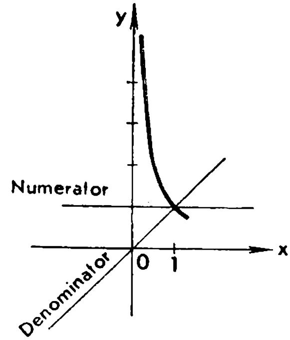

Fig. 26

minator. If $x = 0$ then $y_1 = y_2 = 1$, whence $y = 1$. For $x \neq 0$ the numerator is less than the denominator and the quotient is less than unity. Since both the numerator and the denominator are always positive, the quotient is positive too; hence, the graph is enclosed within a band between the abscissa axis and the line $y = 1$. When x increases without limit, the denominator also increases without limit while the numerator remains constant; therefore the quotient approaches zero. All this results in the graph of the quotient shown in Fig. 23. The curve thus obtained is the same as that plotted point-by-point (Fig. 3).

In the method of graphical division special role is played by those values of x with which the denominator becomes zero. If this is not accompanied by the numerator also turning zero, the quotient becomes infinite. To realize the meaning of this expression let us plot the curve

$$y = \frac{1}{x} \tag{10}$$

We already know the graphs of both the numerator and the denominator (Fig. 24). For $x = 1$ we have $y_1 = y_2 = 1$, whence $y = 1$. For $x > 1$ the numerator is smaller than the denominator and the quotient is less than 1, similarly to the foregoing example, infinite rise of x results in the quotient approaching zero and we obtain the part of the curve corresponding to the values $x > 1$ (Fig. 25).

Let us consider now the range of x values between 0 and 1. When x approaches zero from the side of 1, the denominator approaches zero while the numerator remains equal to 1. Therefore the quotient increases without limit, exceeding

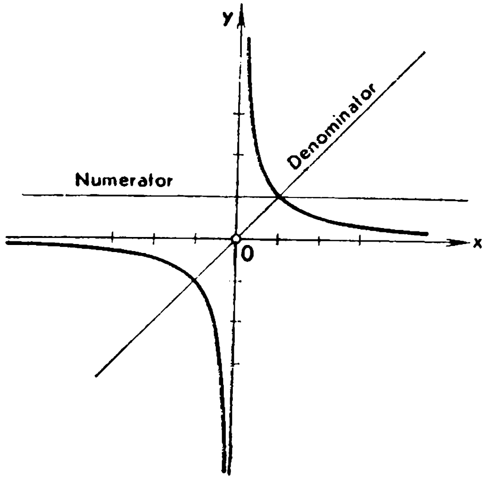

Fig. 27

however large number, given sufficiently small values of x, and we obtain the branch going off to infinity (Fig. 26). If $x < 0$ the denominator and consequently the whole quotient become negative. The general shape of the graph is presented in Fig. 27.

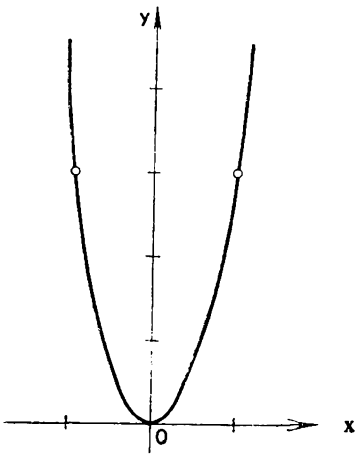

Fig. 28

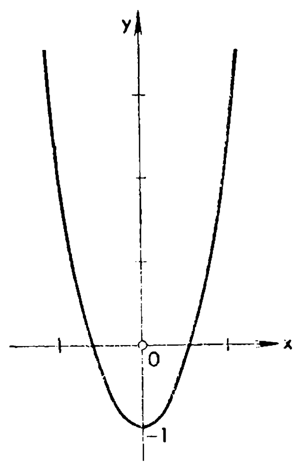

Fig. 29

Now we are ready to tackle the problem of plotting the curve discussed at the beginning of this section

$$y = \frac{1}{(3x^2-1)^2} \qquad (11)$$

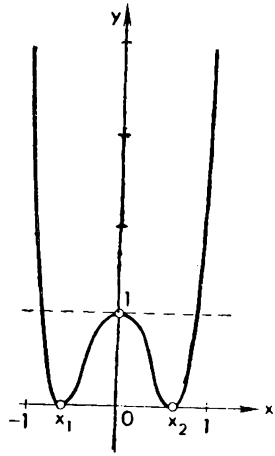

Fig. 30

We shall first plot the graph of the denominator. The curve $y_1 = 3x^2$ is the "triple" reference parabola (Fig. 28). Subtraction of unity means shifting the graph by one unity downward (Fig. 29). The curve intersects the x-axis at two points which can easily be found by setting $3x^2 - 1$ equal to zero: $x_{1,2} = \pm\sqrt{1/3} = \pm 0.577\ldots$. Let us square the obtained graph. The ordinates of the points x_1 and x_2 will remain equal to zero. All other ordinates will be positive and the graph will be located above the abscissa axis. At the point $x = 0$ the ordinate will be equal to $(-1)^2 = 1$, and this is the maximum ordinate within the interval from x_1 to x_2. Outside this interval the curve will steeply rise upward on both sides of the plot (Fig. 30).

The graph of the denominator is thus plotted. The dotted line in the same drawing shows the graph of the numerator $y_2 = 1$. Now we only have to divide the numerator by the denominator. Since both the numerator and the denominator are always positive, the quotient will be positive and the whole graph will be located above the abscissa axis. For $x = 0$ the numerator and the denominator are equal, and their ratio is equal to 1. Let us move rightward from point O along the abscissa axis. The numerator remains equal to 1 while the denominator decreases; consequently, the quotient increases from its value 1. When we reach the point $x_2 = 0.577\ldots$, the denominator becomes equal to zero. This means that by this moment the value of the quotient will go to infinity (Fig. 31). To the right of the point x_2 the denominator will rapidly change in a reverse way from the value 0 to 1 and then will grow unboundedly. The graph of the quotient, on the contrary, will return from infinity to 1, will intersect the line $y = 1$ at the same point

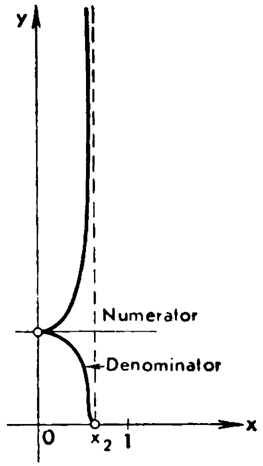

Fig. 31

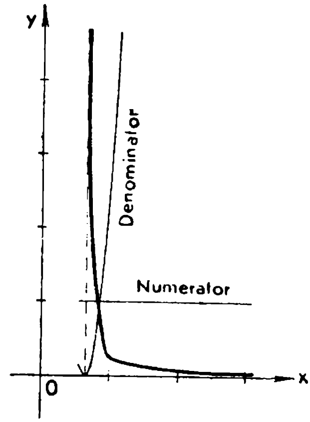

Fig. 32

as the graph of the denominator, and will then approximate without limit to zero (Fig. 32).

The pattern on the left-hand side of the ordinate axis will be the same (Fig. 33).

We have marked on this graph the points corresponding to integer values $x = 0, 1, 2, 3, -1, -2, -3$. These are the very points which we selected while plotting the graph "point-by-point" in page 10. But the true shape of the graph differs considerably from that given in Fig. 5.

We see that in reality the curve does not decrease smoothly from the value 1 (for $x = 0$) to the value 1/4 (for $x = 1$) and to still smaller values, but instead goes up to infinity. Here we can also locate the point with the coordinates $x = 1/2$, $y = 16$ which could not be forced onto the former, erroneous graph but fits perfectly the new, correct graph.

We have discussed the simplest operations which can be performed with graphs. More precisely, we started off with the simplest equation $y = x$ and then used the four arithmetic operations: addition, subtraction, multiplication and division.

