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THE GEOMETRY

OF THE

COMPLEX DOMAIN

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BY

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PREFACE

The problem of representing imaginary elements in algebra and geometry has claimed the attention of mathematicians for centuries. Even the Greeks were dimly conscious of the necessity of finding some solution. Starting with the properties of conjugate diameters of an ellipse, they surmised that similar properties must hold in the case of the hyperbola, the only difficulty involved being that of statement. With the rise of algebra, the complex roots of real equations clamoured more and more insistently for recognition. So long as no formula was known for the solution of an equation of degree higher than the second, quadratic equations with imaginary roots might be dismissed as involving inherent contradictions; such a simple procedure could not be retained after the advance of the science had led to the solution of the cubic. That arch-rogue Cardan recognized that the classical formula which he stole from Tartaglia would involve imaginary numbers in the very case where the equation itself had three real roots.

The general philosophic difficulties inherent in trying to find a real meaning for the symbol $\sqrt{-1}$, general questions as to whether graphical symbols should be looked upon as representing numbers or quantities, and what might be the ultimate distinction between a quantity and a number, do not lie within the scope of the present work. Broadly speaking, the bulk of the volume, with the exception of the last chapter, is devoted to the consideration of two main problems.

(A) A system of objects called 'points' is so given that each is determined by the values of a fixed number of real parameters. If these parameters take on not real but complex values, they fail to correspond to points in the original domain. What sort of real objects may then be put into correspondence with them?

(B) In a system of points determined by a number of complex parameters, a sub-system is taken whose elements depend in specified fashion on a fixed number of real parameters; what are the geometrical properties of the sub-system?

As an example of (A) we may ask how to find a geometrical representation of the complex points of a line, a circle, or a plane. Question (B) leads to mathematical considerations of a very different order. We usually assume that whatever is true in the real domain is true in the complex one also;* the properties of the complex portion of a curve are inferred from those of its real trace. If we are asked for our grounds for this erroneous belief, we are inclined to reply 'Continuity' or 'Analytic continuation' or what not. But these vague generalities do not by any means exhaust the question. There are more things in Heaven and Earth than are dreamt of in our philosophy of reals. What, for instance, can be said about the totality of points in the plane such that the sum of the squares of the absolute values of their distances from two mutually perpendicular lines is equal to unity? This is a very numerous family of points indeed, depending on no less than three real parameters, so that it is not contained completely in any one curve, nor is any one curve contained completely therein; it is an absolutely different variety from any curve or system of curves in the plane.

The material dealing with problem (A) is largely historical in nature, and is of real importance in mathematical history. There are also in existence a number of scattered monographs dealing with one phase or another of problem (B). The present work is, apparently, the first attempt that has been made to coordinate this material and present a consistent account of

* Large works of a decidedly uncritical sort have been written to develop this thesis. For instance, Hatton, *The Imaginary in Geometry*, Cambridge University Press, 1919.

the whole subject. It is hoped that enough new results are included to save it from the reproach of being merely a work of compilation. Much more could have been written on certain phases of the subject, but, as the late Jules Tannery said in the preface to a work by one of his pupils, 'Un petit livre est rassurant.'

Every student of geometry in the complex domain will find that he is forced to refer continually to the work of two admirable contemporary geometers, Professor Corrado Segre of Turin, and Professor Eduard Study of Bonn. The names of both appear incessantly throughout this book; the author had the rare privilege to be the pupil of each of these masters. Geographical separation has cut him off from the one, the inexorable logic of history has impeded his communion with the other. But his sense of obligation has never wavered, and he begs to offer the present work as a small token of admiration and esteem.

Cambridge, U.S.A.
1921.

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CHAPTER I

THE REPRESENTATION OF THE BINARY DOMAIN

The first writer to make a serious attempt to give a geometrical interpretation of the complex roots of a real quadratic equation was John Wallis.* The learned Oxonian approached the problem as follows.† An indicated square root of a negative number is, on its face, an absurdity, since the square of any number, positive or negative, is itself positive.‡ But this contradiction is entirely on a par with the more familiar one where we speak of negative numbers, for what can be more absurd than to speak of a number that is less than nothing? Now it is well known that this latter difficulty or contradiction disappears entirely when we represent our positive and negative numbers by points on two opposite scales. Since, therefore, 'Quodque in rectis admitti solet lineis: pariter in planis superficiebus (eodem ratione) admitti debet', § a proper study of the geometry of the plane should solve all our difficulties. As an illustration, consider the following example. ||

Suppose that, in one place, 30 acres of land have been reclaimed from the sea, and that, in another, the sea has taken 20 acres from us. What has been our gain? Evidently 10 acres, i.e. 1600 square perches, the equivalent of a square

* The best historical account of the subject-matter of the present chapter is that of Ramorino, 'Gli elementi imaginarii nella geometria', Battaglini's *Giornale di matematica*, vols. xxxv and xxxvi, 1897 and 1898. See also Beaman, 'A Chapter in the History of Mathematics', *Proceedings American Association for the Advancement of Science*, vol. xlvi, 1897, and Matzka, *Versuch einer richtigen Lehre von der Realität der vorgeblich imaginären Grössen*, Prag, 1850, pp. 137–47; Hankel, *Vorlesungen über complexe Zahlen*, Leipzig, 1869, p. 19.

† See his *Algebra*, Oxford, 1685. The present Author has seen only the Latin edition of 1693. The detailed references which follow are to this edition.

‡ Ch. 66. § p. 287. || Ibid.

40 perches on a side. If, however, in still another place, the sea deprives us of 20 acres more, our gain is now -10 acres, or -1600 square perches. We could not properly say that we had gained the equivalent of a square either 40 perches or -40 perches on a side, but on whose side was

$$\sqrt{-1600} = 40\sqrt{-1}.$$

This ingenious but scarcely convincing example is followed in succeeding chapters by others of a more serious nature. The writer gives example after example of geometrical constructions suggested by quadratic equations of negative discriminant. Perhaps the most elegant is the following; we use the original notation.*

Let us investigate the geometrical significance of the equation

$$aa \mp ba + œ = 0.$$

We shall assume that

$$œ > 0,$$

for this may always be established by adding a positive quantity to the roots. Let C be the middle point of a segment $A\alpha$ of length b. Erect a perpendicular to $A\alpha$ at C and lay off the length $CP = \sqrt{œ}$. Let us then construct a right triangle with CP as one side, and the other side $PB = \frac{1}{2}b$.

In the case where $\frac{1}{4}bb > œ$, PB will be the hypotenuse, and B will take either of two positions on $A\alpha$ whose distances from A are the roots of the quadratic equation above. When, however, $\frac{1}{4}bb < œ$, CP must be the hypotenuse and the points B will not lie on the line $A\alpha$. The geometrical construction is equally real in both cases, the only distinction being that in the first case we get a point on the line $A\alpha$ and in the second we do not.

Let us push the matter a little further, using a more modern form of notation. Let the roots of the equation in the second case be $p+qi$, $p-qi$. Then, if we take A as origin, and $A\alpha$ as axis of reals, we represent the first given complex values by the point whose coordinates are

$$\left(p\left[1+\frac{q}{\sqrt{p^2+q^2}}\right], \quad \frac{q^2}{\sqrt{p^2+q^2}}\right).$$

* pp. 290, 291.

We see that for each finite value $p+qi$ there is a definite point in the upper half-plane. Conversely, suppose that we have a point B in this half-plane. Since $AC = PB$, if we overlook the size of $\measuredangle PBC$ and let C slide along the X axis, we see that P will trace a parabola with B as focus, and the Y axis as directrix. On the other hand, if we

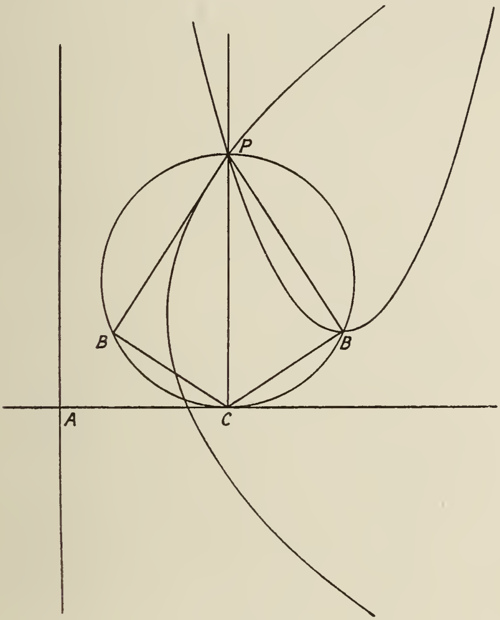

FIG. 1.

disregard the size of PB and remember that, as C slides, the $\measuredangle PBC = \frac{\pi}{2}$, we see that P will trace a parabola with vertex at B and a vertical axis. These two parabolas will have two real intersections on opposite sides of the vertical line at C. Each point B will thus appear once as a point $p+qi$ and once as a point $p-qi$.

It is clear that as a means of representing all complex values, Wallis's method leaves something to be desired.* On

* For a critical study of Wallis's other constructions see Enneström, 'Die geometrische Darstellung imaginärer Grössen bei Wallis,' *Bibliotheca Mathematica*, Series 3, vol. vii, 1906, pp. 263 ff.

the other hand, there is no reason to think that such a representation was what Wallis had primarily in mind. The question which he asked himself was, 'What geometrical constructions are called for by the general quadratic equation?' He answered this interesting question with abundant insight and skill.

A period of sixty-five years elapsed after the publication of Wallis's *Algebra* before any other mathematician attacked the problem of representing complex numbers, the next rash man being one Heinrich Kühn, who published *Meditationes de quantitatibus imaginariis construendis, et radicibus imaginariis exhibendis*, in 1753.* This writer says † 'that he was led to consider the problem of complex quantities some fifteen years before, when Euler had invited him to find the cube of $-1 \pm \sqrt{-3}$'.

Kühn begins with what we call pure imaginary numbers. Let us construct four squares, each with two sides along the coordinate axes, the numerical length of each side being a. Since positive and negative lengths are determined for each axis, the writer makes the bizarre assumption that the areas of these various squares are severally ‡

$$a \times a, \quad -a \times (+a), \quad (-a) \times (-a), \quad +a \times (-a).$$

If, then, the area of a square be $-a^2$, the proper expression for one side or root is $\sqrt{-a^2}$.

'Secundum □ tum β habet latitudinem priuatiuam $Pr = -a$, et longitudinem positiuam $PQ = +a$, adeoque eius area $= Pr \cdot PQ = -a \cdot +a = -a^2$, eiusque latus seu radix, cum nec per sola $PQ (= +a)$ nec per sola $Pr (= -a)$ exprimi posset, sed utriusque dimensionis simul ratio habenda sit, recte exprimetur per $\sqrt{(Pr \cdot PQ)} = \sqrt{(-PR \cdot PQ)} = \sqrt{-a \cdot +a}$ seu breuitatis gratia per $+\sqrt{-a^2}$.

* *Novi commentarii Academiae Petropolitanae*, vol. iii, Petrograd, 1753. A few details about Kühn are given by Cantor, *Geschichte der Mathematik*, second edition, Leipzig, 1901, vol. iii, p. 726.

† p. 170.

‡ This same idea appears in Wallis, loc. cit., p. 287. Did Kühn take it outright?

§ Ibid., pp. 172, 173.

'Hic satis speciose obiici potest eiusmodi rationis radices $x = \mp\sqrt{-a^2}$, e.g. $\mp\sqrt{-9}$ esse mere imaginarias, impossibiles et inassignabiles propterea quod ex $-a^2$ nullo modo radix quadrata extrahi possit, nec enim eam esse $-a$ nec $+a$ cum $-a \cdot -a$ item $+a \cdot +a$ det quadratum positiuum $= +a^2$ atque adeo omnia quadrata realia aut assignabilia esse positiua. At non difficilis ad ista est responsio. Praeterquam enim quod calculus ex datis possibilibus aut realibus profectus, et ex axiomatibus indubiis conuenienter tractatus, nullo modo ad impossibilia, ad non realia aut assignabilia deducere posset, minus recte etiam supponi videtur, omnia quadrata realia esse positiua.' *

The reasoning is simply this. A square can be so placed that whereas one side lies along a positive length, another lies along a negative length. The area of the square must then be taken as negative, and as every square must have a 'radix seu latus' the latter must be the indicated square root of a negative quantity.

Kühn next turns to the discussion of the affected quadratic. He gives a fearfully long and involved discussion, covering no less than twenty-three pages, the upshot of which may be stated as follows: †

If we have given two squares, positive or negative, we may construct a third square equal to their sum or difference. We can likewise construct the half sum and half difference. Suppose that we have the equation

$$x^2+px+q = 0,$$

to construct positive or negative squares x_1^2, x_2^2, where

$$\tfrac{1}{2}(x_1^2+x_2^2) = \frac{p^2}{2} - q; \quad \tfrac{1}{2}(x_1^2-x_2^2) = p\sqrt{\tfrac{1}{4}p^2-q}.$$

If, therefore, $p^2 > 4q$ the two squares are positive, and their sides, the roots of the quadratic equation, are known. When, however, $p^2 < 4q$ the construction cannot be actually effected we can find a positive square equal to $\frac{1}{4}p^2$ and a negative one equal to $\frac{1}{4}p^2-q$ in the sense given above, and even though x cannot actually be found as the 'radix seu latus' of any positive square, still it should be looked upon as known.

* pp. 176, 177.

† pp. 196, 197, 198.

Unless this interpretation of Kühn's work be grossly unfair, he represents a retrogression rather than an advance, as compared with Wallis, and certainly is far from deserving the mead of praise which has been bestowed upon him as the first to find a geometrical interpretation for complex numbers.* Who then, deserves the credit for this discovery?

Caspar Wessel was born June 8, 1745, at Josrud, Norway, and died in 1818. By profession he was a surveyor, and is said to have achieved some distinction in his work. He also studied law, passing the examination in Roman law in 1778. His title to fame as a mathematician rests on a single work, written when he had attained the substantial age of fifty-two, and entitled *Om Directionens analytiske Betegning*. This was presented to the Royal Danish Academy in 1797, published in their Memoirs in 1799, and then allowed to sink into restful oblivion for ninety-eight years, till discovered by some curious antiquary, and republished in French on the hundredth anniversary of its birth.† The fundamental idea of the memoir is to develop a system of vector analysis, a system of algebraic operations with vectors. The internal evidence would seem to show that the representation of complex numbers appeared of secondary importance to the writer, although he says ‡

'Ce qui m'a donné l'occasion de l'écrire c'est que je cherchais une méthode qui me permît d'éviter les opérations impossibles; l'ayant découverte, je l'ai employée pour me convaincre de la généralité de certaines formules connues.'

How shall we build up a calculus of vectors? The value of a vector shall be taken as depending on its length and direction, so that two vectors are equal when, and only when, their lengths are equal, and their directions identical. § The method

* Matzka, loc. cit., p. 139.

† The translation was by Thiele and Valentiner and entitled *Essai sur la représentation analytique de la direction*, Copenhagen, 1897. This is about the only available source of information about Wessel, and the following page references are thereto.

‡ Ibid., p. 5.

§ No writer before Mourey in 1828 seems to have clearly grasped the idea that the equality of vectors needed to be defined.

of adding vectors is practically imposed upon us by the nature of the problem; we reduce to a common origin, and add by the parallelogram construction.

The first real difficulty appears when we attempt to define the product of two vectors. Let us quote Wessel verbatim: *

'Le produit de deux segments doit, sous tous les rapports, être formé avec l'un des facteurs de la même manière que l'autre facteur est formé avec l'autre segment positif ou absolu qu'on a pris égal à 1, c'est à dire que:

1° Les facteurs doivent avoir une direction telle qu'ils puissent être placés dans le même plan que l'unité positive.

2° Quant à la longueur, le produit doit être à l'un des facteurs comme l'autre facteur est à l'unité.

3° En ce qui concerne la direction du produit, si l'on fait partir de la même origine l'unité positive, les facteurs, et le produit, celui-ci doit être dans le plan de l'unité et des facteurs, et doit dévier de l'un des facteurs d'autant de dégrés, et dans le même sens, que l'autre facteur dévie de l'unité.'

This excellent definition calls for one or two remarks. To begin with, it never occurred to Wessel that the product of two vectors might be something different from a vector, which explains the reason why he could not reach the wealth of results afterwards attained by the followers of Grassmann and of Hamilton. Secondly he, like others who followed him, assumes that the operation which converts the unit vector into a given vector must, necessarily, be defined as a rotation through a certain angle, and an alteration of the length in a certain ratio. He might with equally sound logic, though far less mathematical success, have defined it in any one of a number of other ways. He might, for instance, have said that a certain amount had been added to or subtracted from the length (instead of from the logarithm thereof) or that the angle with the unit vector had been altered in a given ratio. Wessel's choice was the right and proper one, but in no sense the only one open to him.

After laying these foundations, Wessel is able to build up his structure rapidly. If we denote the four unit vectors laid off on the axes by 1, ϵ, -1, and $-\epsilon$ respectively, the law of

* p. 9.

multiplication shows that $\epsilon^2 = -1$, so that we may replace ϵ by $\sqrt{-1}$. The standard vector can be written $u + v\epsilon$ and the law of multiplication, joined with the trigonometric formulae for the functions of the sums and differences of angles, gives

$$(a + b\epsilon)(c + d\epsilon) = (ac - bd) + (ad + bc)\epsilon,$$

$$\frac{a + b\epsilon}{c + d\epsilon} = \frac{ac + bd}{a^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\epsilon,$$

$$(\cos v + \sin v\epsilon)^{\frac{n}{m}} = \cos\frac{n}{m}v + \sin\frac{n}{m}v\epsilon.$$

The expression $(\cos v + \sin v\epsilon)^{\frac{1}{m}}$ has m values, to wit

$$\cos\frac{v}{m} + \sin\frac{v}{m}\epsilon,\ \cos\frac{v + \pi}{m} + \sin\frac{v + \pi}{m}\epsilon \ldots$$

$$\cos\frac{v + (m-1)\pi}{m} + \sin\frac{v + (m-1)\pi}{m}\epsilon.^{*}$$

The last development in this part of the essay is interesting; it must be remembered that in Wessel's time nobody bothered about the convergence of series. Let x be a complex number †

$$(1 + x)^m = 1 + \frac{mx}{1} + \frac{m(mx - 1)}{1 . 2}x^2 + \ldots + \frac{m!}{k!(m-k)!}x^k.$$

Let $l = a + b\epsilon = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n}\ldots$

$$(1 + x)^m = 1 + ml + \frac{m^2l^2}{2!} + \frac{m^3l^3}{3!}1\ldots$$

$$= e^{me} = e^{ma + mb\epsilon}.$$

Hence $l = \log_e(1 + x)$.

The author adds:

'Je présenterai une autre fois, si l'Académie me le permet, les preuves de ces théorèmes.'

The Academy was doubtless willing, but the proofs were never presented.

* p. 15. It is astounding that a man of such mathematical knowledge and insight should have confused π and 2π.

† Ibid., p. 16.

The latter portions of Wessel's classic are not concerned with the representation of the usual complex numbers, and do not, therefore, concern us in the present work. It is perfectly clear from what we have quoted that he has all of the essentials of the usual method of representation. It is worth noting that he represents a complex number by the vector whose components are the real part and the coefficient of the imaginary unit, rather than by the point with these two as Cartesian coordinates; i.e. he uses a vector whose initial point is the origin, instead of using its terminal point only.

It was perhaps fortunate for the progress of mathematical science, if not for the fame of Wessel, that during the hundred years when his memoir slept, other writers, independent of him, attacked the same problem, and achieved the same results. The first of these successors was Jean Robert Argand, who was born in 1768 in Geneva, but who passed the better part of his obscure life as a humble book-keeper in Paris. In 1806 he published a short memoir entitled: 'Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.'* Before publication, Argand wisely showed his work to Legendre. The great arithmetician gave him some advice about it, and, some time later, mentioned the memoir in a letter to a man named Français. After the latter's death his brother, J. B. Français, saw the letter, and starting therefrom developed the subject according to his own ideas, publishing a short note entitled: 'Nouveaux principes de géométrie de position, et interprétation géométrique des symboles imaginaires.' †

Français' publication came presently to the notice of Argand, who wrote a note to the author giving an account of the history of his own researches. He followed this by sending a development of his ideas to the same journal, and a copy thereof to Français. Finally, in the same number there was

* The author has only seen the second edition, with preface and appendices by Houel, Paris, 1874. We shall presently refer to this work by page number.

† Gergonne's *Annales de Mathématiques*, vol. iv, 1813, pp. 61 ff.

published a short note from Français, couched in the following terms.*

. . . 'Je viens de recevoir à l'instant le Mémoire de M. Argand que j'ai lu avec autant d'intérêt que d'empressement. Il ne m'a pas été difficile d'y reconnaître le développement des idées dans la lettre de M. Legendre à feu mon frère et il n'y a pas le moindre doute qu'on ne doive à M. Argand la première idée de représenter géométriquement les quantités imaginaires. C'est avec bien de plaisir que je lui en fais hommage, et je me félicite de l'avoir engagé à publier ses idées dans l'ignorance où j'étais de leur publication antérieure.'

How many cases are there in the history of mathematics where a question of priority has been settled with such courtesy and good feeling?

Argand begins by considering that negative numbers are related to positive ones, not only through numerical ratio, but also through a reversal of direction.† This being so, the problem of finding a mean proportional between two quantities with opposite signs requires us to find the square root of the product of their numerical values, and a direction which is a mean between their two directions, i.e. which is perpendicular to them. Quantities which correspond to horizontal directions are called 'prime' quantities, those which correspond to vertical directions are 'median' quantities. Instead of writing $a\sqrt{-1}$ and $-a\sqrt{-1}$ he writes, $\sim a$, $\dotplus a$.

Argand next takes up the rules for adding and multiplying directed quantities. The rules for addition are obvious enough. With regard to multiplication, he reasons much as Wessel does. Since the product is to each factor as the other factor is to the prime vector, the tensors of the factor vector factors must be multiplied, and their angles added. This leads quickly to De Moivre's theorem

$$\cos na \sim \sin na = (\cos a \sim \sin a)^n.$$

Expanding on the right, we get

$$\cos^n a - \frac{n(n-1)}{1\cdot 2}\cos^{n-2} a \sin^2 a + \ldots$$

$$\sim \left(n\cos^{n-1} a \sin a \frac{n(n-1)(n-2)}{3!}\cos^{n-3} a \sin^3 a + \ldots\right)\cdot$$

* Ibid., p. 98.

† See his memoir, cit., p. 4.

Now let n become infinitely great, while $na = x$, a constant.

$$\cos^k a = \left(\cos\frac{x}{n}\right)^k,$$

and the limit of this is unity. Similarly

$$\sin^k a \,.\, n(n-1)\ldots(n-k+1)$$

$$= \left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^k \frac{n(n-1)\ldots(n-k+1)}{n^k} x^k \ldots$$

and the limit of this is x^k; hence

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} \ldots \quad \sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} \ldots.$$

This is but one of a number of similar developments which, with certain complicated trigonometric identities, constitute the bulk of the remainder of the essay. He closes with a proof of the fundamental theorem of algebra, which runs as follows *

Let $Y(X) = X^n + aX^{n-1} + bX^{n-1} + \ldots + fX + g = 0$;

the author adds:

'Il faut observer que les lettres $a, b, \ldots g$ ne sont point restreintes à ne représenter que des nombres primes (réels) comme cela a lieu à l'ordinaire.'

We next give to X the value $p + \rho i$, and develop by Taylor's theorem

$$Y(p+\rho i) = Y(p) + i\rho Q + i^2\rho^2 R + \ldots.$$

Then says the author:

'Si l'on suppose i infiniment petit, les termes affectés de i^2, i^3, &c., disparaissent, et l'on a

$$Y(p+\rho i) = Y(p) + i\rho Q.$$

He then shows that $Y(p+\rho i)$ can usually be made smaller in absolute value than $Y(p)$, and so concludes that the function, for some value of the argument, must reach its minimum value 0. Of course such a proof is in no sense mathematically

* pp. 58, 59.

valid. What is of interest is the statement that the usual proofs deal only with the case where the coefficients are real.

We may dismiss Argand with the remark that his ideas are essentially those of Wessel, but that his development of the subject is less careful than that of the long-neglected Scandinavian surveyor.

The year 1806, in which Argand's first memoir appeared, gave birth likewise to another longer and more ambitious essay on the same subject. One William Morgan presented to the Royal Society a formidable monograph entitled: * 'Mémoire sur les quantités imaginaires', by the Abbé Buée. It is vain to speculate after this lapse of time as to why such a memoir was accepted by the Society. Was the good Abbé an *émigré* whom the British delighted to honour the year that they defeated his non-emigrated countrymen at Trafalgar? Some such reason there must have been, as the intrinsic worth of the memoir would never recommend it for publication. †

The fundamental idea of Buée is that in arithmetic we consider not merely numbers, but numbers affected by certain qualities. A number prefixed by + is an additive number, one prefixed by − is subtractive. If the prefix be $\sqrt{-1}$ the quality is neither additive nor subtractive, but a mean between the two:

> 'Ce signe mis devant a (a signifie une ligne ou une surface) veut dire qu'il faut donner à a une relation perpendiculaire à celle qu'on lui donnerait si l'on avait simplement $+a$ ou $-a$.' ‡

Very good so far, although the reference to a surface is not perfectly clear. We find presently:

> 'Il faut distinguer la perpendicularité indiquée par ce signe de celle qu'indiquent les signes sin et cos . . . Sin et cos sont des signes artificiels, $\sqrt{-1}$ est un signe naturel, puisqu'il est une conséquence des signes + et − ....

* *Philosophical Transactions*, 1806.

† An extensive review by Peacock will be found in the *Edinburgh Review*, vol. xii, 1808. The reviewer singles out for attack the only really valuable feature in the article, showing that he has utterly failed to grasp the important underlying thought.

‡ p. 28.

'Quoique la perpendicularité soit proprement la seule qualité indiquée par le signe $\sqrt{-1}$ on peut lui faire signifier *au figuré* une qualité toute différente, pourvu qu'on puisse raisonner sur cette qualité comme on raisonnerait sur la perpendicularité même. Par exemple, si $+s$ représente une somme possédée, et $-s$ la même somme due, $s\sqrt{-1}$ peut représenter la même somme ni possédée, ni due.*

'Si, par exemple, j'exprime un temps futur par t, et un temps passé par $-t$, $t\sqrt{-1}$ ne peut rien signifier, puisque le présent, qui est la qualité moyenne entre le futur et le passé, n'est qu'un instant indivisible et qui n'a d'autre expression que 0.'

The author goes on to say, however, that when one uses the term 'present' in the sense of present week or present month, then, if the next period be t while the last is $-t$, the present one is composed of $t\frac{\sqrt{-1}}{2}$ and $-t\frac{\sqrt{-1}}{2}$.†

... There is worse yet to follow. We read eighteen pages later‡:

'Par conséquent, si $(-1+\sqrt{-\frac{1}{2}})$ indique une seule ligne, l'une des quantités qui composent $(-1+\sqrt{-\frac{1}{2}})$ indique la longueur de cette ligne, et l'autre l'épaisseur de son point extrême.

Finally, we find seventeen pages further still §:

$$\sqrt{-1} = \frac{e^{\frac{\pi}{2}\sqrt{-1}} - e^{-\frac{\pi}{2}\sqrt{-1}}}{2} = e^{\frac{\pi}{2}\sqrt{-1}}$$

$$(\sqrt{-1})^n = e^{n\frac{\pi}{2}\sqrt{-1}} = e^{\left(\frac{\pi}{2}+\frac{\pi}{2}+\dots+\frac{\pi}{2}\right)\sqrt{-1}} = e^{(90^\circ+90^\circ+\dots+90^\circ)\sqrt{-1}}$$

The author then asks himself in despair what

$$(90^\circ+90^\circ+\dots+90^\circ)\sqrt{-1}$$

may mean anyway. All of which was published in the Philosophical Transactions of the Royal Society of London in the year of our Lord 1806, and of the reign of His Most Gracious Majesty George III, 47.

* p. 30. † pp. 34, 35.
‡ p. 50. § p. 67.

It is a curious fact connected with the history of attempts to give a geometrical representation of complex numbers, that not only did different mathematicians arrive independently at identical conclusions, but that on two occasions two independent publications appeared in the same year. Argand and Buée both wrote in 1806, two other writers published in 1828. The first of these was a certain C. V. Mourey.* This author writes with a notable exuberance, and dedicates his work 'Aux amis de l'évidence,' but he is by no means lacking in penetration and mathematical insight. At the very outset he points out that there are different ways in which two quantities may be said to be equal. Thus, two non-directed segments are considered equal if their lengths be equal, but two directed segments (*chemins*) are looked upon as equal only when there is identity both of length and of direction.†

In studying vectors with different directions one looks particularly at the angle, called by this author the 'angle directif', which one vector makes with another. This is indicated by a subscript equal in magnitude to the given angle in the system where a right angle is the unit, so that if AC and AB be equal in length but at right angles to one another we have such an equation as

$$AC = AB_1.$$

More generally, if AB, AD, AF, and AH be four unit vectors laid off on the axes, we have:

$$AB = 1,\quad AD = 1_1,\quad AF = 1_2 = -1,\quad AH = 1_3 = -1_1.$$

In general

$$1_{2p+q} = 1_{2p} \cdot 1_q.$$

Comparing the angles of vectors with different initial points, we find ‡

$$(AD + DE)_r = AD_r + DE_r.$$

The author then develops what he calls 'directive multi-

* *La vraie théorie des quantités négatives et des quantités prétendues imaginaires*, Paris, 1828. The present author has only seen the second edition, Paris 1861.

† p. 7. ‡ p. 33.

plication'. Such a number as $(\frac{9}{4})_{\frac{2}{3}}$ is defined as an operator which alters the length of a vector in the ratio 9 : 4 and swings it through two-thirds of a right angle, i.e. $\frac{9}{4}e^{\frac{\pi i}{3}}$. If, then, the number $(a^m)_r$ be defined as the same as $a_r{}^m$ the numbers $1, 1_{\frac{4}{n}}, 1_{\frac{8}{n}}\ 1_{\frac{4(m)}{n}}$ have all the same nth power, namely unity, and the author exclaims *:

'Voilà les racines de l'unité, voilà les quantités prétendues imaginaires.†

The last serious problem which Mourey takes up is the fundamental theorem of algebra. His proof shows a very considerable amount of acumen. Translated into modern notation, he starts with n points of the z plane,

$$z_1, z_2, \dots z_n,$$

and proves it possible to find such a point z that

$$(z-z_1)(z-z_2)\dots(z-z_u) = u+vi,$$

where $u+v_i$ is a given complex value. The proof consists in a careful watching of the argument of each of these complex factors, as z traces a closed circuit in the plane. It is rather curious that Mourey nowhere lays stress upon the fact that every vector in the plane can be expressed linearly in terms of the vectors 1 and 1_1.

The other writer who published in the year 1828 was the Rev. John Warren.‡ The word 'quantity', as used by Warren, is defined explicitly as meaning a vector. The essay is written with a certain workmanlike thoroughness which suggests the professional mathematician, and is closer in spirit to Wessel's work than to that of any other previous writer on the subject. Moreover, Warren sees the necessity for discussing certain important points which had been previously neglected; e.g. on pp. 74 ff., he inquires just which value of $(1+b)^{\frac{m}{n}}$ is represented by the usual binomial expansion, when b is a complex number, and $\frac{m}{n}$ a fraction.

* p. 32. † p. 45.

‡ *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, Cambridge, 1828.

We come at last to the year 1831. On April 15 of that year Gauss presented to the Royal Society of Göttingen a short essay entitled: 'Theoria residuorum biquadraticorum, commentatio secunda.' *

The fundamental idea of Gauss may be expressed in the following terms:

Let a set of objects A, B, C, &c., be arranged in such a scale that we can say that the relation or transfer from A to B is the same as that from B to C or from C to D, &c. Each of these relations may be expressed by the symbol $+1$, and if the inverse relations be considered, as that from B to A, we represent it by the symbol -1. If our system of objects extend indefinitely in either direction, then any one of our integers, positive or negative, will express the relation of any one of our objects, chosen as the first, to some one other object of the series.

Suppose, next, that instead of having a single series of objects, we have a series of series, and the relation of any object in one series to the corresponding object in the one or other of the next adjacent series be expressed by the symbol i or $-i$. The four fundamental relations are 1, i, -1, and $-i$. We then look upon our system of objects as arranged like the points of a plane lattice (they can always be put into one to one correspondence with such a system of points). The system will be carried into itself by a rotation through 90° about any one of its points, the relation 1 will be carried into the relation i, while this latter is carried into the relation -1.

'Das heisst, aber, in der Sprache der Mathematiker, $+i$ ist mittlere Proportionalgrösse zwischen $+1$ und -1 oder entspricht dem Zeichen $\sqrt{-1}$. . . . Hier ist, also, die Nachweisbarkeit einer anschaulichen Bedeutung von $\sqrt{-1}$ vollkommen gerechtfertigt, um diese Grösse in das Gebiet der Gegenstände der Arithmetik zuzulassen'. . . . 'Hätte man $+1$, -1, $\sqrt{-1}$ nicht positive, negative, imaginäre (oder gar unmögliche) Einheit, sondern etwa directe, inverse, laterale Einheit genannt,

* See his *Collected Works*, Gottingen, 1878, vol. ii, especially pp. 174 and 178.

so hätte von einer solchen Dunkelheit kaum die Rede sein konnen.' *

Several remarks are in order with regard to this beautiful little memoir. To begin with, it appeared comparatively late in time, but the author says,† that traces of the same idea are to be found in his memoir of 1799 dealing with algebra. Here is an important point: Is the statement well founded? The memoir in question is his dissertation,‡ and contains on the constructive side a proof that every real polynomial

$$Ax^n + Bx^{n-1} + \ldots + Kx + L$$

is divisible into factors of the types

$$x - r\cos\phi \quad \text{and} \quad xx - 2r\cos\phi\, x + rr.$$

The whole process is perfectly blind and meaningless without the clue that the complex value $x + y\sqrt{-1}$ corresponds to the point whose cartesian coordinates are (x, y) and polar coordinates (r, ϕ). We may, then, say that Gauss's claim is amply borne out. Secondly, it is noticeable that in the memoir of 1831 he confines his attention to complex integers, but that was because his interest for the moment was in a purely arithmetical question, and does not at all affect the general question. Thirdly, Gauss, in contradistinction to Wessel, Argand, Buée, Mourey, and Warren, thinks of the point (x, y), not of the vector from the origin to that point. This is surely the way that we do at present, and since our mathematical speech has so far solidified that it is too late to call the representation after Wessel, it seems better to associate the name of Gauss therewith, rather than that of Argand.

It is not to be supposed that even the publication of Gauss's memoir of 1831 put an end to experimentation in representing complex numbers. The standard, however, was set, and we can refer to what has been done since in most cursory fashion.

To begin with, our modern treatment of complex numbers has a twofold aspect, the geometrical representation in the

* pp. 177, 178. † p. 175.

‡ 'Demonstratio nova theorematis omnem functionem algebricam rationalem integra unius variabilis in factores reales primi vel secundi gradus resolvi posse', *Collected Works*, vol. iii, pp. 3–31.

complex plane, and the arithmetical theory of operations with number pairs. This latter falls outside the scope of the present work, but we may mention in passing that it sprang into being fully armed, not from the head of Jove, but from that of that extraordinary genius Sir William Rowan Hamilton.* Secondly, we notice that the Gauss representation contains one serious gap. The Gauss plane, as originally defined, is not a perfect continuum; nothing is said of the infinite region thereof. If each finite point is to represent a finite value of the complex variable, then, if the correspondence is to be perfect in every respect, we must consider the infinite domain as a single point corresponding to the value of z which satisfies the equation $\frac{1}{z} = 0$. This is exactly what we do when we study the geometry of the inversion group. It must be conceded, however, that the idea of a single point at infinity lacks intuitive force. The difficulty is overcome by the admirable expedient of projecting the plane stereographically upon a sphere. The correspondence of real point and complex number is then perfect and complete. It is not absolutely certain to whom is due the credit for this device. We are probably safest in following Neumann, and ascribing it to Riemann.†

'Erwähnen muss ich dabei jedoch eines Gedankens, der mir aus Riemann's Vorlesungen durch mündliche Ueberlieferung zu Ohren kam, und der auf meine Darstellung von nicht geringem Einfluss wurde. Dieser Gedanke besteht in der Projection der auf der Horizontalebene ausgebreiteten Functionswerthe nach einer Kugelfläche hin.'

Another point to be noted is that a geometer familiar with the principle of duality in the projective geometry of the plane comes very naturally to the idea of representing a complex number, not by a point, but by a line in the plane. The

* See his remarkable and too little known memoir, 'Theory of Conjugate Functions, or Algebra of Couples', *Transactions of the Royal Irish Academy*, vol. xvii, 1837.

† Neumann, *Vorlesungen über Riemann's Theorie Abelscher Integrale*, Leipzig. 1865, p. vi, foot-note.

attempt to do this has already been made,* but the results are neither simple nor attractive. The reason for this ill-success is instructive. Let the complex number

$$\zeta = u + vi$$

be represented by the real line

$$ux + vy + 1 = 0.$$

We see that every finite value of ζ gives one real line, but that all lines through the origin will correspond to the infinite value. The connectivity of the projective plane of points and of the plane of lines is such as to preclude the possibility of a one to one correspondence with the totality of values of a single complex variable, but the usual method of assuming only a single infinite point is preferable to having all lines through the origin correspond to the single infinite value of the variable. Secondly, in the Gauss plane the general analytic transformation $z' = f(z), \quad \bar{z}' = \bar{f}(\bar{z})$

has the absolute differential invariant

$$\frac{dz'\delta\bar{z}' + \delta z' d\bar{z}}{2\sqrt{dz'd\bar{z}'}\sqrt{\delta z'\delta\bar{z}'}} = \frac{dz\,d\bar{z} + d\bar{z}\,\delta z}{2\sqrt{dz\,d\bar{z}}\sqrt{\delta z\,\delta\bar{z}}}.$$

This represents twice the cosine of the angle of the tangents to two curves intersecting at z, and its invariance proves that every transformation of this sort is a conformal one. But the corresponding expression in the ζ plane gives the cosine of the angle subtended at the origin by the points of contact of the line ζ with two of its envelopes and the group leaving this angle invariant, is of altogether minor importance. Let us point out, in conclusion, that when we come to the problem of representing all the complex points of a real plane, we shall encounter representations of the points of a single real line quite different from anything which we have seen so far. Whatever virtues such methods may possess as parts of a larger whole, for a single line, i.e. for a single complex variable, they all fall hopelessly short of the standard set by the classic method of Wessel, Argand, and Gauss.

* Brill, 'A New Method for the Graphical Representation of Complex Quantities', *Messenger of Mathematics*, Series (2), vol. xvii, 1888.

CHAPTER II

THE GEOMETRY OF THE BINARY DOMAIN

§ 1. The Real Binary Domain.

In the present section we shall give the name *real points* to any set of objects in one to one correspondence with pairs of real homogeneous coordinate values *

$$(X_1,\ X_2)$$

which are not both zero. When we say that the coordinates are homogeneous we mean that the point $(X_1,\ X_2)$ is identical with the point $(rX_1,\ rX_2)$, $r \neq 0$. If four points (X), (Y), (Z), (T) be given, the expression

$$\frac{|XY| \cdot |ZT|}{|XT| \cdot |ZY|} = \frac{\begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \cdot \begin{vmatrix} Z_1 & Z_2 \\ T_1 & T_2 \end{vmatrix}}{\begin{vmatrix} X_1 & X_2 \\ T_1 & T_2 \end{vmatrix} \cdot \begin{vmatrix} Z_1 & Z_2 \\ Y_1 & Y_2 \end{vmatrix}} = (XZ,\ YT), \qquad (1)$$

shall be called a *cross ratio* of the four. We leave to the reader the task of verifying the familiar theorem that four distinct points have at most six distinct cross ratios, which make a group of values such as

$$L,\quad \frac{1}{L},\quad 1-L,\quad \frac{1}{1-L},\quad \frac{L-1}{L},\quad \frac{L}{L-1}.$$

The first two and the last two points in the notation above shall be said to form *pairs*. Notice that when the two members of one and only one pair are interchanged, a cross

* In the present work we shall consistently use large letters to indicate real quantities, and small ones to indicate complex ones. We shall make one exception only to this rule, namely, we shall use the letters u, x, v, and w as real curvilinear parameters in differential expressions. It looks too bizarre to write the partial derivative of a small letter with regard to a large one. We shall use such classical notations as $re^{i\phi}$ where r and ϕ are real.

ratio is replaced by its reciprocal. If two points of a pair tend to coalesce, the members of the other pair remaining fixed, the two cross ratios associated with this pairing, approach the value 1. If two not paired points tend to approach, the other two remaining in place, the cross ratio will either approach 0 or become infinite. When three points are given, there is a unique fourth point which will make with them any assigned cross ratio other than unity, zero, or infinity. If we inquire as to the possibility that two of the six associated cross ratio values should be identical, we find that, in the real case, this can only happen in the case of the triad of associated values

$$-1, \ \tfrac{1}{2}, \ 2.$$

When this arises, we shall say that each pair is *harmonically separated* by the other, the relation between the two being entirely symmetrical. The points (1, 0) and (0, 1) are called the *zero points*, while (1, 1) is the *unit point*.

Theorem 1] *The ratio of the coordinates of a point is equal to the cross ratio where the zero points form one pair, and the unit point and given point form the other pair.*

A transformation of the type

$$\begin{aligned} \rho X_1 &= A_{11}X_1' + A_{12}X_2' \\ \rho X_2 &= A_{21}X_1' + A_{22}X_2' \end{aligned} \qquad |A_{ij}| \neq 0 \qquad (2)$$

is called a *collineation*. The inverse is given by

$$\begin{aligned} \sigma X_1' &= -A_{22}X_1 + A_{12}X_2, \\ \sigma X_2' &= A_{21}X_1 - A_{11}X_2. \end{aligned}$$

Theorem 2] *The totality of collineations is a three-parameter group.*

Theorem 3] *Each cross ratio of four given points is invariant under every collineation.*

Theorem 4] *If three points be invariant under a collineation, every point is invariant, and the collineation is the identical one.*

It is clear from this that a collineation is completely determined by the fate of three points. Let us proceed to prove the converse, which is of no small importance:

Theorem 5] *A collineation may be found to carry any three distinct points into any other three such points.*

It is merely necessary to prove that a collineation can be found to carry three arbitrary points (Y), (Z), and (T) into the zero points and the unit point, as the product of this and the inverse of the transformation which carries the other three points into these will accomplish the desired end. The collineation desired is expressed by the equations

$$\begin{aligned} |ZY| X_1 &= |ZT| Y_1 X_1' + |TY| Z_1 X_2', \\ |ZY| X_2 &= |ZT| Y_2 X_1' + |TY| Z_2 X_2'. \end{aligned} \tag{3}$$

Since cross ratios are invariant under every collineation, we see that, in particular, harmonic separation is an invariant relation. We shall now prove the remarkable theorem that the invariance of this one relation is enough to insure that a one to one transformation shall be a collineation.

Theorem 6] *Every one to one transformation of the real binary domain which leaves the relation of harmonic separation invariant, is a collineation.**

Suppose that we have a transformation of this nature which carries the zero points and unit point into three specified points. We may follow this with a collineation which carries them back again, and the product is a transformation of the type described in our theorem, which leaves these three points invariant. Let us prove that it must be the identical transformation, so that the original transformation was the collineation inverse to the one employed.

It will simplify our analysis if we abandon homogeneous co-

* This theorem is due to Von Staudt. See his *Geometrie der Lage*, Nuremberg, 1847, p. 50. His proof is lacking in rigour owing to an inadequate treatment of the question of continuity. The question was debated at some length in the early numbers of the *Mathematische Annalen*, and eventually rigorous proofs were found. That given here is due to Darboux, 'Sur la géométrie projective', *Math. Annalen*, vol. xvii, 1880.

ordinates, and replace $\frac{X_1}{X_2}$ by X. Our transformation will be characterized by the equations

$$X = F(X'),$$
$$F(0) = 0, \quad F(1) = 1, \quad F(\infty) = \infty \cdot \tag{4}$$

Since (X) and (Y) are harmonically separated by $\frac{(X)+(Y)}{2}$ and ∞,

$$F\left(\frac{(X+Y)}{2}\right) = \frac{F(X)+F(Y)}{2},$$
$$F\left(\frac{X}{2}\right) = F\frac{(X+0)}{2} = \frac{1}{2}F(X),$$
$$F(X+Y) = F(X)+F(Y).$$

If R be a rational number

$$F(RX) = RF(X),$$

it is easily proved algebraically that the N. S. condition that there should exist a pair of points separating harmonically both the pair $(1)\,(X)$ and the pair $(0)\,(\infty)$ is that $X > 0$.

Hence, if $X > 0$ then $F(X) > 0$.