The functions $y(x)$ obtained as a result of such operations are presented in the form of a quotient of two polynomials

$$y(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + \ldots + a_nx^n}{b_0 + b_1x + \ldots + b_mx^m}$$

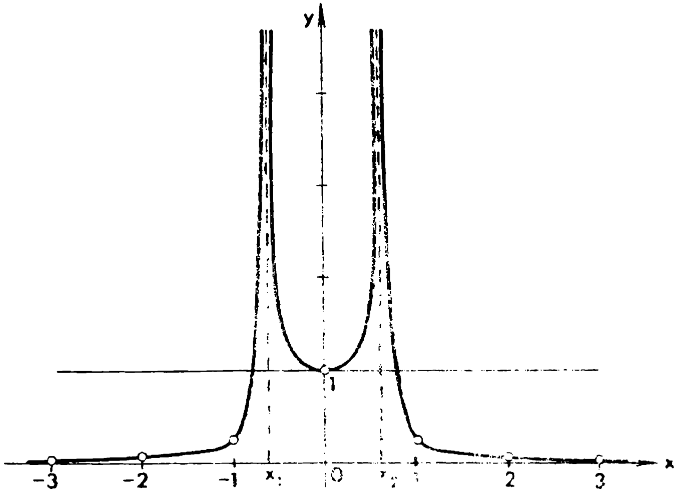

Fig. 33

and are referred to as the *rational* functions of the variable x. (Other functions exist as well in the calculus but even their definition requires a developed theory of real numbers; therefore in the present pamphlet we shall restrict the discussion to rational functions only.)

For those readers who became interested in plotting graphs "by successive operations" we suggest here several problems for practice and self-checking.

Problems

Plot the graphs of the following equations:

1. $y = x^2 + x + 1$. 2. $y = x(x^2 - 1)$. 3. $y = x^2(x - 1)$.

4. $y = x(x - 1)^2$. 5. $y = \frac{x}{x-1}$.

Recommendation. In Problem 5 it is advisable to separate an integer component:

$$\frac{x}{x-1} = 1 + \frac{1}{x-1}$$

6. $y = \frac{x^2}{x-1}$. 7. $y = \frac{x^3}{x-1}$.

Recommendation. Separation of the integer component will also be useful in Problems 6 and 7.

8. $y = \pm\sqrt{x}$.

Recommendation. Square root of x exists when $x \geqslant 0$* and does not exist when $x < 0$.

9. $y = \pm\sqrt{1 - x^2}$.

How can it be proved that the resulting curve is the circle?

Recommendation. The exact definition of a circle should be recalled and then the Pythagorean theorem applied.

10. $y = \pm\sqrt{1 + x^2}$.

Prove that as $x \to \infty$ the branches of this curve approach infinitely closely the bisectrices of the quadrant angles.

Recommendation. $\sqrt{1+x^2} - x = \dfrac{1}{\sqrt{1+x^2}+x}$.

11. $y = \pm x\sqrt{x(1 - x)}$. 12. $y = \pm x^2\sqrt{1 - x}$.

13. $y = \dfrac{1-x^2}{2\pm\sqrt{1-x^2}}$. 14. $y = x^{2/3}(1 - x)^{2/3}$.

Solutions to all problems are given in pp. 48-50.

2. Derivatives

The method of plotting the graph "by successive operations" enables us to obtain the general picture as to the character of the function variation. But these methods become inadequate to solve certain more precisely formulated problems. For example, the curve of a graph (Fig. 34) having dropped down to a certain value y_0, which corresponds to the abscissa x_0, starts rising; the corresponding function $y(x)$ is said *to have a local minimum at the point* x_0. The concept of the *local maximum* has similar meaning; we say that the function $y = y(x)$ has a local maximum at the point x_0 if its graph rises as x increases up to the point x_0 and then begins to slope down (Fig. 35). The question is, what are the exact values of x_0 and y_0?

* This statement is not self-evident at all but requires for its substantiation a well developed theory of real numbers. Its proof can be found in any good textbook on calculus. Here it is only necessary, having assumed the existence of the root, to plot its graph.

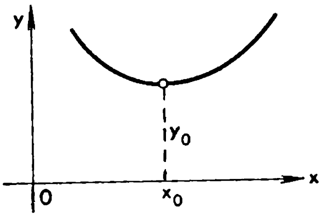

Fig. 34

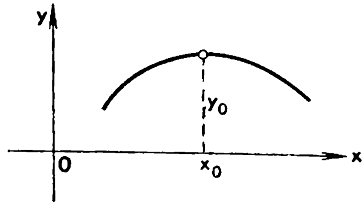

Fig. 35

It is easy to imagine a specific situation in which such a problem would arise. For instance, let the graph of Fig. 34 plot the cost of one ton of some product as function of daily consumption of electric energy. Low daily consumption of energy will mean very slow production of one ton of the product, and the existence of permanent expenditures (cost of manpower, etc.) will result in high cost per ton of the product. High daily consumption of energy will shorten the time of production of one ton of product but the cost per ton will be increased due to the rise in the cost of the energy consumed. This cost per ton will be minimum at a certain value of the daily energy consumption; we are naturally very much interested to know what this minimum cost should be and what daily consumption of energy it corresponds to. Below we shall discuss a similar problem (p. 32).

To obtain an exact answer to the question on the position of the point of local minimum new methods are required, which lead us into the field of the calculus called the *differential calculus.*

The idea of solving the suggested problem is as follows. A *tangent straight line* can be drawn at each point of the graph $y(x)$. The tangent straight line at the point A of the $y(x)$ graph (Fig. 36) is defined as the line α passing through the point A in such a way that the curve $y(x)$ itself, as it approaches the point A, enters into any no matter how small angle with the vertex in A, containing the straight line α, and remains inside it. (In Sec. 1 the abscissa axis was said to be tangent to the reference parabola in just this sense.) An arbitrary straight line β passing through the point A is called the *secant* for the graph $y(x)$; an angle with the vertex in A and the bisectrix β, into which the curve in the vicinity of the point A does not enter, can always be indicated for a secant not coinciding with the tangent (Fig. 37). We shall denote the slope of the tangent at the point (x, y) by $k = k(x)$. This function $k(x)$ is called the *derivative of the function* $y(x)$. (Later we shall prove that if $y(x)$ is a polynomial, then $k(x)$ is also a polynomial, and if $y(x)$ is a rational function, then $k(x)$ is a rational function too, and shall present precise rules for the calculation of $k(x)$.) Let us assume the function $k(x)$ as already found for the specified function $y(x)$. At the sought-for point of the local minimum (x_0, y_0) the tangent line α must be horizontal [reductio ad absurdum (the proof by contradiction): we have seen that the curve $y = y(x)$ must enter into the angle however small containing the line α; if the line α is oblique, we can construct a small angle with α as its bisectrix, whose sides have slopes of the same sign (Fig. 36) and, consequently, the curve $y = y(x)$ cannot have a local minimum at (x_0, y_0)]. Therefore, at the point $x = x_0$ of the local minimum, $k(x_0) = 0$. Thus we have the equation

$$k(x) = 0$$

Generally speaking, this equation can have several solutions. Each of them defines the point (x_0, y_0) on the curve $y = y(x)$ at which the tangent is horizontal; we must find all these solutions and among them single out that which is of interest to us. Thus, once the function $k(x)$ is known, the problem is reduced to solving the algebraic equation.

We shall now pass to plotting the function $k(x)$. First of all let $y = x^2$ be our reference parabola. We want to

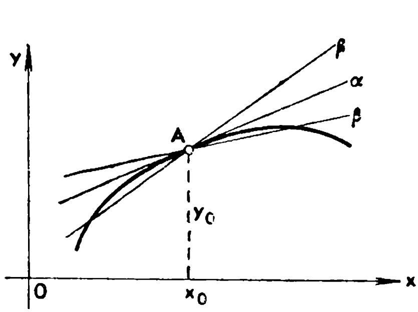

Fig. 36

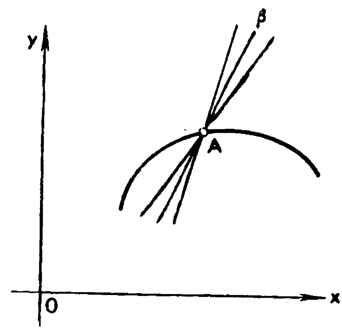

Fig. 37

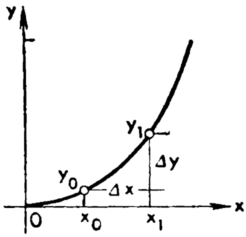

Fig. 38

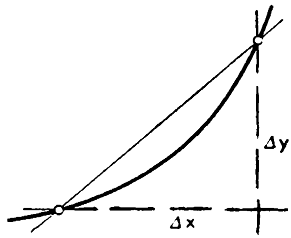

Fig. 39

find the slope of the tangent at the point (x_0, y_0) of this parabola.

Let us denote by Δx and Δy the increments of the abscissa and ordinate of the parabola when we move from the point (x_0, y_0) to the point (x_1, y_1) in its vicinity:

$$x_1 = x_0 + \Delta x, \qquad y_1 = y_0 + \Delta y$$

(Fig. 38). Since

$$y_0 = x_0^2, \quad y_0 + \Delta y = (x_0 + \Delta x)^2$$

we find by subtraction that

$$\Delta y = 2x_0 \Delta x + (\Delta x)^2$$

Let us draw a secant line through two points (x_0, y_0) and (x_1, y_1) (Fig. 39). Its slope is obviously equal to

$$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x$$

and its equation in the complete form is (see Eq. (4'))

$$y = (2x_0 + \Delta x)(x - x_0) + y_0 \tag{12}$$

Let us reduce Δx down to zero (Fig. 40). Secant (12) correspondingly rotates around the point (x_0, y_0) and, when Δx becomes zero, takes the position described by the equation

$$y_{\tan} = 2x_0 (x - x_0) + y_0 \tag{*}$$

This straight line, resulting from secant rotation, *is the sought-for tangent to the parabola* $y = x^2$ *at the point* (x_0, y_0).

Let us prove this statement. The equation of the curve $y = x^2$ can be rephrased in the form

$$y = y_0 + 2x_0 (x - x_0) + (x - x_0)^2 =$$
$$= y_0 + [2x_0 + (x - x_0)] (x - x_0)$$

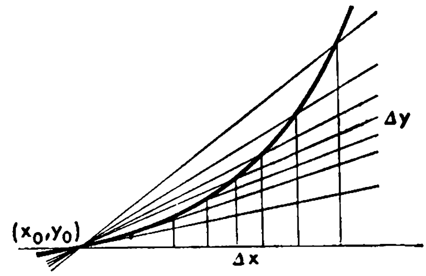

Fig. 40

thus it is apparent that with small deviations of x from x_0 the curve passes within the angle however small, formed by the straight lines

$$y = y_0 + (2x_0 \pm \varepsilon)(x - x_0)$$

which is what will be the case if we assume that $-\varepsilon < x - x_0 < \varepsilon$. Since the straight line (*) lies within this angle for any ε, then according to the above stated definition it is the sought-for tangent.

The slope of the tangent proved to be equal to $2x_0$; thus, the derivative of the function $y = x^2$ is

$$k(x) = 2x$$

Let us see what we obtain by this method for a function $y = P(x)$, where $P(x)$ is the n-th degree polynomial

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n \qquad (13)$$

We again connect by the secant the point (x_0, y_0), at which we have to draw the tangent, with a closely located point $(x_0 + \Delta x,\ y_0 + \Delta y)$ of the curve. We obtain

$$y_0 + \Delta y = P(x_0 + \Delta x) =$$
$$= x_0 + a_1(x_0 + \Delta x) + \ldots + a_n(x_0 + \Delta x)^n \qquad (14)$$

Let us denote by ε_1 any sum of terms containing Δx to the first and higher power, by ε_2 any sum of terms containing Δx to the second and higher power.

Since

$$y_0 = a_0 + a_1 x_0 + \ldots + a_n x_0^n$$

we obtain by removing the brackets in Eq. (14) by the Newton formula* and by subtracting Eq. (13) from Eq. (14)

$$\Delta y = (a_1 + 2a_2x_0 + \ldots + na_nx_0^{n-1})\,\Delta x + \varepsilon_2 \tag{15}$$

Further on, the slope of the secant is obtained by dividing Δy by Δx. Since $\varepsilon_2 : \Delta x = \varepsilon_1$, its expression has the form

$$\frac{\Delta y}{\Delta x} = a_1 + 2a_2x_0 + \ldots + na_nx_0^{n-1} + \varepsilon_1$$

The complete equation of the secant is as follows:

$$y = (a_1 + 2a_2x_0 + \ldots + na_nx_0^{n-1} + \varepsilon_1)(x - x_0) + y_0$$

When we assume that $\Delta x = 0$, ε_1 becomes zero and we obtain the equation of the tangent

$$y_{\tan} = (a_1 + 2a_2x_0 + \ldots + na_nx_0^{n-1})(x - x_0) + y_0$$

Consequently the expression for the slope of the tangent line is

$$k = a_1 + 2a_2x_0 + \ldots + na_nx_0^{n-1} \tag{16}$$

When x_0 is fixed, k is a number; if x_0 is varied, this number will also vary and we shall obtain the function giving the values of corresponding slopes of tangents to the curve $y = P(x)$ at its various points. As we already said, this function is the derivative of $P(x)$; it is denoted as $P'(x)$.

The obtained formula can be written in the form

$$P'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1} \tag{16'}$$

The rule of forming $P'(x)$ out of $P(x)$ is quite simple: in the sum (13) each x^k is replaced by kx^{k-1}.

In particular, the derivative of a constant (i.e. of a function which for all values of x takes on the same value $y = a_0$) is equal to 0. In this case, however, this is also obvious geometrically: the tangent to the plot of the function $y = a_0$ is horizontal at each point!

Returning to the general case, we shall emphasize the equality stemming from Eq. (15):

$$P(x + \Delta x) = P(x) + P'(x)\,\Delta x + \varepsilon_2 \tag{17}$$

---

* The Newton formula: for any k and any u and v

$$(u+v)^k = u^k + \frac{k}{1}u^{k-1}v + \frac{k(k-1)}{1\cdot 2}u^{k-2}v^2 + $$
$$+ \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}u^{k-3}v^3 + \ldots + \frac{k(k-1)}{1\cdot 2}u^2v^{k-2} + \frac{k}{1}uv^{k-1} + v^k.$$

Let us consider a similar problem about a tangent for the general rational function

$$y=\frac{P(x)}{Q(x)} \tag{18}$$

where $P(x)$ and $Q(x)$ are polynomials.

Using Eq. (17) we obtain

$$y_0+\Delta y=\frac{P(x_0+\Delta x)}{Q(x_0+\Delta x)}=\frac{P(x_0)+P'(x_0)\,\Delta x+\varepsilon_2}{Q(x_0)+Q'(x_0)\,\Delta x+\varepsilon_2} \tag{19}$$

Subtracting Eq. (18) from Eq. (19), we obtain

$$\Delta y=\frac{P(x_0)+P'(x_0)\,\Delta x+\varepsilon_2}{Q(x_0)+Q'(x_0)\,\Delta x+\varepsilon_2}-\frac{P(x_0)}{Q(x_0)}=$$

$$=\frac{[P'(x_0)\,Q(x_0)-Q'(x_0)\,P(x_0)]\,\Delta x+\varepsilon_2}{Q(x_0)\,[Q(x_0)+Q'(x_0)\,\Delta x+\varepsilon_2]} \tag{20}$$

Hence the slope of the secant is

$$\frac{\Delta y}{\Delta x}=\frac{P'(x_0)\,Q(x_0)-Q'(x_0)\,P(x_0)+\varepsilon_1}{Q^2(x_0)\,\varepsilon_1} \tag{21}$$

The complete equation of the secant takes the form

$$y=\frac{P'(x_0)\,Q(x_0)-Q'(x_0)\,P(x_0)+\varepsilon_1}{Q^2(x_0)\,\varepsilon_1}(x-x_0)+y_0$$

Let us assume that $Q(x_0)\neq 0$ and $\Delta x=0$, we obtain the equation of the tangent

$$y_{\tan}=\frac{P'(x_0)\,Q(x_0)-Q'(x_0)\,P(x_0)}{Q^2(x_0)}(x-x_0)+y_0$$

The slope of the tangent when $x=x_0$ is thus equal to

$$y'(x_0)=\frac{P'(x_0)\,Q(x_0)-Q'(x_0)\,P(x_0)}{Q^2(x_0)}$$

This formula gives the rule for calculating the derivative of the quotient $\frac{P(x)}{Q(x)}$:

$$\left(\frac{P}{Q}\right)'=\frac{P'Q-Q'P}{Q^2} \tag{22}$$

Let us consider several examples of applying all the obtained formulas.

1. What two positive numbers, whose sum is equal to c, yield the maximum product?

This problem has an elementary algebraic solution. Not resorting to this solution, we shall use our general method.

If one of these numbers is x and the other is $c - x$, we must find the maximum of the function

$$P(x) = x(c - x) = -x^2 + cx$$

We have

$$P'(x) = -2x + c$$

and, setting the derivative equal to zero, we find the solution

$$x = c/2, \quad c - x = c/2$$

The next problem, though similar to the first in its form, has no elementary solution.