If $X > 0$ the N. S. condition that there should be a pair separating harmonically both $(0)\,(X)$ and $(\infty)\,(RX)$ is

$$RX > X.$$

Hence, in this case $\quad F(RX) > F(X).$

It appears that $F(X)$ must increase with X so that it is a continuous function. Hence, for all values of X and R

$$F(RX) = RF(X) = XF(R),$$
$$F(X) = X.$$

There is another form in which our collineation can be put, which is of importance with regard to what comes later. We have merely to eliminate ρ from the equations (2). We obtain a bilinear form in the variables (X) and (X') which may be expressed in the Clebsch-Aronhold symbolic notation *

$$(A_1X_1 + A_2X_2)\,(A_1'X_1' + A_2'X_2') \equiv (AX)\,(A'X') = 0. \tag{5}$$

* The more usual form of the symbolic notation is $A_x A'_{x'}$. The form here used is, however, preferable.

The letters A are meaningless except in the form of product $A_i A_j'$.

Theorem 7] *If a single pair of points correspond interchangeably in a collineation, that is true of every pair of points, and the transformation is involutory with a period two.*

We shall usually speak of such a transformation as an *involution.* The analytic condition is easily found to be

$$A_{11} + A_{22} = 0.$$

Theorem 8] *Two pairs of corresponding points will always determine an involution, and there exists just one involution which will interchange the members of any two pairs.*

§ 2. The Complex Binary Domain, Collineations and Anti-collineations.

The universe of discourse for the rest of the present chapter is the complex binary domain. This is a system of objects called *points* in one to one correspondence with pairs of homogeneous coordinate values (x_1, x_2) not both zero. The point with conjugate imaginary coordinates $(\bar{x}_1, \bar{x}_2)$ shall be called the *conjugate* point. A cross ratio of four points (x), (y), (z), (t) will be given by the expression

$$\frac{|xy| \cdot |zt|}{|xt| \cdot |zy|} = (xz, yt). \tag{6}$$

The remarks in the first section about the six cross ratios of four given points all hold good in the complex domain. There is, however, another case where four points have less than six different cross ratios, namely the equi-harmonic case, where one of the six values is an imaginary cube root of -1. Theorem 1] holds equally in the complex domain.

Definition. A system of collinear points of such a nature that:

A. The cross ratios of any four are real.

B. There exists a point of the system making with three given points thereon any given real cross ratio other than zero, one, and infinity, shall be called a *chain*.*

It is clear that three distinct points can belong to only one chain. The chain which contains the points (y), (z), and (t) may be expressed in the parametric form

$$|yz|x_1 = |tz|y_1X_1 + |yt|z_1X_2,$$
$$|yz|x_2 = |tz|y_2X_1 + |yt|z_2X_2. \tag{7}$$

Theorem 9] *Three distinct points will belong to one and only one chain.*

Theorem 10] *The chain determined by the zero points and the unit point is the real domain.*

We shall define as a *collineation* any transformation of the type

$$\rho x_1 = a_{11}x_1' + a_{12}x_2' \qquad |a_{ij}| \neq 0.$$
$$\rho x_2 = a_{21}x_1' + a_{22}x_2' \tag{8}$$

It is seen at once that theorems 2], 3], 4], and 5] apply equally in the complex domain.

Theorem 11] *A collineation carries a chain into a chain.*

Theorem 12] *A collineation may be found to carry any given chain into the real domain.*

Besides collineations, there is a second type of transformations which is fundamental in our work. These are called *anti-collineations*,† and are given by equations of the type

$$\rho x_1 = a_{11}\bar{x}_1' + a_{12}\bar{x}_2',$$
$$\rho x_2 = a_{21}\bar{x}_1' + a_{22}\bar{x}_2'. \tag{9}$$

It is evident that any anti-collineation can be factored into the product of a collineation and the interchange of conjugate imaginary points.

Theorem 13] *An anti-collineation is completely determined by the fate of any three points.*

* First defined and studied by Von Staudt in his *Beiträge zur Geometrie der Lage*, Part II, Nuremberg, 1858, pp. 137 ff.

† The name is, apparently, due to Segre. See his fundamental article, 'Un nuovo campo di ricerche geometriche', *Atti della R. Accademia delle Scienze di Torino*, vol. xxv, 1889, p. 291.

Theorem 14] *An anti-collineation may be found to carry any three points into any other three.*

Theorem 15] *The cross ratio of any four points is carried by an anti-collineation into the conjugate imaginary value.*

Theorem 16] *An anti-collineation will carry a chain into a chain.*

Theorem 17] *The product of a collineation and an anti-collineation is an anti-collineation; the product of two anti-collineations is a collineation.*

We are now able to prove another very fundamental proposition:

Theorem 18] *Every continuous one to one transformation of the binary domain which leaves harmonic separation invariant is either a collineation or an anti-collineation.*

The method of proof is entirely analogous to that used for 6]. We may follow our transformation with a collineation which restores the zero points and the unit point to their original positions, and we merely need to prove that this product transformation is either the identical transformation, or the interchange of each point with its conjugate. Let the transformation be characterized by the equations

$$x = f(x'),$$

$$f(0) = 0, \quad f(1) = 1, \quad f(\infty) = \infty.$$

We see by the reasoning used in 6] that for each rational real point we shall have

$$X = f(X).$$

Since the transformation is assumed to be continuous, this identity must hold for every real point.* As before, we have

$$f(x) + f(y) = f(x+y),$$

$$f(X) + f(Yi) = f(X + Yi),$$

$$f(-z) = -f(z).$$

* Segre, loc. cit., p. 288, expresses the opinion that it is likely that the requirement of continuity may be dropped, as it may be deduced from the invariance of harmonic separation, but confesses his inability to find a proof. The present author is of the same opinion, and has had the same ill success.

Since z and $-z$ are harmonically separated by x and y, when, and only when,
$$xy = z^2,$$
and since Yi and $-Yi$ are harmonically separated by Y and $-Y$, while
$$f(Y) = Y,\quad f(-Y) = -Y,$$
$$[f(Yi)]^2 = -f(Yi)[f(-Yi)]$$
$$= -[f(Y)]^2$$
$$= (Yi)^2,$$
$$f(Yi) = \pm Yi,$$
$$f(X+Yi) = X \pm Yi.$$
Since
$$f(Yi+Y'i) = f(Yi) + f(Y'i),$$
if
$$f(Yi) = Yi,$$
$$f(Y'i) = Y'i.$$
Hence, either
$$f(z) \equiv z,$$
or
$$f(z) \equiv \bar{z}.$$

The difference between a collineation and an anti-collineation comes out in the clearest possible manner when we represent our complex binary domain in the Gauss plane. Employing a form of notation more familiar in the theory of functions, the general collineation may be written

$$z' = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \qquad a_{11}a_{22} - a_{12}a_{21} \neq 0. \tag{10}$$

This is a real direct circular transformation which carries a point into a point, and a circle or line into a circle or line, while angles are preserved both in numerical magnitude and sign. We may pass continuously from this transformation to the identical one. On the other hand, an anti-collineation takes the form

$$z' = \frac{a_{11}\bar{z} + a_{12}}{a_{21}\bar{z} + a_{22}} \qquad a_{11}a_{22} - a_{12}a_{21} \neq 0. \tag{11}$$

This appears as a real indirect circular transformation, differing from the first by the fact that angles are reversed in algebraic sign, and we cannot pass over continuously to the identical transformation. We see also from 12] that a chain of points in the complex domain will appear as a circle or line in the Gauss plane, and, conversely, every circle or line in this plane (by definition there is no line at infinity) will correspond to a chain.*

Theorem 19] *Every one to one transformation of the points of a complex line that carries a chain into a chain is either a collineation or an anti-collineation.*

It will be sufficient to prove that every one to one transformation of the Gauss plane that carries points into points, and circles and lines into circles and lines, must be of type (10) or type (11).† Let us suppose that our transformation carries a point O into a point O'.

We may precede our transformation by an inversion with O as centre, and follow it by one with O' as centre, thus getting a new transformation answering the given description, that carries points into points, lines into lines, and circles into circles. Parallelism will be an invariant property here, hence a parallelogram will go into a parallelogram, and a rectangle, which is a parallelogram inscriptible in a circle, will go into a rectangle. A square, which is a rectangle with mutually perpendicular diagonals, will go into a square. If a square $ABCD$ go into a square $A'B'C'D'$, we may find a transformation composed of a rigid motion, a central similitude, and, when the corresponding vertices follow in reverse sense of rotation, a reflection in a straight line, which will carry $A'B'C'D'$ back into $ABCD$. The net result will be a transformation of our given type that leaves the corners of a square in place. It appears that every square sharing two vertices with the given square will also stay in place, as will each of the four squares into which the given one may be subdivided.

* The literature of circular transformations in the plane is of course enormous. For an elaborate recent discussion see the author's *Treatise on the Circle and the Sphere*, Oxford, 1916, chap. vii, where many further references will be found.

† This proof is taken from Swift, 'On the Condition that a Point Transformation of the Plane be a Projective Transformation', *Bulletin American Math. Soc.*, vol. x, 1904; also Darboux, loc. cit.

In fact every point whose distances from two adjacent sides of the square are rational in terms of the length of a side will be invariant. If we can prove that our transformation is continuous, it will follow that every point is invariant. Now the necessary and sufficient condition that a point should be between two others is that every line through it should intersect in two points every circle through the two, and this is an invariant condition. Hence betweenness is invariant, the transformation is continuous, and every point is fixed. Our original transformation has in this way been factored into others, all of which are of type (10) or (11).

The cross ratio of four complex points has a real significance in the Gauss plane, which must now claim our attention. If the points correspond to the complex coordinate values z_1, z_2, z_3, and z_4, and be represented by the real points P_1, P_2, P_3, and P_4 respectively, and if we write

$$\frac{z_1 - z_2}{z_1 - z_4} \times \frac{z_3 - z_4}{z_3 - z_2} = re^{i\theta},$$

$$r = \frac{P_1P_2 \times P_3P_4}{P_1P_4 \times P_3P_2}, \quad \theta = \measuredangle P_4P_1P_2 - \measuredangle P_3P_1P_2.$$

It is time to take up the question of classifying collineations and anti-collineations. For this purpose we must look for the fixed points. (8) will carry a point into itself if ρ be a root of the equation

$$\begin{vmatrix} a_{11} - \rho & a_{12} \\ a_{21} & a_{22} - \rho \end{vmatrix} = 0.$$

When this equation has equal roots, there is but one fixed point, otherwise two (except, naturally, in the case of the identical collineation). Choosing the one fixed point as the unit point (1, 0) we find that $a_{11} = a_{22}, a_{21} = 0$. Our transformation can never be involutory, and each point is harmonically separated from the one invariant point by its mate in the given transformation and in its inverse. This property is characteristic of the present type of collineation, which is said to be *parabolic*.

When the transformation is not parabolic, if the two fixed points be (y) and (z) it may be written in the form

$$\frac{|xy|\cdot|x'z|}{|xz|\cdot|x'y|} = re^{i\theta}. \tag{12}$$

Theorem 20] *In a non-parabolic collineation, each cross ratio formed by a pair of corresponding points with the two fixed points is constant.*

We recognize the following types of non-parabolic collineations :

Loxodromic	$r^2-1\neq 0$,	$\theta-k\pi\neq 0$,	k integral.
Hyperbolic		$\theta-k\pi=0$	„
Elliptic		$r^2-1=0$	
Involutory		$r^2-1=\theta-k\pi=0$	k integral.

Theorem 21] *The necessary and sufficient condition that a collineation should be involutory is that corresponding points should be harmonically separated by the invariant points.*

We turn aside to give a very general theorem about transformations.

Let A be a fixed element for a transformation T_1 while a second transformation T_2 carries A into B. Then B is a fixed element in the transformation $T_2T_1T_2^{-1}$. If our transformations T_1 and T_2 be commutative, $T_2T_1T_2^{-1}$ is identical with T_1, so that B is also a fixed element of the transformation T_1. Hence either A is identical with B, and so is fixed for T_2, or the latter changes A, one fixed element of T_1, into B, another fixed element.

Theorem 22] *If two transformations be commutative, then each will permute among themselves all of the fixed elements of any kind in the other.*

If two collineations be commutative, each will leave invariant, or interchange, the fixed points of the other; if neither be involutory, they must have the same fixed points.

We leave to the reader the task of showing that this necessary condition is also sufficient.

Suppose that a collineation T is characterized by the fact that it carries ABC into $A'B'C'$ respectively. Let I be the involution which interchanges A and B', B and A'. The product is a transformation where A and B correspond to B and A respectively, and is, hence, an involution

$$II = 1, \quad IT = JT, \quad JJ = 1.$$

Hence $$T = IJ.$$

Theorem 23] *Every collineation can be factored into two involutions.*

There is another analytic form for collineation which brings out their invariants in satisfactory shape; this is the complex symbolic form, corresponding to the real form (5), namely,

$$(ax)(a'x') \equiv (\dot{a}x)(\dot{a}'x') = 0. \qquad (13)$$

The relative invariants are

$$|aa'| = \alpha, \qquad (14)$$

which vanishes when the collineation is involutory, and the discriminant

$$|a\dot{a}| \cdot |a'\dot{a}'| = \Delta_a, \qquad (15)$$

which vanishes when the collineation is degenerate. The fixed points are found from the equation

$$(ax)(a'x) = 0.$$

The condition for a parabolic collineation will be

$$\alpha^2 - 2\Delta_a = 0.$$

When this condition is not satisfied, we may choose the roots of our quadratic equation as the zero points; the collineation takes the simple form:

$$\rho x_1 = a_{11} x_1', \quad \rho x_2 = a_{22} x_2',$$

and the invariant cross ratio, which is characteristic of the transformation

$$\frac{\alpha + \sqrt{\alpha^2 - 2\Delta_a}}{\alpha - \sqrt{\alpha^2 - 2\Delta_a}}. \qquad (16)$$

Suppose that we have a second collineation

$$(bx'')\,(b'x') = 0,$$

the products are

$$|\,a'b'\,|\,(ax)\,(bx'') = 0, \quad |\,ab\,|\,(a'x)\,(b'x'') = 0.$$

The first of these may be written

$$|\,a'b'\,|\cdot|\,ab\,|\cdot|\,xx''\,| + |\,a'b'\,|\,(bx)\,(ax'')$$
$$= \tfrac{1}{2}\,|\,a'b'\,|\cdot|\,ab\,|\cdot|\,xx''\,| + \tfrac{1}{2}\,|\,a'b'\,|\,[(ax)\,(bx'') + (bx)\,(ax'')].$$

The latter part is unaltered when we interchange (x) and (x''), hence the N. S. condition that the product of the two in either order should be involutory is

$$|\,ab\,|\cdot|\,a'b'\,| = 0. \qquad (17)$$

Under these circumstances, the two collineations are said to be harmonic.

The product of one and the inverse of the other is involutory if

$$|\,ab'\,|\cdot|\,a'b\,| = 0. \qquad (18)$$

Here the two are said to be *orthogonal.*

There is an analytic form for our anti-collineations which corresponds to that for collineations, namely,

$$(ax)\,(\bar{a}'\bar{x}') = (\bar{a}\bar{x})\,(a'x') = 0. \qquad (19)$$

If there be a fixed point for the anti-collineation its coordinates will satisfy the equations

$$(ax)\,(\bar{a}'\bar{x}) = (a'x)\,(\bar{a}\bar{x}) = 0.$$

When the original anti-collineation is not involutory, these two are distinct, and the fixed points, if there be any, are roots of the equation

$$|\,\bar{a}\bar{a}'\,|\,(ax)\,(a'x) = 0. \qquad (20)$$

We shall reach this same equation if we seek a pair of points which are interchanged in our anti-collineation. The roots are the points which are invariant in that collineation which is the square of the given anti-collineation.

Theorem 24] *In a non-involutory anti-collineation either one point or two points are invariant, and no points are* *interchanged, or two points are interchanged, and none invariant.*

These three types are called respectively *parabolic, hyperbolic*, and *elliptic* anti-correlations. There remains the case of the involutory anti-collineation or *anti-involution*. Here the fixed points satisfy the equation

$$(ax)(\bar{a}\bar{x}) = 0. \tag{21}$$

We shall define as the discriminant of this

$$| a\dot{a} | \cdot | \bar{a}\bar{\dot{a}} | \equiv A, \tag{22}$$

which cannot vanish if the anti-involution be a proper transformation.

The left-hand side of the essentially real expression (21) is bilinear in the conjugate imaginary variables (x) $(\bar{x})$, and comes under a general type which we shall now define.

Definition. A form which is bilinear in a set of homogeneous variables $u_0, u_1, u_2 \ldots u_n$, and their conjugates $\bar{u}_0, \bar{u}_1, \bar{u}_2 \ldots \bar{u}_n$, and which, when multiplied by a constant non-vanishing factor is essentially real for all sets of conjugate imaginary values of the variables is called a *Hermitian Form.** Our form (21) is certainly under this head. Conversely, if the bilinear form

$$\Sigma\, a_{ij} u_i \bar{u}_j$$

be Hermitian, we must have

$$a_{ji} = \rho \bar{a}_{ij}, \quad \rho\bar{\rho} = 1.$$

If $\rho = 1$ we may write at once

$$(au)(\bar{a}\bar{u}).$$

If $\rho = e^{i\theta}$ we have merely to multiply through by $e^{-\frac{i\theta}{2}}$ to reduce to this form.

Returning to the binary Hermitian form (21), let us transform it through the collineation

$$(cx)(c'x') = 0,$$

or the anti-collineation

$$(cx)(\bar{c}'\bar{x}') = 0.$$

* Hermite, 'Sur la théorie des formes quadratiques', *Crelle's Journal*, vol. xlvii, 1854, pp. 345 ff.

The result in either case may be written

$$|\,ac\,|\cdot|\,\bar{a}\bar{c}\,|\,(c'x')\,(\bar{c}'\bar{x}'),$$

the discriminant is

$$|\,ac\,|\cdot|\,\dot{a}\dot{c}\,|\cdot|\,c'\dot{c}'\,|\cdot|\,\bar{a}\bar{c}\,|\cdot|\,\dot{\bar{a}}\dot{\bar{c}}\,|\cdot|\,\bar{c}'\dot{\bar{c}}'\,|$$
$$=\tfrac{1}{4}\,|c\dot{c}\,|\cdot|\,c'\dot{c}'\,|\cdot|\,\bar{c}\dot{\bar{c}}\,|\cdot|\,\bar{c}'\dot{\bar{c}}'\,|\;|\,a\dot{a}\,|\cdot|\,\bar{a}\dot{\bar{a}}\,|.$$

It will be noted that the part of this product which depends upon (c) is essentially positive. Hence the sign of the discriminant of the binary Hermitian form is invariant for collineations and anti-collineations. When the intrinsic sign is positive and the coefficients of $x_1\bar{x}_1$ and $x_2\bar{x}_2$ are real, the form is said to be *definite*, otherwise *indefinite*. *Two points* (x) *and* (x') *are said to be conjugate with regard to the* Hermitian form (21) if they correspond in the anti-involution (19). If we take two such points as unit points of our coordinate system, the Hermitian form takes the canonical shape

$$A_1 x_1\bar{x}_1 \pm A_2 x_2\bar{x}_2.$$

We see thus, that when the form is definite, it vanishes for no point; when it is indefinite, it vanishes for a simply infinite set of points, i.e. a set depending on one real parameter. We shall return to this form in a moment, but pause first to look at the Hermitian form of vanishing discriminant. We write this in non-symbolic notation

$$B_{11}x_1\bar{x}_1 + b_{12}x_1x_2 + \bar{b}_{12}x_2\bar{x}_1 + B_{22}x_2\bar{x}_2, \quad B_{11}B_{22} = b_{12}\bar{b}_{12}.$$

We may assume that both B_{11} and B_{22} are positive.

Let $$b_1 = \sqrt{B_{11}}\,e^{i\phi}, \qquad b_2 = \sqrt{B_{22}}\,e^{i\psi}$$

We may choose ϕ and ψ so that

$$b_1\bar{b}_2 = b_{12}, \qquad b_2\bar{b}_1 = \bar{b}_{12}.$$

We see therefore that the form divides into two conjugate imaginary linear factors.

Theorem 25] *A binary Hermitian form of vanishing discriminant is rationally factorable, and vanishes for a single point.*

We return to the indefinite Hermitian form

$$A_1 x_1 \bar{x}_1 - A_2 x_2 \bar{x}_2,$$

Writing $\quad x_1' = \sqrt{A_1}\, x_1,\; x_2' = \sqrt{A_2}\, x_2,$

we get the canonical equation

$$x_1 \bar{x}_1 = x_2 \bar{x}_2,$$

which shows that the cross ratio of the four points

$$(1, 1), (1, -1), (e^{i\theta}, 1), (x_1' x_2')$$

is real, i.e. we have a chain. Conversely, if we divide one of our equations (7) by the other and eliminate $X_1 : X_2$ between that and the conjugate equation we shall have a Hermitian form equated to zero.

Theorem 26] *An indefinite binary Hermitian form will vanish for all points of a chain.*

Theorem 27] *In an anti-involution, either there are no invariant points, or all points of a chain are invariant.*

Let us look for invariant chains under an anti-involution. If the chain

$$(bx)\,(\bar{b}\bar{x}) \equiv (\dot{b}x)\,(\bar{\dot{b}}\bar{x}) = 0$$

be invariant for the anti-involution

$$(ax)\,(\bar{a}\bar{x}') = (ax')\,(\bar{a}\bar{x}) = 0,$$

it must be identical with the chain

$$|\, ab \,|\cdot|\, \bar{a}\bar{b} \,|\cdot(\dot{a}x')\,(\bar{\dot{a}}\bar{x}') = 0,$$

$$|\, \bar{a}\bar{b} \,|\,(\bar{\dot{a}}\bar{x}') \equiv |\, \bar{\dot{a}}\bar{a} \,|\,(\bar{b}\bar{x}') + |\, \bar{a}\bar{b} \,|\,(\bar{\dot{a}}\bar{x}'),$$

$$|\, ab \,|\,(\dot{a}x')\,|\, \bar{\dot{a}}\bar{a} \,|\,(\bar{b}\bar{x}') = -\,|\, \dot{a}b \,|\,(ax')\,|\, \bar{\dot{a}}\bar{a} \,|\,(\bar{b}\bar{x}')$$

$$= -\tfrac{1}{2}\,|\, a\dot{a} \,|\cdot|\, \bar{a}\bar{\dot{a}} \,|\,(bx')\,(\bar{b}\bar{x}').$$

The transformed chain may thus be written

$$-\tfrac{1}{2}\,|\, a\dot{a} \,|\cdot|\, \bar{a}\bar{\dot{a}} \,|\,(bx')\,(\bar{b}\bar{x}') + |\, ab \,|\cdot|\, \bar{a}\bar{b} \,|\,(\dot{a}x')\,(\bar{\dot{a}}\bar{x}') = 0.$$

The condition reduces to

$$D_{ab} \equiv |\, ab \,|\cdot|\, \bar{a}\bar{b} \,| = 0. \tag{23}$$

Theorem 28] *In every anti-involution there will be a system of invariant chains depending on two real parameters.*

Theorem 29] *The necessary and sufficient condition that a chain should be invariant for an anti-involution is that the anti-involution associated with the given chain should be commutative with the given anti-involution.*

We leave the proofs of these simple theorems to the reader. Two chains which are related in this way are said to be *orthogonal.*

Theorem 30] *An anti-involution will appear in the Gauss plane as an inversion in a circle, or the product of such an inversion with a reflection in its centre. Orthogonal chains will be represented by orthogonal lines or circles.*

Let us extend our inquiry by looking for invariant chains under a collineation. The general non-parabolic collineation may be written

$$x_1 = re^{\iota\theta} x_1', \quad x_2 = x_2'.$$

Let the chain be

$$A_{11} x_1 \bar{x}_1 + a_{12} x_1 \bar{x}_2 + \bar{a}_{12} x_2 \bar{x}_1 + A_{22} x_2 \bar{x}_2 = 0.$$

The transformed chain is

$$r^2 A_{11} x_1' \bar{x}_1' + re^{\iota\theta} a_{12} \bar{x}_1' x_2' + re^{-\iota\theta} \bar{a}_{12} \bar{x}_1' x_2 + A_{21} x_2' \bar{x}_2' = 0.$$

The original chain will be invariant if

1) $$A_{11} = A_{22} = 0, \quad e^{r\iota\theta} = 1.$$

2) $$a_{12} = \bar{a}_{12} = 0, \quad r = 1.$$

In the first case we have a hyperbolic collineation, and the chain passes through the fixed points, in the second case we have an elliptic collineation and the chain is orthogonal to every chain through the fixed points. In this case it shall be said to be *about* the two points. A typical parabolic collineation may be written

$$x_1 = x_1' + b_{12} x_2', \quad x_2 = x_2'.$$

The transformed chain is, in this case,

$$A_{11} x_1' \bar{x}_1' + (A_{11} \bar{b}_{12} + a_{12}) x_1' \bar{x}_2' + (A_{11} b_{12} + \bar{a}_{12}) x_2' \bar{x}_1' + (A_{11} b_{11} \bar{b}_{12} + a_{12} b_{12} + \bar{a}_{12} \bar{b}_{12} + A_{22}) x_2' \bar{x}_2' = 0.$$

For an invariant chain it is necessary and sufficient that

$$A_{11} = a_{12}b_{12} + \bar{a}_{12}\bar{b}_{12} = 0.$$

We have an infinite number of invariant chains. All pass through the single fixed point, and no two have any other point in common. We next take the hyperbolic anti-collineation

$$x_1 = b_{11}\bar{x}_1', \quad x_2 = b_{22}\bar{x}_2', \quad b_{11}\bar{b}_{11} \neq b_{22}\bar{b}_{22}.$$

The transformed chain is

$$A_{11}\, b_{11}\, \bar{b}_{11}\, x_1'\, \bar{x}_1' + \bar{a}_{12}\, \bar{b}_{11}\, b_{22}\, x_1'\, \bar{x}_2'$$
$$+ a_{12}\, b_{11}\, \bar{b}_{22}\, x_2\, \bar{x}_1' + A_{22}\, b_{22}\bar{b}_{22}\, x_2'\, \bar{x}_2' = 0.$$

For an invariant chain we must have

$$A_{11} = A_{22} = 0, \quad a_{12} = \sqrt{b_{11}\, \bar{b}_{22}}.$$

There are thus two mutually orthogonal chains through the fixed points. An elliptic anti-involution may be written

$$x_1 = b_{12}\bar{x}_2', \quad x_2 = b_{21}\bar{x}_1', \quad b_{12}\bar{b}_{12} \neq b_{21}\bar{b}_{21}.$$

The transformed chain is

$$A_{22}b_{21}\bar{b}_{21}x_1'\bar{x}_1' + a_{12}\bar{b}_{12}b_{21}x_1'\bar{x}_2' + \bar{a}_{12}b_{12}\bar{b}_{21}x_2\bar{x}_1'$$
$$+ A_{11}b_{12}\bar{b}_{12}x_2'\bar{x}_2' = 0.$$

There is only one invariant chain, namely,

$$\sqrt{b_{21}\, \bar{b}_{21}}\, x_1\, \bar{x}_1 - \sqrt{b_{12}\bar{b}_{12}}\, x_2\, \bar{x}_2 = 0.$$

Lastly, take the parabolic case

$$x_1 = \bar{x}_1' + b_{12}\bar{x}_{12}', \quad x_2 = \bar{x}_2'.$$

The transformed chain is

$$A_{11}x_1'\bar{x}_1' + (A_{11}b_{12} + \bar{a}_{12})\, x_1'\bar{x}_2' + (A_{11}\bar{b}_{12} + a_{12})\, x_2\bar{x}_1'$$
$$+ (A_{11}b_{12}\bar{b}_{22} + a_{12}b_{12} + \bar{a}_{12}\bar{b}_{12} + A_{22})\, x_2'\bar{x}_2' = 0.$$

There is but a single invariant chain, namely,

$$i\,(x_1\bar{x}_2 - x_2\bar{x}_1) + \frac{i}{2}(\bar{b}_{12} - b_{12}) = 0.$$

The study of these invariant chains is important, as it enables us to answer completely the question as to what conditions are necessary and sufficient if two of our transformations are to be commutative. We see from 22] that they must have the same system of invariant points and invariant chains; let the reader show by taking up case after case that these necessary conditions are also sufficient:

Theorem 31] *The necessary and sufficient condition that two collineations should be commutative is that they should have the same fixed points, or that one should be involutory, interchanging the fixed points of the other.*

Theorem 32] *The necessary and sufficient condition that a collineation and an anti-collineation, neither of which is involutory, should be commutative, is that they should be hyperbolic, elliptic, or parabolic together, with the same fixed or interchanging points, and, in the parabolic case, that the chain fixed in the anti-collineation should be among the fixed chains of the collineation. If the collineation be involutory, the anti-collineation must be hyperbolic with the same fixed points, or elliptic with interchanging points interchanged in the collineation, or involutory, keeping fixed or interchanging the fixed points of the collineation.*

Theorem 33] *The necessary and sufficient condition that two anti-collineations, neither of which is involutory, should be commutative, is that they should be elliptic, hyperbolic, or parabolic together, with the same fixed or interchanging points, and the same invariant chains. An anti-involution will be commutative with any anti-collineation whose fixed or interchanging points and chains are fixed or interchanged by it.* The necessary and sufficient condition that two anti-involutions should be commutative is that their product should be an involution.*

* The last two theorems are incorrectly given in the author's *Treatise on the Circle and the Sphere*, cit. p. 328, although the error is mentioned in the table of errata. For a correct statement in a less geometrical form see Benedetti, 'Sulla teoria delle forme iperalgebriche', *Annali della R. Scuola normale di Pisa*, vol. viii, 1899, pp. 60 ff.

Given a non-parabolic collineation

$$x_1 = re^{i\theta}x_1', \quad x_2 = x_2',$$

this can be factored into

$$x_1 = rx_1'', \qquad x_2 = x_2'',$$

$$x_1'' = e^{i\theta}x_1', \quad x_2'' = x_2'.$$

The first is the product of the anti-involutions

$$x_1 = \sqrt{r}\bar{x}_2, \quad x_2 = \bar{x}_1, \quad \dot{x}_1 = \bar{x}_2'', \quad \dot{x}_2 = \sqrt{r}\bar{x}_1'',$$

while the second is the product of

$$x_1'' = e^{i\phi}\bar{x}_1, \quad x_2'' = \bar{x}_2, \quad \dot{x}_1 = e^{i(\phi-\theta)}\bar{x}_1', \quad \dot{x}_2 = \bar{x}_2'.$$

The parabolic case can be treated as the limit of the hyperbolic case:

Theorem 34] *A loxodromic collineation is the product of four anti-involutions, a non-loxodromic one is the product of two.*

If we take an anti-collineation which is not involutory and precede it by an anti-involution which leaves its invariant chain in place, the product is a collineation which has an invariant chain, and so is not loxodromic:

Theorem 35] *A non-involutory anti-collineation can be factored into three anti-involutions.*

Before leaving our study of anti-collineations, it is worth while to find a few of their invariants.* Given the anti-collineation

$$(ax)(\bar{a}'\bar{x}') \equiv (\dot{a}x)(\bar{\dot{a}}'\bar{x}') = 0.$$

If $$\alpha_1 \equiv | a\dot{a} | \cdot | \bar{a}'\bar{\dot{a}}' | = 0,$$

the transformation is improper. The fixed points of the squared transformation come from

$$| \bar{a}'\bar{\dot{a}} | (ax)(\dot{a}'x) = 0.$$

* For the remainder of the present section cf. Benedetti, loc. cit., pp. 63 ff. This writer has an insatiable appetite for finding complete systems of invariants of these forms and transformations. See also Peano, 'Formazioni invariantive delle corrispondenze', Battaglino's *Giornale di matematica*, vol. xx, 1882, pp. 81 ff.

This will vanish identically if the given transformation be an anti-involution. If, on the other hand,

$$\alpha_0 \equiv | a\dot{a}' | \cdot | \bar{a}'\bar{\dot{a}}' | = 0,$$

the original transformation had a period four. The condition for a parabolic anti-collineation is

$$| \bar{a}'\bar{\dot{a}} | \cdot | \bar{b}'\bar{\dot{b}} | [| a\dot{a}' | \cdot | b\dot{b}' | - 2 | ab | \cdot | \dot{a}'\dot{b}' |] = 0,$$

where (a), $(\dot{a})$, (b), $(\dot{b})$, are all equivalent letters. The first part of these is α_0^2. To find the second part, we note that

$$| \bar{a}'\bar{\dot{a}} | \cdot | \bar{b}'\bar{b} | = | \bar{a}'\bar{b}' | \cdot | \bar{\dot{a}}\bar{\dot{b}} | - | \bar{\dot{a}}\bar{b}' | \cdot | \bar{a}'\bar{b} |,$$

$$| ab | \cdot | \dot{a}'\dot{b}' | \cdot | \bar{a}'\bar{\dot{a}} | \cdot | \bar{b}'\bar{\dot{b}} | = | ab | \cdot | \dot{a}'\dot{b}' | \cdot | \bar{\dot{a}}\bar{b}' | \cdot | \bar{a}'\bar{b} |.$$

Hence, the second part is

$$- | ab | \cdot | \bar{a}'\bar{b}' | \cdot | \dot{a}'\dot{b}' | \cdot | \bar{a}'\bar{\dot{b}} | = -\alpha_1\bar{\alpha}_1,$$

and the condition for a parabolic anti-collineation is

$$\alpha_0^2 - \alpha_1\bar{\alpha}_1.$$

In the case of an anti-involution $\alpha_0 = 0$ and $\alpha_1 = A$ is the only invariant. Let us look for concomitants of the two anti-involutions

$$(ax)(\bar{a}\bar{x}') = (\dot{a}x)(\bar{\dot{a}}\bar{x}') = 0,$$

$$(bx)(\bar{b}\bar{x}') = (\dot{b}x)(\bar{\dot{b}}\bar{x}') = 0.$$

$$A = | a\dot{a} | \cdot | \bar{a}\bar{\dot{a}} |, \quad B = | b\dot{b} | \cdot | \bar{b}\bar{\dot{b}} |,$$

$$D_{ab} = | ab | \cdot | \bar{a}\bar{b} |.$$

A, B, and D_{ab} are only symbolic products of conjugate imaginary factors and are not, therefore, necessarily positive, though they are necessarily real. In fact, if the Hermitian forms be written with coefficients of $x_1\bar{x}_1$ real, there will be two invariant chains if

$$A < 0, \quad B < 0.$$

We may reduce these to

$$ix_1\bar{x}_2' - ix_2\bar{x}_1' = 0, \quad (bx)(\bar{b}\bar{x}') = 0,$$

$$A = -2, \quad B = 1 \, | b\dot{b} | \cdot | \bar{b}\bar{\dot{b}} |, \quad D_{ab} = i(b_1\bar{b}_2 - \bar{b}_1 b_2).$$

The first transformation leaves the chain of reals in place. If it contain two members of the other chain they will be given by

$$b_1\bar{b}_1X_1^2+(b_1\bar{b}_2+\bar{b}_1b_2)X_1X_2+b_2\bar{b}_2X_2^2=0.$$

For real solutions

$$AB-D_{ab}^2 \geqq 0, \quad A<0, \quad B<0. \qquad (24)$$

Let the reader show that the angle of the circles in the Gauss plane which represent these chains will be given by

$$\cos\theta=\frac{D_{ab}}{\sqrt{A}\sqrt{B}}.$$

We have already seen that the condition for orthogonal chains is

$$D_{ab}=0. \qquad (23)$$

Take three anti-involutions

$$(ax)(\bar{a}\bar{x}')=0, \quad (bx)(\bar{b}\bar{x}')=0, \quad (cx)(\bar{c}\bar{x}')=0.$$

If they be not linearly dependent we may find an anti-involution

$$(dx)(\bar{d}\bar{x}')=0,$$

commutative with all three. We must eliminate d and $\bar{d}$ from the equation, and from

$$|ad|\cdot|\bar{a}\bar{d}|=|bd|\cdot|\bar{b}\bar{d}|=|cd|\cdot|\bar{c}\bar{d}|=0.$$

The result is

$$\begin{vmatrix} a_1\bar{a}_1 & a_1\bar{a}_2 & a_2\bar{a}_1 & a_2\bar{a}_2 \\ b_1\bar{b}_1 & b_1\bar{b}_2 & b_2\bar{b}_1 & b_2\bar{b}_2 \\ c_1\bar{c}_1 & c_1\bar{c}_2 & c_2\bar{c}_1 & c_2\bar{c}_2 \\ x_2\bar{x}_2' & -x_2\bar{x}_1' & -x_1\bar{x}_2' & x_1\bar{x}_1' \end{vmatrix}=0.$$

Developing by Laplace's method

$$|ac|\cdot|\bar{a}\bar{b}|(bx)(\bar{c}\bar{x}')-|ab|\cdot|\bar{a}\bar{c}|(cx)(\bar{c}\bar{x}')=0.$$

The discriminant

$$|dd|\cdot|\bar{d}\bar{d}|$$

can be written as the symbolic product of the matrices, namely,

$$\left\|\begin{matrix} a_1\bar{a}_1 & a_1\bar{a}_2 & a_2\bar{a}_1 & a_2\bar{a}_2 \\ b_1\bar{b}_1 & b_1\bar{b}_2 & b_2\bar{b}_1 & b_2\bar{b}_2 \\ c_1\bar{c}_1 & c_1\bar{c}_2 & c_2\bar{c}_1 & c_2\bar{c}_2 \end{matrix}\right\| \times \left\|\begin{matrix} \dot{a}_2\bar{\dot{a}}_2 & -\dot{a}_2\bar{\dot{a}}_1 & -\dot{a}_1\bar{\dot{a}}_2 & \dot{a}_1\bar{\dot{a}}_1 \\ \dot{b}_2\bar{\dot{b}}_2 & -\dot{b}_2\bar{\dot{b}}_1 & -\dot{b}_1\bar{\dot{b}}_2 & \dot{b}_1\bar{\dot{b}}_1 \\ \dot{c}_2\bar{\dot{c}}_2 & -\dot{c}_2\bar{\dot{c}}_1 & -\dot{c}_1\bar{\dot{c}}_2 & \dot{c}_1\bar{\dot{c}}_1 \end{matrix}\right\|$$

$$= D_{abc} \equiv \begin{vmatrix} A & D_{ab} & D_{ac} \\ D_{ba} & B & D_{bc} \\ D_{ca} & D_{cb} & C \end{vmatrix}.$$

Thus the three chains

$$(ax)(\bar{a}\bar{x}) = 0, \quad (bx)(\bar{b}\bar{x}) = 0, \quad (cx)(\bar{c}\bar{x}) = 0,$$

$$A < 0, \qquad B < 0, \qquad C < 0,$$

will have a common point without being linearly dependent, when and only when

$$D_{abc} = 0. \tag{25}$$

§ 3. Chains.

We saw in Theorems 9] and 10] that three distinct points will belong to only one chain. Through two given points there will pass an infinite number of chains. An arbitrary chain through the points (y) and (z) may be expressed parametrically

$$x_i = X_1\rho y_i + X_2\sigma z_i, \tag{26}$$

where ρ and σ are two complex multipliers, constant for the chain. Take the anti-involution

$$(ax)(\bar{a}\bar{x}') = 0.$$

This will interchange the points (y) and (z) above if

$$(ay)(\bar{a}\bar{z}) = (a\bar{z})(\bar{a}\bar{y}) = 0.$$

The point (x) given here will be changed into the point x' where

$$x_i' = Y_1\rho y_1 + Y_2\sigma z_i; \quad X_1 Y_1 \rho\bar{\rho}(y\bar{y}) + X_2 Y_2 \sigma\bar{\sigma}(z\bar{z}) = 0.$$

Theorem 36] *If an anti-involution interchange a single pair of points of a chain, the whole chain is invariant in the* *anti-involution, and if the latter have a chain of fixed points, this latter is orthogonal to the given chain.**

Theorem 37] *An involution leaves invariant every chain through its fixed points and every chain about them.*

Theorem 38] *Through a given point will pass just one chain about two given points.*

We see, in fact, that there is but one anti-involution that leaves one point invariant and interchanges two others. Let the reader next prove:

Theorem 39] *Through two given points will pass one chain invariant in a given anti-involution, and only one when the given points are not interchanged in the transformation.*†

Theorem 40] *There is just one anti-involution which interchanges two given points, and leaves invariant a given chain not through either point. If there be a chain about the points orthogonal to the given chain, it is invariant for the transformation.*

Theorem 41] *If a chain be invariant for an anti-involution, the transformation of its points is projectively equivalent to an involution in the real binary domain.*

Suppose that we have two given points and a chain which contains neither and is not about the two. There is one chain through the given points, orthogonal to the given chain, and one anti-involution exchanging them and leaving the given chain invariant. This latter will leave invariant all chains through the two points as well as the given chain. The anti-involution associated with the chain through the two points orthogonal to the given chain will leave invariant all chains about the two points, and also the given chain:

Theorem 42] *If a chain contain neither of two given points, and if it be not about them, it will be met by chains through the given points in pairs of points of one involution, and by chains about them in pairs of points of another.*

* The remaining theorems, which are merely restatements of familiar theorems about orthogonal circles in the Gauss plane, will be found in an article by Young, 'The Geometry of Chains on the Complex Line', *Annals of Math.*, Series (2), vol. xi, 1909, pp. 37 ff.

† Incorrectly given. Ibid., p. 41.

The chains about the unit points will appear in the Gauss plane as the circles whose common centre is the origin. Two of these will touch every circle not of the system, and every line not through their common centre:

Theorem 43] *If a chain do not pass through two given points, and be not about them, then two chains about the points will be tangent to it. If, furthermore, there be a chain about the two points orthogonal to the given chain, there will be two chains through the two points tangent to that chain.*

§ 4. Hyperalgebraic forms.

Our binary complex domain is a one-parameter system, when considered in terms of complex variables, but a two-parameter system when considered in terms of real variables. We may thus find therein families of points depending analytically on a single real parameter; the chain is the first example. Every real analytic curve in the Gauss plane will represent such a system, and every such system will be represented by a real curve.

Definition. A system of points in any continuum, whose coordinates are analytic functions, not all constants, of a single real parameter, shall be called a *thread.*

When we use homogeneous coordinates we shall assume that the ratios are not all constants either. If the functions involved be algebraic, we shall say that the thread is algebraic. Such a thread will appear in the Gauss plane as the real algebraic curve

$$F(X, Y) = 0,$$

where the coefficients are all real. Writing

$$X = \frac{x_1\bar{x}_2 + x_2\bar{x}_1}{2\,x_2\bar{x}_2}, \quad Y = -i\,\frac{x_1\bar{x}_2 - x_2\bar{x}_1}{2\,x_1\bar{x}_2},$$

we have

$$\sum \frac{n\,!}{p\,!\,(n-p)\,!}, \quad \frac{n\,!}{q\,!\,(n-q)\,!},$$

$$a_{pq}x_1^p x_2^{n-p}\bar{x}_1^q\bar{x}_2^{n-q} = 0, \quad a_{qp} = \bar{a}_{pq},$$

which may be written in the symbolic form

$$(ax)^n (\bar{a}\bar{x})^n = 0. \tag{27}$$

The left-hand side of this is called a *hyperalgebraic form*. The number $2n$ is called the *order* of the thread; it has a significance which we shall discover later. Conversely, if we start with our equation (7) and put

$$\frac{x_1}{x_2} = X + iY, \quad \frac{\bar{x}_1}{\bar{x}_2} = X - iY,$$

$$F(X, Y) = 0,$$

where the coefficients are real. Suppose that we can find a set of values satisfying this equation (27) for which

$$a_i (ax)^{n-1} (\bar{a}\bar{x})^n \not\equiv 0, \quad \bar{a}_i (ax)^n (\bar{a}\bar{x})^{n-1} \not\equiv 0.$$

Putting $\quad x_1 = x, \quad x_2 = 1, \quad \bar{x}_1 = \bar{x}, \quad \bar{x}_2 = 1,$

$$f(x, \bar{x}) = 0, \quad \frac{\partial f}{\partial x} \neq 0, \quad \frac{\partial f}{\partial \bar{x}} \neq 0,$$

whence $\quad F(X, Y) = 0, \quad \dfrac{\partial F}{\partial X} \neq 0, \quad \dfrac{\partial F}{\partial Y} \neq 0.$

Now, if a real algebraic plane curve contain a single real point which is not singular, it contains a single infinite set of such points. The point (x) which satisfies the inequalities above shall also be said to be *not singular* for the form:

Theorem 44] *If a single point can be found whose coordinates reduce to zero a given hyperalgebraic form, and if the point be not singular for that form, then the form equated to zero will give the equation of a thread. Conversely, every algebraic thread is determined by equating a hyperalgebraic form to zero.*

The difference between a singular and a non-singular point is easily seen in the case of a binary Hermitian form. If, when the coefficient of $x_1\bar{x}_1$ is real the discriminant be negative, there is a thread with a singly infinite set of non-singular points. If the discriminant be zero, there is, as we saw in 25], only one point and this is singular.

Let us now ask the question: What points does the chain (26) share with the algebraic thread (27)? Substituting

$$[\rho\,(ay)\,X_1 + \sigma\,(az)\,X_2]^n\,[\bar\rho\,(\bar a\bar y)\,X_1 + \bar\sigma\,(\bar a\bar z)\,X_2]^n = 0. \quad (28)$$

Theorem 45] *The number of points common to a chain and an algebraic thread is in no case greater than the total order of the thread, except in the case where the chain is completely contained in the thread.*

The total coefficient of ${X_1}^{2n-k}\,{X_2}^k$ in the expanded equation (28) is

$$\frac{n!}{k!\,(n-k)!}\,\rho^n\bar\rho^{n-k}\sigma^k(ay)^n(\bar a\bar y)^{n-k}(\bar a\bar z)^k + \frac{n\cdot n!}{(k-1)!\,(n-k+1)!}$$
$$\cdot\,\rho^{n-1}\sigma\bar\rho^{n-k+1}\bar\sigma^{k-1}(ay)^{n-1}(az)\,(\bar a\bar y)^{n-k+1}(\bar a\bar z)^{k-1} + \ldots \quad (29)$$

Let us inquire under what circumstances this will vanish for all sets of conjugate imaginary values $\rho\,\bar\rho$, $\sigma\,\bar\sigma$, $(z)\,(\bar z)$.

Fundamental Lemma.

If for all sets of conjugate imaginary values

$$x_1 x_2 \ldots x_m, \quad \bar x_1 \bar x_2 \ldots \bar x_m,$$

the analytic function

$$f\,(x_1 x_2 \ldots x_m, \quad \bar x_1 \bar x_2 \ldots \bar x_m)$$

is equal to zero, then this function vanishes identically for all sets of values of the variables involved.

Let us prove this lemma by mathematical induction. If the function

$$f\,(x_1, \bar x_1)$$

vanishes for all sets of conjugate imaginary values $x_1, \bar x_1$, putting $x_1 = X + iY$ we get $\phi\,(X, Y) = 0$ for all real values X, Y. But an analytic function of X and Y which vanishes for all real values must have vanishing partial derivatives of every order at each real point and, hence, by Taylor's theorem, be identically zero. If, then, our fundamental lemma be assumed for $m-1$ pairs of conjugate values, it holds for the last also.

Returning to our equation (27), we see that this will vanish for all sets of conjugate imaginary values,

$$\rho\bar{\rho},\ \sigma\bar{\sigma},\ z_1\bar{z}_1,\ z_2\bar{z}_2,$$

if $(ay)^{n-r}(\bar{a}\bar{y})^{n-s}a_1^{\lambda}a_2^{r-\lambda}\bar{a}_1^{\mu}\bar{a}_2^{s-\mu} \equiv 0, \quad r+s=k.$

We shall have furthermore

$$(ay)^{n-r}(\bar{a}\bar{y})^{n-s}a_1^{\lambda}a_2^{r-\lambda}\bar{a}_1^{\mu}\bar{a}_2^{s-\mu} = 0, \quad r+s \leqq k, \qquad (30)$$

so that the total coefficient of $X_1^{2n-k+s}X_2^{k-s}$ will also vanish. Hence the left side of (27) is divisible by X_2^{k+1}, or every chain through (y) will have $k+1$ intersections with the thread at that point. These conditions are both necessary and sufficient. The point (y) shall be said, under these circumstances, to have the multiplicity $k+1$. If $k \geqq 1$ it may be an isolated point of the thread:

Theorem 45] *A point (y) will have the multiplicity $k+1$ for the thread (27) when and only when*

$$(ay)^{n-r}(\bar{a}\bar{y})^{n-s}a_1^{\lambda}a_2^{r-\lambda}\bar{a}_1^{\mu}\bar{a}_2^{s-\mu} = 0, \quad r+s \leqq k, \qquad (30)$$

yet these equations do not hold for all values of $r+s \leqq k+1$.

We are easily led from these equations to a form of polarization in our hyperalgebraic forms. We define as the (p,q)th polar of (y) the expression

$$\frac{p!\,q!}{(n!)^2}\left(y\frac{\partial}{\partial x}\right)^p\left(\bar{y}\frac{\partial}{\partial\bar{x}}\right)^q(ax)^n(\bar{a}\bar{x})^n = (ay)^p(\bar{a}\bar{y})^q(ax)^{n-p}(\bar{a}\bar{x})^{n-q}.$$

It is to be noted that only in the case where $p=q$ is this hyperalgebraic. Equating this to zero we get

$$(ay)^p(\bar{a}\bar{y})^q(ax)^{n-p}(\bar{a}\bar{x})^{n-q} = (ay)^q(\bar{a}\bar{y})^p(ax)^{n-q}\bar{a}\bar{x}^{n-p} = 0;$$

eliminating (x) and $(\bar{x})$ in turn we get

$$(cx)^{(n-p)^2+(n-q)^2} = 0, \quad (\bar{c}\bar{x})^{(n-q)^2+(n-p)^2} = 0.$$

If there be any pairs of conjugate imaginary values which will reduce to zero the (p,q)th polar and its conjugate where $p \neq q$, there are roots of these equations. A point where coordinates reduce to zero the (p,q)th polar of a given point shall be said to be *on that polar.*

Theorem 46] *There are at most* $(n-p)^2+(n-q)^2$ *points on the* (p, q)*th polar of a given point, when* $p \neq q$, *unless the point have a multiplicity of at least* $2n+1-p-q$, *in which case this polar vanishes identically.*

Theorem 47] *If a point have a multiplicity* $k+1$ *for a hyperalgebraic thread, it will have at least a multiplicity* $k+1-p-q$ *for the* (p, q)*th polar of an arbitrary point.*

There is much more interest in those polars which are hyperalgebraic, i.e. where $p = q$. If there be a single non-singular point on such a polar, there is a whole thread. The $(n-1, n-1)$th polar is called the *chain polar.* If it contain any non-singular point, it will contain a chain of points, called the *polar chain.* The chain polar of (y) will, by 25], vanish for only one point if

$$(ay)^{n-1}(\dot{a}y)^{n-1}\bar{a}\bar{y}^{n-1}(\bar{\dot{a}}\bar{y})^{n-1}\,|\,a\dot{a}\,|\cdot|\,\bar{a}\bar{\dot{a}}\,| = 0.$$

It is evident that the chain polar gives rise to an anti-involution. The locus of points so situated that the anti-involutions associated with their chain polars with regard to the two hyperalgebraic forms

$$(ax)^n(\bar{a}\bar{x})^n, \quad (bx)^m(\bar{b}\bar{x})^m,$$

are commutative, will be given by the equation

$$|\,ab\,|\cdot|\,\bar{a}\bar{b}\,|\cdot(ax)^{n-1}(bx)^{m-1}(\bar{a}\bar{x})^{n-1}(\bar{b}\bar{x})^{m-1} = 0.$$

Of course this equation may not be satisfied by any points at all. It will contain any point there may be whose polar chains are orthogonal.

Let us now suppose that (y) is a simple point of our thread (27). In the expression (28) the coefficient of X_1^{2n} will vanish automatically. That of $X_1^{2n-1}X_2$ will be

$$\rho^{n-1}\bar{\rho}^{n-1}(ay)^{n-1}(\bar{a}\bar{y})^{n-1}\ [\rho\bar{\sigma}(ay)(\bar{a}\bar{z})+\bar{\rho}\sigma(\dot{a}z)(\bar{a}\bar{y})].$$

The equation $\qquad (ay)^{n-1}(\bar{a}\bar{y})^n(ax) = 0$

has only one root, namely $\quad (x)=(y).$

Hence the coefficient of $X_1^{2n-1}X_2$ cannot vanish independently of ρ and σ. If

$$\rho\bar{\sigma}(ay)^n(\bar{a}\bar{y})^{n-1}(\bar{a}\bar{z})+\bar{\rho}\sigma(ay)^{n-1}(az)(\bar{a}\bar{z})^n = 0 \qquad (31)$$

the coefficient of $X_1^{2n-1} X_2$ will vanish. There is but one chain of the type (26) where this is the case:

Theorem 48] *A non-singular point of an algebraic thread will be connected with every other point of the line by just one chain tangent to the thread.*

The chain polar of (y) is given by the equation

$$(ay)^{n-1}(\bar{a}\bar{y})^{n-1}(ax)(\bar{a}\bar{x}) = 0.$$

If (z) be a point of this polar chain, and if we write the condition that all points of the chain (26) be included therein, we find (31) again:

Theorem 49] *The polar chain of a non-singular point of an algebraic thread is tangent to the thread at that point.*

We may obtain still further light on the (p, q)th polar form when we represent our binary domain on the Riemann sphere. Let us write

$$sX_0 = x_1\bar{x}_1 + x_2\bar{x}_2, \quad sX_1 = x_1\bar{x}_1 - x_2\bar{x}_2, \quad sX_2 = x_1\bar{x}_2 + x_2\bar{x}_1,$$
$$sX = -i(x_1\bar{x}_2 - x_2\bar{x}_1).$$

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 \equiv (CX)^2 = 0.$$

In plane coordinates,

$$-U_0^2 + U_1^2 + U_2^2 + U_3^2 \equiv (\Gamma U)^2 = 0.$$

Our thread (27) becomes the surface,

$$[(a_1\bar{a}_1 + a_2\bar{a}_2) X_0 + (a_1\bar{a}_1 - a_2\bar{a}_2) X_1 + (a_1\bar{a}_2 + a_2\bar{a}_1) X_2$$
$$+ i(a_1\bar{a}_2 - a_2\bar{a}_2) X_3]^n \equiv AX^n = 0.$$

The quadric polar of (Y) will be

$$[(a_1\bar{a}_1 + a_2\bar{a}_2) Y_0 + (a_1\bar{a}_1 - a_2\bar{a}_2) Y_1 + (a_1\bar{a}_2 + a_2\bar{a}_1) Y_2$$
$$+ i(a_1 a_2 - a_2\bar{a}_1) Y_3]^{n-2}.$$

$$[(a_1\bar{a}_1 + a_2\bar{a}_2) X_0 + (a_1\bar{a}_1 - a_2\bar{a}_2) X_1 + (a_2\bar{a}_2 + a_2\bar{a}_1) X_2$$
$$+ i(a_1\bar{a}_2 - a_2\bar{a}_1) X_3]^2 \equiv (AY)^{n-2}(AX)^2 = 0.$$

The condition for apolarity with $(\Gamma U)^2$ is

$$(AY)^{n-2}(A\Gamma)^2 = 0.$$

$$[(a_1\bar{a}_1 + a_2\bar{a}_2) Y_0 + (a_1\bar{a}_1 - a_2\bar{a}_2) Y_1 + (a_1\bar{a}_2 + a_2\bar{a}_1) Y_2$$
$$+ i(a_1\bar{a}_2 - a_2\bar{a}_1) Y_3]^{n-2}.$$

$$[-(a_1\bar{a}_1 + a_2\bar{a}_2)^2 + (a_1\bar{a}_1 - a_2\bar{a}_2)^2$$
$$+ (a_1\bar{a}_2 + a_2\bar{a}_1)^2 - (a_1\bar{a}_2 - a_2\bar{a}_1)^2] = 0,$$

and this condition is certainly satisfied, as the second factor vanishes identically. The two quaternary quadratic forms are, thus, apolar. Moreover, it is not hard to show that through the total intersection of an algebraic surface and a quadric there will pass but one surface of the same order as the given one which possesses the property that the quadric polar of an arbitrary point is apolar to the given quadric. The substitution of quaternary for binary coordinates gives just this surface through the curve on the Riemann sphere which represents the given thread. Lastly, we notice that the (p, p)th polar of (y) with regard to (27) becomes

$$[(a_1\bar{a}_1+a_2\bar{a}_2)\,Y_0+(a_1\bar{a}_1-a_2\bar{a}_2)\,Y_1 + (a_1\bar{a}_2+a_2\bar{a}_1)\,Y_2+i(a_1\bar{a}_2-a_2\bar{a}_1)\,Y_3]^p$$
$$[(a_1\bar{a}_1+a_2\bar{a}_2)\,X_0+(a_1\bar{a}_1-a_2\bar{a}_2)\,X_1 + (a_1\bar{a}_2-a_2\bar{a}_1)\,X_2+i(a_1\bar{a}_2-a_2\bar{a}_1)\,X_3]^{n-p}=0,$$
$$(AY)^p\,(AX)^{n-p}=0,$$

which is the pth polar of (Y) with regard to the algebraic surface. We thus reach the capital proposition:*

Theorem 50] *The quaternary form which corresponds by* (31) *to the thread* (27) *gives such a surface through the curve on the Riemann sphere that represents the thread, that the quadric polar of an arbitrary point is apolar to the sphere. At the same time the* (p,p)*th polar of a point in the binary domain will correspond to the intersection of the Riemann sphere with the* p*th polar of the corresponding point.*

It would lead us too far afield to go further into the invariant theory. We turn instead to the task of finding a geometrical significance for $2n$. We reach this by a system of successive approximations. Let us start with

$$(ay)\,(\bar{a}\bar{y})\,(ax)\,(\bar{a}\bar{x})=0. \tag{32}$$

The discriminant of this is

$$(ay)\,(\dot{a}y)\,(\bar{a}\bar{y})\,(\bar{\dot{a}}\bar{y})\;|\,a\dot{a}\,|\cdot|\,\bar{a}\bar{\dot{a}}\,|.$$

* This is the fundamental theorem in an admirable article by Kasner, 'The Invariant Theory of the Inversion Group', *Transactions American Math. Soc.*, vol. i, 1900, p. 44.

If there be a thread of points (y) for which this expression vanishes, then the discriminant of the chain in (x) cannot have a constant sign, or there are an infinite number of points (y), each of which corresponds to a chain of points. If there be any points (y) which lie on the chains which correspond to them, they will belong to a thread whose equation is

$$(ax)^2 (\bar{a}\bar{x})^2 = 0.$$

Conversely, this last equation gives rise, through polarization, to the relation (32).

Let us next suppose that we have set up a relation, called an *equipolarization*, of the following type:

A) Each thread of order $2(n-1)$ of an infinite system corresponds to a point in a region depending analytically on two real parameters, and, conversely, each point of this region shall correspond to a thread of this order.

B) An infinite number of points lie on the corresponding threads, but these points do not all lie on a thread of order $2(n-1)$ or less.

C) The $(1, 1)$th polar of (z) with regard to the thread corresponding to (y) is identical with the $(1, 1)$th polar of (y) with regard to the thread corresponding to (z).

Let us see analytically what these conditions involve.

Clearly by A) and C) the equation of the thread corresponding to (y) must be bilinear in (y) and $(\bar{y})$ say,

$$\sum l^{(p,q)}_{i,\,j,\,i',\,j'} y_p \bar{y}_q x_1{}^i x_2{}^j \bar{x}_1{}^{i'} \bar{x}_2{}^{j'}.$$

$$p = 1,\ 2, \quad q = 1,\ 2, \quad i+j = n-1, \quad i'+j' = n-1.$$

Since this is essentially real

$$\sum_{p,\,q} l^{(p,q)}_{i,\,j,\,i',\,j'} y_p \bar{y}_q = \sum_{p,\,q} \bar{l}^{(p,q)}_{i',\,j',\,i,\,j} y_q \bar{y}_p.$$

Our form may thus be written

$$\sum_{p,\,q} y_p \bar{y}_q (l^{(p)} x)^{n-1} (\bar{l}^{(q)} \bar{x})^{n-1}.$$

Condition C) gives

$$\sum y_p \bar{y}_q (l^{(p)} z)(\bar{l}^{(q)} \bar{z})(l^{(p)} x)^{n-2} (\bar{l}^{(q)} \bar{x})^{n-2}$$
$$\equiv \sum z_p \bar{z}_q (l^{(p)} y)(\bar{l}^{(q)} \bar{y})(l^{(p)} x)^{n-2} (\bar{l}^{(q)} \bar{x})^{n-2}.$$

Putting $$(z) = x,$$

$$\sum_{p,\,q} y_p \bar{y}_q (l^{(p)} x)^{n-1} (\bar{l}^{(q)} \bar{x})^{n-1}$$
$$\equiv \sum_{p,\,q} x_p \bar{x}_q (l^{(p)} y)(\bar{l}^{(q)} \bar{y})(l^{(p)} x)^{n-2} (\bar{l}^{(q)} \bar{x})^{n-2}.$$

Consider the algebraic thread

$$\sum_{p,\,q} x_p \bar{x}_q (l^{(p)} x)^{n-1} (\bar{l}^{(q)} \bar{x})^{n-1} = 0.$$

This is the locus of points lying on the corresponding threads of order $2(n-1)$ and the (1, 1)th polar of (y) with regard thereto is a linear combination of these two forms which we have proved equal, and so proportional to either.

Now one of these terms gives the thread corresponding to (y).

Theorem 51] *The order of an algebraic thread exceeds by two, twice the number of successive equi-polarizations necessary to construct the thread.**

In order to reach another characteristic of a thread we introduce the following transformation of our complex binary domain. This consists in establishing a one to one correspondence between the points of our domain and the chains which include one of the zero points. We do this by writing

$$y_2 \bar{y}_1 x_1 \bar{x}_2 + y_1 \bar{y}_2 x_2 \bar{x}_1 + 2 y_2 \bar{y}_2 x_2 \bar{x}_2 = 0.$$

If we let the zero points correspond to one another, then every other point of the domain corresponds to a definite chain through the special zero point. Conversely, every chain through this special zero point will correspond to a definite point in the plane. The relation between (x) and (y) is reciprocal. We shall call this a *point reciprocation*. Since

* This method of defining order was first given by the author in an article, 'Meaning of Plucker's equations for a real Algebraic Curve', *Rendiconti del Cercolo Matematico di Palermo*, vol. xl, 1915.

the product of two conjugate imaginary numbers is essentially real, we see that no point can lie on the chain which corresponds to it. Geometrically, the chain corresponding to (y) passes through $(1, 0)$, is orthogonal to the chain connecting (y) to the zero points, and contains the mate of (y) in the anti-involution

$$x_1\bar{x}_1' + x_2\bar{x}_2' = 0.$$

Let the reader show that a polar reciprocation in the Gauss plane with regard to a self-conjugate imaginary circle will correspond to a point reciprocation in the binary domain. If (y) trace a thread, the corresponding chain will envelop a thread, and the relation between the two is reciprocal.

Definition. The maximum order for any point reciprocal of a given thread is called its *class*.

A point of a thread which has the property that no other points are in the vicinity shall be called a *conjugate point*. It must be at least a double point of the thread, as we see from the Gauss representation. If all tangent chains at double point are tangent to one another, it shall be called a *cusp*. A chain through an arbitrary point O shall be called a *conjugate tangent*, when the corresponding point in one, and, hence, in any point reciprocation where O plays the special rôle, is a conjugate point. The meaning of these terms will appear clearly when we set up in double column the correspondence between these objects and their representatives in the Gauss plane:

Binary Domain.	Gauss Plane.
Algebraic thread.	Real algebraic curve.
Order.	Order.
Point reciprocation.	Polar reciprocation.
Class.	Class.
Conjugate point.	Real conjugate point.
Cusp.	Cusp.
Conjugate tangent chain through special point.	Real conjugate double tangent.
Osculating chain through special point.	Real inflectional tangent.

If, then, for the sake of simplicity of statement, we limit ourselves to such threads that neither they nor their point reciprocals have worse singularities than double points, we may deduce from Klein's famous identity for real curves:*

Theorem 52] *The order of a thread, plus the maximum number of osculating chains through an arbitrary point and twice the number of conjugate tangent chains through such a point, is equal to the class, plus the number of cusps and twice the number of conjugate points.*

There is one more form of transformation which must be mentioned in conclusion. One method of reaching a chain is through an anti-involution. In the same way the thread (27) may be reached from the involutory transformation:

$$(ax)^n(\bar{a}\bar{x}')^n = 0. \qquad (33)$$

This shall be called a *symmetry in the given thread.* The geometrical significance of this in the complex binary domain is not particularly simple or interesting; the case is quite different for the corresponding transformation of the Gauss plane. Let us write

$$x_1 \equiv z \equiv X + iY, \quad x_2 = 1; \quad \bar{x}_1 = \bar{z} = X - iY, \quad \bar{x}_2 = 1.$$
$$x_1' \equiv z' \equiv X' + iY', \quad x_2' = 1; \quad \bar{x}_1' = \bar{z}' = X' - iY', \quad \bar{x}_2' = 1.$$
$$f(z, \bar{z}') = 0.$$

Since $\bar{z}'$ is an analytic function of z we have a real inversely conformal transformation of the plane that leaves the given curve invariant, in fact all points of the curve remain in place. Such a transformation is called *Schwarzian symmetry.*† Let us prove that there exists one, and only one, such transformation in any given real curve.

It is evident that a real conformal collineation will carry one such symmetry into another. Our proof will consist in showing that we can find a real direct conformal transforma-

* Klein, 'Eine neue Relation zwischen den Singularitäten einer algebraischen Curve', *Math. Annalen*, vol. x, 1876. For a proof that there is no other such relation see the Author, 'The Characteristic Numbers of a Real Algebraic Plane Curve', *Rendiconti del Cercolo Matematico di Palermo*, vol. xlii, 1917.

† Schwarz, *Mathematische Abhandlungen*, vol. ii, Berlin, 1890, pp. 151 ff.

tion that will carry any real analytic curve into the axis of reals, and that there is but one Schwarzian symmetry in that axis. If the given curve be expressed in the form

$$X = f(t), \quad Y = \phi(t),$$

the real direct conformal transformation desired is

$$X + iY = f(X' + iY') + i\phi(X' + iY').$$

The product of two symmetries leaving invariant all points of the real axis would be a real directly conformal transformation leaving all real points in place. If this be

$$z' = y(z), \quad \bar{z}' = \bar{y}(\bar{z}),$$

then $$z' - \bar{z}' = y(z) - \bar{y}(\bar{z}) = z - \bar{z},$$

i.e. $$y(z) - z = \bar{y}(\bar{z}) - \bar{z}.$$

But, by our fundamental lemma, this gives

$$y(z) \equiv 0.$$

Theorem 53] *A symmetry in an algebraic thread will correspond in the Gauss plane to a Schwarzian symmetry in the corresponding real curve.*

The Schwarzian symmetry is best described geometrically in terms of the minimal lines of the Gauss plane or Riemann sphere. If two points be the reflections of one another in a real line, say the real axis, the minimal lines through them intersect in pairs on that real axis. This property is, however, invariant for a real direct conformal collineation. Hence, a Schwarzian symmetry consists essentially in interchanging points whose minimal lines meet in pairs on the given real curve. A direct analytic proof will come at once from the equations above. It is this method of treating the subject that lies at the basis of recent work in conformal geometry.*

* Cf. Kasner, 'Conformal Geometry', *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge, 1913, vol. ii, p. 81, and Pfeiffer, 'Conformal Geometry of Analytic Arcs', *American Journal of Mathematics*, vol. xxxvii, 1915.

CHAPTER III

THE REPRESENTATION OF POINTS ON A CURVE

THE first writer who undertook to give a real representation of complex points on any locus other than a straight line was the Abbé Buée, to whom we paid our respects in the first chapter. He devoted but little attention to the subject, his reasoning being substantially as follows: *

If a plane curve have an axis of symmetry, a line running clear of the curve, perpendicular to this axis, will meet the curve in pairs of conjugate imaginary points. The plane of the curve being

$$z = 0,$$

we may represent the complex point $(x, yi, 0)$ by the real point $(x, 0, y)$. Consider a complex branch of the real circle.

$$x^2 + y^2 = a^2, \quad z = 0,$$

will be represented by the real hyperbola

$$x^2 - z^2 = a^2, \quad y = 0.$$

A far better discussion of the interpretation of complex points of a real curve was given a few years later by a very different grade of geometer, Jean Victor Poncelet. There is an excellent discussion of the complex points of a real conic in the first section of his classic, *Traité des propriétés projectives des figures.*† It is to be noted that Poncelet preceded Mourey and Warren. His method is purely synthetic, and he gives not a little attention to exposing his views as to the philosophical significance of imaginary elements, which do not concern us here. We prefer to employ a little simple analysis to

* Loc. cit., pp. 79 ff.

† First edition, Paris, 1822; second edition, Paris, 1865. Subsequent references are to the latter—the first part of the work is uniform in the two.

express his main idea. We shall confine our attention to central conics. If we take any pair of conjugate diameters of such a conic as axes, the equation will take the form:

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

If we take an interior point Q with the abscissa x the corresponding ordinate will be

$$y = \frac{b}{a} \sqrt{\pm (a^2 - x^2)}.$$

If, however, Q be an exterior point, which we may assume without restriction to lie upon the x axis, we may determine on a line through this point parallel to the conjugate diameter two points whose distance therefrom is

$$y = \frac{b}{a} \sqrt{\pm (x^2 - a^2)}.$$

The segment determined by these points is called an *ideal chord* of the conic, its extremities represent the imaginary intersections of the line with the conic, and the totality of all such points for all chords with this given direction will be the curve

$$\frac{x^2}{a^2} \mp \frac{y^2}{b^2} = 1,$$

which is called a *supplementary* for the given direction. Each central conic has a singly infinite set of supplementaries, each having double contact therewith at the extremities of the diameter conjugate to the direction of the corresponding ideal chords. The author is thus able to prove that two real conics which are not tangent have two common chords, real or ideal.* He reaches the circular points at infinity by the following simple reasoning:† . . . Suppose that we have two ellipses which are similar and similarly placed. If we take parallel diameters of the two, the corresponding supplementary hyperbolas are also similar and similarly placed, and so have two common points at infinity, which represent the imaginary

* Loc. cit., p. 33. † Ibid., p. 47.

infinite intersections of the ellipses. But any two circles are similar, and similarly placed, ellipses. Hence any two circles have the same infinite points.

We shall encounter the fundamental idea of the Poncelet supplementaries again and again in the present work. We leave it for the moment, after calling attention to the fact that the point midway between the extremities of an ideal chord is the same as that midway between the imaginary points represented, and the distance of the real representing points is the square root of minus the distance of the imaginary points. The real points are conjugate with regard to the conic, and are the only conjugate pair on their line whose middle point is on the conjugate diameter.

While Poncelet was developing these sound ideas with regard to imaginary points on curves, the subject was being studied under less happy auspices in England. The first writer to be mentioned in this connexion is Gregory.* Let us explain his plan. We start with the curve

$$f(x, y) = 0.$$

Let (x) take a complex value. We represent this by a point in the (x, z) plane by the Gauss scheme. As a second Gauss plane, we take that through the origin perpendicular to the line from the origin to this first point, and take the (y) axis as axis of reals in this plane. Lastly we find the reflection of the origin in the line connecting the two representing points in these two Gauss planes, and that shall be the point to represent the complex point (x, y). Analytically, if our complex point be

$$x = r(\cos\phi + i\sin\phi), \quad y = s(\cos\psi + i\sin\psi),$$

the representing point is

$$X = r\cos\phi - s\sin\phi\sin\psi,$$

$$Y = s\cos\psi,$$

$$Z = r\sin\phi + s\cos\phi\sin\psi.$$

If we substitute the trigonometric values of x and y in the equation of the curve and split into real and imaginary parts,

* 'On the existence of Branches of Curves in Several Planes', *Cambridge Mathematical Journal*, vol. i, 1839.

we have five equations from which we can eliminate r, s, ϕ, and ψ, leaving

$$F(X, Y, Z),$$

and this real surface will represent the curve in question. Our confidence in Gregory is somewhat shaken by his geometric proof* that

$$\sin\left(e^{\frac{ip}{q}\pi} X\right) = e^{\frac{ip}{q}\pi} \sin X.$$

A refinement upon these clumsy developments was introduced by Walton.† He represents the complex x in the Gauss plane X, Z, and the complex y in the Y, Z Gauss plane. The representing point is the reflection of the origin in the line connecting the points chosen in these two Gauss planes. Analytically the complex point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2,$$

is represented by the real point

$$X = X_1, \quad Y = Y_1, \quad Z = X_2 + Y_2.$$

The real surface corresponding to a given curve is given by eliminations similar to those used in the other case. It is a curious fact that these early writers, one and all, seem to overlook the very arbitrary element involved in their various methods of representation, and to have believed that the scheme was imposed by the nature of the problem. Thus we find Walton saying:‡

'To talk of curves having branches out of their planes would be a very strange mode of speaking, if it be thereby signified that a curve can both lie entirely in one plane and be partly out of it; and even if such a paradox were not implied, it could hardly be called a good expression. However, there can be no impropriety in saying that a curve lies partly in one plane and partly in another.'

* Loc. cit., p. 264.

† 'On the General Theory of Loci of Curvilinear Intersection', ibid., vol. ii, p. 85; 'On the General Interpretation of Equations between Two Variables', ibid., p. 103; 'On the General Theory of Multiple Points', ibid., p. 155; 'On the Extension of possible Asymptotes to impossible Branches of Curves', ibid., p. 236; 'On the Doctrine of Impossibles in Algebraic Geometry', *Cambridge and Dublin Mathematical Journal*, vol. vii, 1852.

‡ 'On the Doctrine of Impossibles', cit. p. 239.

Gregory and Walton have had one lineal successor in recent times. It is curious to see a mathematician of incomparably higher standing reverting two score years later to their general line of thought. This writer was Appell, who proposed* to represent the complex point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2,$$

by the real point

$$X = X_1, \quad Y = Y_1, \quad Z = \sqrt{X_2^2 + Y_2^2}.$$

It can hardly be said that Appell was happily inspired in this. The radical attached to Z is a blemish, and there is no distinction between the points which represent two conjugate imaginary points.

There is another writer whose ideas followed somewhat similar lines, namely Bjerknes.† His scheme is more ambitious and less simple. We start with the complex point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2,$$

which lies on the real curve

$$\phi(x, y) = 0.$$

The totality of points satisfying this equation will depend on two real parameters, and may be represented as the totality of points of the plane. To get varieties in the total locus which depend on a single real parameter, we introduce a single real equation, called the 'abscissa equation',

$$f(X_1, X_2) = 0.$$

We write also $$x + iy = X + iY$$

where X and Y are real. This involves the additional equations $$X = X_1 - Y_2, \quad Y = Y_1 + X_2.$$

Eliminating y between the two equations where it appears gives us a complex equation in the arguments x, X, and Y. Splitting into real and imaginary parts, we have two real

* 'Sur une représentation des points imaginaires en géométrie plane', Grunert, *Archiv der Mathematik*, vol. lxi, 1878.

† *Ueber die geometrische Representation der Gleichungen zwischen zwei veränderlichen reellen oder complexen Grössen*, Christiania, 1859.

equations in the arguments X, Y, X_1, and X_2. Eliminating X_1 and X_2 between these and the abscissa equation, we get

$$F(X, Y) = 0.$$

The geometrical meaning of the various algebraic processes may be explained as follows. The point

$$X = X_1 - Y_2, \quad Y = Y_1 + X_2$$

is the point which is carried into the Gauss representative of the given point, when the line connecting the latter with its conjugate is carried by a real motion into the axis of reals. We shall encounter this point frequently in the next chapter when we come to study the representation of Laguerre. The point (X_1, X_2) is the representative according to the same scheme of the projection of the given point upon the axis of x. Bjerknes' problem is to study the locus traced by one of these representing points, while the other traces a given real abscissa curve.

As an example, let us take the circle

$$x^2 + y^2 = 1$$

with the abscissa curve $\quad X_2 = 0.$

We find $\quad X^2 + Y^2 = 1,$

so that the representing curve is the real part of the given curve. Had we taken the abscissa equation

$$X_2 = RX_1,$$

we should have found

$$(X + iY)^2 - 2x(X + iY) + 1 = 0,$$

$$2X_1(X - RY) = X^2 - Y^2 + 1,$$

$$X_1(Y + RX) = XY,$$

$$X^2 + Y^2 = \frac{Y + RX}{Y - RX}.$$

It is clear that there is a very arbitrary element in this representation. The real point (X, Y) will represent every complex point (x, y) for which

$$X = X_1 - Y_2, \quad Y = Y_1 + X_2.$$

Bjerknes himself is at great pains to point this out, e.g.*

$$y = Rix, \quad X_2 = 0.$$

Here x and iy are real, so that, regardless of R, the representing curve is

$$Y = 0.$$

Given $\qquad b^2x^2 + a^2y^2 = a^2b^2, \quad X_2 = 0.$

If $\qquad x^2 \leqq a^2, \quad Y_2 = 0, \quad b^2X^2 + a^2Y^2 = a^2b^2.$

If $\qquad x^2 > a^2, \quad Y_1 = 0, \quad Y = 0.$

Such are the general outlines of the method which Bjerknes develops in a sixty-page pamphlet. It is certainly a curious fact that a pupil of Riemann's who had heard the master's lectures on Abelian functions,† including, presumably, an account of the surfaces indissolubly associated with his name, should have cared to develop independently a representation of so inferior a sort. The vice is ineradicable. All work with the complex variable

$$x + iy,$$

where x and y are complex, involves the great difficulty that the equation

$$x + iy = x' + iy'$$

does not, necessarily, involve

$$x = x', \quad y = y'.$$

For this reason we content ourselves with mentioning by name only a successor of Bjerknes who developed the same ideas from this analytic point of view.‡

Before taking up the work of the next writer, we shall make some further remarks about the methods of Gauss and Poncelet. It was the fate of both of these geometers to have their ideas rediscovered from time to time. The first rediscoverer of Poncelet deserves warm praise for pointing out the close connexion between complex points and elliptic involutions, the fundamental idea of the Von Staudt theory

* Loc. cit., p. 8.

† Cajori, *History of Mathematics*, second edition, New York, 1919, p. 421.

‡ MacBerlin, 'Om Komplexa Koordinater inom plana Geometrien', *Acta Universitatis Lundensis*, vol. ix, 1872.

which is developed at length in the last chapter of the present work. Let us give some geometrical developments which lead to this connexion.

Let the real line which connects two conjugate imaginary points be taken as the x axis in a real plane through it. A real involution thereon will be given by the real bilinear equation

$$Axx' + B(x+x') + C = 0.$$

When

$$B^2 < AC,$$

this involution is elliptic, and the search for its imaginary double points leads to the quadratic equation

$$Ax^2 + 2Bx + C = 0,$$

whose roots are conjugate imaginaries. Conversely, if we start with the conjugate imaginary points

$$x = P + Qi, \quad \bar{x} = P - Qi,$$

they are the double points of the elliptic involution

$$xx' - P(x+x') + P^2 + Q^2 = 0.$$

There is a perfect one to one correspondence between the elliptic involutions of collinear points in a real plane and the pairs of conjugate imaginary points of that plane.

The importance of this correspondence for some of our work can scarcely be overstated. For instance, in the Gauss representation, the conjugate imaginary points $(P+Qi, 0)$, $(P-Qi\ 0)$ are represented by the real points (P, Q) and $(P, -Q)$. Let us pass a circle through these real points. The point $(P, 0)$ has the power $-Q^2$ with regard to this circle. If, therefore, x and x' be the abscissae of the intersections of this circle with the x axis,

$$(x-P)(x'-P) + Q^2 = 0,$$

and this reduces immediately to the equation above.

The Gauss method consists in representing a pair of conjugate imaginary points by the real points common to the coaxal circles cutting the axis of reals in the elliptic involution associated with the imaginary points. The representing points *are the real points of the lines connecting the imaginary points with a determined circular point at infinity and the intersection of the plane with the real circle common to the conjugate imaginary spheres of zero radius whose centres are the given imaginary points. The relation between conjugate imaginary points and their real representatives will be unaltered by a real motion of the plane.*

This last remark leads to Laguerre's extension of the Gauss method, which will be studied at length in the next chapter. For the moment we prefer to consider the involution a little more closely. Let us ask which of the circles in this coaxal system has the smallest radius. Evidently that one whose centre is $(P, 0)$, the point midway between our representing points, and also midway between the given imaginary points. The square of the diameter is $4Q^2$, which is the negative of the square of the distance between the given complex points. This circle cuts the line in the closest pair of points of the given involution. Conversely, if an elliptic involution be given, there is just one pair which divides harmonically the infinite point and its mate in the involution, and these are the closest pair of the involution. If two real points be given, they are the closest pair of that elliptic involution which includes them as one pair, and their mid-point with the infinite point of their line as another pair.

Each pair of finite real points will be the closest pair of an elliptic involution and may be taken to represent the conjugate imaginary double points of that involution; each pair of conjugate imaginary double points may be so represented. When the conjugate imaginary points lie on a conic in a real plane, the involution is one of conjugate points with regard to that conic, and this is the method of Poncelet.

The writer who first presented the Poncelet method from this point of view was Paulus. His work is clean cut and well written.* The same, unfortunately, cannot be said about

* 'Ordnungselement der einförmigen involutorischen Gebilde', *Grunerts Archiv*, vol. xxi, 1853, and 'Ueber uneigentliche Punkte und Tangenten', ibid., vol. xxii, 1854.

the next rediscoverer, F. Maximilien Marie.* This fearful and wonderful man gave a good part of his life to championing his views about complex numbers, as we may easily learn from the 344 pages of autobiography which he appends to his *Théorie des Fonctions.* He possessed the knack of quarrelling with his contemporaries almost to the point of genius. Many and varied were his griefs against Poinsot, Chasles, Hermite, Briot, Cauchy, Sturm, Puiseux, Bonnet, de Tilly, and Darboux. Cauchy had the extraordinary patience to receive Marie at his house every Tuesday for the better part of a year,† but when Cauchy failed to refer to Marie's work in sufficiently laudatory terms in a report read to the Académie des Sciences the latter made up his mind never to cross that threshold again.‡ He never forgave Chasles for including Poncelet's name, but omitting his own, from the 'Rapport sur le progrès en géométrie', and complained to the Minister of Public Instruction against this piece of flagrant injustice. His quarrels with Puiseux and Briot had to do with the periodicity of certain integrals, and the limits of convergence of Taylor's series.

It is hard to take such a man seriously enough to find out what he has to say, especially when his own views are expressed by the pleasantly frank statement, 'J'ai toujours eu beaucoup de peine à lire les ouvrages de mathématiques'.§ Nevertheless it would be a great mistake to pass him over in silence, for his method of representation is of fundamental importance, and he carried it much farther than any of his predecessors.

How can we represent the complex point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2$$

by a real point invariantly connected with it for all real motions of the plane? or, more generally, for every real affine collineation? Such a collineation will carry the real line which connects conjugate imaginary points into the real line

* *Théorie des fonctions de variables imaginaires*, Paris, 1874–6; *Réalisation et usage des formes imaginaires en géométrie*, Paris, 1891, besides numerous earlier 'articles'.

† *Théorie des fonctions*, Part III, p. 71. ‡ Ibid., p. 77.

§ Ibid., p. 78.

connecting the transformed points. We may find a real affine collineation to carry any real line and real point not on it into any other such point and line. Hence, if a point be invariantly connected with a line, it must be on that line. The representing point must, therefore, have the coordinates

$$X = X_1 + kX_2, \quad Y = Y_1 + kY_2.$$

Now, says Marie,* 'Il n'y aura aucun avantage à donner à k une valeur autre que 1'. Accordingly we so take it, and write

$$X = X_1 + X_2, \quad Y = Y_1 + Y_2.$$

It is evident that we have here a simple and workable method for representing all the finite complex points in the plane, and we shall so consider it in greater detail in the next chapter. But Marie's own interest lay chiefly in representing the functions of a single complex variable, i.e. the points of a single curve, so we have introduced him at this point.

Let us see how all this connects up with Poncelet. Let us take one more point, namely,

$$X' = X_1 - X_2, \quad Y' = Y_1 - Y_2.$$

The points (x, y) and $(\bar{x}, \bar{y})$ lie on the same line as (X, Y) and (X', Y') and have the same middle point, while the square of the distance in one case is the negative of what it is in the other, which agrees precisely with what was said on p. 70. Or we may return to our previous analysis. Our conic referred to a pair of conjugate diameters is

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

If we take the conjugate imaginary complex points (the axes are oblique)

$$x = X_1, \quad y = iY_2; \quad \bar{x} = X_1, \quad \bar{y} = -iY_2,$$

they will be represented by the real points

$$X = X_1, \quad Y = Y_2, \quad X' = X_1, \quad Y' = -Y_2,$$

which are seen to be conjugate with regard to the conic, and to lie on the supplementary

$$\frac{X^2}{a^2} \mp \frac{Y^2}{b^2} = 1.$$

* *Réalisation et usage*, cit. p. 5.

Marie, instead of calling this a supplementary, uses the overworked word 'conjugate'. He has a general definition for a conjugate which, although rather vague, may be expressed in about the following terms.* Given the curve

$$f(x, y) = 0.$$

Every real finite point in the plane will represent at least one complex point on the curve. Let us try to resolve the curve into a series of loci, each of which is given by the variation of a single real parameter. The real branch of the curve, when this exists, is such a locus. Can we find a system of points on the curve whose ordinates all become real, after a real rotation of the axes? If the ordinate of a complex point be real, the line connecting it with the conjugate point must be parallel to the y axis. The system in question will then be such a one that the lines connecting each point with its conjugate has a fixed direction, i.e. we must have the pairs of intersections with a pencil of real parallel lines. We see at once that in the case of a conic this gives a Poncelet supplementary. Analytically

$$Y_2 = RX_2.$$

Let us note in conclusion that Marie marks a great advance over Poncelet or Paulus in that he distinguishes between the representation of a complex point and that of its conjugate.

The methods of representation which we have so far discussed consist one and all in representing complex points by real ones. It is easy to see, however, that we may obtain simple representations of complex points in the plane by means of real lines in space. The first writer to enter on this path seems to have been Weierstrass.† The idea which he threw out, without apparently attaching much importance thereto, was to represent the complex point

$$(X_1 + iX_2,\ Y_1 + iY_2, 0)$$

by the line connecting the real points

$$(X_1, X_2, 0), \quad (Y_1, Y_2, K).$$

* *Réalisation et usage*, cit. p. 5.

† See his *Collected Works*, vol. iv, Leipzig, 1892, p. 323.

In this system every finite complex point is represented by a single real line not lying in or parallel to the (x, y) plane, and, conversely, every such line will represent a single complex point. This idea was worked out independently and in great detail by Van Uven.* Geometrically, we project our complex point orthogonally upon the x axis, and represent in the (x, y) plane as a Gauss plane; it is also projected orthogonally on the line

$$x = z - k = 0,$$

and this point is represented in the $z = k$ plane as a Gauss plane. The representing line joins these two representing points.

The first thing to be noticed here is that the correspondence is not perfect, when infinite values are included, owing to the special rôle of lines lying in or parallel to the (x, y) plane. We shall encounter in the next chapter a considerable number of other attempts to represent the points of the complex plane by real lines in space, and there will always be exceptional elements. This difficulty is inevitable. The totality of complex points is carried into itself by the totality of transformations of the type

$$x' = f(x, y), \quad y' = \phi(x, y).$$

In terms of real parameters we have

$$X_1' = F_1(X_1, X_2, Y_1, Y_2), \; X_2' = F_2(X_1, X_2, Y_1, Y_2),$$
$$Y_1' = F_3(X_1, X_2, Y_1, Y_2), \; Y_2' = F_4(X_1, X_2, Y_1, Y_2),$$

where the functions $F_1 \ldots F_4$ are not purely arbitrary, but are solutions of certain partial differential equations of the first order, analogous to the Cauchy-Riemann differential equations. On the other hand, a real line in space may be represented by six real homogeneous coordinates X_1, X_2, X_3, X_4, X_5, and X_6, where

$$X_1^2 + X_2^2 + X_3^2 - X_4^2 - X_5^2 - X_6^2 = 0.$$

Putting $X_6 = 1$, and finding X_5 from this equation, we see that the other four X's may be subjected to any real transformation provided the resulting X_5 turns out real, and this

* *Algebraische Strahlcongruenzen und verwandte complexe Ebenen*, Amsterdam, 1911.

will not be ensured by any partial differential equations of the first order.

A second point to consider in connexion with the Weierstrass-Van Uven representation is that since an analytic curve, not a set of parallel lines, makes y an analytic function of x, the relation between the two parallel planes is directly conformal. The converse is true, so that every curve is represented by the lines connecting corresponding points in directly conformally related parallel planes, and every such congruence of lines will represent a curve which is not a line parallel to a certain direction. If this relation between the planes be anything other than a translation, corresponding points in the two planes can only move parallel to one another when they move in a minimal direction. Hence the focal planes through a line of the congruence must pass through the circular points at infinity in the (x, y) plane, and the focal surface must be two cones with these imaginary points as vertices. The converse is also true: every such congruence will establish a real directly conformal relation between the parallel planes.* Van Uven's book of 577 pages is devoted chiefly to the congruences which represent curves of the type

$$y^n = k\,x^{\pm m}.$$

Such prolixity leaves us in a state of bewilderment.†

There are two other writers who use real lines to represent the points of a complex curve, to whom we must pay our respects. The first of these is Henschell.‡ His method seems about as complicated as one could readily wish. We start with a complex point in the (x, y) plane, and through it draw tangents to the circle which that plane cuts from the sphere,

$$X^2 + Y^2 + Z^2 = 1.$$

* Given without proof by Van Uven, loc. cit., p. 16.

† Van Uven's ideas have been much better developed by Wilczynski, 'Line-geometric Representation of Functions of a Complex Variable', *Transactions American Math. Soc.*, vol. xx, 1919. The centre of interest here is in the focal surfaces of certain congruences of lines; the representations for complex values play a subordinate rôle.

‡ *Versuch einer räumlichen Darstellung complexer ebener Gebilde.* Dissertation, Weimar, 1892.

The points of contact are projected stereographically from (0, 1, 0) upon the x axis and represented in the (x, z) plane as a Gauss plane. The representing points are then projected back stereographically upon the same sphere, and joined by a real line.