2. What two positive numbers, whose sum is equal to c, have the following property: the cube of the first number times the square of the second yields the maximum possible value?

Here we must find the maximum of the function

$$P(x) = x^3(c - x)^2 = x^5 - 2cx^4 + c^2x^3$$

We know that the derivative of the function must be zero at the point of the maximum.

We calculate the derivative

$$P'(x) = 5x^4 - 8cx^3 + 3c^2x^2$$

Setting it equal to zero, we have an obvious solution $x = 0$. Seeking solutions for $x \neq 0$, we have to solve the quadratic equation

$$5x^2 - 8cx + 3c^2 = 0$$

Its solution is

$$x_{1,2} = \frac{4c \pm \sqrt{16c - 15c}}{5} = \frac{4c \pm c}{5}$$

Thus the tangent to the graph of the function $y = P(x)$ is horizontal at the points $x_1 = 0$, $x_2 = \frac{3}{5}c$ and $x_3 = c$. The values of x_1 and x_3 yield zero for the value of $P(x)$ while x_2 yields a positive value $P(x_2) = (3/5)^3(2/5)^2c^5$. Hence, the numbers sought for are $x_2 = 3c/5$, $c - x_2 = 2c/5$.

3. At what angles does the curve $y = P(x) = x(x - 1) \times \times (x - 2)(x - 3)$ (see Fig. 21) intersect the x-axis? Obviously, we shall mean by the angle between the axis and the curve the angle between the axis and the tangent to the curve (at the point of intersection of the curve with the axis).

Solution. Removing the brackets we obtain

$$y = x^4 - 6x^3 + 11x^2 - 6x$$

whence according to Eq. (16′)

$$y' = 4x^3 - 18x^2 + 22x - 6$$

The curve $y = P(x)$ intersects the x-axis at the points 0, 1, 2 and 3. Successive substitution in the above formula yields

$$y'(0) = -6, \; y'(1) = 2, \; y'(2) = -2, \; y'(3) = 6$$

These numbers are slopes of tangent lines at the points indicated, i.e. the tangents of the angles in question.

4. At what points will the tangent line to the curve of Problem 3 be horizontal?

The slope of the tangent line at the points where it is horizontal is zero. Setting the expression for the derivative equal to zero, we obtain the equation

$$4x^3 - 18x^2 + 22x - 6 = 0$$

This equation has an obvious solution $x_1 = 3/2$ (from the drawing, on the basis of symmetry arguments). Taking out the factor $x - 3/2$ we obtain

$$4x^3 - 18x^2 + 22x - 6 = 4(x - 3/2)(x^2 - 3x + 1)$$

Now it remains to solve the quadratic equation $x^2 - 3x + 1 = 0$. The solution yields

$$x_{2,3} = \frac{3 \pm \sqrt{5}}{2} = 1.5 \pm 1.118 \ldots$$

The corresponding ordinates can easily be calculated:

$$y_1 = \frac{3}{2} \cdot \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{9}{16}$$

$$y_{2,3} = \frac{3 \pm \sqrt{5}}{2} \cdot \frac{1 \pm \sqrt{5}}{2} \cdot \frac{-1 \pm \sqrt{5}}{2} \cdot \frac{-3 \pm \sqrt{5}}{2} =$$

$$= \frac{1}{16}(5 - 9)(5 - 1) = -1$$

5. We know that velocity of a motor ship is expressed as function of the cost of fuel consumed per hour, p roubles, by the formula

$$v = c\frac{p}{p+1} \qquad (23)$$

This formula is illustrated by the graph (Fig. 41). The graph agrees with a natural assumption that at the beginning, i.e. at comparatively low expenditures for fuel, the velocity of the motor ship increases in proportion to the increase in fuel consumption, then the rate of velocity increase slows down, and no rise in fuel supply can force the ship to move faster than certain limiting velocity c.

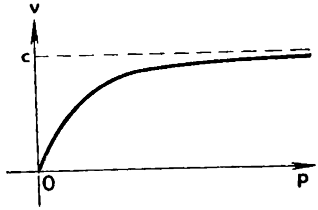

Fig. 41

Aside from expenditures for fuel there are permanent expenditures that take q roubles per hour. What should the velocity of steaming from the port A to the port B, located at a distance of s km from A, be to realize the minimum cost of the cruise?

Solution. Let v be the velocity and $T = s/v$ the time of travelling. Cost of fuel is calculated as follows: cost per hour is obtained by inverting formula (23)

$$p = \frac{v}{c-v} \tag{24}$$

and total cost P is then found by multiplying Eq. (24) by the time $T = s/v$; $P = \frac{s}{c-v}$. Permanent expenditures Q come to $qT = q\frac{s}{v}$. The total expenditure R is equal to

$$R = P + Q = \frac{s}{c-v} + q\frac{s}{v} = s\left(\frac{1}{c-v} + \frac{q}{v}\right) \tag{25}$$

The graph of this function is of the form shown in Fig. 42. To find the velocity v corresponding to the minimum total cost, we set the derivative of R over v equal to zero

$$R'(v) = s\left(\frac{1}{(c-v)^2} - \frac{q}{v^2}\right) = 0$$

Whence

$$v^2 - q(c-v)^2 = 0, \quad v^2 = q(c-v)^2$$

$$v = \sqrt{q}(c-v) = \sqrt{q}c - \sqrt{q}v^*$$

$$v = \frac{\sqrt{q}}{1+\sqrt{q}} c \text{ km/h} \tag{26}$$

* One solution, corresponding to the equation

$$v = -\sqrt{q}(c-v)$$

is rejected as meaningless, since the right-hand side is negative ($v < c$).

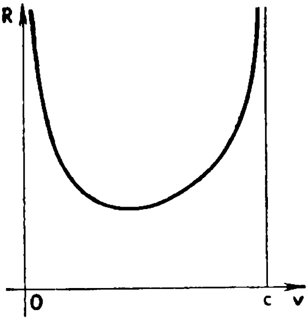

Fig. 42

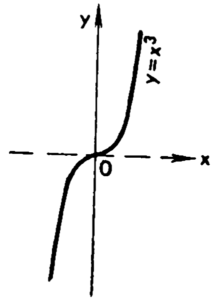

Fig. 43

Substituting Eq. (26) into Eq. (25), we find also the total expenditures for the most economical cruise:

$$R = s\,(1 + \sqrt{q})\,\left(\frac{1}{c} + \sqrt{q}\right)$$

6. What is the tangent to the curve $y = x^3$ for $x = 0$?

We have $y' = 3x^2$, which for $x = 0$ obviously yields $y' = 0$, so that the x-axis is the tangent (Fig. 43).

We see that in this case the tangent passes from one side of the curve onto the opposite side: when $x > 0$ the curve is above the tangent while for $x < 0$ it goes under the tangent line. The points of the graph at which the tangent line passes from one side of the curve to the other are called the *points of inflection* (Fig. 44). Thus, the same value $x = 0$ determines the point of inflection for the family of curves

$$y = Cx^3 + x$$

for various values of C (Fig. 45).

Indeed, we have $y'(0) = 1$, so that the equation of the tangent line drawn through the point (0, 0) takes the form

$$y_{\tan} = x$$

therefore the difference

$$y - y_{\tan} = Cx^3$$

alters sign when we pass from negative to positive values of x.

Let us state the problem of finding the inflection point of the graph of a given function $y = f(x)$.