The last writer whose name should be mentioned in this connexion is Vivanti,* who represented the point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2, \quad z = 0$$

by means of the line whose equations are

$$X_1 X + X_2 Z = 1,$$
$$Y_1 Y + Y_2 Z = 1.$$

Geometrically, this amounts to the following. The complex point is projected orthogonally on each of the axes, and the points so reached are represented in the (x, z) and (y, z) planes as Gauss planes. Two spheres are constructed whose diameters are the segments bounded by the origin and the representing points. These spheres intersect in a circle whose inverse in the unit sphere about the origin is the line in question.

The thoughtful reader will have been much surprised that no mention has been made of Riemann in the present chapter, except for a brief reference in connexion with Bjerknes. This is indeed no oversight. On the contrary, the chapter deals primarily with the unsuccessful attempts of others to solve a problem which Riemann solved completely. The subject of Riemann surfaces is, however, so universally recognized as a corner-stone of modern analysis, and so carefully explained in every good text-book on the theory of functions of a complex variable, that it would be an impertinence to take it up here. We make an exception to this rule only to mention one short article which has all the originality of its brilliant author, Klein.†

Suppose that we should undertake to represent, not the complex points of a curve, but the complex tangents thereto;

* 'Preliminari pello studio delle funzioni', *Rendiconti del Cercolo Matematico di Palermo*, vol. ix, 1895.

† 'Eine neue Art Riemannscher Flächen', *Math. Annalen*, vol. vii, 1874.

the one problem passes over into the other by a polar reciprocation. The curve being real, each real tangent might be represented by its point of contact, each complex one by its real point. Through each real point in certain two-dimensional regions will pass several complex tangents, conjugate imaginary in pairs. We make the correspondence one to one by replacing the plane by a Riemann surface, there being as many sheets over each point as there are complex tangents to the curve. The curve itself will be a curve of junction for pairs of sheets. The same is true of an inflexional tangent, or a conjugate tangent, i.e. a double tangent with conjugate imaginary points of contact. An ordinary double tangent will be represented by its points of contact.

Let us consider two examples. Take a real ellipse. From an interior point will radiate two conjugate imaginary tangents to the curve. The Riemann surface will consist of two elliptical disks, covering the area within the curve, and joined along the curve. A slightly more elaborate surface is called for by the curve

$$y^2 = (x^2-1)(x-2).$$

The real part of the curve is an oval from $x = -1$ to $x = 1$, and an open branch beginning with $x = 2$ and running off to the right indefinitely. This branch has two finite real inflexions, and one inflexion at infinity at the end of the y axis. There will be a small region bounded by the curve and the finite inflexional tangents, from each of whose points six real tangents can be drawn to the curve, and the same will be true of the regions bounded by the curve a, finite tangent, and the infinite tangent. Hence there will be six sheets of the Riemann surface over the oval, four over the region outside the oval, but not separated therefrom by an inflexional tangent, two over the region reached where one such tangent is crossed, and none over the three regions where six real tangents are concurrent.

CHAPTER IV

THE REPRESENTATION OF POINTS OF A PLANE

§ 1. Representation by means of point-pairs.

We had frequent occasion to remark in the course of the last chapter that the methods for representing the points of a curve, or at least some of them, were perfectly adequate to represent all the points of a complex plane. The reason for explaining them in that chapter, instead of waiting until the present one, was that the writers who first discovered them were more interested in the more restricted problem. We now return to these methods and consider them from the broader point of view, and in comparison with other methods which have been devised for representing all the points of a complex plane.

We showed on p. 75 that the usual Gauss representation of the complex points of a single line could be described in such geometrical terms as to suggest an immediate extension to the representation of all finite points of the plane, and mentioned in that connexion the name of Laguerre. This admirable geometer seems to have been the first writer to really apprehend the scope and meaning of the problem.* His ideas were greatly developed by two others. Gaston Tarry studied the elementary properties of the representation with great patience and a wealth of detail.† Eduard Study reworked the whole subject in its wider aspects, bringing to the discussion that profundity of vision which is characteristic of all of his mathe-

* 'Sur l'emploi des imaginaires en géométrie', *Collected Works*, Paris, 1905, vol. ii, pp. 88 ff.

† Tarry's papers are found under a variety of titles in the *Proceedings of the Association française pour l'Avancement des Sciences*, Toulouse, 1887, Oran, 1888, Paris, 1889, and Marseilles, 1891.

matical work.* Tarry's treatment is entirely synthetic and highly painstaking. The thoughtful reader will see, however, that he might have saved himself, and his readers, a great amount of labour by a better grasp of the modern abstract idea of geometry. He starts out by defining as a complex point a pair of real points with an order of preference between the two. Then he defines the modulus and argument of the distance of two complex points. A certain transformation of the real plane is defined as a complex line, and the modulus and argument of the logarithm of the angle of two complex lines are also defined. It is very easy to supply the analytic work which justifies these definitions, and which Tarry omits. Fortified thus, he goes through a good proportion of the theorems of elementary geometry, showing that they hold equally well in the complex domain so defined. If, however, he had merely shown that the fundamental assumptions of plane geometry, exclusive of those of order, hold in the complex domain also, no further discussion would have been needful.

As an introduction to the Laguerre method, let us repeat in greater detail what we said on p. 75 about the Gauss representation. This latter consists essentially in replacing each complex point of the x axis by the real point of the line connecting it with the circular point at infinity $(1, i, 0)$. The conjugate imaginary point is represented in the same way. Taking the conjugate imaginary points together, their minimal lines intersect in two real points. If these be taken in one order, they lie respectively on the first and second minimal lines through the first complex point; if taken in the reverse order they lie on the first and second minimal lines through the conjugate imaginary point, a minimal line being called *first* when it passes through the circular point at infinity whose coordinates are given above. When stated in this way, the x axis drops out of sight, its only rôle being to connect the conjugate imaginary points.

Definition. In the Laguerre system, each real point is represented by itself, each finite complex one by a first and

* *Ausgewählte Gegenstände der Geometrie*, Leipzig, 1911.

second real point lying respectively on the first and second minimal line through the given point. The same pair of real points in the reverse order will represent the conjugate imaginary point.

Suppose, conversely, that we have a pair of real points, called, respectively, the first and second. The first minimal line through the first point will meet the second minimal line through the second point in precisely that complex point which will be represented by the given real pair in the present system.

The Laguerre representation is here developed on the hypothesis of a line at infinity. We might, however, have extended our finite plane to be a perfect analytic continuum, by adjoining a single real infinite point, through which passed a pair of conjugate imaginary minimal lines. This is the continuum of the geometry of inversion, and is indistinguishable descriptively from the Euclidean sphere.* As before, there is a first and second minimal line through each point without exception.

Theorem 1] *The Laguerre representation is without any exception in the finite domain, and has no exception at all if that domain be extended to be the perfect continuum of the geometry of inversion. In this latter case it is equivalent to representing each real point of a sphere by itself, and each complex point by the ordered pair of points of contact of the two tangent planes to the sphere which pass through the given point and its conjugate.*

Since every circular transformation carries a minimal line into a minimal line, we have

Theorem 2] *The relation of a complex point to its Laguerre representatives is unaltered by a real direct circular transformation of the plane.*

* The literature of the subject of infinite regions is large and rather controversial. For two good discussions see Beck, 'Ein Gegenstück zur projektiven Geometrie', *Grunerts Archiv der Mathematik*, Series 3, vol. xviii, 1911, and Bôcher, 'Infinite Regions of various Geometries', *Bulletin American Math. Soc.*, vol. xx, 1914.

It is rather curious that previous writers seem to have paid but little attention to this peculiarity of the Laguerre representation. Let us also underline the merit mentioned in theorem 1], namely, that there are no exceptions.

Enough of a general nature has now been said about the Laguerre method: it is time to come to grips with it analytically. Let the complex point be

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2. \tag{1}$$

This lies on the two lines

$$\begin{vmatrix} x' & y & 1 \\ X_1 + iX_2 & Y_1 + iY_2 & 1 \\ 1 & \pm i & 0 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ X_1 \pm Y_2 & Y_1 \pm X_2 & 1 \\ 1 & \pm i & 0 \end{vmatrix} = 0.$$

We may therefore represent it by the two real points

$$X = X_1 - Y_2, \quad Y = Y_1 + X_2, \quad X' = X_1 + Y_2, \quad Y' = Y_1 - X_2. \tag{2}$$

$$\begin{aligned} X &= \frac{x+\bar{x}}{2} + i\frac{y-\bar{y}}{2}, \quad & Y &= \frac{y+\bar{y}}{2} - i\frac{x-\bar{x}}{2}, \\ X' &= \frac{x+\bar{x}}{2} - i\frac{y-\bar{y}}{2}, \quad & Y' &= \frac{y+\bar{y}}{2} + i\frac{x-\bar{x}}{2}. \end{aligned} \tag{3}$$

The point midway between the given point and its conjugate is (X_1, Y_1), the slope of the line connecting them is $\frac{Y_2}{X_2}$, and their distance is

$$2i\sqrt{X_2^2 + Y_2^2}.$$

From these facts we reach

Theorem 3] *In the Laguerre representation we pass from two conjugate imaginary points to their representatives by rotating their segment through an angle of* 90° *and multiplying the distance between them by* $-i$.

Our next task shall be to find an expression for the distance of two points in terms of their representatives. The line connecting a pair of complex points meets that which connects their conjugates in a real point (unless the four be collinear). This point may be finite or infinite; the two cases must be handled separately. When the real point is finite, we may take it as the origin. The lines separating these conjugate imaginary lines harmonically will form an elliptic involution which will share one pair with the involution of mutually orthogonal lines, and we take this pair for our axes. The two given complex points may then be written

$$(X_1+iX_2, \quad RX_2-RiX_1)\,(\dot{X}_1+i\dot{X}_2, \quad R\dot{X}_2-Ri\dot{X}_1).$$

Their distance will be given by

$$d=\sqrt{1-R^2}\,\sqrt{(X_1-\dot{X}_1)^2-(X_2-\dot{X}_2)^2+2i(X_1-\dot{X}_1)(X_2-\dot{X}_2)}.$$

When the line connecting the given points has a real direction it is at the same pure imaginary distance as its conjugate from a real line which we may take as the x axis. Our points may be written

$$(X_1+iX_2, \quad iY_2)\,(\dot{X}_1+i\dot{X}_2, \quad iY_2).$$

$$d=\sqrt{(X_1-\dot{X}_1)^2-(X_2-\dot{X}_2)^2+2i(X_1-\dot{X}_1)(X_2-\dot{X}_2)}.$$

This is the limiting case of the other when $R=0$.

In the first case the first point is represented by A and A', the second by B and B', where

$$A=((1+R)X_1, \ (1+R)X_2), \quad A'=((1+R)X_1, \ -(1-R)X_2).$$

$$B=((1+R)\dot{X}_1, \ (1+R)\dot{X}_2), \quad B'=((1+R)\dot{X}_1, \ -(1-R)\dot{X}_2).$$

Similarly in the second case,

$$A=((X_1-Y_2), \ X_2), \quad A'=((X_1+Y_2), \ -X_2).$$

$$B'=((\dot{X}_1-Y_2), \ \dot{X}_2), \quad B'=((\dot{X}_1+Y_2), \ -\dot{X}_2).$$

If 2θ be the angle which AB makes with $A'B'$, we find in both cases the general formula

$$d^2=AB\,.\,A'B'\,[\cos 2\theta+i\sin 2\theta]. \tag{4}$$

Theorem 4] *If two complex points be represented in the Laguerre system by the pairs AA' and BB', then the modulus of the distance of the complex points is the geometric mean between AB and $A'B'$, and the argument of this complex distance is the angle between the lines AB and $A'B'$.**

* This formula seems to have been incorrectly given by Laguerre, op. cit., vol. ii, p. 97. He uses A where we use A' and vice versa.

Theorem 5] *If two complex points be represented by the real point-pairs* AA' *and* BB', *the line connecting them will have a real direction if* $(AB)^2 = (A'B')^2$. *The square of their distance will be real if* AB *be parallel to* $A'B'$ *while it is pure imaginary when these two lines are mutually perpendicular.**

Theorem 6] *The totality of points of a minimal line will be represented by pairs consisting of a single fixed point for the first (second) member, and all points of the plane for the second (first) member.*

We find furthermore from our equations above that

$$\frac{AB}{A'B'} = \frac{1+R}{1-R}.$$

This shows that the points of a general line correspond to a conformal collineation of the plane, and, since all parts of the X axis are invariant, the collineation is inversely conformal, i.e. the sense of angles is reversed. The ratio of stretching is $\frac{1-R}{1+R}$. This will be equal to unity only in the case of a line of real direction. It is the negative of the ratio of the ordinates of corresponding real points, i.e. it is the ratio of the parts into which a segment connecting corresponding real points is divided by the x axis. We define this latter as the *axis* of the transformation.

Theorem 7] *The totality of points of a non-minimal line will be represented by an inversely conformal collineation of the plane. The segments connecting corresponding points will be divided by the axis of the collineation into two parts, whose ratio is equal to the ratio of stretching. This ratio is equal to unity when, and only when, the line has a real direction.*

Let us find the interpretation in the real domain of the angle of two complex lines. If two lines, real or imaginary, meet in a point P, while a minimal line meets them in R and R', then if their angle be θ, we prove by the law of cosines:

$$e^{-i\theta} = \frac{PR'}{PR}.$$

* Study, loc. cit., p. 21.

Suppose, then, that we have two complex lines intersecting in a point which is represented by the real points O and O'. Their intersections with a minimal line will be represented by the pairs AO and BO. The absolute value of the squares of the distances from the intersection of the given lines to their intersections with this minimal line will by 4] be $\frac{BO}{AO}$, and this is the ratio of the ratios of similitude in the two collineations associated with the given lines. As for the argument of the complex angle, that is one-half of the $\measuredangle BOA$, i.e. the angle of the bisectors if the angles $\measuredangle BOO'$ and $\measuredangle AOO'$, and these bisectors are the axes of the collineations.

Theorem 8] *The angle of two intersecting lines is* $\sqrt{-1}$ *multiplied by the logarithm of a complex number, whose modulus is the square root of the ratio of the ratios of similitude in the two conformal collineations representing the given lines, while its argument is the angle of these two axes.**

The fact that the relation of a complex point to its Laguerre representative is invariant for a real direct circular transformation suggests the idea that a complex circle must be represented by a transformation which will be almost as simple as a conformal collineation. Such is the case. Let I and J be the circular points at infinity. The N. S. condition that four points $P_1P_2P_3P_4$ should be concyclic or collinear is that the cross ratios of the lines connecting them with I and J should be equal.

$$I(P_1P_2, P_3P_4) = J(P_1P_2, P_3P_4).$$

If P_1 be represented by A_1A_1', &c., the points IA_1P_1 being collinear,

$$I(A_1A_2, A_3A_4) = J(A_1'A_2', A_3'A_4').$$

Since the conjugate imaginary points are also concyclic,

$$I(A_1'A_2', A_3'A_4') = J(A_1A_2, A_3A_4).$$

If, then,

$$I(A_1A_2, A_3A_4) = J(A_1A_2, A_3A_4),$$

$$I(A_1'A_2', A_3'A_4') = J(A_1'A_2', A_3'A_4').$$

Hence the A's and the A''s are concyclic together, or the transformation is a circular one.

* Tarry, Article of 1889, cit. p. 87.

There remains the question as to whether this circular transformation is directly or inversely conformal. If the given circle be real or self-conjugate imaginary, the given complex point and also its conjugate lies thereon, and the real points are mutually inverse in this circle. Such an inversion is inversely conformal. But we can pass from any complex circle to a real one by a continuous change of the coefficients, whereas we cannot pass continuously from an inversely conformal transformation to a directly conformal one in this way. Hence the original circular transformation was inversely conformal. Lastly, since three complex points will determine a circle or line, and an inversely conformal circular transformation is determined by the fate of three points, so every transformation of this sort will represent a line or circle.

Theorem 9] *In the Laguerre representation the points of a not null circle will be represented by a real inversely conformal circular transformation of the plane, and every such transformation will correspond to a line or circle. A real or self-conjugate imaginary circle will correspond to the inversion in itself.*

A chain of points, as defined in Ch. II, is characterized by the fact that each cross ratio of any four is real. We may extend this concept as follows. We shall mean by a *cross ratio* of four points of a unicursal curve the cross ratio of four special adjoint curves of a pencil passing through them, i. e. the cross ratio of the four corresponding points of a line to which the curve is birationally equivalent. Thus, if the curve be expressed parametrically in the form

$$x = \frac{f(t)}{\phi(t)} = y = \frac{\psi(t)}{x(t)},$$

where the functions involved are polynomials, we mean by the cross ratio such an expression as

$$\frac{(t_1-t_2)\,(t_3-t_4)}{(t_1-t_4)\,(t_3-t_2)}.$$

Definition. A system of points on a unicursal algebraic curve shall be said to form a *chain* if the cross ratios of any four be real, and if there be a point of the system making with any three an assigned real cross ratio.

Suppose, in particular, we have a chain of points on a circle. If we letter these points as before, we must have

$$I(P_1P_2, P_3P_4) = I(A_1A_2, A_3A_4) = J(A_1A_2, A_3A_4)$$
$$I(A_1'A_2', A_3'A_4') = J(A_1'A_2', A_3'A_4').$$

Theorem 10] *A chain of points on a line or circle will appear in the Laguerre representation as two real lines or circles which correspond in an inversely conformal circular transformation.*

Enough has now been said about the Laguerre representation; let us turn to the closely allied method which we have already discussed, and which from now on we shall refer to as the Marie representation. The first writer to see in this a means of representing all the points of a plane seems to have been Mouchot.* His treatment is entirely synthetic, but is most easily understood if we rewrite the equations given in the last chapter when discussing Marie:

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2.$$

$$X = X_1 + X_2, \quad Y = Y_1 + Y_2, \quad X' = X_1 - X_2, \quad Y' = Y_1 - Y_2. \tag{5}$$

$$X = \frac{x+\bar{x}}{2} - i\frac{(x-\bar{x})}{2}, \quad Y = \frac{y+\bar{y}}{2} - i\frac{(y-\bar{y})}{2}, \tag{6}$$

$$X' = \frac{x+\bar{x}}{2} + i\frac{(x-\bar{x})}{2}, \quad Y' = \frac{y+\bar{y}}{2} + i\frac{(y-\bar{y})}{2}.$$

Each finite complex point will thus give a real point-pair (A, A') and conversely. With this much analysis somewhere in the back of his head, Mouchot starts bravely to follow the synthetic road, somewhat after the fashion of Tarry. Unfortunately his zeal exceeds his discretion when it comes to carrying through his Cartesian reform, and he soon is in over his depth. Thus, after defining a complex point as a real point-pair, he defines a complex segment as a real segment-pair.

* *La réforme Cartésienne étendue aux diverses branches de mathématiques pures*, Paris, 1876, and *Les branches de la géométrie supérieure*, Paris, 1892. Subsequent references are to this latter book, the only one seen by the author.

We are then told that two segments with the same initial point are 'homogeneous',* and presently:

L'ensemble de tous les segments rectilignes, homogènes, groupés dans un plan autour d'un point, forme, dans le sens le plus général du mot, une droite ayant ce point pour origine. †

One is at liberty to put pretty nearly any desired construction on these words. The most natural would seem to be that a complex line is represented by the totality of point-pairs with one fixed member, which is very far from being the case. No, we must class Mouchot's work as misdirected effort; the best judgement on it is that given by the late Jules Tannery in his review of the second book: ‡

M. Mouchot, dont le nom nous est bien connu, grâce à ses belles recherches sur l'utilisation de la chaleur solaire, est de ceux qui suivent avec persévérance leurs propres idées, et qui se préoccupent plus du progrès qu'elles font dans leur esprit que de la façon dont pensent les autres.'

It is indeed fortunate that abler geometers than Mouchot have devoted their attention to the Marie representation. It was familiar to Tarry, and incidentally used by him. More recently, it was used as the basis of an exhaustive but disappointing study by Davis.§ Curiously enough, both Tarry and Davis modified the method in such a way that instead of using the two real points given by (5) they took one of these points and the point midway between the two. This variation introduced an injurious asymmetry. To find the real significance and efficacy of the Marie representation it is better to turn to the article by Study recently quoted. Instead of following his lead, however, let us try to grasp the details of this representation by following as closely as possible the various steps which we took in the case of the Laguerre method.

Theorem 11] *We pass from the Marie to the Laguerre representation by rotating each point-pair around the middle point through an angle of* 90°.

* Ibid., p. 22. † Ibid., p. 24.

‡ *Bulletin des Sciences mathématiques*, vol. xxvii, 1892, p. 237.

§ 'The Imaginary in Geometry', *University of Nebraska Studies*, vol. x, No. 1, 1910.

Let us next look at the distance formula. If the line connecting the complex points be, as before,

$$y = RiX,$$

$$d = \sqrt{1-R^2}\,\sqrt{(X_1-\dot{X}_1)^2-(X_2-\dot{X}_2)^2+2i(X_1-\dot{X}_1)(X_2-\dot{X}_2)},$$

$$A = ((X_1+X_2),\ R\,(X_1-X_2)),\quad A' = ((X_1-X_2),\ -R\,(X_1+X_2)),$$

$$B' = (\dot{X}_1+\dot{X}_2),\ R\,(\dot{X}_1-\dot{X}_2)),\quad B' = ((\dot{X}_1-\dot{X}_2),\ -R\,(\dot{X}_1+\dot{X}_2)),$$

$$(AB)^2 = (1+R^2)\,[(X_1-\dot{X}_1)^2+(X_2-\dot{X}_2)^2 + 2\,(1-R^2)\,(X_1-\dot{X}_1)\,(X_2-\dot{X}_2),$$

$$(A'B')^2 = (1+R)^2\,[(X_1-\dot{X}_1)^2+(X_2-\dot{X}_2)]^2 - 2\,(1-R^2)\,(X_1-\dot{X}_1)\,(X_2-\dot{X}_2),$$

$$\tfrac{1}{2}\,[(AB)^2-(A'B')^2] = 2\,(1-R^2)\,(X_1-\dot{X}_1)\,(X_2-\dot{X}_2).$$

If, as before, the lines AB and $A'B'$ make an angle 2θ,

$$\cos 2\theta = \frac{(1-R^2)\,[(X_1-\dot{X}_1)^2-(X_2-\dot{X}_2)^2]}{AB\cdot A'B'},$$

$$d^2 = (AB)\,(A'B')\cos 2\theta + \tfrac{1}{2}\,[(AB)^2-(A'B')^2].$$

Theorem 12] *If two complex points be represented by the real point-pairs* (AA'), (BB') *respectively, while the lines* AB, $A'B'$ *make an angle* 2θ, *then the real part of the square of the distance of the complex points is* $AB\times A'B'\times\cos 2\theta$, *while the pure imaginary part is* $\frac{1}{2}[(AB)^2-(A'B')^2]$.

Theorem 13] *The necessary and sufficient condition that the square of the distance of the complex points discussed in* 12] *should be real is that* $(AB)=(A'B')$; *the square of the distance will be pure imaginary if the line* AB *is perpendicular to the line* $A'B'$. *The line connecting the complex points will have a real direction if* AB *be parallel to* $A'B'$.

We may set forth the relation between these two methods of representing complex points in triple column as follows:

Complex points.	Laguerre representation.	Marie representation.
Real square distance	$AB \perp A'B'$	$AB \perp A'B'$
Pure imaginary squared distance	$AB \parallel A'B'$	$(AB)=(A'B')$
Connecting line, real direction	$(AB)=(A'B')$	$AB \parallel A'B'$

Let us see what sort of a transformation of the plane will correspond to a straight line. If the line be real, it is clear that the representing points will be thereon. If it be complex, with a complex direction, we see from the equations above

$$RX' = Y, \quad Y' = -RX.$$

This is an affine transformation, whose square is a reflection in the origin. The Jacobian is equal to unity, so that the transformation is directly equi-areal. Lastly,

$$R^2X'^2 + Y'^2 = R^2X^2 + Y^2,$$

so that corresponding points lie at the extremities of conjugate diameters of the same ellipse whose axes have a fixed ratio, and lie along the axes of coordinates. When the line connecting the complex points has a real direction it is geometrically evident that the transformation is merely a reflection in a real line parallel thereto.

Theorem 14] *The Marie representation of a real line is the totality of real point-pairs thereon. The representation of a complex line with a finite real point is an affine collineation of period four which is equi-areal. Corresponding points are the extremities of conjugate diameters of the same ellipse of a concentric system of similar and similarly placed ellipses. The representation of a complex line with a real direction is a reflection in a real line parallel thereto.**

§ 2. Representation by means of lines.

We saw in the last chapter how the real lines of space could be utilized to represent the complex points of the plane, and, in fact, we indicated four methods for accomplishing this representation. We return now to this same question, and give other methods better suited to the study, not of an individual curve, but of the whole plane. We begin with a plan worked out in considerable detail by Duport.† The scheme

* Study, loc. cit., p. 22.

† 'Sur un mode particulier de représenter les imaginaires', *Annales de l'École normale*, Série 2, vol. ix, 1880.

is simplicity itself after what we have learned about the Laguerre representation. We take as our complex plane

$$z = 0,$$

project the first Laguerre representative on $z+1 = 0$ and the second on $z-1 = 0$, and connect by a straight line. The method is seen to bear a certain relationship to the Weierstrass-Van Uven method discussed in the last chapter. It is a shade less simple analytically, but more symmetrical and superior geometrically. The formulae are as follows. We start with the complex point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2, \quad z = 0,$$

and represent it by the real line:

$$X = X_1 + Y_2 Z, \quad Y = Y_1 - X_2 Z.$$

Theorem 15] *The relation of complex point to representing line is invariant for every rotation of space around a line of given direction, and every translation perpendicular to that direction. The interchange of conjugate imaginary points will appear as a reflection in a certain plane perpendicular to this direction.*

We find at once from 5]

Theorem 16] *The lines which represent two points in the Duport system will intersect when, and only when, the square of the distance of those points is real.*

Since the distance of two points is equal to that of their orthogonal projections on any plane parallel to their line, we have from 4]

Theorem 17] *The modulus of the square of the distance of two points is equal to the product of the distances which the representing lines in the Duport system cut on a certain pair of parallel planes; the argument of this square is equal to the angle of the lines along which these two distances are measured.*

Let us seek the system of lines which represent points on the line

$$y = RiX.$$

The representing line will join the points

$$((1+R)X_1,\ (1+R)X_2),\ -1)\quad ((1-R)X_1,\ -(1-R)X_2,\ 1).$$

It will also contain the points

$$\left(0,\ \frac{R^2-1}{R}X_2,\ \frac{1}{R}\right)\left((1-R^2)X_1,\ 0,\ R\right).$$

These latter lie on two fixed mutually perpendicular lines, which are parallel to the bisectors of the angle of the given line and its conjugate, while the product of their distances from the given complex plane is unity.

Theorem 18] *The points of a non-minimal line of imaginary direction are represented in the Duport system by the lines which meet the two skew lines which are obtained by lifting the bisectors of the angle of the given complex line and its conjugate perpendicular to the given plane two distances whose squares are the negatives of the tangents of the halves of the two angles formed by these complex lines. The points of a line of real direction will be represented by the lines parallel to a given plane and intersecting a given line. The points of a minimal line will be represented by a bundle of concurrent lines. The points of a real line will be represented by the totality of real lines cutting the given one orthogonally. In every case the representing lines will establish an inversely conformal collinear relation between two parallel planes.*

Suppose, now, that we have a general analytic curve in our plane. If we replace an infinitesimal arc of the curve by a piece of its tangent, and apply 18] we see that the focal points of a line in general position are at two distances from the given plane whose product is unity, and its focal planes meet the latter in mutual perpendicular lines:

Theorem 19] *The congruence of lines in the Duport system which represents a curve which is not a minimal line is of such a nature that the product of the distances of the focal points of a line in general position from the given plane is unity, while the focal planes cut that plane in two mutually perpendicular lines.* *

* Duport, loc. cit., p. 322.

It is a curious fact that if we seek the best interpretation in line geometry of the fundamental metrical invariants of the complex plane, we are led naturally to the non-Euclidean metric, both in the complex plane and in space. The first writer to perceive this was Klein; it is worth while to give the details of an elegant representation first exhibited by him.*

Suppose that our complex plane is that of projective geometry, with a Cayleyan system of measurement based upon an absolute conic. Each point of the plane will correspond to the two points where its polar meets that conic, which we shall take as self-conjugate imaginary. Through the conic we shall pass a real quadric with imaginary rulings, which quadric shall be the absolute quadric for a three-dimensional non-Euclidean metric of hyperbolic type. Each point of the conic will correspond to a real point of the quadric, namely, that on the imaginary generator of the first system passing through the point of the conic; the conjugate imaginary point will correspond to the real point on the generator of the second system through the given point. Each point of the plane not on the conic will correspond to two real points of the quadric, or, better, to the real line joining them. Conversely, each real line meeting the quadric twice in real points will determine two points of the conic and a point off the conic where their tangents meet. The only exception occurs in the case of a point of the conic. Each such point shall be set in correspondence with the pencil of tangents to the quadric at the real point of the first generator through it.

Let us next exhibit the analytic foundation for this system. The complex plane being

$$X_0 = 0,$$

while the absolute quadric is

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0,$$

* 'Eine Uebertragung des Pascalschen Sechsecks auf Raumgeometrie', *Math. Annalen*, vol. xxii, 1883. It is curious that the article immediately preceding, which is by the same author, gives another representation connected with the Von Staudt theory which we discuss in our last chapter.

we may express the latter in the parimetric form

$$X_0 = \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2, \quad X_1 = \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2, \quad X_2 = \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1,$$
$$X_3 = i[\xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1].$$

It is to be noticed that the given complex plane is in the ultra-infinite domain with regard to a dweller within the Absolute, and it is only for points within the Absolute that the usual real metrical assumptions, excluding the parallel axiom, are valid. Consequently it appears a sort of poet's licence to speak of distances in the given plane. But we may come right back to earth by replacing points and distances in this ultra-infinite plane by the corresponding planes and dihedral angles through the absolute pole of this plane, which is a perfectly good actual point.

If, in the equations above, we treat (ξ) and $(\bar{\xi})$ as independent variables, then, while one remains constant, the other traces a generator of the Absolute. The coordinates of the intersection of a first generator, $\xi =$ constant, with the given plane will be

$$\rho y_0 = 0,$$
$$\rho y_1 = -2\xi_1\xi_2,$$
$$\rho y_2 = \xi_1^2 - \xi_2^2,$$
$$\rho y_3 = i(\xi_1^2 + \xi_2^2).$$

If (y) lie on the polar of (x), we have

$$(x_1 + ix_3)\xi_1^2 - 2x_1\xi_1\xi_2 - [x_2 - ix_3]\xi_2^2 = 0.$$

If (x) be given, and the roots of this equation be $(\eta_1\eta_2)(\zeta_1\zeta_2)$, while

$$x_1^2 + x_2^2 + x_3^2 + r^2 = 0,$$

$$\eta_1\zeta_1 = -(x_2 - ix_3), \quad \eta_1\zeta_2 + \eta_2\zeta_1 = 2x_1, \quad \eta_2\zeta_2 = x_2 + ix_3,$$
$$\eta_1\zeta_2 - \eta_2\zeta_1 = r.$$

Let the representing points be (Y) and (Z).

Then the Plücker coordinates of the line joining them are

$$P_{ij} = Y_iZ_j - Y_jZ_i.$$

Thus, in detail,

$$P_{01} = 2(\eta_2\zeta_1\bar{\eta}_2\bar{\zeta}_1 - \eta_1\zeta_2\bar{\eta}_1\bar{\zeta}_2),$$

$$P_{02} = (\eta_1\zeta_1 - \eta_2\zeta_2)(\bar{\eta}_1\bar{\zeta}_1 - \bar{\eta}_2\bar{\zeta}_2) + (\eta_1\zeta_2 - \eta_2\zeta_1)(\bar{\eta}_1\bar{\zeta}_2 - \bar{\eta}_2\bar{\zeta}_1),$$

$$P_{03} = i[(\eta_1\zeta_1 + \eta_2\zeta_2)(\bar{\eta}_1\bar{\zeta}_2 - \bar{\eta}_2\bar{\zeta}_1) - (\eta_1\zeta_2 - \eta_2\zeta_1)(\bar{\eta}_1\bar{\zeta}_1 + \bar{\eta}_2\bar{\zeta}_2)],$$

$$P_{23} = 2i[-\eta_1\zeta_2\bar{\eta}_2\bar{\zeta}_1 + \eta_2\zeta_1\bar{\eta}_1\bar{\zeta}_2],$$

$$P_{31} = -i[(\eta_1\zeta_1 - \eta_2\zeta_2)(\bar{\eta}_1\bar{\zeta}_2 - \bar{\eta}_2\bar{\zeta}_1) - (\eta_1\zeta_2 - \eta_2\zeta_1)(\bar{\eta}_1\bar{\zeta}_1 - \bar{\eta}_2\bar{\zeta}_2)],$$

$$P_{12} = [(\eta_1\zeta_1 + \eta_2\zeta_2)(\bar{\eta}_1\bar{\zeta}_2 - \bar{\eta}_2\bar{\zeta}_1) + (\eta_1\zeta_2 - \eta_2\zeta_1)(\bar{\eta}_1\bar{\zeta}_1 + \bar{\eta}_2\bar{\zeta}_2)],$$

$$P_{01} + iP_{23} = -2\bar{r}x_1,$$

$$P_{02} + iP_{31} = -2\bar{r}x_2,$$

$$P_{03} + iP_{12} = -2\bar{r}x_3.$$

These formulae exhibit in the clearest fashion how we use a real line to represent a complex point. Let us proceed to develop those metrical relations between lines which correspond to metrical relations between points in the complex plane.*

The equation of our Absolute quadric being

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0,$$

if the space constant of measurement be i, we have for the distance of (Y) and (Z)

$$\cosh d = \frac{-Y_0Z_0 + Y_1Z_1 + Y_2Z_2 + Y_3Z_3}{\sqrt{-Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2}\sqrt{-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2}}.$$

The polar of (P) with regard to the Absolute will be

$$Q_{0i} = P_{jk}, \quad Q_{jk} = -P_{0i}, \quad i, j, k = 1, 2, 3.$$

Two lines in general position will have two common perpendiculars, which are mutually absolute polar. One perpendicular will thus inevitably be in the actual or finite domain. The distances which our lines determine on their common perpen-

* Sketched without proof by Study, 'Nicht-euklidische und Liniengeometrie', *Jahresbericht der deutschen Mathematikervereinigung*, vol. xi, 1902. In more detail in the author's *Elements of Non-Euclidean Geometry*, Oxford, 1908 pp. 116 ff.

diculars shall be called the *distances* of the lines. That which is on the ultra-infinite perpendicular can be interpreted in terms of a finite angle if desirable. If we write for convenience

$$f(P, P') = \sum_{i=1}^{i=3} (P_{0i}P'_{jk} + P_{jk}P_{0i}'),$$

$$\phi (P, P') = \sum_{i=1}^{i=3} (-P_{0i}P_{0i}' + P_{jk}P_{jk}'),$$

we find, after rather a tedious reckoning, that *

$$-\sinh d_1 \sinh d_2 = \frac{f(P,P')}{\sqrt{\phi (P,P)}\sqrt{\phi (P',P')}},$$

$$\cosh d_1 \cosh d_2 = \frac{\phi (P,P')}{\sqrt{\phi (P,P)}\sqrt{\phi (P',P')}}.$$

If two points (x) and (x') of the complex plane $x_0 = 0$ have the distance d, then

$$\cosh d = \frac{x_1x_1' + x_2x_2' + x_3x_3'}{\sqrt{x_1^2 + x_2^2 + x_3^2}\sqrt{x_1'^2 + x_2'^2 + x_3'^2}}$$

$$= \frac{\phi (P, P') - if (P, P')}{\sqrt{\phi (P,P)}\sqrt{\phi\phi'(P',P')}}.$$

Theorem 20] *In the Klein-Study representation the real part of the hyperbolic cosine of the distance of two complex points will be equal to the product of the hyperbolic cosines of the distances of the corresponding real lines, while the pure imaginary part will be equal to the product of the hyperbolic sines of the distances.*

The points of a line in our complex plane are the totality of points conjugate to a given point with regard to the Absolute conic of the plane. The polar of (x') is

$$x_1'x_1 + x_2'x_2 + x_3'x_3 = 0,$$

which involves

$$\phi (P_1P') = f(P_1P') = 0.$$

* Ibid., pp. 111, 112.

Theorem 21] *The points of a line not tangent to the absolute conic will appear in the Klein-Study representation as the system of actual lines cutting an actual line at right angles; the points of a tangent line will appear as a bundle of Lobachewski parallels which are concurrent on the Absolute.*

There is just one other method of representing complex points by real lines which we will touch upon in closing. The credit, or discredit, for discovering this seems to be due to the Author.* The fundamental idea is simplicity itself. We have merely to project our given plane from an outside point upon an imaginary plane (i.e. one which does not contain the conjugate of a generic point) and use as representing line that which connects the projected point to its conjugate. Analytically, let us start with the plane

$$x_0 = 0,$$

and project from the point (1, 0, 0, 0) upon the plane

$$x_0 + ix_1 = 0.$$

The point $(X_1 + iY_1,\quad X_2 + iY_2,\quad X_3 + iY_3)$

will be represented by the line

$$P_{01} = (X_1^2 + Y_1^2), \qquad P_{23} = X_2 Y_3 - X_3 Y_2,$$
$$P_{02} = (X_1 X_2 + Y_1 Y_2), \quad P_{31} = X_3 Y_1 - X_1 Y_3,$$
$$P_{03} = (X_1 X_3 + Y_1 Y_3), \quad P_{12} = X_1 Y_2 - X_2 Y_1.$$

This method suffers in practice from the great number of exceptional points. The real line common to the two planes well represent all of its own complex points, all lines which intersect this real line represent no points at all. The points of a line will usually be represented by the lines of a linear congruence with conjugate imaginary directrices. The characteristic feature of the article mentioned is that it is entirely synthetic and projective. A complex point is defined as an elliptic involution according to the Von Staudt practice, to be explained in the last chapter, and each complex con-

* 'A Pure Geometrical Representation of all points of the Projective Plane', *Transactions American Math. Soc.*, vol. i, 1900. Cf. also Juel, 'Über einen neuen Beweis der Kleinschen Relation zwischen Singularitäten, &c.', *Math. Annalen*, vol. lxi, 1905.

struction called for is worked out in real terms. This excessive purity of method makes the article rather dull reading.

§ 3. Other Representations.

It is evident that besides pairs of points and individual lines there are other real geometrical figures which may be used to represent the complex points of the plane. We shall give examples of two or three such systems of representation in this concluding section.

We begin with a very obvious procedure which is usual in the theory of functions. Taking our complex point

$$x = X_1 + iX_2, \quad y = Y_1 + iY_2, \tag{1}$$

we shall represent it by the real point of a four-dimensional projective space which has the coordinates

$$X_0 = 1, \quad X_1 = X_1, \quad X_2 = X_2, \quad X_3 = Y_1, \quad X_4 = Y_2.$$

To be specific, we assume that if the right-hand side of the first equation were replaced by 0, we should have the equation of the hyperplane at infinity. There will be a one to one correspondence between the finite real points of this S_4 and the finite points of c, our complex plane. The geometrical significance of the correspondence may be described in the following terms.*

Let our given plane be determined in S_4 by the equations

$$x_2 = x_4 = 0.$$

We connect the complex point

$$x_0 = 1, \quad x_1 = X_1 + iX_2, \quad x_2 = 0, \quad x_3 = Y_1 + iY_2, \quad x_4 = 0,$$

with the infinite complex line having the equations

$$x_0 = 0, \quad x_1 + ix_2 = 0, \quad x_3 + ix_4 = 0.$$

The connecting plane will have one real point, whose coordinates are given above, and this is taken to represent the complex point in question. The representation will not only

* Segre, 'Le rappresentazioni reali delle forme complesse', *Math. Annalen*, vol. xl, 1892.

break down when the projecting plane contains a whole line of real points, i.e. when it lies with its conjugate in a real S_3, and this can only happen when it is at infinity.

However useful this method may be from the point of view of the analyst, as a geometrical method for representing complex points it falls behind others which we have seen. Not only are there many exceptional elements, but we are forced to go outside of our own S_3.

If we had followed a strictly historical order in the present chapter, we should have come much earlier to the next method, which was devised by Sophus Lie.* We start with a Cartesian S_3 and seek to represent all points of

$$y = 0.$$

To do so, we allow the complex point

$$x = X_1 + iX_2, \quad y = 0, \quad z = Z_1 + iZ_2$$

to be represented by the real point

$$X = X_1, \quad Y = X_2, \quad Z = Z_1,$$

to which is attached a weight Z_2. In this way we represent complex non-weighted points of the plane, by real weighted points of space. The geometrical connexion may be explained as follows:

The given complex point is projected orthogonally upon the x axis, and the Gauss representative is found in the (x, y) plane. A perpendicular is dropped from there upon the plane, parallel to the (x, y) plane which passes half-way between the given complex point and its conjugate. The foot of this perpendicular is the point sought. The points of a two-parameter family, as a curve, will be represented by the points of a surface on which lies a one-parameter family of isobaric curves. The most interesting of these is the system of points of weight 0, which Lie calls the *Null Strip*.† As an example let us take a straight line, not parallel to the z axis

$$(B_1 + iB_2)(Z_1 + iZ_2) = (X_1 + iX_2) - (A_1 + iA_2).$$

* 'Ueber die Darstellung des Imaginären in der Geometrie', *Crelles Journal*, vol. lxx, 1869.

† Ibid., p. 346.

The representing surface is the plane

$$B_1(X-A_1)+B_2(Y-A_2) = (B_1^2+B_2^2)Z.$$

The isobarics are the lines where this plane meets the planes

$$B_2(X-A_1)-B_1(Y-A_2) = -(B_1^2+B_2^2)Z.$$

These are the lines of steepest slope, if the (x, y) plane be looked on as horizontal. The null strip is the line given by

$$B_2(X-A_1)-B_1(Y-A_2) = 0.$$

On the other hand, the given line has the equations

$$x-(B_1+iB_2)z = A_1+iA_2,$$

$$y = 0.$$

The given line and the null strip lie in the plane

$$x+iy-(B_1+iB_2)z = A_1+iA_2,$$

and this contains one of the circular points at infinity of the (x, y) plane. Conversely, every finite real line will determine with this circular point a complex plane which will meet the (x, y) plane in a complex line not parallel to the (x) axis. We thus pass over naturally to the idea of looking upon the Lie representation, much as Klein looked upon that associated with his name, to wit, as a means of representing the complex lines in the plane by the real lines in space.*

If the complex line rotate about a fixed point, the plane connecting it with a fixed circular point will pass through a fixed complex line, and the real representing line will meet this complex line and its conjugate. The lines through a point in the plane will be represented by the real lines of an elliptic linear congruence in space.

Considered merely as a means of representing complex elements, it is clear that Lie's method falls well below some others. It is an interesting historical fact, however, that it

* For a discussion of the Lie method from this point of view see Smith, 'On Sophus Lie's Representation of Imagineries in Plane Geometry', *Annals of Mathematics*, Series 2, vol. iii, 1901–2. Compare with this excellent exposition a most obscure article by Busche, 'Ueber eine reelle Darstellung der imaginären Gebilde in der Geometrie', *Crelles Journal*, vol. cxxii, 1900.

was writing upon this subject that put Lie on the track of some of his most important geometrical ideas.*

We saw in 1] that the Laguerre representation is without any exception if our complex space be that of the geometry of inversion, but none of the methods so far given avoids exceptional elements when applied to projective space. We shall give, in conclusion, one method which has the noteworthy virtue of having no exception whatever for this same domain. It is the method of Segre.† We start with the conjugate imaginary points

$$(x_1, \quad x_2, \quad x_3) \quad (\bar{x}_1, \quad \bar{x}_2, \quad \bar{x}_3).$$

We write the nine equations

$$\begin{aligned} \sqrt{2}\,X_{11} &= x_1\bar{x}_1, & X_{23} &= x_2\bar{x}_3+x_3\bar{x}_2, & iX_{32} &= x_3\bar{x}_2-x_2\bar{x}_3, \\ \sqrt{2}\,X_{22} &= x_2\bar{x}_2, & X_{31} &= x_3\bar{x}_1+x_1\bar{x}_3, & iX_{12} &= x_1\bar{x}_3-x_3\bar{x}_1, \\ \sqrt{2}\,X_{33} &= x_3\bar{x}_3, & X_{12} &= x_1\bar{x}_2+x_2\bar{x}_1, & iX_{21} &= x_2\bar{x}_1-x_1\bar{x}_2. \end{aligned} \tag{7}$$

These nine homogeneous quantities (X) will give the points of a four-dimensional variety in S_8. They are connected by equations of three types

$$X_{ij}^2+X_{ji}^2-2\,X_{ii}X_{jj}=0, \quad X_{ij}X_{jk}-X_{kj}X_{ji}=\sqrt{2}\,X_{jj}X_{ki},$$
$$X_{ij}X_{jk}+X_{ji}X_{jk}=\sqrt{2}\,X_{jj}X_{ik}. \tag{8}$$

Of course, these equations are not all independent; let us determine the order of the four-dimensional variety which they determine. For this purpose we must treat (x) and $(\bar{x})$ as independent variables, and consider four linear equations

$$\sum_{i,j=1}^{i,j=3} A_{ij}X_{ij} = \sum_{i,j=1}^{i,j=3} B_{ij}X_{ij} = \sum_{i,j=1}^{i,j=3} C_{ij}X_{ij} = \sum_{i,j=1}^{i,j=3} D_{ij}X_{ij} = 0.$$

If the first equation be omitted, and (x) be eliminated from the last three we get a cubic in (x). Omitting the second equation likewise, we get a second cubic in (x). These cubics have nine intersections, of which some are to be rejected as

* Smith, loc. cit., p. 164.

† Segre, *Rappresentazioni*, cit. p. 422.

they correspond to setting the third and fourth equations above proportional to one another. The statement that the third and fourth equations in $(\bar{x})$ are proportional leads to two quadratic equations, one of which says that the coefficients of $\bar{x}_i$ and $\bar{x}_k$ are proportional in the two equations, and the other that the two coefficients of $\bar{x}_j$ and $\bar{x}_k$ are proportional. These quadratics in (x) have four common solutions, of which one is extraneous as it comes from equating the coefficient of $\bar{x}_k$ to zero in both equations. Hence there are three values of (x) which will make the last two equations in $(\bar{x})$ proportional, or the four equations have six common solutions. This is the maximum number, and as we can reach it in a particular case, it is general. The points of our plane are represented without exception by the real points of an S_4^6. As for the quadratic equations above, let us write the equation

$$\begin{vmatrix} \sqrt{2}X_{11} & X_{12}-iX_{21} & X_{31}+iX_{13} \\ X_{12}+iX_{21} & \sqrt{2}X_{22} & X_{23}-iX_{32} \\ X_{31}-iX_{13} & X_{23}+iX_{32} & \sqrt{2}X_{22} \end{vmatrix} = 0. \qquad (9)$$

Then the left-hand sides of the equations (8) are the partial derivatives of the left-hand side of this equation:

Theorem 22] *In the Segre representation, each point of a complex plane is represented, without exception, by a real point of a variety of four dimensions and the sixth order in a space of eight dimensions. This variety is composed of the singular points of a hypersurface of the third order.*

Suppose that we have a line and its conjugate:

$$u_1x_1+u_2x_2+u_3x_3 = 0, \quad \bar{u}_1\bar{x}_1+\bar{u}_2\bar{x}_2+\bar{u}_3\bar{x}_3 = 0.$$

Still treating (x) and $(\bar{x})$ as independent variables, let $(\bar{x})$ remain fixed, while (x) takes values linearly dependent on two; the same will be true of the coordinates (X_{ij}). A similar result will hold when (x) is fixed. We have thus a surface with two sets of rulings.

Theorem 23] *A line in the projective plane will be represented by a real quadric surface with imaginary generators.*

A chain upon our line may be expressed in the form:

$$x_i = X_1 y_i + X_2 z_i, \quad \bar{x}_i = X_1 \bar{y}_i + X_2 \bar{z}_i.$$

This establishes a projective relation between the two sets of generators of the quadric.

Theorem 24] *A chain will be represented in the Segre system by the points of a conic.*

If $(\bar{x})$ remain fixed, while (x) varies freely, the point (X_{ij}) will trace an imaginary plane. If (x) remain fixed, while $(\bar{x})$ varies freely, (X_{ij}) will trace the conjugate imaginary plane.

Theorem 25] *Through each real point of the representing S_4^6 will pass two conjugate imaginary planes of the variety. A collineation of the complex plane will appear in S_8 as a real collineation, permuting among themselves the planes of each of these systems.*

We shall learn in the next chapter how to interpret a collineation that interchanges the two systems. Remembering that, if any two planes be given, the sum of the number of dimensions of their common variety and of the smallest space which includes them both (non-intersection being counted as a space of -1 dimensions) is equal to 4, we reach:

Theorem 26] *Two planes of the same system of S_4^6 have no common point, and lie in an S_5; two of different systems have a common point and lie in an S_4.*

If we take two planes of the same system corresponding to two fixed values for (x), their S_5 contains ∞^1 planes, obtained by giving to (x) any value linearly descendent, on the two given values. This variety of the S_5 is of the third order, as we see when we seek the values of (x) from the equations

$$(ux) = \sum_{i,j} A_{ij} X_{ij} = \sum_{i,j} B_{ij} X_{ij} = \sum_{i,j} C_{ij} X_{ij} = 0.$$

We have thus two systems of cubic varieties, obtained by leaving one of the sets of variables (x) and $(\bar{x})$ perfectly free, while the others are constrained by a linear relation.

Theorem 27] *Two S_3^3's of S_6^4 which belong to the same system have a common plane; two which belong to different systems have a common quadric.*

It is worth noticing in conclusion that we may pass from this representation to two of our previous ones in the following manner. We pick out a quadric of S_4^6. Through it will pass an S_3^3 of each system, and these lie in conjugate imaginary S_5's. These will meet a real S_4 in conjugate imaginary lines. Each point of S_4^6 will determine with the conjugate imaginary S_5's two conjugate imaginary S_6's, which meet the real S_4 in conjugate imaginary planes through the conjugate imaginary lines. These imaginary planes will have one real point in common. This is described at the beginning of the present section. Suppose, secondly, that we have a system of conjugate imaginary planes through two conjugate imaginary lines l, $\bar{l}$ of a real S_4. Let us take a real line r not in S_4, but meeting it in a real point. Together they determine a real S_5. The conjugate imaginary planes of S_4 through l and $\bar{l}$ will determine with r conjugate imaginary four-dimensional spaces which will meet any real S_3 of S_5 in conjugate imaginary planes, sharing a common line. This real line may be taken to represent the real point in S_4 where the conjugate imaginary planes through l and $\bar{l}$ meet one another. We are thus back on what amounts essentially to the last representation in the last section.

CHAPTER V

THE TERNARY DOMAIN: ALGEBRAIC THEORY.

§ 1. Chain Figures.

In our study of the binary domain we found that the chain played a very central and important rôle. The coordinates of the points of a chain were linearly dependent, with real multipliers, on those of two members of the system. This feature will characterize a chain in a space of any number of dimensions. In the plane we have a figure dual to the chain which must now be defined.

Definition. A system of concurrent and coplanar lines, of such a nature that the cross ratios of any four are always real, while the system includes that line which makes any assigned real cross ratio with any three of its members, shall be called a *line chain.*

It is clear that the properties of a line chain are absolutely dual to those of the point chain developed in Ch. II, and that the easiest way to construct a line chain is to draw lines from the points of a point chain to some point not collinear with them.

Definition. A system of points in any number of dimensions whose independent coordinates are analytic functions of two real variables and no less shall be called a *congruence.*

Definition. A congruence shall be called a *chain congruence* if

a) the points be not all collinear,

b) a line connecting two points of the congruence contain a chain thereof,

c) two chains of the congruence have always just one common point.

We see to begin with that all points of a chain congruence must be in one plane, otherwise we could easily find non-intersecting chains of the congruence. The totality of real points of a plane will give the simplest example of a chain congruence. More generally, if (y) (z) (t) be three non-collinear points of the system,

$$x_i = X_1 \rho y_i + X_2 \sigma z_i + P_3 \tau t_i; \quad \bar{x}_i = X_1 \bar{\rho} \bar{y}_i + X_2 \bar{\sigma} \bar{z}_i + X_3 \bar{\tau} \bar{t}_i,$$

where (x) alone is variable, will constitute a part at least of a chain congruence. Now a chain connecting an arbitrary point (r) of the plane, with the point (x) will be

$$x_i' = Y_1 \lambda r_i + X_1 \rho y_i + X_2 \sigma z_i + X_3 \tau t_i,$$

and this will share a point with the chains of the congruences obtained by putting $X_1 = 0$, $X_2 = 0$, $X_3 = 0$ above when, and only when, (r) is expressible in this general form; hence these equations give us the complete chain congruence.

We may draw still further conclusions from these equations. For if (x) be any point of the plane not collinear with two of the points (y), (z), or (t), we may put $X_1 = X_2 = X_3 = 1$, and solve for ρ, σ, and τ, so that there is a chain congruence which includes any four points, no three of which are collinear. On the other hand, four such points could not belong to two different chain congruences, for the same would be true of the diagonal points of their complete quadrangle. Hence the two congruences would determine the same chain on each side of this quadrangle, and their chains on any line would have six common points, and so be identical.*

Theorem 1] *Four coplanar points, whereof no three are collinear, belong to just one chain congruence.*

Theorem 2] *Every chain congruence is projectively equivalent to the real domain of the plane.*

It appears from this that the form for the chain congruence given above is perfectly general. It can, however, be simpli-

* The first writer to discuss the chain congruence seems to have been Segre. See his fundamental article, 'Un nuovo campo', cit. pp. 433 ff. For a different treatment see Young, 'Planar chains, and their Associated Projectivities', *Transactions American Math. Soc.*, vol. xi, 1910.

fied in the case of a particular congruence by imagining that the constant multipliers have been *swallowed* by the homogeneous point coordinates. Thus the general chain congruence may be written

$$x_i = X_1 y_i + X_2 z_i + X_3 t_i\,;\ \bar{x}_i = X_1 \bar{y}_i + X_2 \bar{z}_i + X_3 \bar{t}_i, \quad (1)$$

If we solve the first three equations for X_1, X_2, and X_3, and substitute in the last three, we get

$$| a_{ij} | \bar{x}_i = \sum_{j,\,k} \bar{a}_{ij} A_{kj} x_j$$

$$| a_{ij} | \equiv | yzt | \quad (2)$$

Definition. A system of coplanar non-concurrent lines of such a nature that:

A) A line chain of the system passes through the intersection of each two lines.

B) Two line chains of the system have always a common line and shall be called a *chain congruence of lines.*

Theorem 3] *Four coplanar lines, whereof no three are concurrent, belong to just one chain congruence of lines.*

Theorem 4] *Every chain congruence can be carried on by a collineation into the domain of all real lines, and by a correlation into that of all real points.*

Theorem 5] *The lines containing chains of a chain congruence form a chain congruence of lines, while the vertices of chains of lines in such a congruence form a chain congruence.*

We saw in 24] of the last chapter that in the Segre system a chain is represented by a conic. What will represent a chain congruence? It will be sufficient to consider the real domain. If in equation (7) of that chapter we have $x = \bar{x}$, three of the coordinates will vanish identically, while the other six are proportional to constant multiples of the terms of the general ternary quadratic form.

Theorem 6] *In the Segre representation, each chain congruence will correspond to a Veronese surface of the fourth order* *lying in a space of five dimensions. There are ∞^2 planes which meet such a surface in conics, and each of these will correspond to a chain of the congruence. Every real quadric of S_4^6 will either contain a single real point of such a surface, or meet it in a real conic. There are ∞^2 real quadrics which do the latter.**

Let us turn momentarily from the chain congruence to see what happens in equations (1) if

$$|yzt| = |\bar{y}\bar{z}\bar{t}| = 0.$$

If there exist three such real multipliers, Y_1, Y_2, and Y_3, that

$$Y_1 y_i + Y_2 z_i + Y_3 t_i \equiv 0,$$

we have merely a chain of points. If not, we may write

$$(vx) \equiv X_1(vy) + X_2(vz) + X_3(vt) = 0,$$

$$(\bar{v}\bar{x}) \equiv X_1(\bar{v}\bar{y}) + X_2(\bar{v}\bar{z}) + X_3(\bar{v}\bar{t}) = 0.$$

Solving these, we shall find a point of the system on any line in the plane so that we have all points of the line common to (y), (z), and (t). There is, however, one point of the line which has ∞^1 determinations. We write first

$$\frac{X_1 y_1 + X_2 z_1 + X_3 t_1}{X_1 y_2 + X_2 z_2 + X_3 t_2} = \frac{x_1}{x_2}, \quad \frac{X_1 \bar{y}_1 + X_2 \bar{z}_1 + X_3 \bar{t}_1}{X_1 \bar{y}_2 + X_2 \bar{z}_2 + X_3 \bar{t}_2} = \frac{\bar{x}_1}{\bar{x}_2}.$$

If these equations in (X) be not independent, but equivalent,

$$\frac{(\bar{y}_1\bar{x}_2 - \bar{y}_2\bar{x}_1)}{(y_1 x_2 - y_2 x_1)} = \frac{(\bar{z}_1\bar{x}_2 - \bar{z}_2\bar{x}_1)}{(z_1 x_2 - z_2 x_1)} = \frac{(\bar{t}_1\bar{x}_2 - \bar{t}_2\bar{x}_1)}{(t_1 x_2 - t_2 x_1)}.$$

Now, by the fundamental identity of the binary invariant theory,

$$(\bar{y}_1\bar{x}_2 - \bar{y}_2\bar{x}_1)(\bar{z}_1\bar{t}_2 - z_2\bar{t}_1) + (\bar{z}_1\bar{x}_2 - \bar{z}_2\bar{x}_1)(\bar{t}_1\bar{y}_2 - \bar{t}_2\bar{y}_1)$$
$$+ (\bar{t}_1\bar{x}_2 - \bar{t}_2\bar{x}_1)(\bar{y}_1\bar{z}_2 - \bar{y}_2\bar{z}_1) \equiv 0.$$

* Segre, *Rappresentazioni reali*, cit. p. 430. For an account of the surface see Veronese, 'Le superficie omoloidi, etc.', *Atti della R. Accademia dei Lincei*, series III, vol. xix, pp. 188 ff.

Hence, in the present case,

$$(y_1x_2-y_2x_1)(\bar{z}_1\bar{t}_2-\bar{z}_2\bar{t}_1)+(z_1x_2-z_2x_1)(\bar{t}_1\bar{y}_2-\bar{t}_2\bar{y}_1)$$
$$+(t_1x_2-t_2x_1)(\bar{y}_1\bar{z}_2-\bar{y}_2\bar{z}_1)=0.$$

$$\frac{1}{\rho}x_i = y_i(\bar{z}_1\bar{t}_2-\bar{z}_2\bar{t}_1)+z_i(\bar{t}_1\bar{y}_2-\bar{t}_2\bar{y}_1)+t_i(\bar{y}_1\bar{z}_2-\bar{y}_2\bar{z}_1). \qquad (3)$$

These equations hold while $i=1$, or 2. There will thus be an infinite number of sets of real values X_1, X_2, X_3 where

$$X_1y_1+X_2z_1+X_3t_1=\rho\,[y_1(\bar{z}_1\bar{t}_2-\bar{z}_2\bar{t}_1)$$
$$+z_1(\bar{t}_1\bar{y}_2-\bar{t}_2\bar{y}_1)+t_1(\bar{y}_1\bar{z}_2-\bar{y}_2\bar{z}_1)].$$

$$X_1y_2+X_2z_2+X_3t_2=\rho\,[y_2(\bar{z}_1\bar{t}_2-\bar{z}_2\bar{t}_1)$$
$$+z_2(\bar{t}_1\bar{y}_2-\bar{t}_2\bar{y}_1)+t_2(\bar{y}_1\bar{z}_2-\bar{y}_2\bar{z}_1)].$$

But since $\quad |yzt| = |\bar{y}\bar{z}\bar{t}| = 0,$

$$X_1y_3+X_2z_3+X_3t_3=\rho\,[y_3(\bar{z}_1\bar{t}_2-\bar{z}_2\bar{t}_1)$$
$$+z_3(\bar{t}_1\bar{y}_2-\bar{t}_2\bar{y}_1)+t_3(\bar{y}_1\bar{z}_2-\bar{y}_2\bar{z}_1)].$$

It appears, then, that in equations (3) we may allow the subscripts to take all three values, and these equations give the coordinates of the singular point of their representation.*

Let us turn back to our chain congruence, for we are far from having finished with it. The real domain suggests immediately the symmetrical and involutory transformation between conjugate imaginary points. We get from 2]:

Theorem 7] *There is associated with every chain congruence an involutory transformation of the plane, which is not a collineation, but which carries points into points and lines into lines without exception, and leaves all points of the congruence, but no other points, invariant.*

Theorem 8] *A line either contains a chain of a chain congruence or a single point thereof where it meets the corresponding line in the involutory transformation associated with the congruence.*

Theorem 9] *The necessary and sufficient condition that a line should contain a chain of a chain congruence is that it*

* Segre, *Nuovo campo*, cit. p. 434, points out the existence of this point, without finding its coordinates.

should contain one pair, and hence an infinite number of pairs of corresponding points in the associated transformation.

Theorem 10] *Through each point of the plane of a chain congruence, which is not itself a point of the congruence, will pass just one line containing a chain of the congruence. Through every point of the congruence will pass a chain of such lines.*

Suppose that we have two chain congruences; will they necessarily have any common elements? We may safely assume that one congruence is the real domain, while the other takes the form (2). Writing the equations

$$\frac{x_1}{\bar{x}_1} = \frac{x_2}{\bar{x}_2} = \frac{x_3}{\bar{x}_3}$$

we get three real conics in the (X) plane, with, usually, three intersections, of which at least one must be real.

Theorem 11] *Two chain congruences in the same plane must have at least one common point.*

We pass to the next larger linear system. Let us imagine that (y), (z), (s), and (t) are four points, no three collinear, and that these coordinate values are not linearly dependent in terms of real multipliers. We write

$$x_i = X_1 y_i + X_2 z_i + X_3 s_i + X_4 t_i.$$

Here is a set of points depending on three real parameters. Let us see whether we can find a line, all of whose points are included in the system. For such a line (u) we have

$$X_1(uy) + X_2(uz) + X_3(us) + X_4(ut) = 0,$$
$$X_1(\bar{u}\bar{y}) + X_2(\bar{u}\bar{z}) + X_3(\bar{u}\bar{s}) + X_4(\bar{u}\bar{t}) = 0,$$

and these equations must be equivalent in order to have ∞^2 common solutions. Now, by a familiar ternary identity

$$(\bar{u}\bar{t})\,|\bar{y}\bar{z}\bar{s}| \equiv (\bar{u}\bar{y})\,|\bar{z}\bar{s}\bar{t}| - (\bar{u}\bar{z})\,|\bar{y}\bar{s}\bar{t}| + (\bar{u}\bar{s})\,\bar{y}\bar{z}\bar{t}\,|.$$

If, thus, we write

$$\frac{(uy)}{(\bar{u}\bar{y})} = \frac{(uz)}{(\bar{u}\bar{z})} = \frac{(us)}{(\bar{u}\bar{s})} = \frac{(ut)}{(\bar{u}\bar{t})}$$

each of these expressions is equal to

$$\frac{(uy)\,|\,\bar{z}\bar{s}\bar{t}\,| - (uz)\,|\,\bar{y}\bar{s}\bar{t}\,| + (us)\,|\,\bar{y}\bar{z}\bar{t}\,|}{(\bar{u}\bar{y})\,|\,\bar{z}\bar{s}\bar{t}\,| - (\bar{u}\bar{z})\,|\,\bar{y}\bar{s}\bar{t}\,| + (\bar{u}\bar{s})\,|\,\bar{y}\bar{z}\bar{t}\,|}.$$

Hence

$$(uy)\,|\,\bar{z}\bar{s}\bar{t}\,| - (uz)\,|\,\bar{y}\bar{s}\bar{t}\,| + (us)\,|\,\bar{y}\bar{z}\bar{t}\,| - (ut)\,|\,\bar{y}\bar{z}\bar{s}\,| = 0,$$

and (u) goes through a fixed point. Conversely, if (u) go through this point, and if

$$\frac{(uy)}{(\bar{u}\bar{y})} = \frac{(uz)}{(\bar{u}\bar{z})} = \frac{(us)}{(\bar{u}\bar{s})},$$

then each of these expressions is equal to $\frac{(ut)}{(\bar{u}\bar{t})}$, and all points of (u) belong to the congruence. We may multiply (u) and $(\bar{u})$ by such multipliers $e^{i\theta}$ and $e^{-i\theta}$ that it is safe to assume

$$(uy) = (\bar{u}\bar{y}) = U_1, \quad (uz) = (\bar{u}\bar{z}) = U_2, \quad (us) = (\bar{u}\bar{s}) = U_3.$$

$$|\,yzs\,|\,u_i = U_1\alpha_i + U_2\beta_i + U_3\gamma_i,$$

$$|\,\bar{y}\bar{z}\bar{s}\,|\,\bar{u}_i = U_1\bar{\alpha}_i + U_2\bar{\beta}_i + U_3\bar{\gamma}_i.$$

Hence (u) belongs to a chain congruence of lines. It goes through the singular point above if

$$U_1\left[\,|\,\bar{z}\bar{s}\bar{t}\,| - \frac{|\,\bar{y}\bar{z}\bar{s}\,|}{|\,yzs\,|}(\alpha t)\right] - U_2\left[\,|\,\bar{y}\bar{s}\bar{t}\,| - \frac{|\,\bar{y}\bar{z}\bar{s}\,|}{|\,yzs\,|}(\beta t)\right]$$
$$+ U_3\left[\,|\,\bar{y}\bar{z}\bar{t}\,| - \frac{|\,\bar{y}\bar{z}\bar{s}\,|}{|\,yzs\,|}(\gamma t)\right] = 0.$$

The conjugate equation is identically satisfied. Hence there are ∞^1 lines forming a chain congruence through this point.

Theorem 12] *The system of points linearly dependent in terms of real multipliers on four given points, of which no three are collinear, is either a chain congruence or a chain of lines.*

§ 2. Linear Transformations.

Suppose that we have a continuous one to one transformation of the plane that carries points on a line into points on a line. Can we give an analytic form for every such transformation?

To begin with, the transformation will carry a complete quadrilateral into a complete quadrilateral, hence harmonic separation is invariant. It will follow, from Ch. I, theorem 19], that the relation between a line and its transform is either a collineation or an anti-collineation. If we follow our transformation by a collineation which replaces the zero points and the unit point where they were, we have a new transformation which leaves these four points invariant. It will likewise leave invariant the diagonal points of their complete quadrangle, and so every real point on a side of this quadrangle. Hence every real line will have six real fixed points, of which at least four are distinct; every real point is fixed. Suppose that there is a complex point which is invariant. All real points and one complex point will be invariant on the line connecting it with its conjugate. Hence every point of that line is invariant. Lastly, if we take any complex point of the plane, and connect it with two real points, not on the line last drawn, nor on the line connecting the given point with its conjugate, we have two lines each with a real fixed point, and an imaginary fixed point on the line recently drawn. Hence the lines are fixed and their intersection is fixed, i.e. the transformation is the identical one. When, on the other hand, there is no fixed complex point, the transformation must consist in the interchange of conjugate imaginary points, for such is the case on the real line connecting a point with its conjugate. It appears, thus, that our original transformation was either a collineation, or the product of a collineation and an interchange of conjugate imaginary points. This latter is called an *anti-collineation*, and is written

$$\rho x_i' = \sum_j a_{ij}\bar{x}_j, \quad |a_{ij}| \neq 0. \tag{4}$$

Theorem 13] *Every one to one continuous transformation of the plane which carries collinear points into collinear points is either a collineation or an anti-collineation.**

It is worth noticing that when we limit ourselves to the real plane, we may drop the requirement of continuity, and

* Segre, *Nuovo campo*, cit. p. 291.

announce, in virtue of Ch. I. 6], that every one to one transformation of the real plane that carries collinear points into collinear points is a collineation.

Theorem 14] *The totality of all collineations and anti-collineations is an eight-parameter group, the collineations forming an eight-parameter sub-group. Every transformation of the general group will carry a chain into a chain, and a chain congruence into a chain congruence.*

Theorem 15] *An anti-collineation of the plane will appear in the Segre representation as a collineation in S_8, which interchanges the two systems of planes of S_4^6.**

Among the anti-collineations the most interesting are the involutory ones or anti-involutions. In such transformation, every line connecting a pair of conjugate points is invariant, and the points thereon are transformed by a binary anti-involution, so that, since there is an invariant point, there is an invariant chain. The lines bearing these invariant chains will form a chain congruence of lines, and the invariant point will form a chain congruence. We have thus the involutory transformation associated with a chain congruence which we have already studied in the present chapter. If, in an anti-involution the points (1, 0, 0) and (0, 1, 0) be invariant, the transformation may be written

$$x_1' = -\frac{a_{13}}{\bar{a}_{13}}\bar{x}_1 + a_{13}\bar{x}_3,$$

$$x_2' = -\frac{a_{23}}{\bar{a}_{23}}\bar{x}_2 + a_{23}\bar{x}_3,$$

$$x_3' = \bar{x}_3.$$

Here the points (x) and (x') may be taken arbitrarily, neither being on the line of the invariant points, and the coefficients found to fit.

Theorem 16] *An anti-involution may be found to leave two chosen points invariant, and to interchange any other two points, neither of which is collinear with the first two.*

* Segre, *Rappresentazioni reali*, cit. p. 425.

If two pencils of lines be anti-projective, their common line being self-corresponding, then we may find an anti-involution to leave their vertices in place and interchange the intersections of two pairs of corresponding lines. The anti-projective relation between the two pencils will thus be carried over invariant. On a line which connects the intersection of corresponding lines of the pencils to the transformed intersection, we shall have a chain of points where corresponding lines meet. The lines of two such chains will meet in an invariant point, which belongs to both chains.

Theorem 17] *If two pencils of lines be anti-projective, while their common line is self-corresponding, the locus of their points of intersection is a chain congruence.*

We leave to the reader the task of proving the following theorem, which is outside the proper subject-matter of the present chapter:

Theorem 18] *In a space of* $2n$ *dimension every anti-involution will leave invariant a variety depending linearly, with real parameters, on* $2n+1$ *points. In a space of* $2n+1$ *dimensions, an anti-involution will either leave no points invariant, or else a variety linearly dependent in terms of real parameters on* $2n+2$ *points.**

A real collineation will leave the real domain invariant. This raises the general question of when a collineation will leave a chain congruence in place. If the congruence be the real domain, the collineation must be real, but such a collineation may leave other chain congruences invariant besides the real domain. If the characteristic equation of the collineation have distinct roots, there are three distinct fixed points, of which at least one is real, and the other two are real or conjugate imaginary. Hence, as the real domain may be taken to represent any invariant chain, the fixed points of the collineation are either points of the invariant chain, or points interchanged in the associated anti-involution. Now look at

* Cf. Sforza, 'Contributo alla geometria complessa', *Giornale di matematiche*, vol. xxx, 1892, p. 166 ff.

the converse. If the collineation have three real fixed points, it may be written

$$x_1' = A_1 x_1, \quad x_2' = A_2 x_2, \quad x_3' = A_3 x_3.$$

An arbitrary chain congruence through these points may be written

$$x_1 = \alpha_1 X_1, \quad x_2 = \alpha_2 X_2, \quad x_3 = \alpha_3 X_3,$$

and this will be found invariant for our collineation.

Suppose, secondly, that the characteristic equation has a pair of conjugate imaginary roots. We may reduce the collineation to

$$x_1' = A_1 x_1, \quad x_2' = R\cos\Theta x_2 + R\sin\Theta x_3,$$
$$x_3' = -R\sin\Theta x_2 + R\cos\Theta x_3.$$

This will leave invariant every chain congruence of the form

$$x_1 = \alpha_1 X_1,$$
$$x_2 = S\cos\Phi\alpha_2 + S\sin\Phi\alpha_3,$$
$$x_3' = -S\sin\Phi\alpha_2 + S\cos\Phi\alpha_3.$$

Theorem 19] *If a collineation with three and only three fixed points leave a chain congruence invariant, then either the three points belong to the congruence, or one belongs thereto, and the other two are interchanged in the associated anti-involution. These necessary conditions for an invariant congruence are also sufficient.*

We might, in similar fashion, examine the types of congruence invariant under the other types of collineation.* We will not, however, follow out this idea, but rather develop another type of linear transformation.

Suppose that we have a continuous one to one transformation that carries a point into a line, and a range of collinear points into a pencil of concurrent lines. The product of this and a correlation is clearly a collineation or an anti-collineation. Hence the original transformation was either a correlation, or a transformation which we shall define as an *anti-correlation*, and express in the form:

$$\rho u_i = \sum_j a_{ij}\bar{x}_j, \quad |a_{ij}| \neq 0.$$

* Young, *Planar Chains*, cit. pp. 287 ff.

The most interesting anti-correlations are the anti-polarities. Here, it is easy to see, we are safe to assume that

$$a_{ji} = \bar{a}_{ij}.$$

If there be any points which lie on the corresponding lines, their coordinates will satisfy the equation

$$\sum a_{ij} x_i \bar{x}_j = 0, \quad a_{ji} = \bar{a}_{ij}, \quad | a_{ij} | \neq 0. \tag{5}$$

The left side of this equation was defined on p. 45, as a Hermitian form. The equation can be expressed most neatly by employing the Clebsch-Aronhold symbolism, and writing

$$(ax)(\bar{a}\bar{x}) = 0, \quad | aa'a'' | \cdot | \bar{a}\bar{a}'\bar{a}'' | \neq 0. \tag{6}$$

If we subject our variables (x) and $(\bar{x})$ to the collineation

$$x_i' = \sum_j b_{ij} x_j, \quad \bar{x}_i' = \sum_j \bar{b}_{ij} \bar{x}_j,$$

we find that

$$\sum a_{ij} x_i' \bar{x}_i' = 0,$$

$$| a_{ij} | = | b_{ij} | \cdot | \bar{b}_{ij} | \cdot | a_{ij}' |.$$

Let us turn aside for a moment and study the cases where the discriminant vanishes. If the rank of the matrix be 2, then the equations

$$\sum_j a_{ij} \bar{x}_j = 0$$

have one and only one common solution. If this be the point $(0, 0, 1)$, the equation becomes

$$A_{11} x_1 \bar{x}_1 + a_{12} x_1 \bar{x}_2 + \bar{a}_{12} x_2 \bar{x}_1 + A_{22} x_2 \bar{x}_2 = 0.$$

This will give a system of lines through the singular point which, as we saw in Ch. I, either meet the line $x_3 = 0$ in a chain or not at all. When the rank of the matrix is unity, we may reduce to

$$x_3 \bar{x}_3 = 0.$$

Theorem 20] *If a ternary Hermitian form be equated to zero, then if the rank of the matrix be two, the locus so defined is either a chain of lines, or a single point; if the rank be unity, the locus is a line.*

Conversely, we see that the line-chain

$$u_i = Lv_i + Mw_i$$

may be written equally well

$$(vx)(\bar{w}\bar{x})-(wx)(\bar{v}\bar{x}) = 0.$$

Let us return to the form (5) or (6). The expressions

$$\sum a_{ij}y_i\bar{x}_j, \quad \sum a_{ij}x_i\bar{y}_j,$$

or their symbolic equivalents $(ay)(a\bar{x})$, $(ax)(\bar{a}\bar{y})$ vanish together. Either equated to zero gives the polar line to (y) in the anti-polarity. We see that if (x) be on the anti-polar of (y), then (y) is on the anti-polar of (x). We may thus find a self-conjugate triangle, as in the case of a conic, and, with this as coordinate triangle, reduce the equation to the form

$$A_1x_1\bar{x}_1+A_2x_2\bar{x}_2+A_3x_3\bar{x}_3 = 0.$$

If the three A's have the same sign, this equation can have no solution, otherwise there will be a three-parameter system of points whose coordinates satisfy the equation.

Theorem 21] *If a ternary Hermitian form of non-vanishing discriminant be equated to zero, either there is no point whose coordinates satisfy the equation, or there is a system depending on three real parameters.*

Theorem 22] *Sylvester's law of inertia holds for ternary Hermitian forms.**

§ 3. Hyperconics.

The locus of all points whose coordinates satisfy an equation of the type (5) or (6), where there are any such points, shall be called a *hyperconic*. We may reduce such a locus to the canonical form

$$x_1\bar{x}_1+x_2\bar{x}_2-x_3\bar{x}_3 = 0.$$

It will be noticed that two sides of the coordinate triangle meet the variety in chains of points, while the third side does not meet it at all. On the other hand, the line $x_2 = x_3$ meets it in a single point. More generally, if we express the line from (y) to (x) in the parametric form

$$x_1 = \xi_1y_i+\xi_2z_i, \quad \bar{x}_1 = \bar{\xi}_1\bar{y}_i+\bar{\xi}_2\bar{z}_i,$$

* Segre, *Nuovo campo*, cit. p. 605, note. See also an article by the author, 'Geometry of Hermitian Forms', *Transactions American Math. Soc.*, vol. xxi, 1920, p. 46.

and substitute in (6), we get

$$\xi_1\bar{\xi}_1 (ay)(\bar{a}\bar{y}) + \xi_1\bar{\xi}_2 (ay)(\bar{a}\bar{z}) + \xi_2\bar{\xi}_1 (az)(\bar{a}\bar{y}) + \xi_2\bar{\xi}_2 (az)(\bar{a}\bar{z}) = 0.$$

The discriminant of this binary Hermitian form is

$$\tfrac{1}{2} \mid aa'u \mid \cdot \mid \bar{a}\bar{a}'\bar{u} \mid, \quad (uy) = (uz) = 0.$$

Since the coefficient of $\xi_1\bar{\xi}_1$ is real, we see from the reasoning that led up to II. 25] that when this last expression is positive, there are no solutions of the binary equation, and the line fails to meet the hyperconic. When this expression is negative, there is a chain of points of intersection. When it vanishes, there is but a single point. Let the reader remember that this discriminant is only a symbolic product; it is not really the product of two conjugate imaginary factors.

Theorem 23] *A straight line will meet a hyperconic, either in a chain of points, or in a single point, or not at all.*

Theorem 24] *The necessary and sufficient condition that a line should meet a hyperconic in a single point is that it should contain its pole in the corresponding anti-polarity.*

We shall, naturally, call such a line a *tangent*, although it is not the limiting position of a secant. The tangential equations corresponding to (5) and (6) are

$$\sum A_{ij} u_i \bar{u}_j = 0. \tag{7}$$

$$\mid aa'u \mid \cdot \mid \bar{a}\bar{a}'\bar{u} \mid = 0. \tag{8}$$

Theorem 25] *There are no curves which are entirely contained in a hyperconic.*

It is evident, in fact, that no variety which fails to intersect a straight line can contain a curve. This is rather surprising when we reflect that a curve is a two-parameter system of points, while a hyperconic is a three-parameter system. In the next chapter we shall handle this same question in all generality.

Let us look for a moment at the question of the intersection of a hyperconic and a chain congruence, which we may safely take as the real domain. The polar lines of the points of the congruence will generate a chain congruence of lines. The lines containing chains of the given congruence will, by 5], generate a chain congruence of lines, and this, by 11], will share at least one line with the congruence of polars. Let us take this as the line $x_3 = 0$. Suppose, first, that this line is a tangent to the hyperconic. We may take the point of contact, which must be real, as (0, 1, 0). The equation of the hyperconic will then be

$$A_{11}x_1\bar{x}_1 + a_{13}x_1\bar{x}_1 + \bar{a}_{13}x_3\bar{x}_1 + a_{23}x_2\bar{x}_3 + \bar{a}_{23}\bar{x}_2x_3 + A_{33}x_3\bar{x}_3 = 0.$$

The real points, of which there is surely one, lie on a conic whose discriminant is

$$-A_{11}\frac{(a_{23}+\bar{a}_{23})^2}{4}.$$

If

$$A_{11} = 0,$$

the discriminant of our hyperconic vanishes, and this we exclude explicitly. If

$$a_{23} + \bar{a}_{23} = 0,$$

the real part of the hyperconic is on the locus given by

$$A_{11}x_1^2 + (a_{13}+\bar{a}_{13})\,x_1x_3 + A_{33}x_3^2 = 0,$$

and so is either two chains, one chain, or a single point. On the other hand, if our line $x_2 = 0$ be not tangent, we may take its pole as (0, 0, 1) and write

$$A_{11}x_1\bar{x}_1 + a_{12}x_1\bar{x}_2 + \bar{a}_{12}x_2\bar{x}_1 + A_{22}x_2\bar{x}_2 + A_{33}x_2\bar{x}_3 = 0.$$

This may have a real thread, or no real points at all, as we see by studying the two hyperconics

$$x_1\bar{x}_1 + x_2\bar{x}_2 - x_3\bar{x}_3 = 0,$$

$$x_1\bar{x}_1 + 2i(x_1\bar{x}_2 - x_2\bar{x}_1) + x_2\bar{x}_2 + x_3\bar{x}_3 = 0.$$

Theorem 26] *A hyperconic will share with a chain congruence, either a thread on a non-degenerate conic, or a pair of chains, a single chain, a single point, or no point at all.*

We may study the intersections of a hyperconic with a conic by expressing the latter parametrically,

$$x_i = \xi_1^2\alpha_i + 2\xi_1\xi_2\beta_i + \xi_2^2\gamma_i.$$

The results do not, however, seem to be particularly interesting.* It is better to see how our hyperconic will appear in our various representations of the plane. To begin with, we remark that in (5) the terms

$$a_{ij}x_i\bar{x}_j+\bar{a}_{ij}x_j\bar{x}_i$$

could equally well be written

$$\tfrac{1}{2}(a_{ij}+\bar{a}_{ij})(x_i\bar{x}_j+x_j\bar{x}_i)-\tfrac{1}{2}[i(a_{ij}-\bar{a}_{ij})][i(x_i\bar{x}_j-x_j\bar{x}_i)],$$

so that our equation is linear, with real coefficients, in terms of the nine variables

$$x_i\bar{x}_i,\quad x_i\bar{x}_j+x_j\bar{x}_i,\quad i(x_i\bar{x}_j-x_j\bar{x}_i).$$

Remembering formulae (7) of the last chapter, we reach

Theorem 27] *A hyperconic will appear in the Segre representation as the intersection of* S_4^6 *with a real hyperplane, and every such intersection will correspond to a hyperconic.*

Theorem 28] *Through eight points in general position there will pass just one hyperconic.*

It is interesting to see how a hyperconic will appear in the Klein-Study representation. The hyperconic

$$\sum A_i x_i\bar{x}_i=0$$

will give the line complex

$$\sum_{i=1}^{i=3} A_i(p_{0i}^2+p_{jk}^2)=0.$$

This is a Battaglini complex.† It is the locus of lines where the tangent planes to the quadrics

$$-u_0^2+u_1^2+u_2^2+u_3^2=0,$$

$$\sum A_i(u_0^2+u_1^2+u_2^2+u_3^2)-2\sum A_i u_0^2=0$$

form a harmonic set.

If, therefore, we take a hyperbolic system of measurement, where the first of these quadrics is the Absolute, we reach the pretty theorem

Theorem 29] *A hyperconic will appear in the Klein-Study representation as a Battaglini complex of lines through which*

* Cf. Guareschi, 'Geometria di una forma quadratica e di una forma d'Hermite', *Atti della R. Accademia delle Scienze di Torino*, vol. xli, 1906.

† Cf. Jessop, *Treatise on the Line Complex*, Cambridge, 1903, p. 133.

the pairs of tangent planes to a quartic are mutually perpendicular in a hyperbolic system of measurement.

Returning to the Segre system, we shall define as *apolar* two hyperconics
$$(ax)(\bar{a}\bar{x}) = 0, \quad (bx)(\bar{b}\bar{x}) = 0,$$
where
$$|abb'|\cdot|\bar{a}\bar{b}\bar{b}'| = 0.$$

The totality of hyperconics, apolar to a given hyperconic, if degenerate loci be properly counted, will appear in S_8 as the totality of hyperplanes through a given point. They will be the totality of hyperconics in which can be inscribed triangles which are self-conjugate with regard to the given conic, or the totality with regard to which some circumscribed triangle is self-conjugate. The two hyperconics given above have, like two conics, four independent invariants.* We find the covariants by the Clebsch principle for passing from binary to ternary invariants.† Thus the envelope of lines meeting them in two mutually orthogonal chains is, by what precedes Ch. II. 30],
$$|abu|\cdot|\bar{a}\bar{b}\bar{u}| = 0.$$

The corresponding locus is
$$|\alpha\beta x|\cdot|\bar{\alpha}\bar{\beta}\bar{x}| = 0. \tag{9}$$

By employing a familiar ternary identity, we may reduce this to
$$|abb'|\cdot|\bar{b}\bar{a}\bar{a}'|\cdot(a'x)(\bar{b}'\bar{x}) = 0.$$

It is on the whole more interesting to consider the intersections of two hyperconics than their concomitants. Consider the pencil of bilinear forms
$$L\sum a_{ij}x_i\bar{x}_j + M\sum b_{ij}x_i\bar{x}_j;$$
they have the characteristic equation ‡
$$\begin{vmatrix} La_{11}+Mb_{11} & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot & La_{22}+Mb_{22} & \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & La_{33}+Mb_{33} \end{vmatrix} = 0. \tag{10}$$

* Cf. Benedetti, loc. cit., p. 98.

† Cf. Grace and Young, *Algebra of Invariants*, Cambridge, 1903, p. 265.

‡ For a discussion of the elementary divisors of these forms see Loewy, 'Ueber Schaaren reeller quadratischer und Hermitescher Formen', *Crelles Journal*, vol. cxxii, 1900.

The equation is real, and must have one real root.* If this correspond to a bilinear form which vanishes for a single point only, the hyperconics

$$\sum a_{ij}x_i\bar{x}_j = 0, \quad \sum b_{ij}x_i\bar{x}_j = 0$$

may perhaps intersect in this one point, but they can certainly not intersect in any other point.

Suppose, first, that our characteristic equation has three distinct real roots. The two hyperconics have a common self-conjugate triangle. Two sides of this triangle must cut chains from either conic, while one side runs clear of each. Suppose, first, that it is the same side which runs clear in each case. We may reduce our hyperconics to the form

$$x_1\bar{x}_1 + x_2\bar{x}_2 - x_3\bar{x}_3 = 0, \quad A_1x_1\bar{x}_1 + A_2x_2\bar{x}_2 - A_3x_3\bar{x}_3 = 0.$$

Let us suppose, to be definite, that $A_1 > A_2$. Then, if $A_2 > A_3$, or $A_3 > A_1$, there is only one chain of lines linearly dependent on the two hyperconics, and they have no common point. But if $A_1 > A_3 > A_2$, then three real chains will pass through their intersections.

Suppose, secondly, that no side fails to meet both hyperconics. We may reduce them to the forms

$$x_1\bar{x}_1 + x_2\bar{x}_2 - x_3\bar{x}_3 = 0, \quad A_1x_1\bar{x}_1 - A_2x_2\bar{x}_2 + A_3x_3\bar{x}_3 = 0.$$

If $A_2 < A_3$ there is one chain; if $A_2 > A_3$ there are three.

Theorem 30] *If two chains of lines have different vertices, and neither include the joining line, then their points of intersection lie on a third line chain. If two chains lie on different lines, and neither include the intersection, the lines connecting their points will pass through the points of another chain.*

Suppose, now, that our characteristic equation (10) has one real and two imaginary roots. These latter will not yield Hermitian forms, but two bilinear forms, which we may write

$$\sum c_{ij}x_i\bar{x}_j = 0, \quad \sum \bar{c}_{ji}x_i\bar{x}_j = 0.$$

We may, by a collineation of the plane, so arrange matters

* Our discussion of the intersections of two hyperconics is based on Segre, *Nuovo campo*, cit. *Atti della R. Accademia delle Scienze di Torino*, vol. xxvi, 1890, pp. 40 ff.

that the one real root of the characteristic equation shall lead us to the point (0, 0, 1). Then the only terms in x_3 will be $x_3\bar{x}_3$, and as the conjugate imaginary bilinear forms above have vanishing discriminants, their terms in x_1 and x_2 must be factorable, and we may write them

$$(\alpha x_1 + \beta x_2)(\bar{\gamma}\bar{x}_1 + \bar{\delta}\bar{x}_2) + k x_3 \bar{x}_3 = 0,$$

$$(\gamma x_1 + \delta x_2)(\bar{\alpha}\bar{x}_1 + \bar{\beta}\bar{x}_2) + \bar{k} x_3 \bar{x}_3 = 0.$$

If, then, we put

$$\frac{\alpha x_1 + \beta x_2}{x_3} = l, \quad \frac{\gamma x_1 + \delta x_2}{x_3} = m, \quad l\bar{m} + k = 0,$$

these equations represent two anti-projective pencils of lines, whose intersections satisfy the equations above. The vertices are the points $(\beta, -\alpha, 0)$ $(\gamma, -\delta, 0)$, the common line being not self-corresponding, as it was in 17]. It is to be emphasized that this state of affairs will arise whenever we have two conjugate imaginary bilinear forms of vanishing discriminant. Be it noticed also that the two lines

$$\alpha x_1 + \beta x_2 = 0, \quad \gamma x_1 + \delta x_2 = 0$$

will meet hyperconics linearly dependent on the given forms, each in a single point. Hence these hyperconics have double contact. Conversely, suppose that we have two hyperconics with double contact. We may reduce them to

$$a_{12}x_1\bar{x}_2 + \bar{a}_{12}x_2\bar{x}_1 + A x_3\bar{x}_3 = 0, \quad b_{12}x_1\bar{x}_2 + b_{12}x_2\bar{x}_1 + B x_3\bar{x}_3 = 0.$$

The characteristic equation will have one real and two conjugate imaginary roots, and there is no common self-conjugate triangle:

Theorem 31] *If two hyperconics have a common self-conjugate triangle, they either have no common points, or they meet in the intersections of two line chains with no common line and different vertices. If they have double contact they have no common self-conjugate triangle, but determine auto-projective pencils, with no self-corresponding line, about the points of contact. Conversely, two such pencils will give the desired intersections. These are the only cases where the characteristic equation has distinct roots.*

Suppose, next, that the equation has a single root, and a double one that does not annul all the first minors. Each of these must be real, and the latter is the limit of two approaching distinct real roots. The equations of the hyperconics may then be written

$$A_1 x_1 \bar{x}_1 + A_2 x_2 \bar{x}_2 + x_2 \bar{x}_3 + x_3 \bar{x}_2 = 0,$$
$$B_1 x_1 \bar{x}_1 + B_2 x_2 \bar{x}_2 + x_2 \bar{x}_3 + x_3 \bar{x}_2 = 0,$$
$$A_i - B_i \neq 0.$$

The point (0, 0, 1) is common to the two hyperconics, and is their only common point when $(A_1 - B_1)(A_2 - B_2) > 0$. In the contrary case there is a chain through this point whose intersections with the chain

$$(A_1 B_2 - A_2 B_1) x_2 \bar{x}_2 + (A_1 - B_1)(x_2 \bar{x}_3 + x_3 \bar{x}_2) = 0$$

lie on the two hyperconics. The two chains have a common line.