As can be seen from Fig. 45, if the curve passes from the position "below the tangent" into that "above the tangent" as x increases, then its slope $y'(x)$ on both sides of the point of tangency is greater than exactly at this point:

$$y'(x) > y'(x_0)$$
$$x \neq x_0$$

Similarly, if the curve passes from the position "above the tangent" into the "below-tangent" position as x increases, then $y'(x)$ on both sides of the tangency point is less than exactly at this point:

$$y'(x) < y'(x_0)$$
$$x \neq x_0$$

In the first case the inflection point is that of a local minimum of the function $y'(x)$, and in the second case—that of a local maximum of this function. To find these points we must first calculate the derivative of the function $y'(x)$. The function $(y'(x))'$ is called the *second derivative* of the function $y(x)$ and is denoted as $y''(x)$. Our next step is to find the solution of the equation

$$y''(x) = 0$$

Abscissas of all inflection points sought for are contained among the solutions to this equation. ("Superfluous" solu-

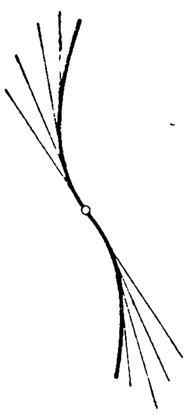

Fig. 44

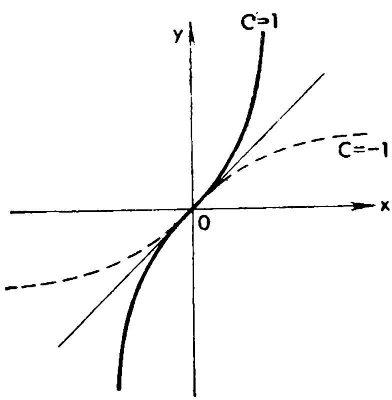

Fig. 45

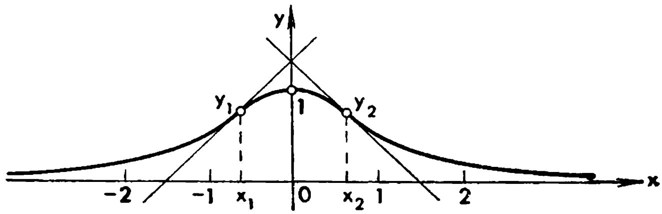

Fig. 46

tions, not determining positions of inflection points, may also appear; of course, such solutions must be rejected*.)

7. Let us find the inflection points of the curve $y = \frac{1}{1+x^2}$ (Fig. 46). The figure prompts us to the conclusion that there must be two such points, situated symmetrically relative to the ordinate axis. We shall find them using the foregoing rules.

We have

$$y' = -\frac{2x}{(1+x^2)^2}$$

$$y'' = (y')' = -\frac{(1+x^2)^2\,2 - 2x\,(4x+4x^3)}{(1+x^2)^4} = \frac{6x^2-2}{(1+x^2)^3}$$

Setting the equation for y'' to zero, we obtain two solutions:

$$x_{1,2} = \pm\sqrt{1/3} = \pm 0.577\ \ldots$$

We see that the differential calculus makes it possible to solve by means of a unified general method a wide variety of problems not solvable within the framework of the elementary mathematics. This is what makes the differential calculus a powerful mathematical tool.

Problems

15. What is the height-to-base diameter ratio for a (cylindrical) can of a given volume, which requires minimum of metal for its production?

---

* For example, we have for the function $y = x^4$:

$$y' = 4x^3,\quad y'' = 12x^2$$

when $x = 0$ we obtain

$$y''(x) = 0$$

however, this point is not the inflection point of the curve $y = x^4$.

16. Square pieces are cut out of a square sheet of iron at its corners and the sheet is then bent along the dashed lines, so that the sheet is converted into a box opened from above (Fig. 47). At what size of the removed square pieces will the volume of the box be maximal?

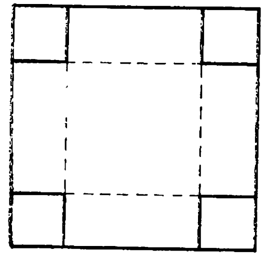

Fig. 47

17. The formula for the derivative of the function $y = \sqrt{f(x)}$ is

$$y' = \frac{f'(x)}{2\sqrt{f(x)}}$$

Having assumed this formula to be valid, solve the following problem. A house is at a distance of h km from a (straight) road (Fig. 48). A person has to walk to the road and then drive to the town which is at a distance (measured along the straight line) of s km from the house. The speed of the pedestrian is u and the speed of the car is v. Find the shortest route.

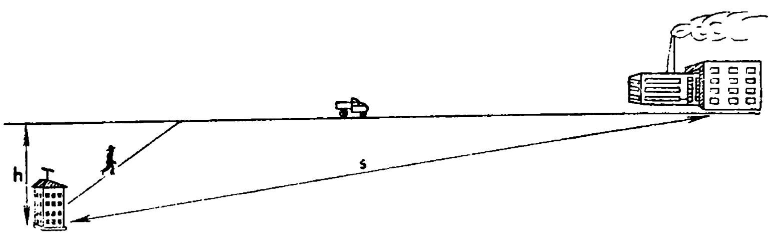

Fig. 48

18. How large should a sector cut out of a circle with the radius R be so that the funnel rolled out of the remainder would have maximum holding capacity?

Numerous problems involving derivatives and their application to various fields of mathematics, physics, etc. can be found in popular books of problems.

3. Integrals

How should we define the area of a plane figure bounded, in the general case, by a non-rectilinear contour? Let us start with the following premises:

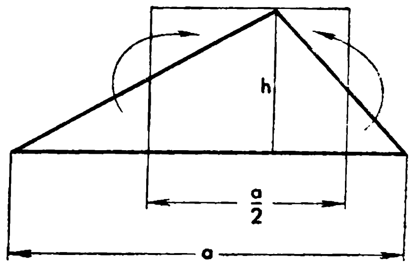

Fig. 49

1. The area of a rectangle with sides a and b is equal to ab.

2. Areas of *equal* (i.e. coinciding when superposed) figures are equal.

3. If the figure Φ is composed of figures $\Phi_1, \ldots, \Phi_n$, then its area is equal to the sum of areas of figures $\Phi_1, \ldots, \Phi_n$.

Hence it follows that the areas of homogeneous figures are equal. Furthermore, a triangle with the base a and height h can be so cut that a rectangle can be constructed with sides $a/2$ and h (Fig. 49); whence the area of the triangle is $ha/2$. Finally, the area of any polygon can be calculated as the sum of areas of triangles composing this polygon (Fig. 50).

4. The area of the figure Φ is smaller than that of any polygon enclosing it, and greater than that of any polygon enclosed by it (Fig. 51).

There is a theorem according to which the *one-to-one correspondence can be set between each plane figure* Φ, *bounded by a not too complicated contour, and a number* $S(\Phi)$, *referred*

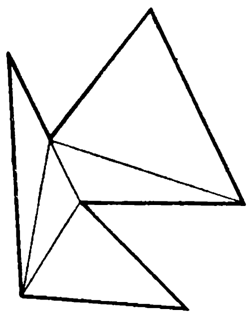

Fig. 50

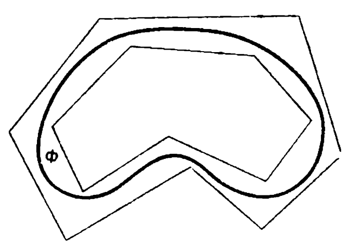

Fig. 51

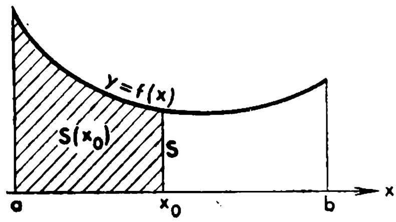

Fig. 52

to as the area of the figure Φ, *so that the conditions* 1-4 *are satisfied*. (The precise statement and the proof of this theorem are too complicated to be presented here; exhaustive discussion of these problems can be found in Lebesque's monograph "On measurement of quantities" ("Sur la mesure des grandeurs", Geneve, Paris, 1956).

In any case, in the indicated sense the area exists for any figure whose contour is composed of finite arcs of graphs of rational functions. Assuming this theorem, we shall discuss the problem of calculating the area of the figure Φ (curvilinear trapezoid) bounded below by a segment of the x-axis from $x = a$ to $x = b$, above by the curve $y = f(x)$ (as usual, $f(x)$ is a rational function), and at the sides by straight lines parallel to the y-axis and passing through the points $x = a$, $x = b$ (Fig. 52).

Let us take a number x_0 located within the range from a to b, and state a similar problem as regards the determination of the area of the figure $\Phi(x_0)$ which differs from the figure Φ in that its right-hand side is formed by a vertical line passing not through the point b, but through x_0. The value of this area depends on the position of the point x_0, i.e. is the function of the argument x_0 and defined on the whole segment $a \leqslant x_0 \leqslant b$, we shall denote this function by $S(x_0)$. It is obvious that $S(a) = 0$ and that $S(b)$ is the sought-for area of the figure Φ. We can approximately trace the graph of this function; in the given case its shape is similar to that shown in Fig. 53.