Theorem 32] *If the characteristic equation of two hyperconics have a double root which does not annul the first minors, then either they have a common tangent at a single common point, or they share the intersections of two line chains with different vertices but a common line.**

If the double root reduce the first minors to zero, there is a line counted twice which has the same pole with regard to each. The intersections will be given by two equations reducible to

$$A_1 x_1 \bar{x}_1 + A_2 x_2 \bar{x}_2 = 0, \quad x_3 \bar{x}_3 = 0.$$

Theorem 33] *If the characteristic equation of two hyperconics have a double root which reduces the first minors to zero, either they have no common point or a common chain of points, with a common tangent at each point of the chain.*

We must, lastly, take up the case of a triple root. When the first minors do not vanish we have the limiting case

* Segre, in the place cited, does not consider multiple roots which do not annul first minors.

of 32]. It will be sufficient here to study the intersections of the loci

$$a_{12}x_1\bar{x}_2 + \bar{a}_{12}x_2\bar{x}_1 + A_{22}x_2\bar{x}_2 + a_{13}x_1\bar{x}_3 + \bar{a}_{13}x_3\bar{x}_1 + a_{23}x_2\bar{x}_3 + \bar{a}_{23}x_3\bar{x}_2 + A_{33}x_3\bar{x}_3 = 0,$$

$$B_{22}x_2\bar{x}_2 + B_{33}x_3\bar{x}_3 = 0,$$

$$a_{12}\bar{a}_{12}B_{33} + a_{13}\bar{a}_{13}B_{22} = 0.$$

Here, since B_{22} and B_{33} must have opposite signs, the line chain must really be there. Every line of this chain, except the tangent, meets the hyperconic in a point, and, hence, in a chain.

Theorem 34] *If the characteristic equation of two hyperconics have a triple root which does not reduce the first minors to zero, they have a common point and a common tangent, and a chain of common points on each other line through the common point, belonging to a line chain which includes the tangent.*

There remains the case of the triple root which reduces all first minors to zero. This is the limiting case of 33]. We may reduce our hyperconics to

$$A_{11}x_1\bar{x}_1 + a_{23}x_2\bar{x}_3 + \bar{a}_{23}x_3\bar{x}_2 + A_{33}x_3\bar{x}_3 = 0,$$

$$A_{11}x_1\bar{x}_1 + a_{23}x_2\bar{x}_3 + \bar{a}_{23}x_3\bar{x}_2 + B_{33}x_3\bar{x}_3 = 0.$$

There is but one point common to the hyperconics.

Theorem 35] *If the characteristic equation of two hyperconics have a triple root which reduces all first minors to zero, they have a single common point, with the same tangent there, and every point on that tangent has the same polar with regard to both hyperconics.*

A system of hyperconics linearly dependent on three shall be called a *net*.* Let the equations of the three be

$$(ax)(\bar{a}\bar{x}) = (bx)(\bar{b}\bar{x}) = (cx)(\bar{c}\bar{x}) = 0.$$

We form the Jacobian

$$|\bar{a}\bar{b}\bar{c}|\,(ax)(bx)(cx) = 0.$$

* Ibid., p. 52.

This cubic is the locus of points whose polars with regard to the three are concurrent; it is also the locus of points of concurrence of such polars. The relation of such pairs of conjugate points will establish an involutory transformation of the cubic into itself, which we shall call the first. There is a second involutory transformation which we reach as follows. Let us write the discriminant of the bilinear form

$$\lambda (ax)(\bar{a}\bar{x}) + \mu (bx)(\bar{b}\bar{x}) + \nu (cx)(\bar{c}\bar{x}) = 0.$$

If this vanish for a set of complex values λ, μ, ν it will vanish for the conjugate imaginary values. We thus obtain a pair of points on the cubic of the sort which we encountered in studying the second case leading up to 31], i.e. double contact. From such a pair of points the thread common to hyperconics of the net is projected by two anti-projective pencils. We have, thus, a second involutory transformation of the cubic into itself. The double points of the first transformation are those which are common to all hyperconics of the net, those of the second are those whence the first set are projected by a set of lines anti-projective with itself, i.e. by a chain of lines. It can be shown that the relation between the two sets of points is a reciprocal one.*

Theorem 36] *If six points be given in a plane, there is always one net of hyperconics through them and, in general, but one such net. In this latter case the locus of points whence these six are projected by the lines of a chain is a cubic curve through the six points, and all others cammon to the hyperconics of the net. This curve is the locus of points with concurrent polars with regard to all hyperconics of the net, and the locus of all such points of concurrence. It is also the locus of pairs of points whence the original six are anti-projectively projected.*

Suppose that we have five points in the plane, no three collinear, and not all belonging to a chain congruence. The hyperconics through them will be linearly dependent on four. In the Segre representation we shall have a system of

* Ibid., p. 56.

hyperplanes linearly dependent on four, that is to say, passing through a linear S_4, which will meet $S_4{}^6$ in a sixth point. Hence all hyperconics of the system pass through a sixth point. If two of the points be (0, 1, 0) and (0, 0, 1) the system of bilinear forms will be

$$\alpha\sum a_{ij}x_i\bar{x}_j+\beta\sum b_{ij}x_i\bar{x}_j+\gamma\sum c_{ij}x_i\bar{x}_j+\delta\sum d_{ij}x_i\bar{x}_j=0.$$

The coefficients with two subscripts 2 or 3 are all zero. Can we give to α, β, γ, and δ such a set of values that the polar of $(0, 1, 0)$ is indeterminate, while for the conjugate values the polar of $(0, 0, 1)$ is indeterminate? If we can do so, then by the reasoning used in 31] the points which reduce both forms to zero, including the four remaining points common to all of our hyperconics, will be anti-projectively projected from the two given points. We must have

$$\alpha\, a_{13}+\beta\, b_{13}+\gamma\, c_{13}+\delta\, d_{13}=0$$
$$\alpha\, a_{22}+\beta\, b_{22}+\gamma\, c_{22}+\delta\, d_{22}=0$$
$$\bar{\alpha}\, a_{12}+\bar{\beta}\, b_{12}+\bar{\gamma}\, c_{12}+\bar{\delta}\, d_{12}=0$$
$$\bar{\alpha}\, a_{32}+\bar{\beta}\, b_{32}+\bar{\gamma}\, c_{32}+\bar{\delta}\, d_{32}=0$$
$$\alpha\, a_{21}+\beta\, b_{21}+\gamma\, c_{21}+\delta\, d_{21}=0.$$

Here there are but three linear equations to determine α, β, γ, and δ, so that the conditions can be fulfilled. We thus reach a remarkable result.*

Theorem 37] *Given five points in a plane, no three of which are collinear, nor do the five belong to a chain congruence. Every hyperconic through the five points passes through a sixth. The relation of the six is perfectly symmetrical; each set of four are anti-projectively projected from the other two.*

An example of such a set will be given by the points

$(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$ $(1, 1, 1)$ (y_1, y_2, y_3)

$$\left(\frac{\bar{y}_2-\bar{y}_3}{\begin{vmatrix} y_2y_3 \\ \bar{y}_2\bar{y}_3\end{vmatrix}},\ \frac{\bar{y}_3-\bar{y}_1}{\begin{vmatrix} y_3y_1 \\ \bar{y}_3\bar{y}_1\end{vmatrix}},\ \frac{\bar{y}_1-\bar{y}_2}{\begin{vmatrix} y_1y_2 \\ \bar{y}_1\bar{y}_2\end{vmatrix}}\right).$$

* For the first part of this see Segre, *Rappresentazioni reali*, cit. p. 436. The second part comes with the aid of the simple analysis above from his *Nuovo*

§ 4. The Hermitian Metrics.

As the study of the projective properties of conic sections leads us by one of the most natural roads to the classical systems of non-Euclidean geometry, so the theorems just given concerning hyperconics lead us to a system of measurement whose name is given in the heading of this section.*

We start with a Hermitian form, which we shall call the *fundamental form*

$$x_1\bar{x}_1 + x_2\bar{x}_2 + x_3\bar{x}_3 \equiv (x\bar{x}). \tag{11}$$

If (x) and (y) be two points, we define as their distance, d, where

$$\cos d = \frac{\sqrt{(x\bar{y})}\ \sqrt{(y\bar{x})}}{\sqrt{(x\bar{x})}\ \sqrt{(y\bar{y})}}. \tag{12}$$

This system of measurement shall be called a *Hermitian metric of the elliptic type*. If we replace d by $\frac{d}{k}$ and x_3 by kx_3 we find

$$k \sin\frac{d}{k} = \frac{\sqrt{\begin{vmatrix} x_2 x_3 \\ \bar{y}_2\bar{y}_3 \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_2\bar{x}_3 \\ y_2 y_3 \end{vmatrix} + \begin{vmatrix} x_3 x_1 \\ \bar{y}_3\bar{y}_1 \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_3\bar{x}_1 \\ y_3 y_1 \end{vmatrix} + \frac{1}{k^2}\begin{vmatrix} x_1 x_2 \\ \bar{y}_1\bar{y}_2 \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_1\bar{x}_2 \\ y_1 y_2 \end{vmatrix}}}{\sqrt{\frac{1}{k^2}(x_1\bar{x}_1 + x_2\bar{x}_2) + x_3\bar{x}_3}\ \sqrt{\frac{1}{k^2}(y_1\bar{y}_1 + y_2\bar{y}_2) + y_3\bar{y}_3}}.$$

The limit of this as k becomes infinite is

$$d' = \frac{\sqrt{\begin{vmatrix} x_2 x_3 \\ \bar{y}_2\bar{y}_3 \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_2\bar{x}_3 \\ y_2 y_3 \end{vmatrix} + \begin{vmatrix} x_3 x_1 \\ \bar{y}_3\bar{y}_1 \end{vmatrix} \cdot \begin{vmatrix} \bar{x}_3\bar{x}_1 \\ y_3 y_1 \end{vmatrix}}}{\sqrt{x_3\bar{x}_3}\ \sqrt{y_3\bar{y}_3}}, \tag{12'}$$

and this we call the *parabolic type* of Hermitian metric. If we write

$$\frac{x_1}{x_3} = x; \quad \frac{x_2}{x_3} = y; \quad \frac{y_1}{y_3} = x'; \quad \frac{y_2}{y_3} = y',$$

campo, cit. vol. xxvi, p. 57, note. In view of this it is curious to read in a noteworthy article by Study, 'Kürzeste Wege im Komplexen Gebiete', *Math. Annalen*, vol. lx, 1905, p. 348, 'Merkwürdiger Weise scheint dieser einfache Satz der Aufmerksamkeit der Geometer bisher entgangen zu sein'. Few geometers set a higher value on Segre's work than Study did.

* The first writer to treat this subject was Fubini; see his short note, 'Sulle metriche definite da una forma hermitiana', *Atti del R. Istituto Veneto*, vol. lxiii, 1903–4, pp. 501 ff.

we have a non-homogeneous form for our distance expressions,

$$\cos d = \frac{\sqrt{x\bar{x}' + y\bar{y}' + 1}\ \sqrt{\bar{x}x' + \bar{y}y' + 1}}{\sqrt{x\bar{x} + y\bar{y} + 1}\ \sqrt{x'\bar{x}' + y'\bar{y}' + 1}}, \tag{13}$$

$$d' = \sqrt{(x' - x)(\bar{x}' - \bar{x}) + (y' - y)(\bar{y}' - \bar{y})}. \tag{13'}$$

In the infinitesimal domain the differentials of arc are given by

$$ds^2 = \frac{dx\,d\bar{x} + dy\,d\bar{y} + (x\,dy - y\,dx)(\bar{x}\,d\bar{y} - \bar{y}\,d\bar{x})}{(x\bar{x} + y\bar{y} + 1)^2}, \tag{14}$$

$$ds'^2 = dx\,d\bar{x} + dy\,d\bar{y}. \tag{14'}$$

Let the reader convince himself that these expressions for distance are real, and that in the elliptic case

$$\cos^2 d < 1$$

the angles of lines shall be defined by the expressions

$$\cos\theta = \frac{\sqrt{(u\bar{v})}\ \sqrt{(\bar{u}v)}}{\sqrt{(u\bar{u})}\ \sqrt{(v\bar{v})}}, \tag{15}$$

$$\cos\theta' = \frac{\sqrt{u_1\bar{v}_1 + u_2\bar{v}_2}\ \sqrt{\bar{u}_1v_1 + \bar{u}_2v_2}}{\sqrt{u_1\bar{u}_1 + u_2\bar{u}_2}\ \sqrt{v_1\bar{v}_1 + v_2\bar{v}_2}}. \tag{15'}$$

The distance from a point to a line shall be defined as the distance to the foot of the perpendicular thereon. We find

$$\sin d = \frac{\sqrt{(ux)}\ \sqrt{(\bar{u}\bar{x})}}{\sqrt{(u\bar{u})}\ \sqrt{(x\bar{x})}}, \tag{16}$$

$$d = \frac{\sqrt{(ux)}\ \sqrt{(\bar{u}\bar{x})}}{\sqrt{u_1\bar{u}_1 + u_2\bar{u}_2}\ \ \sqrt{x_3\bar{x}_3}}. \tag{16'}$$

Distances will be invariant under a group of collineations which leave, in the elliptic case, the fundamental form invariant, except for a constant factor. What is the N.S. condition that a collineation should do this? Let it be written

$$x_i' = \sum_j a_{ij} x_j.$$

The form (11) will be invariant when, and only when,

$$\sum_k a_{ki}\bar{a}_{kj} = 0, \quad i \neq j,$$

$$\sum_k a_{ki}\bar{a}_{ki} = \sum_j a_{kj}\bar{a}_{kj}.$$

The first of these tells us that the terms in two columns of the matrix of the transformation are proportional to the coordinates of two points which are conjugate with regard to the form. A system of solutions of the first may always be made to solve the second also by multiplying through by suitable constant factors. Let us write

$$a_{i1} = a_i, \quad a_{i2} = la_i + mb_i, \quad a_{i3} = r\,|\,\bar{a}_j\bar{b}_k - \bar{a}_k\bar{b}_j\,|,$$

$$l(a\bar{a}) + m(b\bar{a}) = 0.$$

We thus reach the final form*

$$\rho x_1' = a_i x_1 + \frac{[(b\bar{a})\,a_i - (a\bar{a})\,b_i]}{\sqrt{(a\bar{a})\,(b\bar{b}) - (a\bar{b})\,(b\bar{a})}}\,x_2,$$

$$+\frac{\sqrt{(a\bar{a})}\;[\bar{a}_j\bar{b}_k - \bar{a}_k\bar{b}_j]}{\sqrt{(a\bar{a})\,(b\bar{b}) - (a\bar{b})\,(b\bar{a})}}. \quad (17)$$

The parabolic transformation takes a somewhat different form,

$$\begin{aligned} \rho x_1' &= \cos A e^{i\theta_1} x_1 + \sin A e^{i(\theta_1+\phi)} x_2 + a_1 x_3, \\ \rho x_2' &= -\sin A e^{i\theta_2} x_1 + \cos A e^{i(\theta_2+\phi)} x_2 + a_2 x_3, \qquad (17') \\ \rho x_3' &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad a_3 x_3. \end{aligned}$$

We shall call these collineations in the elliptic case *pseudo-orthogonal.* Every such transformation will surely leave one

* The first writer to give a non-explicit expression for this sort of transformation was Loewy, 'Ueber bilineare Formen mit conjugirt imaginären Variabeln', *Nova Acta Leopoldina*, vol. lxxi, 1898. For the elementary divisors of the characteristic equation see the same author, 'Ueber die Transformationen einer Hermiteschen Form in sich selbst', *Göttingische Nachrichten*, 1900. The form here given in the case of n variables was published by the Author, 'Geometry of Hermitian Forms', *Transactions American Math. Soc.*, vol. xxi, 1920. The remaining theorems here given about these transformations are from the same source.

point invariant, and as the coordinates of this point cannot reduce the fundamental form to zero, its polar line with regard thereto will not include it, but will also be invariant. If we take these as a zero point and the opposite side of the coordinate triangle, we see that the problem of determining our pseudo-orthogonal transformation is reduced by the removal of one variable. Continuing thus we reach:

Theorem 38] *Every pseudo-orthogonal collineation will leave in place a triangle which is self-conjugate with regard to the fundamental form.*

The first set of our conditions (17) tells us that the coefficients of a column are proportional to the coordinates of a point, conjugate to that given by another column. If (a), (b), and (c) be the vertices of a triangle which is self-conjugate with regard to the given form, then the matrix

$$\left\| \frac{a_1}{\sqrt{(a\bar{a})}}, \quad \frac{b_2}{\sqrt{(b\bar{b})}}, \quad \frac{c_3}{\sqrt{(c\bar{c})}} \right\|$$

will give a pseudo-orthogonal substitution:

Theorem 39] *A pseudo-orthogonal collineation may be found to carry any triangle which is self-conjugate with regard to the fundamental form into any other such triangle.**

The choice of a self-conjugate invariant triangle will depend upon three complex parameters. On the other hand, if the coordinate triangle remain in place, every transformation of the form

$$x_i' = re_i^{\theta i} x_i, \tag{18}$$

where the multiplier r is the same throughout, will be pseudo-orthogonal.

Theorem 40] *The general pseudo-orthogonal collineation depends upon three complex and three real parameters.*

Are there any one-parameter sub-groups of the pseudo-orthogonal group? Such a group will have the same fixed points as one of its members, so we may assume that the zero

* For a proof of the narrower theorem that these collineations are transitive see Autonne, '*L'Hermitien,*' *Rendiconti del Cercolo Matematico di Palermo*, vol. 16, 1902, p 111.

points remain in place. For a one-parameter group of the type (19) we must have

$$re^{i\theta_j} = f_j(re^{i\theta_i}),$$

the function

$$\phi(z) = \frac{f_j(z)}{z}$$

would have a constant modulus, and so be a constant. For a group depending analytically on one real parameter we should have

$$\theta_i = f_i(t),$$

$$f_i(r) f_i(s) = f_i(r+s),$$

$$f_i(t) = a_i t.$$

We developed in formulae (15) and (15′) the expressions for the angle of two lines. There is another form of angle different from this, which is peculiar to this metric. Consider two threads through the origin

$$x = t\frac{dx}{dt} + \dots, \quad y = t\frac{dy}{dt} + \dots,$$

$$x = t\frac{\delta x}{\delta t} + \dots, \quad y = t\frac{\delta y}{\delta t} + \dots.$$

We have for the distance element

$$ds^2 = \left(\frac{dx}{dt}\frac{d\bar{x}}{d\bar{t}} + \frac{dy}{dt}\frac{d\bar{y}}{d\bar{t}}\right)dt^2 + \dots,$$

$$\delta s^2 = \left(\frac{\delta x}{\delta t}\frac{\delta \bar{x}}{\delta \bar{t}} + \frac{\delta y}{\delta t}\frac{\delta \bar{y}}{\delta \bar{t}}\right)\delta t^2 + \dots,$$

If we seek the square of the third side of this infinitesimal triangle we find

$$d^2 = \left(dt\frac{dx}{dt} - \delta t\frac{\delta x}{\delta t}\right)\left(dt\frac{d\bar{x}}{dt} - \delta t\frac{\delta \bar{x}}{\delta t}\right) + \left(dt\frac{dy}{dt} - \delta t\frac{\delta y}{\delta t}\right)\left(dt\frac{d\bar{y}}{dt} - \delta t\frac{\delta \bar{y}}{\delta t}\right),$$

$$\frac{ds^2 + \delta s^2 - d^2}{2\,ds\,\delta s} = \frac{dx\,\delta\bar{x} + dy\,\delta\bar{y} + \delta x\,d\bar{x} + \delta y\,d\bar{y}}{2\sqrt{dx\,d\bar{x} + dy\,d\bar{y}}\,\sqrt{\delta x\,\delta\bar{x} + \delta y\,\delta\bar{y}}}, \qquad (19)$$

$$= \cos\phi.$$

We shall define this expression as the cosine of the angle which the *direction* $dy : dx$ makes with the *direction* $\delta y : \delta x$.

If we put
$$\frac{dy}{dx} = l, \quad \frac{\delta y}{\delta x} = \lambda,$$
$$\cos\phi = \frac{dx\delta\bar{x}(1 + l\bar{\lambda}) + \delta x d\bar{x}(1 + \lambda\bar{l})}{2\sqrt{dx d\bar{x}(1 + l\bar{l})}\sqrt{\delta x\delta\bar{x}(1 + \lambda\bar{\lambda})}}. \tag{20}$$

For what values of δx and dx will this be a maximum? If we write it in the form
$$\alpha e^{i\psi} + \bar{\alpha} e^{-i\psi},$$
where ψ alone is variable, we see that we obtain a maximum when the two parts of the numerator are equal to one another, and each is equal to the square root of their product, namely,
$$\frac{\sqrt{1 + l\bar{\lambda}}\sqrt{1 + \lambda\bar{l}}}{\sqrt{1 + l\bar{l}}\sqrt{1 + \lambda\bar{\lambda}}}.$$

This, however, is the cosine of the angle of the two lines:

Theorem 41] *If two lines be given through a point, the smallest angle which a direction on one makes with a direction on the other is equal to the angle of the two lines. When these latter are mutually perpendicular, every direction on one is orthogonal to every direction on the other.**

If our angle be taken at a general point, instead of at the origin, we have
$$\cos\phi = \frac{\begin{array}{r} dx\delta\bar{x} + dy\delta\bar{y} + \delta x d\bar{x} + \delta y d\bar{y} + (x dy - y dx) \\ (\bar{x}\delta\bar{y} - \bar{y}\delta\bar{x}) + (x\delta y - y\delta x)(\bar{x} d\bar{y} - \bar{y} d\bar{x}) \end{array}}{\begin{array}{r} 2\sqrt{dx d\bar{x} + dy d\bar{y} + (x dy - y dx)(\bar{x} d\bar{y} - \bar{y} d\bar{x})} \\ \sqrt{\delta x\delta\bar{x} + \delta y\delta\bar{y} + (x\delta y - y\delta x)(\bar{x}\delta\bar{y} - \bar{y}\delta\bar{x})} \end{array}}. \tag{21}$$

In the parabolic case we shall have, similarly,
$$\cos\phi' = \frac{dx\delta\bar{x} + dy\delta\bar{y} + \delta x d\bar{x} + \delta y d\bar{y}}{\sqrt{dx d\bar{x} + dy d\bar{y}}\sqrt{\delta x\delta\bar{x} + \delta y\delta\bar{y}}}. \tag{21'}$$

There is a highly significant connexion between the Hermitian measurement and the Segre representation which

* Study, *Kürzeste Wege*, cit. p. 342. A remarkable connexion is established between angles of direction and distances in a three-dimensional non-Euclidean space.

must now claim our attention. Every pseudo-orthogonal collineation of the plane will correspond in S_8 to a collineation which leaves in place not only S_4^6, but the hyperplane

$$X_{11}+X_{22}+X_{33}=0,$$

as well as all of the hyperquadrics given in IV (7). It will thus leave unaltered every hyperquadric of the form

$$\sum_{ij}(X_{ij}^2+X_{ji}^2-2X_{ii}X_{jj})+\lambda[X_{11}+X_{22}+X_{33}]^2\equiv(AX)^2=0.$$

$$(AX)(AY)\equiv\sum(X_{ij}Y_{ij}+X_{ji}Y_{ji})+(\lambda-1)\sum X_{ii}Y_{ii},$$
$$\equiv(x\bar{y})(y\bar{x})+(\lambda-1)(x\bar{x})(y\bar{y}).$$

$$\frac{1}{2}\left[1-\frac{(AX)(AY)}{\sqrt{(AX)^2}\sqrt{(AY)^2}}\right]=\frac{1}{2\lambda}\left[1-\frac{(x\bar{y})(y\bar{x})}{(x\bar{x})(y\bar{y})}\right].$$

If, thus, we define the Non-Euclidean distance D of (X) and (Y) by

$$\cos D=\frac{(AX)(AY)}{\sqrt{(AX)^2}\sqrt{(AY)^2}},$$

$$\frac{1}{\sqrt{2\lambda}}\sin\frac{1}{2}d=\sin\frac{1}{2}D.$$

We are thus able to interpret our Hermitian metrics in the plane as a classical non-Euclidean metric on S_4^6, a remarkable result.*

It is time to look for geodesic threads in our Hermitian metrics. We begin with one dimension, i.e. the x axis. We have there, in the elliptic case,

$$\cos 2d=\frac{2x'\bar{x}+2x\bar{x}'+(x\bar{x}-1)(x'\bar{x}'-1)}{(x\bar{x}+1)(x'\bar{x}'+1)}.$$

Representing this on the Riemann sphere,

$$X=\frac{x+\bar{x}}{x\bar{x}+1},\quad Y=\frac{i(x-\bar{x})}{x\bar{x}+1},\quad Z=\frac{x\bar{x}-1}{x\bar{x}+1};$$

$$X'=\frac{x'+\bar{x}'}{x\bar{x}+1},\quad Y'=\frac{i(x'-\bar{x}')}{x'\bar{x}'+1},\quad Z'=\frac{x'\bar{x}'-1}{x\bar{x}+1}.$$

* Ibid., p. 356.

The angle subtended at the centre is Θ, when

$$\cos\Theta = XX' + YY' + ZZ' = \cos 2d.$$

In the parabolic case

$$d = \sqrt{(x-x')(\bar{x}-\bar{x}')}.$$

Theorem 42] *The distance of two points of a real line in the elliptic Hermitian metric is equal to one-half of the angle which their representing points on the Riemann sphere subtend at the centre of that sphere; the distance in the parabolic metric is equal to that of the representing points in the Gauss plane.*

On a real line a geodesic thread consists in a particular type of chain; a complex line may always be carried over into a real one by a distance preserving collineation. Hence we have:

Theorem 43] *A geodesic thread on a line is a chain.*

Evidently it is not any arbitrary chain. We shall call such a chain a *normal* one, and seek its geometric definition. The geometric representation on the Riemann sphere of such a chain will be a great circle by 42]. This will correspond to an equation

$$\begin{vmatrix} x+\bar{x} & i(x-\bar{x}) & x\bar{x}-1 \\ x'+\bar{x}' & i(x'-\bar{x}') & x'\bar{x}'-1 \\ x''+\bar{x}'' & i(x''-\bar{x}'') & x''\bar{x}''-1 \end{vmatrix}$$

$$= -2i \begin{vmatrix} x & \bar{x} & x\bar{x}-1 \\ x' & \bar{x}' & x'\bar{x}'-1 \\ x'' & \bar{x}'' & x''\bar{x}''-1 \end{vmatrix} = 0,$$

$$(x\bar{x}'+1)(x'\bar{x}''+1)(x''\bar{x}+1) = (x\bar{x}''+1)(x'\bar{x}+1)(x''\bar{x}'+1),$$

which may be written in invariant form

$$(x\bar{y})(y\bar{z})(z\bar{x}) = (y\bar{x})(z\bar{y})(x\bar{z}). \qquad (22)$$

It is characterized by the fact that its points are, in pairs, conjugate with regard to the fundamental form. The normal chain through (y) and (z) may be written

$$x_i = X_1(y\bar{z})y_i + X_2 z_i, \qquad (23)$$

or, more neatly,

$$x_i = X_1 y_i + X_2 z_i, \quad (y\bar{z}) = (z\bar{y}). \tag{24}$$

In the parabolic case a normal chain will be a chain containing the infinite point of the line in question.

Theorem 44] *In the elliptic case two points which are not conjugate with regard to the fundamental form may be connected by one and only one normal chain; every chain which contains two points which are conjugate with regard to this form is a normal one. In the parabolic case two finite points can always be connected by just one normal chain, while every chain that contains an infinite point is normal.*

Let us next study the trigonometry of the right triangle. We may assume that the vertices are

$$A = (a, 0), \quad B = (0, b), \quad C = (0, 0),$$

$$\cos BC = \frac{1}{\sqrt{1+b\bar{b}}}, \quad \cos CA = \frac{1}{\sqrt{1+a\bar{a}}},$$

$$\cos AB = \frac{1}{\sqrt{1+a\bar{a}}\,\sqrt{1+b\bar{b}}},$$

$$\cos AB = \cos BC \cos CA,$$

$$\sin A = \frac{\sqrt{b\bar{b}(1+a\bar{a})}}{\sqrt{a\bar{a}+b\bar{b}+a\bar{a}b\bar{b}}} = \frac{\sqrt{b\bar{b}}}{\sqrt{1+b\bar{b}}} \div \frac{\sqrt{a\bar{a}+b\bar{b}+a\bar{a}b\bar{b}}}{\sqrt{1+a\bar{a}}\,\sqrt{1+b\bar{b}}},$$

$$= \sin BC \div \sin AB.$$

In the parabolic case

$$AB^2 = BC^2 + CA^2, \quad \sin A = BC \div AB.$$

Theorem 45] *The elliptic Hermitian trigonometry of a right triangle is that of a right triangle in ordinary elliptic measurement; the parabolic Hermitian trigonometry of a right triangle is that of a Euclidean right triangle.*

If, lastly, A, B, C be any three points, we project C orthogonally into a point H of AB

$$AC \geqq AH, \quad BC \geqq BH,$$

$$AC + BC \geqq AH + BH \geqq AB.$$

Theorem 46] *A normal chain is a geodesic thread in the plane; a straight line is a geodesic curve.*

Let us look for two parameter systems or congruences of points which contain a large number of geodesic threads, i.e. of normal chains. A chain congruence contains ∞^2 chains; let us show that this fact characterizes the chain congruence and the straight line (which contains ∞^3 chains).

Suppose that we have a congruence that contains ∞^2 chains, yet which is not a straight line. Then each two points of the congruence, or at least of some domain thereof, may be connected by a chain of the congruence, and the congruence, or a part of it, will be constructed by chains from a fixed point to the points of a chain. We take (1, 0, 0) for this fixed point, and $x_1 = 0$ as the line of the chain. We may then write our congruence in the form

$$x_1 = \phi(R)S, \quad x_2 = R, \quad x_3 = 1;$$

The points $(R_1 S_1)$ and $(R_2 S_2)$ may be connected by a chain, every one of whose points will belong to the congruence. We may thus write

$$\phi(R)S = L\lambda\,[\phi(R_1)S_1] + M\mu\,[\phi(R_2)S_2];$$

$$\bar{\phi}(R_1)S = L\bar{\lambda}\,[\bar{\phi}(R_1)S_1] + M\bar{\mu}\,[\bar{\phi}(R_2)S_2];$$

$$R = L\lambda R_1 + M\mu R_2, \quad R = L\bar{\lambda}R_1 + M\bar{\mu}R_2;$$

$$1 = L\lambda + M\mu, \quad 1 = L\bar{\lambda} + M\bar{\mu}.$$

Then $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$. We may take each $= 1$.

$$M = 1 - L, \quad R = LR_1 + (1-L)R_2;$$

$$\phi\,[LR_1 + (1-L)R_2]\,S = L\phi(R_1)S_1 + (1-L)\phi(R_2)S_2;$$

$$\bar{\phi}\,[LR_1 + (1-L)R_2]\,S = L\bar{\phi}(R_1)S_1 + (1-L)\bar{\phi}(R_2)S_2.$$

Eliminating S we have an identity in R_1 S_1, R_2 S_2 and L.

Putting $\quad S_2 = 0, \quad R_2 = 0, \quad R_1 = 1,$

$$\frac{\phi(L)}{\bar{\phi}(L)} = \frac{\phi(1)}{\bar{\phi}(1)} = \text{const.}$$

We may write $\phi = Te^{i\theta}, \quad \theta = \text{const.}$

Let $ST = V,$

$$x_1 = Ve^{i\theta}, \quad x_2 = R, \quad x_3 = 1.$$

This part of the congruence is a chain congruence.

Theorem 46] *The only irreducible congruences which contain ∞^2 chains are chain congruences.*

A congruence containing ∞^2 normal chains is called a *normal* chain congruence.

Theorem 47] *The only congruences which contain ∞^2 geodesic threads are normal chain congruences. In the elliptic case, every chain congruence which contains the vertices of a triangle, self-conjugate with regard to the fundamental form, is a normal chain congruence. In the parabolic case this is true of every chain congruence which includes a chain on the infinite line.*

In a normal chain congruence there will be a chain of normal chains through each point.

Let us try to find the total length of a chain. Let the chain be on the x axis, and expressed in the form

$$x = \frac{\alpha u}{\beta u + \gamma}, \quad y = 0, \quad u = \bar{u};$$

$$ds = \frac{\sqrt{\bar{\alpha}\alpha\, \gamma\bar{\gamma}}\, du}{\sqrt{[\alpha\bar{\alpha} + \beta\bar{\beta}]\, u^2 + (\beta\bar{\gamma} + \bar{\gamma}\beta)u + \gamma\bar{\gamma}}}; \tag{25}$$

$$\int_{-\infty}^{\infty} ds = \frac{\pi}{\sqrt{1 - \dfrac{(\beta\bar{\gamma} - \gamma\bar{\beta})^2}{4\alpha\bar{\alpha}\gamma\bar{\gamma}}}}.$$

For a normal chain, $\beta = \bar{\beta} = 0$, and the length is π.

We saw in theorem 30] that if two chains lie on different lines, and neither include the point common to their two lines, then a line connecting a point of the one with a point of the other will always contain a point of a third chain, and the relation of the three chains is symmetrical. What will happen when the two given chains are normal ones? What is the analytic equation for the system of all lines meeting the normal chain

$$x_1 = X_1 y_i + X_2 z_i, \quad (y\bar{z}) = (z\bar{y});$$

$$(ux) = (\bar{u}\bar{x}) = 0.$$

$$(uz)(\bar{u}\bar{y}) - (uy)(\bar{u}\bar{z}) = 0.$$

$$\sum A_{ij} u_i \bar{u}_j = 0, \quad A_{ji} = -A_{ij};$$

$$A_{11} + A_{22} + A_{33} = (y\bar{z}) - (z\bar{y}) = 0.$$

We have a Hermitian form in line coordinates to which the fundamental form is apolar. If the fundamental form be apolar to two such Hermitian forms, it will be to any Hermitian form which is a linear combination of them; hence if two of our three chains be normal, the third is also:

Theorem 48] *If two normal chains lie on different lines, neither including the point common to these lines, then every line containing a point of each chain will contain one of a third normal chain.*

We may go a step further in this direction. Let us take a triangle which is self-conjugate with regard to the fundamental Hermitian form as our triangle of reference. There will be on each side of this triangle just one normal chain with regard to which the two vertices are conjugate. Let two of these chains be given by the equations

$$y_1 = y_2\bar{y}_2 - y_3\bar{y}_3 = 0,$$

$$z_2 = z_3\bar{z}_3 - z_1\bar{z}_1 = 0.$$

The line connecting the points (y) and (z) will meet the third side in (x) where

$$x_2 z_1 y_3 = x_1 z_3 y_2, \quad \bar{x}_2\bar{z}_1\bar{y}_3 = \bar{x}_1\bar{z}_3\bar{y}_2, \quad x_1\bar{x}_1 = x_2\bar{x}_2.$$

Hence this third chain is normal, and the vertices are conjugate with regard to it. For the distance of the first two points we have

$$\cos d = \frac{\sqrt{(y\bar{z})}\sqrt{(z\bar{y})}}{\sqrt{(y\bar{y})}\sqrt{(z\bar{z})}} = \tfrac{1}{2}, \quad d = \frac{\pi}{3}.$$

It appears thus, that the line is divided into three equal parts by the three points. Moreover, since

$$(x\bar{y})\,(y\bar{z})\,(z\bar{x}) = (\bar{x}y)\,(\bar{y}z)\,(\bar{z}x),$$

these three points lie in a normal chain. Let us find the angle between the direction of this and one of the given chains. If we write

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3},$$

then, along the normal chain from $(0, y_2, y_3)$ to $(z_1, 0, z_3)$ we may put

$$y = \frac{y_2\bar{y}_3}{u + y_3\bar{y}_3}.$$

At the point $u = 0$ we shall have

$$dx = dx, \quad dy = \frac{-y_2 du}{{y_3}^2\bar{y}_3}.$$

Along the normal chain on the line $x_1 = 0$ we have

$$y_1 = 0, \quad y_2\bar{y}_2 + y_3\bar{y}_3 = 0.$$

$$\delta x = 0, \quad \frac{\bar{y}_2}{\bar{y}_3}\delta y + \frac{y_2}{y_3}\delta\bar{y} = 0.$$

$$dx\,\delta\bar{x} + \delta x\,d\bar{x} + dy\,\delta\bar{y} + \delta y\,d\bar{y} = 0.$$

It appears then from (21) that $\cos\phi = 0$.

Theorem 49] *If a triangle be self-conjugate with regard to the fundamental form in Hermitian metrics of the elliptic type, then there will be on each side one normal chain with regard to which the two vertices are conjugate. A line meeting two of these normal chains will necessarily meet the third. The three points so found are equally spaced and determine a normal chain whose direction makes a right angle with that of each of the original normal chains.**

It is time to go back to the Hermitian trigonometry and the metrical properties of certain simple figures.† We saw

* Study, *Kürzeste Wege*, cit. p. 352.

† The remainder of our work on Hermitian metrics will be found in an article by the author, 'Hermitian Metrics', *Annals of Mathematics*, vol. xxv, 1920.

in theorem 45] that the Hermitian trigonometry of a right triangle is the ordinary elliptic or euclidean trigonometry. Let us pass to the general triangle. Parallel to the concept of normal point chain, we have that of a normal line chain. The lines of such a chain are mutually perpendicular in pairs, and every line chain that contains a perpendicular pair is normal. It is only when three concurrent lines belong to a normal chain that the angle formed by one pair is equal to the sum of the angles of the other two. When will a normal chain of lines meet a transversal in a normal point chain? If the transversal be the x axis, and the vertex of the line chain be the point $(0, c)$, a typical line of the chain may be written

$$X_1\left[\frac{x}{a}+\frac{y}{c}-1\right]+X_2\rho\left[\frac{x}{\bar{b}}+\frac{y}{c}-1\right]=0.$$

The conditions that the line-chain and point-chain be both normal are

$$\rho\left[\frac{1}{b\bar{a}}+\frac{1}{a\bar{c}}-1\right]=\bar{\rho}\left[\frac{1}{a\bar{b}}+\frac{1}{c\bar{c}}-1\right],$$

$$\rho\left[\frac{1}{a\bar{b}}-1\right]=\bar{\rho}\left[\frac{1}{\bar{a}b}-1\right],$$

from which we conclude

$$\rho=\bar{\rho}, \quad a\bar{b}=\bar{a}b.$$

Theorem 50] *A normal line chain will meet a transversal, not through its vertex, in a normal point chain, when, and only when, the line chain contains the perpendicular on the transversal from that vertex. In the parabolic case the transversal is supposed to be a finite line, and the vertex is finite.*

Theorem 51] *A normal point chain will determine a normal line-chain about a point not collinear with it when, and only when, the given chain includes the foot of the perpendicular on its line from the given point. In the parabolic case both point and line are supposed to be finite.*

We see from our formula (18) that when the origin and axes remain fixed we may still transform the plane congruently by the collineation

$$x'=e^{i\theta_1}x, \quad y'=e^{i\theta_2}y.$$

If the vertices of a general triangle be

$$(a, 0)\ (b, 0)\ (0, c),$$

we may carry it into a real triangle in this way when, and only when,

$$a\bar{b} = \bar{a}b.$$

Now in the real domain, the Hermitian metrics are identical with the elliptic or euclidean metrics, so that when this equation holds, the trigonometry of our triangle is the usual elliptic or euclidean trigonometry. It remains to interpret this equation geometrically. A moment's calculation shows that it gives the N. S. condition that the altitudes of the triangle should be concurrent, while the work immediately preceding shows that it expresses the N. S. condition that the foot of one altitude should lie on a normal chain through the vertices. We thus reach the fundamental theorem of Hermitian trigonometry.

Theorem 52] *The necessary and sufficient condition that the Hermitian trigonometry of a triangle should be the same as the corresponding elliptic or euclidean trigonometry is that the altitudes should be concurrent. In this case, and in this case alone, the foot of one altitude, and, hence, of every altitude, lies on a normal chain through two vertices. In this case, and this alone, the triangle can be congruently transformed into a real triangle.*

Let us look at the locus of points at a given distance from a given point. If this be the point (y) we have, in the elliptic case,

$$(x\bar{y})(y\bar{x}) - \cos^2 d\,(x\bar{x})(y\bar{y}) = 0.$$

Here is a Hermitian form with a non-vanishing discriminant, unless $d \cos d = 0$. In the parabolic case we have

$$(x - x')(\bar{x} - \bar{x}') + (y - y')(\bar{y} - \bar{y}') = d^2.$$

Theorem 53] *The locus of all points at a fixed distance not zero, nor, in the elliptic case, congruent to zero* $\left(\textit{modulo } \frac{\pi}{2}\right)$ *from a fixed point is a hyperconic. The characteristic equation* *of this hyperconic, and the fundamental form, in the elliptic case, has a double root which reduces all first minors to zero.*

We shall call this locus a *hypercircle.*

Theorem 54] *In the elliptic case, the envelope of lines making with a fixed line a fixed non-vanishing angle not congruent to zero* $\left(\textit{modulo } \frac{\pi}{2}\right)$ *is a hypercircle; in the parabolic case it is a chain of points on the infinite line.*

Theorem 55] *The polar of a point with regard to a hypercircle is perpendicular to the line connecting that point with the centre. In the elliptic case the product of the tangents of the distances from the centre to a point and its polar is constant; in the parabolic case the product of the distances themselves does not vary.*

We pass to the consideration of the general metrical properties of the hyperconic. We begin in the elliptic case, and look for a canonical form to which the equation of the hyperconic may be reduced by a congruent transformation or change of rectangular axes. If we set up the characteristic equation of the hyperconic and the fundamental form, we see that there must be one real root, giving a point which has the same polar with regard to both forms, and the point itself does not lie on this line. Using this point and line as parts of the coordinate triangle, we may make a first reduction of the equation of our hyperconic to the form

$$A_{11}x_1\bar{x}_1+a_{12}x_1\bar{x}_2+\bar{a}_{12}x_2\bar{x}_1+A_{22}x_2\bar{x}_2+A_{33}x_3\bar{x}_3=0.$$

The essential part of the characteristic equation is now

$$\begin{vmatrix} A_{11}-\rho & a_{12} \\ \bar{a}_{12} & A_{22}-\rho \end{vmatrix}=0,$$

and is bound to have two real distinct roots, since

$$(A_{11}-A_{22})^2+4a_{12}\bar{a}_{12}>0.$$

We are thus able to write our canonical form

$$A_{11}x_1\bar{x}_1+A_{22}x_2\bar{x}_2-A_{33}x_3\bar{x}_3=0. \qquad (26)$$

When the first two coefficients are equal, we have a hypercircle.

Theorem 56] *In the elliptic case, every hyperconic not a hypercircle has three centres, one inside and two outside.*

In the parabolic case, we have a greater variety of choice. If the hyperconic be not tangent to the infinite line, $x_3 = 0$, we may reason as above, and reach the two canonical forms

$$\frac{x\bar{x}}{a^2} + \frac{y\bar{y}}{b^2} = 1, \quad \frac{x\bar{x}}{a^2} - \frac{y\bar{y}}{b^2} = 1. \tag{26'}$$

If it be tangent, we may, by the usual devices of elementary analytic geometry, reduce to

$$y\bar{y} = \alpha x + \bar{\alpha}\bar{x}. \tag{27'}$$

All points of this locus are equidistant from the point $(1+Ri)\frac{\bar{\alpha}}{2}$ and the line $X + (1-Ri)\frac{\bar{\alpha}}{2} = 0$.