If the right-hand abscissa x_0 of the figure $\Phi(x_0)$ is increased by $\Delta x_0 = x_1 - x_0$, the area gains an increment $ABDEC = \Delta S$ (Fig. 54) which is formed by the area of the rectangle $ABDC = y\Delta x_0$ and by the area of the curvilinear

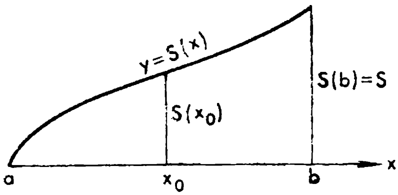

Fig. 53

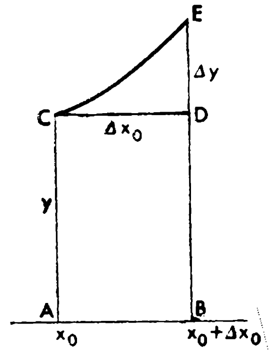

Fig. 54

triangle CDE. The latter area does not exceed $\Delta y \Delta x_0$;* and consequently, in accordance with the foregoing notation it can be denoted by $\varepsilon_1 \Delta x_0$.

Thus

$$S(x_1) - S(x_0) = (y + \varepsilon_1)\,\Delta x_0$$

The equation of a secant of the graph of the curve $y = S(x)$, passing through the points $(x_0, S(x_0))$ and $(x_1, S(x_1))$ is

$$y - S(x_0) = \frac{S(x_1) - S(x_0)}{x_1 - x_0}(x - x_0) = (y(x_0) + \varepsilon_1)(x - x_0)$$

Assuming here, as before, that $x_1 = x_0$, we obtain the equation of the tangent at the point (x_0, y_0):

$$y - S(x_0) = y(x_0)(x - x_0)$$

The slope of the tangent is thus equal to $y(x_0)$. But the slope of the tangent to the graph of the function $y = S(x)$ at the point with the abscissa x_0 is, as we know, the derivative of the function $S(x)$ for $x = x_0$. Thus, we obtain the equality

$$S'(x_0) = y(x_0)$$

Therefore, *to find the function $S(x)$ we have to find the function whose derivative is $y(x)$*, i.e. to perform an operation

* We assume that the function y increases (or decreases) in the range from x_0 to x_1; for the rational function y this interval can always be chosen sufficiently small for this condition to be satisfied.

reciprocal to differentiation and called *integration.* Any function $F(x)$ whose derivative is $y(x)$ is called the *antiderivative of* $y(x)$, or the *integral of* $y(x)$. Let us note that if we have determined one of the antiderivatives $F(x)$ of the function $y(x)$ then any other function of the type $F(x) + C$ (where C is constant) is suitable as an antiderivative of the same function $y(x)$ since the *derivative of a constant is zero.* As we have mentioned, the sought-for function $S(x)$ drops to zero when $x = a$. Therefore, having determined an antiderivative $F(x)$, we can write for the unknown constant C the equation

$$S(a) = F(a) + C = 0, \text{ whence } C = -F(a)$$

Finally, having found the antiderivative function $F(x)$, we shall write the sought-for equation in the form

$$S(x) = F(x) - F(a)$$

and, in particular,

$$S(b) = F(b) - F(a) \qquad (27)$$

which gives the solution of our problem or, more precisely, the solution is reduced to the problem of determining the antiderivative of a given function $f(x)$.

To make possible the application of the general result (27) obtained above we must be able to determine the antiderivative. If the function $y = y(x)$ is a polynomial

$$y = a_0 + a_1 x + \ldots + a_n x^n$$

then one of the antiderivatives can easily be written, viz.:

$$F(x) = a_0 x + a_1 \frac{x^2}{2} + \ldots + a_n \frac{x^{n+1}}{n+1} \qquad (28)$$

Therefore, no substantial difficulties are encountered in calculations of areas of figures bounded (above) by curves of the type $y = P(x)$, where $P(x)$ is a polynomial.

Let, for example, $y = y(x)$ be a linear function (Fig. 55) varying on the segment $a \leqslant x \leqslant b$ from the value p to the value q:

$$y = p + \frac{q-p}{b-a}(x-a)$$

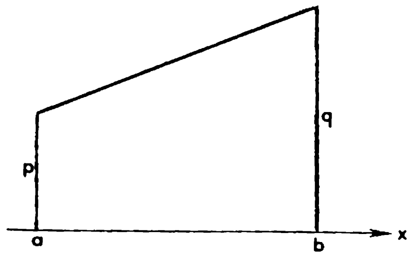

Fig. 55

According to Eq. (28), one of its possible antiderivatives is of the form

$$F(x) = px + \frac{(x-a)^2}{2} \cdot \frac{q-p}{b-a}$$

According to Eq. (27) we have

$$S(b) = F(b) - F(a) = pb + \frac{(b-a)(q-p)}{2} - pa =$$

$$= (b-a)\left(p + \frac{q-p}{2}\right) = (b-a)\left(\frac{p+q}{2}\right)$$

which is identical to the formula for the area of a trapezoid in elementary geometry. In particular, for areas of a rectangle and triangle (particular cases of a trapezoid) we also obtain the formulas of elementary geometry.

However, the method of integration enables us to calculate areas of many non-elementary figures. Let us consider some additional examles.

1. Let us calculate the area OAB (Fig. 56) of the curvilinear triangle bounded below by the segment $0 \leqslant x \leqslant a$ of the x-axis, on the right by the ordinate $x = a$ and above by the curve $y = Cx^n$. The antiderivative of the function $y = x^n$ is

$$F(x) = x^{n+1}/(n+1)$$

therefore according to Eq. (27):

$$S(a) = C\frac{a^{n+1}}{n+1} - C\frac{0^{n+1}}{n+1} = C\frac{a^{n+1}}{n+1} = \frac{a \cdot Ca^n}{n+1}$$

The number Ca^n is the length of the segment AB. Therefore the area of interest, $S(a)$, is $1/(n+1)$-th fraction of the circumscribed rectangle $OABC$.

2. Let us calculate the areas S_1, S_2 and S_3 (Fig. 57) bounded by the curve $y = x(x-1)(x-2)(x-3)$ and by the segments of the x-axis from 0 to 1, from 1 to 2, and from 2 to 3.

Here we have

$$y = P(x) = x^4 - 6x^3 + 11x^2 - 6x$$

and its antiderivative is

$$F(x) = \frac{x^5}{5} - \frac{6x^4}{4} + \frac{11x^3}{3} - \frac{6x^2}{2}$$

Whence

$$S_1 = F(1) - F(0) = \frac{1}{5} - \frac{3}{2} + \frac{11}{3} - 3 = -\frac{19}{30}$$

The minus sign indicates that the area S_1 lies below the x-axis. Then

$$S_2 = F(2) - F(1) = \frac{32}{5} - 24 + \frac{88}{3} - 12 + \frac{19}{30} = \frac{11}{30}$$

and, finally,

$$S_3 = F(3) - F(2) = \frac{243}{5} - \frac{243}{2} + 99 - 27 + \frac{4}{15} = -\frac{19}{30}$$

The last result coincides with S_1; this could be predicted on the basis of symmetry.

3. Let us calculate the area S (Fig. 58) under the curve $y = 1/x^2$ between the lines $x = 1$ and $x = N$, where N is a large number. The antiderivative of $y(x) = 1/x^2$ is,

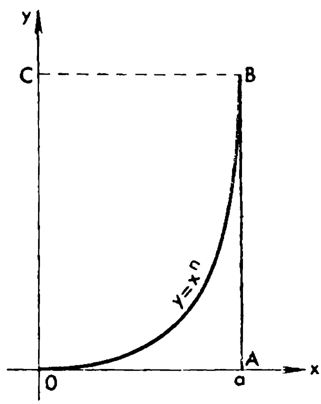

Fig. 56

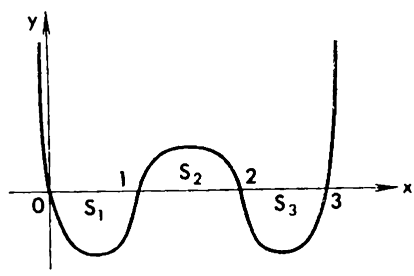

Fig. 57

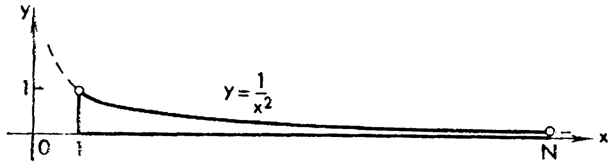

Fig. 58

obviously, $F(x) = -(1/x)$, and we obtain

$$S = F(N) - F(1) = 1 - 1/N \tag{29}$$

It is interesting that at the N however large, this area is smaller than 1. Formula (29) indicates that it is logical to assign to the whole infinitely stretched figure, bounded below by the x-axis, above by the curve $y = 1/x^2$, on the left by the segment of the line $x = 1$ and not bounded on the right-hand side, a finite area, namely the area equal to 1.

We see that calculus of integrals based on the calculus of derivatives enables us to solve in a unified manner a number of problems on areas, which cannot be solved by means of elementary mathematics. However, the problem of area calculation is a comparatively particular problem, only one realization of the general problem of finding the antiderivative function from its derivative. But many problems in mathematics, mechanics, physics, chemistry, biology are reduced to this general problem; the integral calculus makes it possible to solve by means of a general method a great number of problems with most varied specific conditions but with common mathematical essence (for instance, calculation of energy required for launching a satellite; finding the law that governs radioactive decay; quantitative analysis of the course of a chemical reaction or of proliferation of bacteria). Not being also able to describe here all these attractive applications we advise the reader to pay attention to a monograph by G. Phillips "Differential equations", containing a wide variety of problems concerning diverse fields of science and technology and requiring application of the integral calculus.

4. More careful approach will be necessary in the example to follow. The subject is the area S under the same curve $y = 1/x^2$ between the lines $x = a$ and $x = b > a$ (Fig. 59).

Using the same techniques we obtain

$$S = F(b) - F(a) = \frac{1}{a} - \frac{1}{b}$$

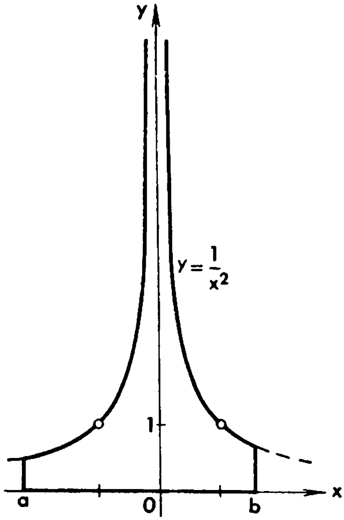

Fig. 59

If a and b are of the same sign, the result is positive; this is quite natural since the curve $y = 1/x^2$ is situated above the x-axis. If, however, a and b have opposite signs, $a < 0$ and $b > 0$, the result unexpectedly turns out negative which contradicts the geometric pattern. The reason is that our application of rule (27) to the antiderivative has been only formal. In reality rule (27) has definite limits as to its application beyond which it is invalidated; unfortunately, here we cannot even indicate precisely these limits since the necessary statements would require the concepts that are not introduced yet.

Besides, more attentive analysis reveals "white spots" in integration of rational functions. Indeed, antiderivatives of certain rational functions can be obtained from the equality

$$\left(\frac{1}{(x-a)^m}\right) = -\frac{m}{(x-a)^{m+1}}$$

stemming from general formula (22). Namely, for the expression

$$y(x) = \frac{b_1}{(x-a_1)^2} + \frac{b_2}{(x-a_2)^3} + \ldots + \frac{b_m}{(x-a_m)^{m+1}} \qquad (30)$$

the antiderivative can be given by the formula

$$F(x) = -\frac{b_1}{x-a_1} - \frac{b_2}{2(x-a_2)^2} - \ldots - \frac{b_m}{m(x-a_m)^m}$$

However, it is not every rational function that can be reduced to form (30). For instance, the antiderivative of the function $1/x$ cannot be obtained in this way. Actually the function $1/x$ has its antiderivative but not only is it nonrational, it even does not belong to a class of elementary functions which are studied in high school. While in the case of differentiation we do not leave the class of rational functions if we start within the class, the reverse operation, viz. integration, necessarily introduces new types of functions. And analysis of these functions calls for quite different general approaches and for absolutely different level of analytical techniques than those we applied throughout this short review.

Therefore, to master properly the techniques of the differential and integral calculus one has to study the preliminary sections of calculus treating real numbers, limits and continuity. These sections contain the necessary fundamentals enabling one to deal with a very wide class of functions that goes far beyond the class of rational functions discussed in this book.

There are many excellent textbooks in which the fundamentals of the calculus are presented.* We hope that beginners among our readers who after reading this pamphlet will be interested in the possibilities of the differential and integral calculus will find an opportunity for a more thorough acquaintance with mathematics which is so useful to other fields of science, and through them to all humanity.

Problems

19. Calculate the area bounded above by the curve $y = x^2 + 1/x^2$, below by the x-axis, and on the right and left by vertical lines intersecting the x-axis at the points $a = 1/2$ and $b = 2$, respectively (Fig. 60).

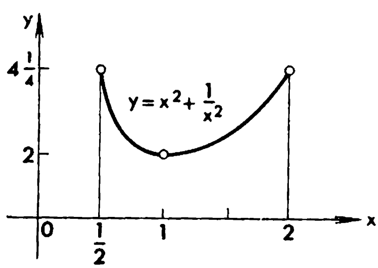

Fig. 60

* For instance: V. I. Smirnov "A Course in advanced mathematics", v. 1; G.M. Fikhtengolts "Fundamentals of calculus", v. 1; A. Ya. Khinchin "Concise course of calculus"; R. Kurant "A Course in differential and integral calculus", v. 1,

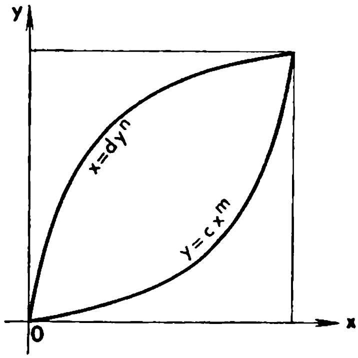

Fig. 61

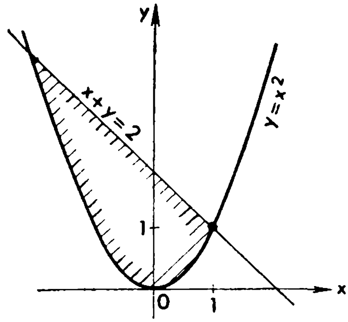

Fig. 62

20. Calculate the area between two curves (Fig. 61):

$$y = cx^m, \quad x = dy^n$$

Recommendation. Use Example 1 of Sec. 3.

21. Calculate the area bounded above by the line $x + y = 2$ and below by the parabola $y = x^2$ (Fig. 62).

Recommendation. Express the area in question as the difference between two areas bounded on both sides by vertical lines.

22. The velocity gained by a falling body in the time t elapsed after falling began is equal to gt ($g = 9.81$ m/s^2). What is the distance covered by the body during this time?

Recommendation. Velocity is the derivative of the displacement with respect to time.