Theorem 57] *There are four types of hyperconic from the point of view of parabolic Hermitian metrics: the hyper-hyperbola with an outside centre, the hyper-ellipse, with an inside centre, the hypercircle, and the hyper-parabola, whose points are equidistant from a fixed point and a fixed line.*

There is a sub-variety of hyper-hyperbola where $A_1 = A_2$. Here the tangents from the centre, called *asymptotes*, are mutually orthogonal in pairs and form a normal line chain. In the elliptic case, consider the asymptotes given by the equation

$$A_2 x_2 \bar{x}_2 - A_3 x_3 \bar{x}_3 = 0.$$

The hyperconic itself may be expressed parametrically in the form

$$x_1 = \frac{e^{i\phi}}{\sqrt{A_1}}, \quad x_2 = \frac{e^{i\chi}\sinh L}{\sqrt{A_2}}, \quad x_3 = \frac{e^{i\psi}\cosh L}{\sqrt{A_3}}.$$

The sine of the distance from this point to the asymptote

$$\frac{e^{i\psi}}{\sqrt{A_3}} x_2 - \frac{e^{i\chi}}{\sqrt{A_2}} x_3 = 0$$

is

$$\frac{\sinh L - \cos L}{\sqrt{P + Q\cosh^2 L + R\sinh^2 L}},$$

an expression which becomes infinitesimal as L increases. A similar result may be found in the parabolic case:

Theorem 58] *As a point of a hyperconic recedes indefinitely from an outside centre, its distance from the nearest asymptote becomes infinitesimal.*

Can we find a point of such a nature that conjugate lines through it are mutually perpendicular in pairs? A line through a centre is perpendicular to a conjugate line when, and only when, the one or the other is an axis, i.e. a line containing two centres. Hence, if there be any point fulfilling our conditions it must be on an axis, but inside the hyperconic. Assuming in (26) that $A_1 > A_2$, consider the point $(0, y_2, y_3)$. An arbitrary line through it has the coordinates

$$(u_1, y_3 - y_2).$$

The perpendicular is

$$\left(-\frac{y_2\bar{y}_2 + y_3\bar{y}_3}{\bar{u}_1},\quad y_3,\quad -y_2\right).$$

These will be conjugate with regard to the hyperconic if

$$-\frac{y_2\bar{y}_2 + y_3\bar{y}_3}{A_1} + \frac{y_3\bar{y}_3}{A_2} - \frac{y_2\bar{y}_2}{A_3} = 0.$$

$$\left(\frac{1}{A_1} + \frac{1}{A_3}\right)y_2\bar{y}_2 + \left(\frac{1}{A_1} - \frac{1}{A_2}\right)y_3\bar{y}_3 = 0.$$

This gives a chain of points, in view of the inequality $A_1 > A_2$, but would not if the inequality were reversed, so that there are no corresponding points on the other axis through the inside centre. We shall call these points *foci*. In the parabolic case, we shall find them in the same way for the central hyperconics. For the hyper-parabola we should have on the x axis the system of points

$$\alpha x_1 + \bar{\alpha}\bar{x}_1 = \bar{\alpha}\bar{\alpha},$$

which is the system found just before 57]. Such points shall be called *foci*.

Theorem 59] *On just one axis of a hyperconic not a hypercircle there is a chain of foci characterized by the fact that conjugate lines through them are orthogonal.*

Consider the focus of the elliptic central conic

$$y_1 = 0, \quad y_2 = \sqrt{A_3(A_1 - A_2)}\,e^{i\theta}, \quad y_3 = \sqrt{A_2(A_1 + A_3)}\,e^{i\psi}.$$

Its polar is called a *directrix.* Its coordinates are

$$v_1 = 0, \quad v_2 = \sqrt{A_2(A_1 - A_2)}\,e^{-i\theta}, \quad v_3 = -\sqrt{A_3(A_1 + A_3)}\,e^{-i\psi}.$$

We thus reach:

Theorem 60] *In the elliptic case the ratio of the sines of the distances from a point of a hyperconic to a focus and to the corresponding directrix is constant, and is the same for all foci. In the parabolic case the ratio of the actual distances is constant, and is equal to unity in the case of the hyperparabola.*

The following system of hyperconics are confocal:

$$\frac{A_1}{A_1 - L}x_1\bar{x}_1 + \frac{A_2}{A_2 - L}x_2\bar{x}_2 + \frac{A_3}{A_3 + L}x_3\bar{x}_3 = 0.$$

From the equations

$$\frac{A_1}{A_1 - L_1}y_1\bar{y}_1 + \frac{A_2}{A_2 - L_1}y_2\bar{y}_2 + \frac{A_3}{A_3 + L_1}y_3\bar{y}_3 = 0,$$

$$\frac{A_1}{A_1 - L_2}y_1\bar{y}_1 + \frac{A_2}{A_2 - L_2}y_2\bar{y}_2 + \frac{A_3}{A_3 + L_2}y_3\bar{y}_3 = 0.$$

We deduce

$$\frac{A_1^2}{(A_1 - L_1)(A_1 - L_2)}y_1\bar{y}_1 + \frac{A_2^2}{(A_2 - L_1)(A_2 - L_2)}y_2\bar{y}_2 + \frac{A_3^2}{(A_3 + L_1)(A_3 + L_2)}y_3\bar{y}_3 = 0,$$

which involves the tangents at (y) to the two hyperconics of the confocal system passing through that point:

Theorem 61] *Through each finite point not on an axis of the hyperconics of a confocal system will pass two of these hyperconics, and these intersect orthogonally.*

By turning to the covariant (9) of a hyperconic and the fundamental form, we reach:

Theorem 62] *The locus of points whence tangents to a* *central hyperconic are mutually perpendicular in pairs is another hyperconic.*

We shall close our discussion of Hermitian metrics with a few differential expressions. Consider the curve

$$y = y(x), \quad \bar{y} = \bar{y}(x).$$

We seek the curvature, defining that function in the usual way as ratio of angle to arc $\frac{dt}{ds}$,

$$ds = \frac{\sqrt{1 + y\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}}{(x\bar{x} + y\bar{y} + 1)} \sqrt{dx\, d\bar{x}}.$$

Equation of the tangent is

$$y'\xi - \eta + (y - xy') = 0.$$

$$\frac{y'}{(y - xy')}\xi + \frac{-1}{(y - xy')}\eta + 1 \equiv u\xi + v\eta + 1 = 0,$$

$$du = \frac{yy''dx}{(y - xy')^2}, \quad dv = -\frac{xy''dx}{(y - xy')^2},$$

$$d\theta = \frac{\sqrt{du\,d\bar{u} + dv\,d\bar{v} + (u\,dv - v\,du)(\bar{u}\,d\bar{v} - \bar{v}\,d\bar{u})}}{(u\bar{u} + v\bar{v} + 1)}$$

$$= \frac{\sqrt{x\bar{x} + y\bar{y} + 1}\sqrt{y''\bar{y}''}}{\sqrt{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}} \sqrt{dx\,d\bar{x}}.$$

$$\frac{d\theta}{ds} = \frac{1}{k} = \frac{\sqrt{y\bar{y}'}}{\left[\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^{\frac{3}{2}}}. \tag{28}$$

In the parabolic case

$$\frac{d\theta}{ds} = \frac{1}{k} = \frac{\sqrt{y'\bar{y}'}}{[1 + y'\bar{y}']^{\frac{3}{2}}} \tag{28'}$$

On the other hand, consider the surface whose distance formula is

$$ds^2 = 2F\,dx\,d\bar{x} = \frac{1 + y'\bar{y}' + (y - \bar{x}y')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2} dx\,d\bar{x}$$

$$= \frac{u\bar{u} + v\bar{v} + 1}{v\bar{v}\,(x\bar{x} + y\bar{y} + 1)^2} dx\,d\bar{x}.$$

The Gaussian curvature is

$$\frac{1}{K} = -\frac{1}{F}\frac{\partial^2 \log F}{\partial x \partial \bar{x}} = \frac{-2y''\bar{y}''}{\left[\frac{1+y'\bar{y}'+(y-xy')(\bar{y}-\bar{x}\bar{y}')}{x\bar{x}+y\bar{y}+1}\right]^3} - 4$$

$$= -\left[\frac{2}{k^2}+4\right].$$

In the parabolic case

$$\frac{1}{K} = -\frac{2}{k^2}.$$

Theorem 63] *The Gaussian curvature of a surface having the same distance element as a given curve in elliptic Hermitian measurement is four less than minus twice the square of the Hermitian curvature. In the parabolic case the difference four is lacking.*

Let us see if we can find a curve of constant curvature? The straight line answers this description; are there any other such curves?

Let the normals at adjacent points A and A' on the curve meet at C, while the tangents at these points meet at D. Owing to the constant curvature the angles $\angle AA'D$ and $\angle A'AD$ are equal, or differ by an infinitesimal of the second order. We may show by two lines of algebraic work that a line through the vertex of a right triangle makes complementary angles with the legs, hence $\angle AA'C = \angle A'AC$, and hence $A'C = AC$, or differs therefrom to an infinitesimal of the second order. Now the usual geometric proof that the differential of arc of the evolute is the same as that of the radius of curvature holds equally in the Hermitian metrics, and as, in this case, the differential of arc of the evolute is of the second order, the evolute must reduce to a single point.

If the normal to a certain curve always pass through the origin, we shall have

$$x\,d\bar{x} + y\,d\bar{y} = 0.$$

$$\bar{x}\,dx + \bar{y}\,dy = 0,$$

$$x\bar{x} + y\bar{y} \quad = k^2.$$

The curve would have to lie on a hypercircle; but the hypercircle contains no curve, for the polar of an inside point does not meet the hypercircle at all.

Theorem 64] *The only curves of constant curvature are straight lines.**

Let us seek, lastly, the differential equation for a geodesic thread on a given curve. We wish to have x and $\bar{x}$ such functions of a real parameter t that the expression

$$\int_a^b \sqrt{2Fx'\bar{x}'}\,dt = \int_a^b R(x, \bar{x}, x', \bar{x}'t)\,dt$$

shall be a minimum. We may treat x and $\bar{x}$ as independent. Thus

$$\frac{\partial R}{\partial x} = \frac{d}{dt}\frac{\partial R}{\partial x'}, \quad \frac{\partial R}{\partial \bar{x}} = \frac{d}{dt}\frac{\partial R}{\partial \bar{x}'}.$$

This gives, in the present case,

$$x'\frac{\partial F}{\partial x} - \bar{x}'\frac{\partial F}{\partial \bar{x}} = \frac{F}{x'\bar{x}'}(x'\bar{x}'' - \bar{x}'x'')$$

$$\text{Let}\quad \log\phi = F.$$

$$x = u + iv, \quad \bar{x} = u - iv.$$

$$v'\frac{\partial\phi}{\partial u} - u'\frac{\partial\phi}{\partial v} = \frac{v'u'' - u'v''}{u'^2 + v'^2},$$

$$v' = u'\frac{dv}{du},$$

$$\frac{d^2v}{du^2} + \left[1 + \left(\frac{dv}{du}\right)^2\right]\left[\frac{dv}{du}\frac{\partial\phi}{\partial u} - \frac{\partial\phi}{\partial v}\right] = 0.$$

In the special case of the line $y = 0$,

$$(x\bar{x} + 1)(x'\bar{x}'' - \bar{x}'x'') + 2x'\bar{x}'(\bar{x}x' - x\bar{x}') = 0,$$

$$x = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha\beta + \gamma\bar{\delta} = \bar{\alpha}\beta + \bar{\gamma}\delta,$$

a normal chain.

§ 5. Hyperalgebraic forms in general.

Suppose that we have an algebraic thread

$$x = x(t), \quad y = y(t).$$

* This theorem is correctly given in the author's article on Hermitian Metrics, but the proof is not above suspicion.

If this be represented by a real curve in any space, the genus of that curve will be an invariant for every one to one algebraic transformation. The transformations on the representing variety will appear as algebraic transformations of the plane, or transformations of a pseudo-algebraic form

$$x' = f(x, y, \bar{x}, \bar{y}),\ y' = \phi(x, y, \bar{x}, \bar{y}),\ \bar{x}' = \bar{f}(\bar{x}, \bar{y}, x, y),$$
$$\bar{y}' = \bar{\phi}(\bar{x}, \bar{y}, x, y).$$

The genus of the curve which represented the thread will be a numerical invariant under these transformations. Every algebraic thread lies on an algebraic curve, and it might seem as though the genus of this curve would be another numerical invariant. Such, however, is not the case. Consider the thread

$$x = \wp(x) + i\wp'(t), \quad y = 0,$$

which lies on a straight line, and the thread

$$x = \wp(t), \quad y = \wp'(t),$$

which lies on an elliptic cubic curve. The one is transformed into the other by the transformation

$$x' = \tfrac{1}{2}(x + \bar{x}), \quad y' = \tfrac{1}{2}(x - \bar{x}).$$

Beside this algebraic invariant, we have certain projective invariants, as the Plücker characteristics of the curve on which the thread lies. Again, if the thread be given by equations

$$F(x, \bar{x}) = 0, \quad \Phi(y, \bar{y}) = 0,$$

the highest degree of either of these equations will be an invariant.

A two-parameter algebraic system will be given by two polynomials

$$f(x, y, \bar{x}, \bar{y}) = \bar{f}(\bar{x}, \bar{y}, x, y) = 0.$$

The degree of these polynomials will be a projective invariant, while two algebraic invariants will be furnished by the algebraic and geometric genera of the algebraic surfaces which represent this system in a four-dimensional space.

The most interesting algebraic varieties are those which depend on three real parameters. Such a variety may be written

$$F(x, y, \bar{x}, \bar{y}) \equiv \bar{F}(\bar{x}, \bar{y}, x, y) = 0,$$

or more neatly in the symbolic form

$$(ax)^n (\bar{a}\bar{x})^n = 0. \tag{29}$$

A point will be non-singular if the three partial derivatives to x_1, x_2, and x_3 and, hence, to $\bar{x}_1, \bar{x}_2, \bar{x}_3$ do not vanish,

$$(ax)^{n-1} (\bar{a}\bar{x})^n a_i \not\equiv 0.$$

Suppose that there is such a point, then, in the real four-dimensional universe

$$\frac{x_1}{x_3} = X_1 + iX_2, \quad \frac{x_2}{x_3} = X_3 + iX_4,$$

we have a real algebraic equation

$$\phi(X_1, X_2, X_3, X_4) = 0,$$

which is satisfied for at least one set of coordinate values, where the first derivatives are not all zero. Hence, by the implicit function theorem, there is a three-parameter real system of points forming a hypersurface in four dimensions whose coordinates satisfy this equation.

Theorem 65] *If there be a single point whose coordinates satisfy a real homogeneous algebraic equation in* (x) *and* $(\bar{x})$ *without being singular, then there is a triply infinite system of such points.*

We shall call a variety of this sort a *hypercurve.*

Theorem 66] *If a chain share with a hypercurve a number of points which exceeds the degree of the equation of the latter, the chain will be completely contained therein.*

Let us find where the general chain from (y) to (x') meets the hypercurve (29)

$$x_i = X_1 \rho y_i + X_2 x_i',$$

$$X_1^{2n} \rho^n \bar{\rho}^n (ay)^n (\bar{a}\bar{y})^n$$
$$+ nX_1^{2n-1} X_2 \rho^{n-1} \bar{\rho}^{n-1} (ay)^{n-1} (\bar{a}\bar{y})^{n-1} [\rho(ay)(\bar{a}\bar{x}') + \bar{\rho}(ax')(\bar{a}\bar{y})]$$
$$+ \frac{n(n-1)}{1 \cdot 2} X_1^{2(n-1)} X_2^2 (ay)^{n-2} (\bar{a}\bar{y})^{n-2} \rho^{n-2} \bar{\rho}^{n-2} [\rho^2 (ay)^2 (\bar{a}\bar{x})^2$$
$$+ \frac{2n}{n-1} \rho\bar{\rho} (ay)(\bar{a}\bar{y})(ax)(\bar{a}\bar{x}) + \bar{\rho}^2 (ax')^2 (\bar{a}\bar{y})] + \ldots = 0. \tag{30}$$

The coefficient of $X_1^{2n-k}X_2^k$ is, dropping the primes from x''s,

$$(ay)^{n-k}(\bar{a}\bar{y})^{n-k}\rho^{n-k}\bar{\rho}^{n-k}\left[\frac{n!}{k!(n-k)!}(ay)^k(\bar{a}\bar{x})^k\rho^k\right.$$

$$+\frac{n.n!}{(k-1)!(n-k+1)!}(ay)^{k-1}(ax)(\bar{a}\bar{y})(\bar{a}\bar{x})^{k-1}\rho^{k-1}\bar{\rho}$$

$$\left.+\ldots\frac{n!}{k!(n-k)!}(ax)^k(\bar{a}\bar{y})^k\bar{\rho}^k\right]. \quad (31)$$

If (y) be on the hypercurve,

$$(ay)^n(\bar{a}\bar{y})^n = 0,$$

and (30) has one root $x_2 = 0$. It will have a second such root if

$$\rho(ay)^n(\bar{a}\bar{y})^{n-1}(\bar{a}\bar{x})+\bar{\rho}(ay)^{n-1}(ax)(\bar{a}\bar{y}) = 0.$$

If neither term vanish alone, then $\frac{\rho}{\bar{\rho}}$ is uniquely determined, and there is just one tangent chain from (y) to (x). If one term vanish the other will. The straight line

$$(ay)^{n-1}(\bar{a}\bar{y})^n(ax) = 0, \quad (32)$$

which is well determined when (y) is not a singular point, shall be called the *tangent* to the variety at (y).

Theorem 67] *If a non-singular point be chosen on an algebraic hypercurve, every point of the plane not on the tangent at that point will be connected therewith by a single chain tangent to the hypercurve at the non-singular point. Every chain on the tangent which passes through the point of contact is tangent to the hypercurve.*

We shall study these tangent chains in greater detail in the next chapter. We define as the (p, q) polar of a point (y) the locus given by

$$(ay)^p(\bar{a}\bar{y})^q(ax)^{n-p}(\bar{a}\bar{x})^{n-q} = 0.$$

If this equation have any solution at all which is not singular, it will give a two-parameter family when $p \neq q$, and a hypercurve when $p = q$.

Theorem 68] *If (x) be on the (p, q) polar of (y), then (y) is on the (q, p) polar of (x).*

The condition that (y) should have a multiplicity $k+1$, i.e. that every chain through it should meet the hypercurve at least $k+1$ times there, but usually not more times, is

$$(ay)^{n-r}(\bar{a}\bar{y})^{n-s}(ax)^r(\bar{a}\bar{x})^s \equiv 0, \quad r+s=k. \tag{33}$$

Theorem 69] *If a point have a multiplicity $k+1$ and if $r+s=k$, its $(n-r, n-s)$ polar is illusory, and it lies on the (r, s) polar of every point.*

The simplest polar is the $(n-1, n-1)$ or hyperconic. This is given by

$$(ay)^{n-1}(\bar{a}\bar{y})^{n-1}(ax)(\bar{a}\bar{x}) = 0. \tag{34}$$

Assuming that (y) is a non-singular point of the hypercurve, this is either a hyperconic with the same tangent at (y) or a chain of lines, or the point (y) alone. The condition for one or the other of these latter cases is by 20]

$$(ay)^{n-1}(a'y)^{n-1}(a''y)^{n-1}(\bar{a}\bar{y})^{n-1})(\bar{a}'\bar{y})^{n-1}(\bar{a}''\bar{y})^{n-1}\,|aa'a''\cdot|\bar{a}\bar{a}'\bar{a}''|$$
$$\equiv K\left|\frac{\partial^2 f}{\partial y_1\partial\bar{y}_1}\quad\frac{\partial^2 F}{\partial y_2\partial\bar{y}_2}\quad\frac{\partial^2 F}{\partial y_3\partial\bar{y}_3}\right| = 0.$$

In non-homogeneous coordinates, if the equation of the variety be

$$F(xy\bar{x}\bar{y}) = 0,$$

this latter hypercurve has the equation

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial x\,\partial F} & \dfrac{\partial^2 F}{\partial x\,\partial \bar{y}} & \dfrac{\partial F}{\partial x} \\ \dfrac{\partial^2 F}{\partial x\,\partial F} & \dfrac{\partial^2 F}{\partial y\,\partial \bar{y}} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial \bar{y}} & F \end{vmatrix} = 0.$$

It will meet the hypercurve in the same points as the *pseudo-Hessian*

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial x\,\partial F} & \dfrac{\partial^2 F}{\partial x\,\partial \bar{y}} & \dfrac{\partial F}{\partial x} \\ \dfrac{\partial^2 F}{\partial y\,\partial F} & \dfrac{\partial^2 F}{\partial y\,\partial \bar{y}} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial \bar{y}} & 0 \end{vmatrix} = 0. \tag{35}$$

It must be noticed that this latter is a covariant for all changes of conjugate imaginary variables $x, y, \bar{x}, \bar{y}$. Let the origin be a point where a three-parameter variety meets the pseudo-Hessian. The tangent at $(x_1 y_1)$ has the equation

$$(x-x_1)\frac{\partial F}{\partial x_1}+(y-\bar{y}_1)\frac{\partial F}{\partial y_1}=0,$$

as we shall see more fully in the next chapter. Let the x axis be the tangent at the origin to the present variety. The equation of the latter will be

$$0 = y x\bar{y} + A_{11} + \bar{x}x + a_{12}x\bar{y} + \bar{a}_{12}y\bar{x} + A_{22}y\bar{y} + c_{11}x^2$$
$$+ 2\dot{c}_{12}xy + \dot{c}_{22}y^2 + \bar{\dot{c}}_{11}\bar{x}^2 + 2\bar{\dot{c}}_{12}\bar{x}\bar{y} + \bar{c}_{22}\bar{y}^2 + \ldots$$

Let us represent the thread cut on the x axis in the Gauss plane.

$$x = \xi + \dot{c}\eta, \quad y = 0.$$

$$0 = A_{11}(\xi^2+\eta^2) + (\dot{c}_{11}+\bar{c}_{11})(\xi^2-\eta^2) + 2\,\dot{c}\,(\dot{c}_{12}-\bar{c}_{12})\xi\eta + -.$$

The simultaneous invariant of the quadratic terms and $\xi^2+\eta^2$ is $2\,A_{11}$ and vanishes when the tangents to the two branches of the curve through the origin of the curve and the Gauss plane cut at right angles, which means, in invariant language, that the tangent chains to the two branches of the thread are mutually orthogonal. But A_{11} vanishes also when, and only when, the origin is on the pseudo-Hessian.

Theorem 70] *The pseudo-Hessian of a three-parameter variety meets it in the singular points, and in the non-singular points where the tangent meets the variety in such a thread that tangent chains to the two branches through the point of contact are mutually orthogonal.*

Theorem 71] *The necessary and sufficient condition that the hyperconic polar of a non-singular point of an algebraic hypercurve should degenerate to a line chain or a single point or line is that this point should lie in the pseudo-Hessian.*

Let us find the geometric significance of the number $2\,n$, the total order of the equation of our hypercurve. Let us call this the *order* of the hypercurve. Let us suppose that we have found a satisfactory definition in the case where the degree is

$2(n-1)$. We set up the following transformation of the plane:

1) Each point in a finite number of four-dimensional connexe regions of the plane corresponds to a hypercurve of order $2(n-1)$ of a certain system, and each such hypercurve corresponds to a point.

2) The (1, 1) polar of (z) with regard to the hypercurve corresponding to (y) is identical to that of (y) with regard to the curve corresponding to (z).

3) A triply infinite set of points lie on the corresponding hypercurves, and these do not generate a hypercurve of order $2(n-1)$ or less.

We may repeat practically word for word the reasoning in Ch. II that led up to 51], merely putting three subscripts to the l's in place of two, thus reaching:

Theorem 72] *The order of an algebraic hypercurve exceeds by two twice the number of transformations of the type given, necessary to generate it.*

This order is, of course, a projective invariant. A second such invariant may be obtained from the algebraic three-parameter variety of tangents to the hypercurve. There does not seem to be any one number that corresponds simply and naturally to the genus of an algebraic plane curve. Each algebraic hypercurve will correspond to an algebraic hypersurface in four dimensions, and such a hypersurface has various genera whose interpretation in the plane does not seem perfectly simple.

CHAPTER VI

DIFFERENTIAL GEOMETRY OF THE PLANE*

§ 1. Congruences of points.

In studying the differential geometry of the complex plane we have to consider one-, two-, and three-parameter systems of points. The first of these are called *threads*, as we saw in the last chapter, and are characterized by equations of the general type:

$$x = x(u),\quad y = y(u),\quad \bar{x} = \bar{x}(u),\quad \bar{y} = \bar{y}(u),\quad u \equiv \bar{u}.$$

By giving to the parameter complex values we obtain the single curve on which the thread lies. No thread could lie on two curves, as these latter will always intersect in discrete sets of points. A thread may also be given by such equations as

$$F(x, \bar{x}) = 0,\quad \Phi(y, \bar{y}) = 0.$$

Of far greater interest than the thread is the congruence given by

$$x = x(u, v),\quad y = y(u, v),\quad \bar{x} = \bar{x}(u, v),$$
$$\bar{y} = \bar{y}(u, v),\quad u = \bar{u},\quad v = \bar{v}.$$

The fundamental question connected with any congruence is this: What is the necessary and sufficient condition that it should be a curve? If a congruence be a curve, the ratio $\frac{dy}{dx}$ must be independent of $\frac{dv}{du}$, and this involves the equation

$$\frac{\partial(x, y)}{\partial(u, v)} \equiv 0.$$

* The major portion of the present chapter is contained in an article by the author, entitled 'Differential Geometry of the Complex Plane', *Transactions American Math. Soc.*, vol. xxii, 1921.

Conversely, if this equation be satisfied for all real values of u and v, we see, by the fundamental lemma of Ch. II, p. 58, that it is satisfied identically.

Theorem 1] *The necessary and sufficient condition that the congruence*

$$x = x(u,v), \quad y = y(u,v), \quad u = \bar{u}, \quad v = \bar{v}$$

*should be a curve is that one at least of the quantities x and y should depend upon two real parameters, and that for all real values of u and v,**

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv 0. \tag{1}$$

In homogeneous coordinates we have the corresponding equation

$$\left| x_1 \frac{\partial x_2}{\partial u} \frac{\partial x_3}{\partial v} \right| = 0.$$

If the congruence be given by the equations

$$f(x,y,\bar{x},\bar{y}) = 0, \quad \bar{f}(\bar{x},\bar{y},x,y) = 0,$$

then

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial \bar{x}} d\bar{x} + \frac{\partial f}{\partial \bar{y}} d\bar{y} = 0,$$

$$\frac{\partial \bar{f}}{\partial x} dx + \frac{\partial \bar{f}}{\partial y} dy + \frac{\partial \bar{f}}{\partial \bar{x}} d\bar{x} + \frac{\partial \bar{f}}{\partial \bar{y}} d\bar{y} = 0.$$

It will be possible to eliminate $d\bar{x}$ and $d\bar{y}$ simultaneously when, and only when,

$$\frac{\partial(f,f)}{\partial(xy)} = \frac{\partial(f,\bar{f})}{\partial(\bar{x}\bar{y})} = 0$$

for all sets of values of x, y, $\bar{x}$, $\bar{y}$ satisfying the equations above.

If a congruence be not a curve, a point where (1) is satisfied shall be called an *unusual* point, otherwise, a *usual* 'one'.

* It is not perfectly clear to whom the credit for this important theorem is due. Study, *Ausgewählte Gegenstände*, cit., p. 43, ascribes it to Segre, *Nuovo campo*, cit., but the latter, though doubtless familiar with the sufficiency of the condition, does not seem to have proved it. Theorem 2 is found there, p. 437.

The singular points where the derivatives vanish will fall in the former category. At a usual point

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv.$$

The equation of the tangent line there will be

$$\left[(\xi - x)\frac{\partial x}{\partial u} + (\eta - y)\frac{\partial y}{\partial u}\right]du + \left[(\xi - x)\frac{\partial x}{\partial v} + (\eta - y)\frac{\partial y}{\partial v}\right]dv = 0,$$

and we thus reach, from what was said at the beginning of Ch. V,

Theorem 2] *If a congruence be not a curve, the tangents at a usual point will generate a line-chain.*

We must now make rather a tiresome digression into the theory of analytic transformations of the plane. If (x, y) and (x', y') be the coordinates of corresponding points, instead of expressing the latter as functions of the former, we may express both as functions of parameters u and v and look at the differential invariants. The relation will be conformal if the minimal directions correspond, i.e. if

$$dx'^2 + dy'^2 = \lambda(dx^2 + dy^2),$$

$$du = \frac{\frac{\partial y}{\partial v}dx - \frac{\partial x}{\partial v}dy}{\frac{\partial(xy)}{\partial(uv)}}, \quad dv = -\frac{\frac{\partial y}{\partial u}dx - \frac{\partial x}{\partial u}dy}{\frac{\partial(xy)}{\partial(uv)}}.$$

$$\frac{\partial(x, y)}{\partial(u, v)}dx' = \frac{\partial(x', y)}{\partial(u, v)}dx - \frac{\partial(x', x)}{\partial(u, v)}dy;$$

$$\frac{\partial(x, y)}{\partial(u, v)}dy' = \frac{\partial(y', y)}{\partial(u, v)}dx - \frac{\partial(y', x)}{\partial(u, v)}dy.$$

The relation will be conformal if

$$\left[\frac{\partial(x', y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y', y)}{\partial(u, v)}\right]^2 = \left[\frac{\partial(x', x)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y', x)}{\partial(u, v)}\right]^2$$

$$\frac{\partial(x', y)}{\partial(u, v)}\frac{\partial(x', x)}{\partial(u, v)} + \frac{\partial(y', y)}{\partial(u, v)}\frac{\partial(y', x)}{\partial(u, v)} = 0.$$

A) Directly conformal—

$$\frac{\partial(x,x')}{\partial(u,v)}=\frac{\partial(y,y')}{\partial(u,v)},\qquad \frac{\partial(x,y')}{\partial(u,v)}=-\frac{\partial(y,x')}{\partial(u,v)}.$$

B) Inversely conformal—

$$\frac{\partial(x,x')}{\partial(u,v)}=-\frac{\partial(y,y')}{\partial(u,v)},\qquad \frac{\partial(x,y')}{\partial(u,v)}=\frac{\partial(y,x')}{\partial(u,v)}.$$

C) Directly equi-areal—

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{\partial(x',y')}{\partial(u,v)}.$$

D) Inversely equi-areal—

$$\frac{\partial(x,y)}{\partial(u,v)}=-\frac{\partial(x',y')}{\partial(u,v)}.$$

A directly equi-areal transformation shall be defined as *special* if

E) $$\frac{\partial(x,y')}{\partial(u,v)}=\frac{\partial(y,x')}{\partial(u,v)}.$$

Equations C) and E) are equivalent to

$$\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}-\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}=0,\qquad \frac{\partial x'}{\partial x}+\frac{\partial y'}{\partial y}=0.$$

Let us find the meaning of these. The corresponding directions $\frac{dy}{dx}$ and $\frac{dy'}{dx'}$ are mutually perpendicular if

$$\frac{\partial x'}{\partial x}dx^2+\left(\frac{\partial x'}{\partial y}+\frac{\partial y'}{\partial x}\right)dxdy+\frac{\partial y'}{\partial y}dy^2=0.$$

The square of the stretching ratio is

$$\frac{\left[\left(\frac{\partial x'}{\partial x}\right)^2+\left(\frac{\partial y'}{\partial x}\right)^2\right]dx^2+2\left[\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}+\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y}\right]dxdy+\left[\left(\frac{\partial x'}{\partial y}\right)^2+\left(\frac{\partial y'}{\partial y}\right)^2\right]dy^2}{dx^2+dy^2}.$$

This will be a maximum or minimum if

$$\left[\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y}+\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial y}\right]\left(dx^2-dy^2\right)$$
$$+\left[\left(\frac{\partial x'}{\partial y}\right)^2+\left(\frac{\partial y'}{\partial y}\right)^2-\left(\frac{\partial x'}{\partial x}\right)^2-\left(\frac{\partial y'}{\partial x}\right)^2\right]dxdy=0.$$

These will be the directions given above when, and only when,

$$\frac{\partial x'}{\partial x}+\frac{\partial y'}{\partial y}=0.$$

A directly equi-areal transformation will be *special* if the directions of maximum and minimum stretching are the directions orthogonal to the corresponding directions.

Let us return to the complex plane. Two conjugate imaginary points have certain invariants of fundamental importance.

$$2\frac{\partial(x,y)}{\partial(u,v)}=J+iJ',$$

$$2\frac{\partial(\bar{x},\bar{y})}{\partial(u,v)}=J-iJ', \qquad (2)$$

$$\frac{\partial(x,\bar{x})}{\partial(u,v)}+\frac{\partial(y,\bar{y})}{\partial(u,v)}=iH,$$

$$\frac{\partial(x,\bar{y})}{\partial(u,v)}-\frac{\partial(y,\bar{x})}{\partial(u,v)}=K.$$

Each of these is absolutely unaltered by a real change of rectangular axes, and multiplied merely by the Jacobian when the real variables u and v are properly replaced by others. We have also an identity which will be occasionally useful. It is the fundamental identity in the invariant theory of binary forms, and amounts in the present case to

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(\bar{x},\bar{y})}{\partial(u,v)}=\frac{\partial(x,\bar{x})}{\partial(u,v)}\frac{\partial(y,\bar{y})}{\partial(u,v)}-\frac{\partial(x,\bar{y})}{\partial(u,v)}\frac{\partial(y,\bar{x})}{\partial(u,v)}. \qquad (3)$$

From this we easily prove that $-H^2$ is the discriminant of the definite form

$$dxd\bar{x}+dyd\bar{y}.$$

In connexion with our study of congruences it is well to bear in mind the Laguerre and Marie representations of complex points which we took up in Ch. IV. Each of these is a representation by means of two real points (X, Y) and (X', Y'). In the Laguerre representation we have

$$X = \frac{x+\bar{x}}{2} + i\frac{y-\bar{y}}{2}, \quad Y = \frac{y+\bar{y}}{2} - i\frac{x-\bar{x}}{2}.$$

$$X' = \frac{x+\bar{x}}{2} - i\frac{y-\bar{y}}{2}, \quad Y' = \frac{y+\bar{y}}{2} + i\frac{x-\bar{x}}{2}.$$

$$\left.\begin{aligned} J+iJ' &= \frac{\partial(X, Y')}{\partial(u,v)} - \frac{\partial(Y, X')}{\partial(u,v)} + i\left[\frac{\partial(X, X')}{\partial(u,v)} + \frac{\partial(Y, Y')}{\partial(u,v)}\right], \\ J-iJ' &= \frac{\partial(X, Y')}{\partial(u,v)} - \frac{\partial(Y, X')}{\partial(u,v)} - i\left[\frac{\partial(X, X')}{\partial(u,v)} - \frac{\partial(Y, Y')}{\partial(u,v)}\right] \\ H &= \left[\frac{\partial(X', Y')}{\partial(u,v)} - \frac{\partial(X, Y)}{\partial(u,v)}\right], \\ K &= \left[\frac{\partial(X', Y')}{\partial(u,v)} + \frac{\partial(X, Y)}{\partial(u,v)}\right]. \end{aligned}\right\} \quad (4)$$

In the Marie representation we have similarly

$$\dot{X} = \frac{x+\bar{x}}{2} - i\frac{x-\bar{x}}{2}. \quad \dot{Y} = \frac{y+\bar{y}}{2} - i\frac{y-\bar{y}}{2}.$$

$$\dot{X}' = \frac{x+\bar{x}}{2} + i\frac{x-\bar{x}}{2}. \quad \dot{Y}' = \frac{y+\bar{y}}{2} + i\frac{y-\bar{y}}{2}.$$

$$\left.\begin{aligned} J+iJ' &= \frac{\partial(\dot{X}, \dot{Y}')}{\partial(u,v)} - \frac{\partial(\dot{Y}, \dot{X}')}{\partial(u,v)} - i\left[\frac{\partial(\dot{X}', \dot{Y}')}{\partial(u,v)} - \frac{\partial(\dot{X}, \dot{Y})}{\partial(u,v)}\right], \\ J-iJ' &= \frac{\partial(\dot{X}, \dot{Y}')}{\partial(u,v)} - \frac{\partial(\dot{Y}, \dot{X}')}{\partial(u,v)} + i\left[\frac{\partial(\dot{X}', \dot{Y}')}{\partial(u,v)} - \frac{\partial(\dot{X}, \dot{Y})}{\partial(u,v)}\right]. \\ H &= \left[\frac{\partial(\dot{X}, \dot{X}')}{\partial(u,v)} + \frac{\partial(\dot{Y}, \dot{Y}')}{\partial(u,v)}\right], \\ K &= \left[\frac{\partial(\dot{X}', \dot{Y}')}{\partial(u,v)} + \frac{\partial(\dot{X}, \dot{Y})}{\partial(u,v)}\right]. \end{aligned}\right\} \quad (5)$$

We obtain from (3) and similar identities

$$H^2-K^2+J^2+J'^2=\left(\frac{\partial(X,X')}{\partial(u,v)}-\frac{\partial(Y,Y')}{\partial(u,v)}\right)^2+\left(\frac{\partial(X,Y')}{\partial(u,v)}+\frac{\partial(Y,X')}{\partial(u,v)}\right)^2$$

$$=\left(\frac{\partial(\dot X,\dot X')}{\partial(u,v)}-\frac{\partial(\dot Y,\dot Y')}{\partial(u,v)}\right)^2+\left(\frac{\partial(\dot X,\dot Y')}{\partial(u,v)}+\frac{\partial(Y,\dot X')}{\partial(u,v)}\right)^2. \quad (6)$$

Suppose that we have a curve in the complex plain. We see from (1) and (4) that in the Laguerre representation

$$\frac{\partial(X,X')}{\partial(u,v)}=-\frac{\partial(Y,Y')}{\partial(u,v)},\quad \frac{\partial(X,Y')}{\partial(u,v)}=\frac{\partial(Y,X')}{\partial(u,v)}.$$

This gives an inversely conformal real transformation by B) unless

$$\frac{\partial(X,Y)}{\partial(u,v)}\frac{\partial X',Y')}{\partial(u,v)}=0.$$

If the first factor vanish, then either (X, Y) is stationary, or moves along a curve. Making the latter assumption, and taking u as the parameter that varies on this curve

$$\frac{\partial X}{\partial v}=\frac{\partial Y}{\partial v}=0,$$

then either $$\frac{\partial X'}{\partial v}=\frac{\partial Y'}{\partial v}=0, \text{ which is inadmissible,}$$

or else $$\left(\frac{\partial X}{\partial u}\right)^2+\left(\frac{\partial Y}{\partial u}\right)^2=0.$$

In either case $$\frac{\partial X}{\partial u}=\frac{\partial Y}{\partial u}=0.$$

It appears that (X, Y) must be fixed, or (x, y) traces a minimal line. Turning to the Marie representation, we have

$$\frac{\partial(\dot X',\dot Y')}{\partial(u,v)}=\frac{\partial(\dot X,\dot Y)}{\partial(u,v)},\quad \frac{\partial(\dot X,\dot Y')}{\partial(u,v)}=\frac{\partial(\dot Y,\dot X')}{\partial(u,v)},$$

a special directly equi-areal transformation unless

$$\frac{\partial(\dot X',\dot Y')}{\partial(u,v)}=\frac{\partial(\dot X,Y)}{\partial(u,v)}=0.$$

Here we may assume

$$\frac{\partial \dot{X}}{\partial v} = \frac{\partial \dot{Y}}{\partial v} = \frac{\partial \dot{X}'}{\partial u} = \frac{\partial \dot{Y}'}{\partial u} = 0.$$

$$\frac{\partial x}{\partial v} + i\frac{\partial \bar{x}}{\partial v} = \frac{\partial y}{\partial v} + i\frac{\partial \bar{y}}{\partial v} = 0.$$

$$\frac{\partial x}{\partial u} - i\frac{\partial \bar{x}}{\partial u} = \frac{\partial y}{\partial u} - i\frac{\partial \bar{y}}{\partial u} = 0.$$

$$\frac{dy}{dx} = \frac{\dfrac{\partial \bar{y}}{\partial u}}{\dfrac{\partial \bar{x}}{\partial u}} = \frac{\dfrac{\partial \bar{y}}{\partial v}}{\dfrac{\partial \bar{x}}{\partial v}} = \frac{d\bar{y}}{d\bar{x}}.$$

Let $$x = p + iq, \quad y = y(p+iq),$$

$$\frac{dy}{dx} = P(p, q) + iQ(p, q), \quad Q \equiv 0.$$

By the Cauchy-Riemann equations

$$\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0. \quad \text{Hence } P = \text{const.},$$

or (x, y) traces a line in the real direction, i.e. parallel to that traced by $(\bar{x}, \bar{y})$. The same is true of $(\dot{X}', \dot{Y})$ and $(\dot{X}, \dot{Y}')$.

Theorem 3] *If a finite curve be not a minimal line, the corresponding pairs of points in the Laguerre representation are connected by a real inversely conformal transformation; if it be not a line with real direction, the corresponding pairs in the Marie representation will be connected by a real special directly equi-areal transformation.*

Theorem 4] *If an inversely conformal real transformation of the plane be given, and each corresponding pair of points be rotated through an angle of 90° about their mid-point, the result will be a real special directly equi-areal transformation.**

* These two splendid theorems are due to Study, *Ausgewählte Gegenstände*, cit., pp. 63, 64. The remainder of this remarkable article is devoted almost entirely to studying the two representations of different types of curves.

We have seen what happens when our two invariants J and J' vanish. Suppose next that

$$iH = \frac{\partial(x, \bar{x})}{\partial(u, v)} + \frac{\partial(y, \bar{y})}{\partial(u, v)} = 0.$$

In the Laguerre representation we have

$$\frac{\partial(X, Y)}{\partial(u, v)} = \frac{\partial(X', Y')}{\partial(u, v)}.$$

If each of these vanish we have

$$K = 0,$$

so that (x, y) and $(\bar{x}, \bar{y})$ are connected by an inversely conformal relation.

Theorem 5] *The relation between conjugate imaginary pairs of points and their Laguerre representatives is such that if either pair generate two finite curves which are not minimal lines, the other pair will be connected by an inversely conformal relation, and conversely.*

We obtain in just the same way

Theorem 6] *The relation between conjugate imaginary pairs of points and their Marie representatives is such that if either pair trace two finite curves which are not parallel lines, the other pair will be connected by a special directly equi-areal relation, and conversely.*

We find from (2) that the condition for a directly conformal relation between (x, y) and $(\bar{x}, \bar{y})$ is

$$H^2 - K^2 + J^2 + J'^2 = 0, \tag{7}$$

and this gives, with the aid of (6),

Theorem 7] *The necessary and sufficient condition that the relation between a congruence, not a curve, and its conjugate should be directly conformal is that the corresponding transformation in the Laguerre or Marie representation should be directly conformal.*

Theorem 8] *If corresponding pairs of points in a real directly conformal transformation be rotated through an* *angle of* 90° *about their mid-point, the resulting transformation is directly conformal.*

Theorem 9] *If in a congruence, not a curve, the equation*

$$iH = 0$$

subsist, while nevertheless conjugate imaginary points are not connected by an inversely conformal relation, then the corresponding transformation in the Laguerre representation is directly equi-areal.

Theorem 10] *If in a congruence, not a curve, the equation*

$$K = 0$$

subsist, while nevertheless conjugate imaginary points are not connected by an inversely conformal relation, then the corresponding transformation in the Laguerre and Marie representations are inversely equi-areal.