Solutions to Problems

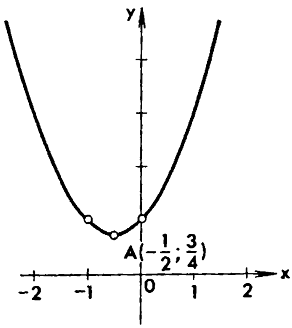

Fig. 63. To problem 1

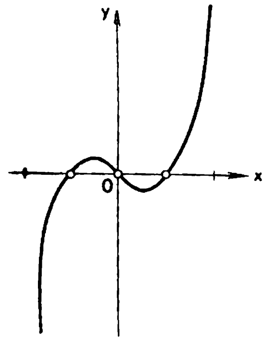

Fig. 64. To problem 2

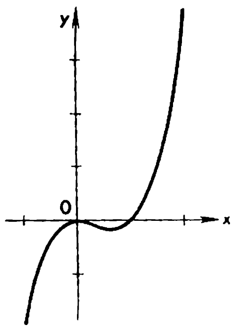

Fig. 65. To problem 3

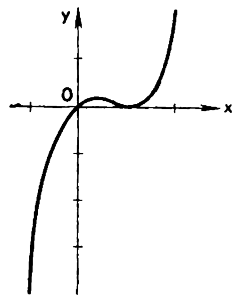

Fig. 66. To problem 4

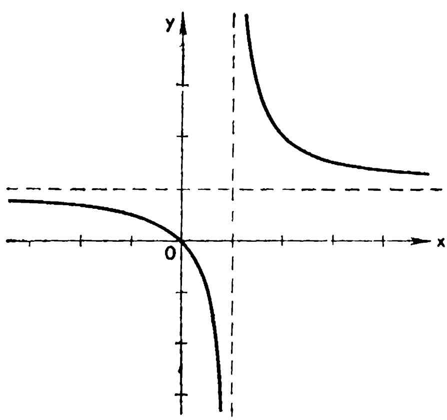

Fig. 67. To problem 5

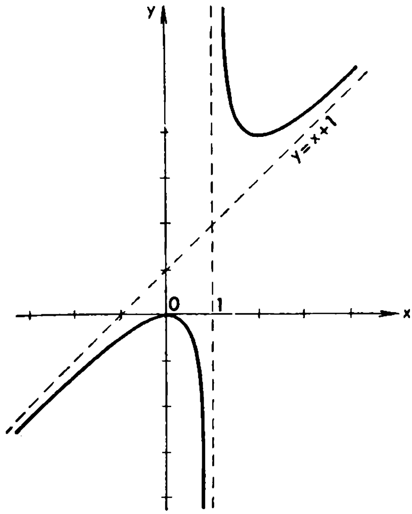

Fig. 68. To problem 6

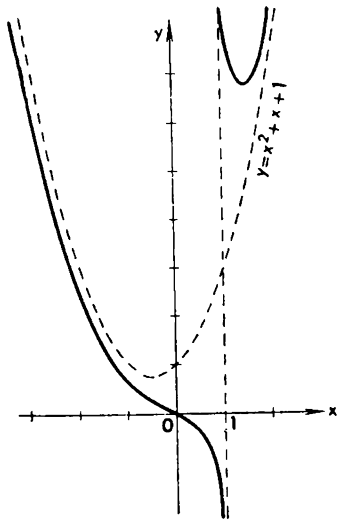

Fig. 69. To problem 7

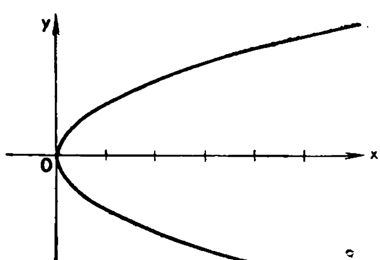

Fig. 70. To problem 8

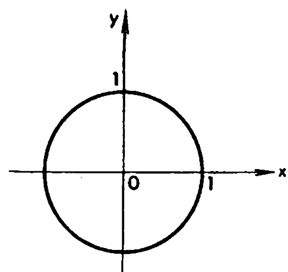

Fig. 71. To problem 9

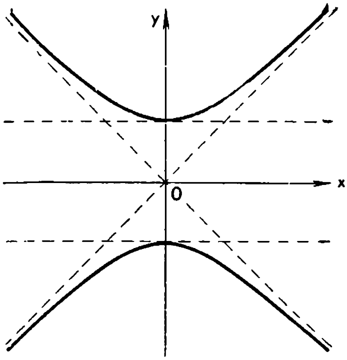

Fig. 72. To problem 10

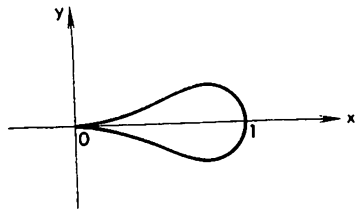

Fig. 73. To problem 11

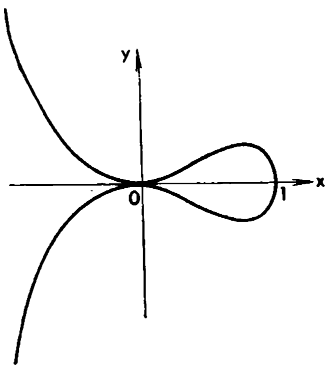

Fig. 74. To problem 12

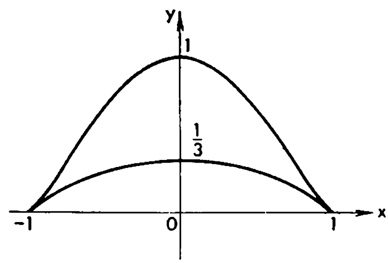

Fig. 75. To problem 13

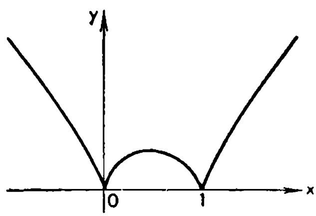

Fig. 76. To problem 14

15. The height of the can must be equal to its diameter.

16. The sides of the removed squares make up 1/6 of the side of the whole square.

17. The sine of the angle between the path of the pedestrian and the normal to the road must be equal to u/v, provided this ratio does not exceed

$$\sqrt{1-\left(\frac{h}{s}\right)^2}$$

Otherwise the shortest route is the path covered on foot along the straight line towards the town.

18. $\alpha = 2\pi\sqrt{\frac{2}{3}}$ radian $\cong 93°$.

19. $S = 33/8$.

20. $S = c^{\frac{n+1}{1-nm}} d^{\frac{m+1}{1-nm}} \left(1 - \frac{1}{n+1} - \frac{1}{m+1}\right)$.

21. $S = 9/2$.

22. $s = (gt)^2/2$.

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is:
Mir Publishers
2 Pervy Rizhsky Pereulok
I-110, GSP, Moscow, 129820
USSR

Printed in the Union of Soviet Socialist Republics

OTHER BOOKS FOR YOUR LIBRARY

THE KINEMATIC METHOD IN GEOMETRICAL PROBLEMS
by YU.I. LYUBICH, L.A. SHOR

When solving a geometrical problem it is helpful to imagine what would happen to the elements of the figure under consideration if some of its points started moving. The relationships between various geometrical objects may then become clear graphically and the solution of the problem may become obvious.
The relationships between the magnitudes of segments, angles and so on in geometrical figures are usually more complicated than the relationships between their rates of change when the figure is deformed. Therefore, in solving geometrical problems one may benefit from a "theory of velocities", i.e. from kinematics.
This little book uses a number of examples to show how kinematics can be applied to problems of elementary geometry, and gives some problems for independent solution. The necessary background information from kinematics and vector algebra is given as a preliminary.
The book is based on lectures given by the authors for school mathematics clubs at the Kharkov State University named after A.M. Gorky. It is intended for high school students.

SOLVING EQUATIONS IN INTEGERS
by A.O. GELFOND

This book is devoted to one of the most interesting branches of number theory, the solution of equations in integers. The solution in integers of algebraic equations in more than one unknown with integral coefficients is a most difficult problem in the theory of numbers. The theoretical importance of equations with integral coefficients is quite great as they are closely connected with many problems of number theory. Moreover, these equations are sometimes encountered in physics and so they are also important in practice. The elements of the theory of equations with integral coefficients as presented in this book are suitable for broadening the mathematical outlook of high-school students and students of pedagogical institutes. Some of the main results in the theory of the solution of equations in integers have been given and proofs of the theorems involved are supplied when they are sufficiently simple.

THE REMARKABLE CURVES
by A.I. MARKUSHEVICH

This small booklet has gained considerable popularity among mathematics enthusiasts. It is based on a lecture delivered by the author to Moscow schoolboys in their early teens, contains a description of a circle, ellipse, hyperbola, parabola, Archimedian spiral, and other curves. The book has been revised and enlarged several times in keeping with the demands from readers, but the characteristic style of the book, which is demonstrative and lucid rather than deductive and dry, has been retained.
The booklet is intended for those who are interested in mathematics and possess a middle standard background.

AREAS AND LOGARITHMS
by A.I. MARKUSHEVICH

The book offers a geometric theory of logarithms, in which (natural) logarithms are represented as areas of various geometrical shapes. All the properties of logarithms, as well as methods of their calculation, are then determined from the properties of the areas. The book also introduces the most simple concepts and properties of integral calculus, without resort to the concept of a derivative.
The book is intended for all mathematics enthusiasts, particularly schoolchildren.

Little Mathematics Library

G. E. SHILOV

CALCULUS OF RATIONAL FUNCTIONS

Mir Publishers · Moscow

The pamphlet "Calculus of Rational Functions" discusses graphs of functions and the differential and integral calculi as applied to the simplest class of functions, viz. the rational functions of one variable. It is intended for pupils of the senior forms and first-year students of colleges and universities.

Mir Publishers
Moscow

The pamphlet "Calculus of Rational Functons" discusses graphs of functions and the differential and integral calculi as applied to the simplest class of functions, viz. the rational functions of one variable. It is intended for pupils of senior forms and first year students of colleges and universities.

Mir Publishers
Moscow