Theorem 11] *If an inversely equi-areal transformation be given in the plane, and each pair of points be rotated through an angle of* 90° *about their mid-point, the resulting transformation is inversely equi-areal.*

The congruence where (x, y) and $(\bar{x}, \bar{y})$ trace curves is a special case of a wider type of congruence, namely, that where a point forming with them a triangle with given angles, or dividing their segment in a fixed ratio, traces a curve. It is not difficult to show that the coordinates of such a point are expressible in the form

$$x' = \frac{x - r\bar{x}}{1 - r} - \rho \frac{y - r\bar{y}}{1 - r}; \quad y' = \rho \frac{x - \bar{x}}{1 - r} + \frac{y - r\bar{y}}{1 - r}.$$

$$\frac{\partial (x'y')}{\partial (uv)} = \alpha [J + iJ'] + \beta [J - iJ'] + \gamma H + \delta K = 0.$$

$$\alpha = \frac{1 + \rho^2}{2}, \quad \beta = \frac{r^2 + \rho^2}{2}, \quad \gamma = i\rho (r - 1), \quad \delta = -(r + \rho^2),$$

$$\delta^2 - \gamma^2 = 4\alpha\beta.$$

This will involve the additional equation

$$\alpha [J - i\bar{J}] + \bar{\beta} [J + i\bar{J}] + \bar{\gamma} H + \bar{\delta} K.$$

The two will be equivalent if

$$\frac{\alpha}{\bar{\bar{\beta}}}=\frac{\beta}{\bar{\bar{\alpha}}}=\frac{\gamma}{\bar{\bar{\gamma}}}=\frac{\delta}{\bar{\bar{\delta}}}.$$

After rather a tedious calculation, we find that if $\rho \neq 0$,

$$r=\frac{\rho}{\bar{\bar{\rho}}}.$$

The most interesting case is where $\rho=0$ and $(x'y')$ is a point dividing the segment from (x,y) to $(\bar{x},\bar{y})$ in a fixed ratio r.

$$\frac{\partial(x,y)}{\partial(u,v)}+r^2\frac{\partial(\bar{x},\bar{y})}{\partial(u,v)}-r\left(\frac{\partial(x,\bar{y})}{\partial(u,v)}-\frac{\partial(y,\bar{x})}{\partial(u,v)}\right)=0.$$

$$\bar{r}^2\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(\bar{x},\bar{y})}{\partial(u,v)}-r\left(\frac{\partial(x,\bar{y})}{\partial(u,v)}-\frac{\partial(y,\bar{x})}{\partial(u,v)}\right)=0.$$

If $$r\bar{r}=1,$$

the point (x',y') is real, and the two equations reduce to

$$e^{-i\theta}(J+J')+e^{i\theta}(J-iJ')=K.$$

When, however, (x',y') is not real,

$$\frac{\partial(x',y')}{\partial(u,v)}=\frac{r}{\bar{\bar{r}}}\frac{\partial(\bar{x},\bar{y})}{\partial(u,v)}.$$

Theorem 12] *If two conjugate imaginary points trace two such congruences, not curves, that a complex point which divides their segment in a fixed ratio traces a curve, then corresponding areas in the two congruences have fixed ratios whose absolute value is unity.*

Let us see under what circumstances a congruence will include a set of threads lying on minimal lines. As (x,y) and $(\bar{x},\bar{y})$ trace two minimal threads, one Laguerre representative must be fixed. This involves

$$\frac{\partial(X,Y)}{\partial(u,v)}\,\frac{\partial(\overline{X},\overline{Y})}{\partial(u,v)}=0.$$

$$H^2-K^2=0. \tag{8}$$

We pass to the more difficult question of finding the N. S.

condition for the existence of a normal net in a congruence. We require the conditions for the subsistence of the equations

$$dx\,\bar{\partial}x + dy\,\bar{\partial}y = 0,$$

$$d\bar{x}\,\bar{\partial}\bar{x} + d\bar{y}\,\bar{\partial}\bar{y} = 0.$$

This means that the Jacobian of the binary quadratic equations,

$$dx^2 + dy^2 = E\,du^2 + 2\,F du dv + G dv^2 = 0,$$

$$d\bar{x}^2 + d\bar{y}^2 = \bar{E}\,du^2 + 2\,\bar{F} du dv + \bar{G} dv^2 = 0,$$

must have real roots. The condition for this is

$$\begin{vmatrix} E & F & G & 0 \\ 0 & E & F & G \\ \bar{E} & \bar{F} & \bar{G} & 0 \\ 0 & \bar{E} & \bar{F} & \bar{G} \end{vmatrix} > 0$$

$$K^2 - H^2 > 0. \qquad (9)$$

$$\frac{\partial(X, Y)}{\partial(u, v)}\;\frac{\partial(X', Y')}{\partial(u, v)} > 0.$$

Theorem 13] *The necessary and sufficient condition that a congruence should include a normal net is that in the Laguerre representation corresponding infinitesimal areas should always have the same sign.*

Theorem 14] *A congruence which contains a system of minimal threads cannot contain a normal net.*

Let us next see under what circumstances the congruence contains a system of threads whose tangents have real directions, so that along them (x, y) and $(\bar{x}, \bar{y})$ move parallel to one another. For this it is necessary and sufficient that in the Marie representation there should be a system of curves parallel to their corresponding curves. We must have

$$dx\,d\bar{y} - dy\,d\bar{x} = 0.$$

$$\left(\frac{\partial x}{\partial u}\frac{\partial \bar{y}}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial \bar{x}}{\partial u}\right) du^2 - \left[\left(\frac{\partial y}{\partial u}\frac{\partial \bar{x}}{\partial v} + \frac{\partial y}{\partial v}\frac{\partial \bar{x}}{\partial u}\right)\right.$$

$$\left. - \left(\frac{\partial x}{\partial u}\frac{\partial \bar{y}}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial \bar{y}}{\partial u}\right)\right] du\,dv + \left[\frac{\partial x}{\partial v}\frac{\partial \bar{y}}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial \bar{x}}{\partial v}\right] dv^2 = 0.$$

The condition for real roots is

$$\left(\frac{\partial(x,\bar{y})}{\partial(u,v)}\right)^2+\left(\frac{\partial(y,\bar{x})}{\partial(u,v)}\right)^2-2\left[\frac{\partial(x,\bar{x})}{\partial(u,v)}\frac{\partial(y,\bar{y})}{\partial(u,v)}+\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(\bar{x},\bar{y})}{\partial(u,v)}\right]>0.$$

We thus get, with the aid of (3),

Theorem 15] *The necessary and sufficient condition that a congruence, not a curve, should contain a system of threads whose tangents have real directions is*

$$K^2-(J^2+J'^2)>0. \tag{10}$$

This is also the N. S. condition that in the Marie representation there should be a set of curves parallel to the corresponding curves.

Let the reader show by similar reckoning

Theorem 16] *The necessary and sufficient condition that a congruence should be such that in the Laguerre representation there should be a set of curves parallel to the corresponding curves is*

$$J^2-(K^2+H^2)>0. \tag{11}$$

When will the distance of two points of a congruence be independent of the path? This is certainly the case where the congruence is a curve. Conversely, if the distance between two points be a function of the values of u and v corresponding to them, the distance element must be a complete differential, i.e.

$$E\,du^2+2F\,du\,dv+G\,dv^2$$

is a perfect square and

$$EG-F^2\equiv\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2=0.$$

Theorem 17] *A curve is the only congruence where two paths of a simply connected region connecting two points have necessarily the same length.*

If a congruence be not a curve there must be a geodesic thread connecting two points. But how shall we define a geodesic thread? If we define it as a path whose first variation is zero we encounter equations in our real parameters

u and v which will not, in general, have real solutions. The most natural thing would seem to be to define as a geodesic a thread where

$$\left|\int \sqrt{E+2Fv'+Gv'^2}\,du\right|$$

is a minimum. Let us write

$$\sqrt{E+2Fv'+Gv'^2}=\phi(u,v,v')+i\psi(u,v,v').$$

We must minimize

$$\left[\int\phi\,du\right]^2+\left[\int\psi\,du\right]^2,$$

$$\int[\delta\phi]\,du\int\phi\,du+\int[\delta\psi]\,du\int\psi\,du=0,$$

$$\left[\frac{\partial\phi}{\partial v}-\frac{d}{du}\left(\frac{\partial\phi}{\partial v'}\right)\right]\int\phi\,du+\left[\frac{\partial\psi}{\partial v}-\frac{d}{du}\left(\frac{\partial\psi}{\partial v'}\right)\right]\int\psi\,du=0.$$

It is by no means easy to see just what can be done with such an equation. Lastly, instead of minimizing the absolute value of the integral, we might have minimized the expression,

$$\int\sqrt{E+2Fv'+Gv'^2}\,\sqrt{\bar{E}+2\bar{F}v'+\bar{b}v'^2}\,du,$$

which is the integral of the absolute value of the differential of arc. Here we are at least face to face with a straightforward problem in the calculus of variations, but it seems a stretch of language to apply the term *geodesic* to a path obtained in this way.

We saw in 2] that the tangent at a usual point of a congruence which is not a curve will generate a line-chain. Let us look more closely at this chain:

$$x'=x(u,v)+(U-u)\frac{\partial x}{\partial u}+(V-v)\frac{\partial x}{\partial v}+\tfrac{1}{2}\Big[(U-u)^2\frac{\partial^2x}{\partial u^2}$$
$$+2(U-u)(V-v)\frac{\partial^2x}{\partial u\,\partial v}+(V-v)^2\frac{\partial^2x}{\partial v^2}\Big]+\dots$$

(12)

$$y'=y(u,v)+(U-u)\frac{\partial y}{\partial u}+(V-v)\frac{\partial y}{\partial v}+\tfrac{1}{2}\Big[(U-u)^2\frac{\partial^2y}{\partial u^2}$$
$$+2(U-u)(V-v)\frac{\partial^2y}{\partial u\,\partial v}+(V-v)^2\frac{\partial^2y}{\partial v^2}\Big]+\dots.$$

This will meet the line

$$y'-y(u,v)=\lambda\,(x'-x(u,v)).$$

$$0=\left[\frac{\partial y}{\partial u}-\lambda\frac{\partial x}{\partial v}\right](U-u)+\left[\frac{\partial y}{\partial v}-\lambda\frac{\partial x}{\partial v}\right](V-v)+\dots$$

$$+\tfrac{1}{2}\left\{\left[\frac{\partial^2 y}{\partial u^2}-\lambda\frac{\partial^2 x}{\partial u^2}\right](U-u)^2+2\left[\frac{\partial^2 y}{\partial u\,\partial v}-\lambda\frac{\partial^2 x}{\partial u\,\partial v}\right](U-u)(V-v)\right.$$

$$\left.+\left[\frac{\partial^2 y}{\partial v^2}-\lambda\frac{\partial^2 y}{\partial v^2}\right](V-v)^2\right\}+\dots. \qquad (13)$$

Similarly

$$0=\left[\frac{\partial \bar{y}}{\partial u}-\bar{\lambda}\frac{\partial \bar{x}}{\partial u}\right](U-u)+\left[\frac{\partial \bar{y}}{\partial v}-\bar{\lambda}\frac{\partial \bar{x}}{\partial v}\right](V-v)+\dots$$

$$+\tfrac{1}{2}\left\{\left[\frac{\partial^2 \bar{y}}{\partial u^2}-\bar{\lambda}\frac{\partial^2 \bar{x}}{\partial u^2}\right](U-u)^2+2\left[\frac{\partial^2 \bar{y}}{\partial u\,\partial v}-\bar{\lambda}\frac{\partial^2 \bar{x}}{\partial u\,\partial v}\right](U-u)(V-v)\right.$$

$$\left.+\left[\frac{\partial^2 \bar{y}}{\partial v^2}-\bar{\lambda}\frac{\partial^2 \bar{x}}{\partial v^2}\right](V-v)^2\right\}+\dots. \qquad (14)$$

We have two curves in the U, V plane, which meet at

$$U=u,\quad V=v.$$

They will touch if

$$\begin{vmatrix} \dfrac{\partial \bar{y}}{\partial u}-\bar{\lambda}\dfrac{\partial \bar{x}}{\partial u}, & \dfrac{\partial \bar{y}}{\partial v}-\bar{\lambda}\dfrac{\partial \bar{x}}{\partial v} \\ \dfrac{\partial y}{\partial u}-\lambda\dfrac{\partial x}{\partial u}, & \dfrac{\partial y}{\partial v}-\lambda\dfrac{\partial x}{\partial v} \end{vmatrix}=0.$$

$$\lambda\bar{\lambda}\frac{\partial(x\bar{x})}{\partial(uv)}-\lambda\frac{\partial(x\bar{y})}{\partial(uv)}-\bar{\lambda}\frac{\partial(y\bar{x})}{\partial(uv)}+\frac{\partial(y\bar{y})}{\partial(uv)}=0. \qquad (15)$$

This is the equation of the line-chain of tangents. It will be unchanged if we replace λ and $\bar{\lambda}$ by their conjugates $\bar{\lambda}$ and λ when, and only when,

$$K=0.$$

It will be unaltered when we replace λ and $\bar{\lambda}$ by the diametral imaginary values $-\dfrac{1}{\bar{\lambda}}$ and $-\dfrac{1}{\lambda}$ when, and only when,

$$H=0.$$

We thus get from 9] and 8]

Theorem 18] *The necessary and sufficient condition that the tangents at an arbitrary usual point of a congruence, not a curve, should have, in pairs, conjugate imaginary slopes, is that the corresponding transformation in the Laguerre representation should be inversely equi-areal; the slopes will be diametral imaginaries in pairs when this transformation is directly equi-areal.*

Let us see whether we can find a tangent that osculates the congruence. For this it is necessary and sufficient that the curves (13) and (14) in the (U, V) plane should osculate one another at $U = u$, $V = v$. If these curves be

$$F(U, V) = 0, \quad \bar{F}(U, V) = 0,$$

the conditions are
$$\frac{\partial(F, \bar{F})}{\partial(u, v)} = 0.$$

$$\frac{\frac{\partial^2 F}{\partial v^2}\left(\frac{\partial F}{\partial v}\right)^2 + \frac{\partial^2 F}{\partial v^2}\left(\frac{\partial F}{\partial u}\right)^2 - 2\,\frac{\partial^2 F}{\partial u\,\partial v}\,\frac{\partial F}{\partial u}\,\frac{\partial F}{\partial v}}{\left[\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2\right]^{\frac{3}{2}}}$$

$$= \frac{\frac{\partial^2 \bar{F}}{\partial u^2}\left(\frac{\partial \bar{F}}{\partial v}\right)^2 + \frac{\partial^2 \bar{F}}{\partial v^2}\left(\frac{\partial \bar{F}}{\partial u}\right)^2 - 2\,\frac{\partial^2 \bar{F}}{\partial u\,\partial v}\,\frac{\partial \bar{F}}{\partial u}\,\frac{\partial \bar{F}}{\partial v}}{\left[\left(\frac{\partial \bar{F}}{\partial u}\right)^2 + \left(\frac{\partial \bar{F}}{\partial v}\right)^2\right]^{\frac{3}{2}}}.$$

It is necessary and sufficient that the following expression should be equal to its conjugate

$$\frac{\left(\frac{\partial^2 y}{\partial u^2} - \lambda\frac{\partial^2 x}{\partial u^2}\right)\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial^2 y}{\partial v^2} - \lambda\frac{\partial^2 x}{\partial v^2}\right)\left(\frac{\partial x}{\partial u} - \lambda\frac{\partial y}{\partial u}\right)^2 - 2\left(\frac{\partial^2 y}{\partial u\,\partial v} - \lambda\frac{\partial^2 x}{\partial u\,\partial v}\right)\left(\frac{\partial y}{\partial u} - \lambda\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial y}\right)}{\left[\left(\frac{\partial y}{\partial u} - \lambda\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right)^2\right]^{\frac{3}{2}}}.$$

The general solution of (15) can be written

$$\lambda = \frac{\dfrac{\partial y}{\partial u} + X\dfrac{\partial y}{\partial v}}{\dfrac{\partial x}{\partial u} + X\dfrac{\partial x}{\partial v}}, \quad X = \frac{\dfrac{\partial y}{\partial u} - \lambda\dfrac{\partial x}{\partial u}}{\dfrac{\partial y}{\partial v} - \lambda\dfrac{\partial x}{\partial v}}.$$

$$\frac{\partial y}{\partial u} - \lambda\frac{\partial x}{\partial u} = X\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right) = \frac{X\dfrac{\partial(xy)}{\partial(uv)}}{\dfrac{\partial x}{\partial u} + X\dfrac{\partial x}{\partial v}}.$$

The expression above becomes

$$\frac{\left(\dfrac{\partial^2 y}{\partial u^2} - \lambda\dfrac{\partial^2 x}{\partial u^2}\right) - 2X\left(\dfrac{\partial^2 y}{\partial u\,\partial v} - \lambda\dfrac{\partial^2 x}{\partial u\,\partial v}\right) + X^2\left(\dfrac{\partial^2 y}{\partial v^2} - \lambda\dfrac{\partial^2 x}{\partial v^2}\right)}{\left(\dfrac{\partial y}{\partial v} - \lambda\dfrac{\partial x}{\partial v}\right)\left(1 + X^2\right)^{\frac{3}{2}}}.$$

Equating this expression to its conjugate, removing the common denominator $(1 + X^2)^{\frac{3}{2}}$, and expressing λ in terms of X, we get the real equation

$$PX^3 + QX^2 + RX + S = 0,$$

which surely will have one real root, perhaps three of them. Hence at an arbitrary usual point there is necessarily one osculating tangent, and there may be three of them. A thread whose tangents osculate the congruence shall be called an *asymptotic thread.*

Theorem 19] *Every congruence, not a curve, contains at least one system of asymptotic threads.*

There remain to be considered the unusual points characterized by

$$\frac{\partial(x, y)}{\partial(u, v)} = 0, \quad \frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)} = 0.$$

Usually these equations are distinct. When, however, they are equivalent we have a thread of unusual points. We thus get from 3]

Theorem 20] *The necessary and sufficient condition that a congruence, not a curve, should contain a thread of unusual* *points is that in the corresponding transformation in the Laguerre representation there should be a curve whose points are inversely conformally transformed.*

§ 2. Three-parameter systems.

There are two natural ways to represent a three-parameter system of points in the plane, namely

$$x = x(u, v, w), \quad y = y(u, v, w);$$
$$\bar{x} = \bar{x}(u, v, w), \quad \bar{y} = \bar{y}(u, v, w) \tag{16}$$
$$F(x, y, \bar{x}, \bar{y}) \equiv \bar{F}(\bar{x}, \bar{y}, x, y) = 0. \tag{17}$$

The fundamental question to be asked about such a system is whether it will contain a curve or not. We have seen examples of both kinds in our study of ternary Hermitian forms in Ch. V. Let us find the N. S. condition that our three-parameter system (16) should contain a curve. Let this curve be characterized by

$$\phi(u, v, w) = \text{const.}$$

$$\frac{\partial \phi}{\partial u} : \frac{\partial \phi}{\partial v} : \frac{\partial \phi}{\partial w} = \frac{\partial w}{\partial u} : \frac{\partial w}{\partial v} : -1.$$

The Jacobian of x and y with regard to u and v under these circumstances must vanish, or

$$\begin{vmatrix} \frac{\partial x}{\partial u} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial u} & \frac{\partial x}{\partial v} + \frac{\partial x}{\partial w}\frac{\partial w}{\partial v} \\ \frac{\partial y}{\partial u} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial u} & \frac{\partial y}{\partial v} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial v} \end{vmatrix} = 0.$$

$$\frac{\partial \phi}{\partial u}\frac{\partial(x, y)}{\partial(v, w)} + \frac{\partial \phi}{\partial v}\frac{\partial(x, y)}{\partial(w, u)} + \frac{\partial \phi}{\partial w}\frac{\partial(x, y)}{\partial(u, v)} = 0.$$

$$\frac{\partial \phi}{\partial u}\frac{\partial(\bar{x}, \bar{y})}{\partial(v, w)} + \frac{\partial \phi}{\partial v}\frac{\partial(\bar{x}, \bar{y})}{\partial(w, u)} + \frac{\partial \phi}{\partial w}\frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)} = 0.$$

The reasoning is reversible, so that it is necessary and sufficient that there should be a function of u, v, and w, whose partial derivatives are proportional to the determinants of the matrix

$$\left\| \begin{matrix} \frac{\partial(x,y)}{\partial(v,w)} & \frac{\partial(x,y)}{\partial(w,u)} & \frac{\partial(x,y)}{\partial(u,v)} \\ \frac{\partial(\bar{x},\bar{y})}{\partial(v,w)} & \frac{\partial(\bar{x},\bar{y})}{\partial(w,u)} & \frac{\partial(\bar{x},\bar{y})}{\partial(u,v)} \end{matrix} \right\|$$

Theorem 21] *If x and y be functions of three real parameters, u, v, and w, and no less, the necessary and sufficient condition that the system so defined should contain one, and, hence, an infinite number of curves is that the Pfaff equation*

$$\begin{vmatrix} du & dv & dw \\ \frac{\partial(x,y)}{\partial(v,w)} & \frac{\partial(x,y)}{\partial(w,u)} & \frac{\partial(x,y)}{\partial(u,v)} \\ \frac{\partial(\bar{x},\bar{y})}{\partial(v,w)} & \frac{\partial(\bar{x},\bar{y})}{\partial(w,u)} & \frac{\partial(\bar{x},\bar{y})}{\partial(u,v)} \end{vmatrix} = 0 \qquad (18)$$

should be integrable.

We must now test for curves when the system is expressed in the form (17). Suppose that the curves, depending on the real parameter R,

$$y = y(x, R), \quad \bar{y} = \bar{y}(\bar{x}, R),$$

are contained in the variety (17),

$$F(x, y(x, R), \bar{x}, \bar{y}(\bar{x}, R)) \equiv 0.$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \equiv \frac{\partial F}{\partial \bar{x}} + \frac{\partial F}{\partial \bar{y}}\frac{d\bar{y}}{d\bar{x}} \equiv 0.$$

$$\frac{\partial^2 F}{\partial x\, \partial \bar{x}} + \frac{\partial^2 F}{\partial y\, \partial \bar{x}}\frac{dy}{dx} + \frac{\partial^2 F}{\partial x\, \partial \bar{y}}\frac{d\bar{y}}{d\bar{x}} + \frac{\partial^2 F}{\partial y\, \partial \bar{y}}\frac{dy}{dx}\frac{d\bar{y}}{d\bar{x}} = 0.$$

$$\frac{\partial F}{\partial x}\frac{\partial F}{\partial \bar{x}}\frac{\partial^2 F}{\partial y\, d\bar{y}} + \frac{\partial F}{\partial y}\frac{\partial F}{\partial \bar{y}}\frac{\partial^2 F}{\partial x\, d\bar{x}} - \frac{\partial F}{\partial x}\frac{\partial F}{\partial \bar{y}}\frac{\partial^2 F}{\partial y\, \partial \bar{x}} - \frac{\partial F}{\partial y}\frac{\partial F}{\partial \bar{x}}\frac{\partial^2 F}{\partial x\, \partial \bar{y}} = 0.$$

This means that for every point where

$$F(x, y, \bar{x}, \bar{y}) = 0$$

the pseudo-Hessian

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x \partial \bar{x}} & \frac{\partial^2 F}{\partial x \partial \bar{y}} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial \bar{x}} & \frac{\partial^2 F}{\partial y \partial \bar{y}} & \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial \bar{x}} & \frac{\partial F}{\partial \bar{y}} & 0 \end{vmatrix} = 0.$$

Conversely, if the pseudo-Hessian vanishes for all points of the variety, the Jacobian in $\bar{x}$ and $\bar{y}$ of

$$F \text{ and } \left(\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right) \text{vanishes.}$$

Hence F, looked upon as a function of $\bar{x}$ and $\bar{y}$, can be written

$$\phi\left(x, y \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right) = 0,$$

and this holds when we treat $\bar{x}$ and $\bar{y}$ as variables independent of x and y.

If, then, we so determine $\bar{x}$ and $\bar{y}$ that

$$\frac{\partial F}{\partial x} = L\frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial \bar{x}} = L\frac{\partial F}{\partial \bar{y}},$$

$$\phi(x, y, L) = 0,$$

and this curve is contained entirely in the variety. We thus get from V. 68]

Theorem 22] *The necessary and sufficient condition that the variety* (17) *should contain a one-parameter family of curves is that the pseudo-Hessian should vanish for all points of the variety. Every tangent line will meet the variety in a thread with a double point at the point of contact; tangent chains to the two branches will be mutually orthogonal.*

The invariants of a three-parameter family of points are less simple and interesting than those of a congruence. The squared differential of arc

$$dx^2 + dy^2$$

is a ternary quadratic form in du, dv, and dw, but its discriminant is equal to zero, since it is merely

$$\left(\frac{\partial (x, y, 0)}{\partial (u, v, w)}\right)^2.$$

Another invariant is the discriminant of the positive form

$$dx\, d\bar{x} + dy\, d\bar{y},$$

and this turns out to be

$$\frac{1}{4}\left[\frac{\partial (x, y, \bar{x})}{\partial (u, v, w)} \frac{\partial (\bar{x}, \bar{y}, x)}{\partial (u, v, w)} + \frac{\partial (x, y, \bar{y})}{\partial (u, v, w)} \frac{\partial (\bar{x}, \bar{y}, y)}{\partial (u, v, w)}\right] > 0,$$

as we see by taking the special case, where

$$x = u + iv, \quad y = w + if\,(uvw).$$

A three-parameter system will always have a one-parameter family of minimal threads. Their differential equation is obtained from

$$dx \pm i dy = 0, \quad d\bar{x} \pm i d\bar{y} = 0,$$

$$\frac{du}{H(v, w) \pm K(v, w)} = \frac{dv}{H(w, u) \pm K(w, u)} = \frac{dw}{H(u, v) \pm K(u, v)}.$$

Let us look at tangents to three-parameter varieties. If we take the variety (17) and replace $x, y, \bar{x}$, and $\bar{y}$ respectively by

$$\frac{x + t\rho x'}{1 + t\rho}, \quad \frac{y + t\rho y'}{1 + t\rho}, \quad \frac{\bar{x} + t\bar{\rho}\bar{x}'}{1 + t\bar{\rho}}, \quad \frac{\bar{y} + t\bar{\rho}\bar{y}'}{1 + t\bar{\rho}},$$

we shall find where an arbitrary chain connecting (x, y) and (x', y') will meet the variety. We get the equation

$$0 = F(x, y, \bar{x}, \bar{y}) + t\left\{\rho\left[(x' - x)\frac{\partial F}{\partial x} + (y' - y)\frac{\partial F}{\partial y}\right]\right.$$
$$\left. + \bar{\rho}\left[(\bar{x}' - \bar{x})\frac{\partial F}{\partial \bar{x}} + (\bar{y}' - \bar{y})\frac{\partial F}{\partial \bar{y}}\right]\right\} + t^2\left[\alpha\rho^2 + t\rho\bar{\rho} + \bar{\alpha}\bar{\rho}^2\right] + \ldots$$

Let (x, y) be a point of the variety. Then the constant term will vanish, and we have one root $t = 0$. For (x', y') in general position we can always determine ρ and $\bar{\rho}$ so that there shall be a second root $t = 0$, i.e. any point in the plane may be connected with (x, y) by a chain tangent there, and usually by one such. But if we have

$$(x'-x)\frac{\partial F}{\partial x}+(y'-y)\frac{\partial F}{\partial y}=0, \tag{19}$$

the conjugate expression will also vanish, and every chain from (x, y) to (x', y') will meet the variety twice at the former point, and so be tangent, unless this latter is a singular point. We shall call this *tangent.* We may thus restate V. 66] in more general form.

Theorem 23] *If a non-singular point be chosen on an analytic three-parameter variety, every point in the plane, not on the tangent thereat, may be connected therewith by a single chain tangent to the variety at the non-singular point. Every chain on the tangent which passes through the point of contact is tangent to the variety at that point. The tangent meets the variety in a thread having a double point at the point of contact; tangent chains to the two branches are mutually orthogonal when the point is on the pseudo-Hessian.*

There are other forms for the equation of the tangent which are worth giving. Let x and y be functions of u, v, and w. Then (17) is an identity in these three variables, giving three equations such as

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial F}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial u}+\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial u}=0.$$

The equation of the tangent will be

$$(x'-x)\frac{\partial(y\bar{x}\bar{y})}{\partial(uvw)}-(y'-y)\frac{\partial(x\bar{x}\bar{y})}{\partial(uvw)}=0. \tag{20}$$

This gives the differential form

$$\frac{dy}{dx}=\frac{\dfrac{\partial(y, \bar{x}, \bar{y})}{\partial(u, v, w)}}{\dfrac{\partial(x, \bar{x}, \bar{y})}{\partial(u, v, w)}}.$$

Comparing with (3) of Ch. V we see that this appears as a singular element in the parametric expression for the pencil of lines through the given point. Expanding dx and dy in terms of du, dv, and dw, we find once more (18), which appears as the differential equation of the tangent. If the variety contain an infinite number of curves, each obtained by putting $w =$ const., then

$$\frac{\partial (x, y)}{\partial (u, v)} = \frac{\partial (\bar{x}, \bar{y})}{\partial (u, v)} = 0,$$

and (18) becomes $\quad dw = 0, \quad w = \text{const.}$

Theorem 24] *If a three-parameter system contain an infinite number of curves, the tangent at any point is the tangent to the curve through that point.*

In homogeneous coordinates, changing u, v, w to $u_1 u_2 u_3$ respectively, the equation of the tangent at (7) will be

$$\sum_{i=1}^{i=3} \left| \bar{y} \frac{\partial \bar{y}}{\partial u_j} \frac{\partial \bar{y}}{\partial u_k} \right| \cdot \left| \frac{\partial y}{\partial u_i} y x \right| = 0. \tag{21}$$

Usually this will give a three-parameter system of lines. An arbitrary line in the plane will determine with any line of the system an infinite number of chains, just one of which will be tangent to the three-parameter line system. There is, however, a single point on each non-singular line of the system, called its *point of contact*, which has the property that every chain through it includes the given line as tangent. The given line appears as a double line in the one-parameter system of lines of the variety through the point of contact. Let us find the point of contact of an arbitrary tangent. We write

$$\sum_{i=1}^{i=3} \left| \bar{y} \frac{\partial \bar{y}}{\partial u_j} \frac{\partial \bar{y}}{\partial u_k} \right| \cdot \left| \frac{\partial y}{\partial u_i} y x \right| \equiv \left| y q x \right| \equiv (a x) = 0. \tag{22}$$

The point of contact will have the equation

$$\sum_{i=1}^{i=3} \left| \bar{a} \frac{\partial \bar{a}}{\partial u_j} \frac{\partial \bar{a}}{\partial u_k} \right| \cdot \left| \frac{\partial a}{\partial u_i} a \omega \right| = 0.$$

$$\frac{\partial a_l}{\partial u_i} = \begin{vmatrix} \frac{\partial y_m}{\partial u_i} & \frac{\partial y_n}{\partial u_i} \\ q_m & q_n \end{vmatrix} + \begin{vmatrix} y_m & y_n \\ \frac{\partial q_m}{\partial u_i} & \frac{\partial q_n}{\partial u_i} \end{vmatrix}$$

$$\left| \frac{\partial a}{\partial u_i}\, a\, \omega \right| = \begin{vmatrix} \left(a \frac{\partial y}{\partial u_i}\right) & (aq) \\ \left(\omega \frac{\partial y}{\partial u_i}\right) & (\omega q) \end{vmatrix} + \begin{vmatrix} (ay) & \left(a \frac{\partial q}{\partial u_i}\right) \\ (\omega y) & \left(\omega \frac{\partial q}{\partial u_i}\right) \end{vmatrix}.$$

By definition $\qquad (aq) = (ay) = 0,$

$$\left| \frac{\partial a}{\partial u_i}\, a\, \omega \right| = \left(a \frac{\partial y}{\partial u_i}\right)(\omega q) - \left(a \frac{\partial q}{\partial u_i}\right)(\omega y).$$

Again

$$\left| \bar{a} \frac{\partial \bar{a}}{\partial u_j} \frac{\partial \bar{a}}{\partial u_k} \right| = \left(\bar{y} \frac{\partial \bar{a}}{\partial u_j}\right)\left(\bar{q} \frac{\partial \bar{a}}{\partial u_k}\right) - \left(\bar{y} \frac{\partial \bar{a}}{\partial u_k}\right)\left(\bar{q} \frac{\partial \bar{a}}{\partial u_j}\right),$$

$$\left(\bar{y} \frac{\partial \bar{a}}{\partial u_j}\right) = \left| \bar{y} \frac{\partial \bar{y}}{\partial u_j} \bar{q} \right| - \left(\bar{a} \frac{\partial \bar{y}}{\partial u_j}\right),$$

$$\left(\bar{q} \frac{\partial \bar{a}}{\partial u_k}\right) = \left| \bar{q} \bar{y} \frac{\partial \bar{q}}{\partial u_k} \right| - \left(\bar{a} \frac{\partial \bar{q}}{\partial u_k}\right),$$

$$\left| \bar{a} \frac{\partial \bar{a}}{\partial u_j} \frac{\partial \bar{a}}{\partial u_k} \right| = \left(\bar{a} \frac{\partial \bar{y}}{\partial u_j}\right)\left(\bar{a} \frac{\partial \bar{q}}{\partial u_k}\right) - \left(\bar{a} \frac{\partial \bar{y}}{\partial u_k}\right)\left(\bar{a} \frac{\partial \bar{y}}{\partial u_j}\right),$$

$$\sum_i \left| \bar{a} \frac{\partial \bar{a}}{\partial u_j} \frac{\partial \bar{a}}{\partial u_k} \right| \cdot \left| \frac{\partial a}{\partial u_i}\, a\, \omega \right| = \left| \left(a \frac{\partial y}{\partial u_1}\right)\left(\bar{a} \frac{\partial \bar{y}}{\partial u_2}\right)\left(\bar{a} \frac{\partial \bar{q}}{\partial u_3}\right) \right| (\omega q)$$

$$+ \left| \left(a \frac{\partial q}{\partial u_1}\right)\left(\bar{a} \frac{\partial \bar{q}}{\partial u_2}\right)\left(\bar{a} \frac{\partial \bar{y}}{\partial u_3}\right) \right| (\omega y).$$

But

$$\left(a \frac{\partial y}{\partial u_i}\right) = \left| yq \frac{\partial y}{\partial u_i} \right| = \left| \bar{y} \frac{\partial \bar{y}}{\partial u_k} \frac{\partial \bar{y}}{\partial u_i} \right| \cdot \left| y \frac{\partial y}{\partial u_i} \frac{\partial y}{\partial u_j} \right|$$

$$- \left| \bar{y} \frac{\partial \bar{y}}{\partial u_i} \frac{\partial \bar{y}}{\partial u_j} \right| \cdot \left| y \frac{\partial y}{\partial u_k} \frac{\partial y}{\partial u_i} \right| = - \left(\bar{a} \frac{\partial \bar{y}}{\partial u_i}\right).$$

Hence the first term on the right vanishes, and the equation of the point of contact is $(\omega y) = 0$.

Theorem 25] *If a three-parameter system of lines have a three-parameter system of tangents, then each point of the first system is the point of contact for its own tangent.**

It may happen, however, that there is only a two-parameter family of tangents. Then each tangent has a single infinite system of points of contact, forming a chain, for we have here the dual of 2].

Theorem 26] *If a three-parameter system of lines have a two-parameter system of tangents, then each tangent has a singly infinite number of points of contact, forming a chain.*

* The author found this theorem and the next, with substantially these proofs, in a set of notes he took on a course of lectures on Complex Geometry, delivered by Professor Study in Bonn in the summer semester of 1904.

CHAPTER VII

THREE-DIMENSIONAL COMPLEX SPACE.

§ 1. Representation of Complex Points.

THE problem of representing the complex points of a three-dimensional space by means of real elements is so much more difficult than the analogous problem for the plane that comparatively little attention has been paid to it. It is necessary to find a six-parameter system of real elements, and associate in a way that is not too artificial with the six-parameter assemblage of complex points. We shall here give only three of the methods that have been suggested. We begin with the Marie system, which carries over with less alteration than any other. The defining equations are, as before,

$$X = \frac{x+\bar{x}}{2} - i\frac{x-\bar{x}}{2}, \quad Y = \frac{y+\bar{y}}{2} - i\frac{y-\bar{y}}{2}, \quad Z = \frac{z+\bar{z}}{2} - i\frac{z-\bar{z}}{2}$$

$$X' = \frac{x+\bar{x}}{2} + i\frac{x-\bar{x}}{2}, \quad Y' = \frac{y+\bar{y}}{2} + i\frac{y-\bar{y}}{2}, \quad Z' = \frac{z+\bar{z}}{2} + i\frac{z-\bar{z}}{2} \quad (1)$$

The representation is perfect for the finite domain, and the relation between representing point and conjugate imaginary pair is invariant for real motions:

Theorem 1] *In the Marie system each complex point of the finite domain is represented by an ordered pair of real points, collinear with the given point and its conjugate having the same mid-point as the latter pair, and separated by a distance whose square is the negative of the square of distance of the conjugate imaginary points. The representing points are the closest pair in the elliptic involution whose double points are the given complex point and its conjugate.*

There is no interest in discussing the point-systems which represent various loci lying in real planes, as this has already been done in Chs. IV and VI. Thus, an imaginary line with a finite real point appears as a real directly equi-areal collineation of the plane. Let us turn to the representation of a complex line that is skew to its conjugate. We may take the common perpendicular to the two, which is surely real, as the z axis, while the real directions bisecting the angles of the complex lines may be taken as those of the x and y axes. The lines may then be written

$$y = Rix, \quad z = Si.$$

$$X' = \frac{1}{R}Y, \quad Y' = -RX, \quad Z' = -Z = -S.$$

Theorem 2] *In the Marie system a complex non-minimal line which is skew to its conjugate will appear as a pair of parallel planes, connected by a directly equi-areal collineation of period four.*

In case the complex line is minimal, we have the simpler case $R^2 = 1$.

Theorem 3] *In the Marie system a minimal line which is skew to its conjugate will be represented by two parallel planes connected by a screw motion about a common perpendicular, the angle of rotation being* $\frac{\pi}{2}$.

Let us consider a general non-minimal space curve. We may express it in the form

$$X = \frac{x(u+iv)+\bar{x}(u-iv)}{2} - i\frac{x(+iv)-\bar{x}(u-iv)}{2}.$$

$$Y = \frac{y(u+iv)+\bar{y}(u-iv)}{2} - i\frac{y(u+iv)-\bar{y}(u-iv)}{2}.$$

$$Z = \frac{z(u+iv)+\bar{z}(u-iv)}{2} - i\frac{z(u+iv)-\bar{z}(u-iv)}{2}.$$

To discover the nature of this surface let us put

$$u+iv = \zeta, \quad u-iv = \bar{\zeta}.$$

We then treat ζ and $\bar{\zeta}$ as independent variables. Our surface is of the general type

$$X = F_1(\zeta) + \phi_1(\bar{\zeta}).$$
$$Y = F_2(\zeta) + \phi_2(\bar{\zeta}).$$
$$Z = F_3(\zeta) + \phi_3(\bar{\zeta}).$$

These equations are characteristic of a translation surface, i.e. one that can be generated in two ways by translating a curve of fixed form so that it always meets a given curve. $(X'Y'Z')$ will trace a second such surface. We have the relations

$$\frac{\partial X}{\partial u} = -\frac{\partial X'}{\partial v}, \quad \frac{\partial X}{\partial v} = -\frac{\partial X'}{\partial u}, \quad \frac{\partial Z}{\partial u} = -\frac{\partial Z'}{\partial v},$$

$$E = G', \quad G = E', \quad F = -F'.$$

We see from these equations that the u and v parameter curves on the two surfaces are isometric, so that $u =$ const. on one is parallel to $v =$ const. on the other at the corresponding point. Conversely, it can be shown that these conditions are sufficient, so that we can state

Theorem 4] *Two real surfaces in one to one correspondence will represent an analytic space curve which is not minimal when, and only when:*

A) *They are translation surfaces with conjugate imaginary generators;*

B) *There is a first and second system of curves on the one so related to a first and second system on the other that at corresponding points the tangent to the first curve on either surface is parallel to the tangent to the second curve on the other;*

C) *The relation is directly equi-areal.**

When the given curve is minimal we have

$$x'^2 + y'^2 + z'^2 = \bar{x}'^2 + \bar{y}'^2 + \bar{z}'^2 = 0.$$

$$\frac{\partial^2 X}{\partial u^2} = -\frac{\partial^2 X}{\partial v^2}, \text{ \&c.} \quad E = G, \quad D'' = -H, \quad F = 0.$$

$$ED'' + GD - 2FD' = 0.$$

* See the dissertation of Graustein, *Eine reelle Abbildung analytischer komplexer Kurven*, Teubner, Leipzig, 1913, p. 66.

Theorem 5] *The surfaces which represent minimal curves are of the type described in* 4], *but they are a minimal surface, and the net of curves is an orthogonal one.**

It is certainly a blemish on this method of representation that use is made of conjugate imaginary systems of curves on the real representing surfaces, and, although this could doubtless be avoided, it seems likely that the resulting statement would be decidedly cumbersome.

We pass next to the Laguerre representation, which takes an entirely different aspect in three dimensions from what it presented in two. Every finite complex point in space is the vertex of a minimal cone going to the circle at infinity; the conjugate imaginary cone will meet this in a real circle, of which the imaginary points are called the *foci*. The line connecting them is the *axis*. There is a one to one correspondence if we distinguish, on the one hand, between the first and second focus, and, on the other, between the circle with one or the other sense of rotation.

Theorem 6] *The Laguerre system in space consists in representing each real point by itself, and each complex one by the real oriented circle of which it is the first focus. The representation is one to one for all finite points, and has no exception whatever if the infinite domain be defined as a single real point, the vertex of a minimal cone.*†

Let us recapitulate certain familiar theorems about circles in space. Each has two foci. The N. S. condition that two circles should be cospherical or coplanar is that their foci should be concyclic or collinear. When this is not the case, the single sphere or plane through their foci is their common orthogonal sphere or plane.

Two non-cospherical circles will usually be cut twice at right angles by two other circles. The axes of these cut the

* Ibid., p. 70.

† Laguerre, 'Sur l'emploi des imaginaires dans la géométrie de l'espace', *Nouvelles Annales de Mathématiques*, Série 2, vol. ii, 1872. Also Molenbroch (better Molenbroek), 'Sur la représentation géométrique des points imaginaires dans l'espace', ibid., Série 3, vol. x, 1891. The treatment here given is from the author's *Treatise on the Circle and the Sphere*, Oxford, 1916, pp. 535 ff.

axes of the given circles, as well as their polars with regard to the common orthogonal sphere. The circles cutting twice orthogonally are those whose foci are the intersections of the lines last found and the sphere.

It may happen, however, that the axes of the two circles, and their polars, with regard to the orthogonal sphere, are four generators of a regulus, so that they are cut by an infinite number of lines generating the conjugate regulus. Each regulus will contain two generators of the sphere, i.e. two skew minimal lines. The given circles are then said to be *paratactic*.* Two real circles are paratactic when, and only when, their foci lie in pairs on two aggregate imaginary skew minimal lines. They are cut twice at right angles by an infinite number of circles generating a cyclide, the given circles belonging to the conjugate generation. If the given paratactic circles be real the same is true of the circles cutting them twice orthogonally. Lastly, if two circles be so related that their first and second foci lie on two intersectional minimal lines they are tangent, and conversely. A circle is paratactic to an oriented line if its first and second foci lie on the first and second minimal planes through that line.

The best way to discuss these relations analytically is to take a system of pentaspherical coordinates. We write

$$\rho x_0 = j\,(x^2+y^2+z^2+t^2), \quad \rho x_1 = (x^2+y^2+z^2-t^2),$$

$$\rho x_2 = 2xt, \quad \rho x_3 = 2yt, \quad \rho x_4 = 2zt.$$

$$j^2 = -1. \tag{2}$$

$$(xx) \equiv \sum_{i=0}^{i=4} x_i^2 = 0.$$

Every sphere or plane will have a linear equation in (x), and every such equation represents a sphere or plane. A point (x') lies on the circle whose foci are (x) and $(\bar{x})$ if

$$(xx') = (\bar{x}x') = 0.$$

* The first mention of the paratactic relation between two circles seems to be in an article by the author, 'A study of the Circle-Cross', *Transactions American Math. Soc.*, vol. xiv, 1913.

A chain of points on a circle, which may be defined for present purposes as the inverse of a linear chain, may be expressed in the form

$$\sigma x'_1 = T^2\alpha_1 + TB_1 + \gamma i.$$

Theorem 7] *A chain of points on a line or circle will appear in the Laguerre representation as a series of circles generating a surface of the eighth order, which may, in special cases, degenerate to a Dupin cyclide, cylinder of revolution, sphere or plane.*

Theorem 8] *A real chain will be represented by the totality of oriented circles which have that line as axis.*

Theorem 9] *A real or self-conjugate imaginary circle will be represented by the totality of circles orthogonal to a non-tangent system of coaxal spheres.*

If a complex circle be given which is not its own conjugate, then if the first and second foci of a circle lie on this circle and its conjugate this real circle is paratactic to that real circle whose foci are the first foci of the two given circles, as well as to that one whose foci are their second foci.

Theorem 10] *The points of a complex circle which is not its own conjugate, and which does not lie in a minimal plane, will be represented by the totality of real oriented circles paratactic to two circles which are not paratactic or tangent to one another.*

Theorem 11] *The points of an imaginary circle in a minimal plane will be represented by the real oriented circles paratactic to a real line, and a real oriented circle not coplanar or paratactic therewith.*

Theorem 12] *The points of a non-minimal finite line which is skew to its conjugate will be represented by the real oriented circles paratactic to two real skew lines.*

Theorem 13] *The points of a minimal line skew to its conjugate will be represented by the system of oriented circles paratactic to two real paratactic circles.*

What sort of a circle congruence will represent a general curve in space?

A non-minimal curve has an osculating circle at every point which is not singular. Conversely, if a congruence of points possess the property that if a point in general position be connected with two near-by points by a circle, this circle will approach a definite limiting position as the near-by points approach the given point as a limit, there is surely a tangent at every point and the orthogonal projection of the congruence on any plane is a curve (if the congruence itself be not a set of lines). Hence the congruence itself is a curve. If thus P, $P+\Delta P$, $P+\Delta' P$ be three adjacent points on the curve, $\overline{P}$, $\overline{P}+\Delta\overline{P}$, $\overline{P}+\Delta'\overline{P}$ their conjugates, and if F_1F_2 are the foci of the circle through the first three points, while $\bar{F}_1\bar{F}_2$ are those of the circle through the other three, the real circles whose foci are $F_1\bar{F}_1$, and $F_2\bar{F}_2$ respectively will approach definite limiting positions, no matter how $P+\Delta P$ and $P+\Delta' P$ approach P as a limit in the congruence.

Theorem 14] *A congruence of oriented circles will represent a non-minimal curve when, and only when, the circles paratactic to three adjacent circles approach definite limiting positions in each case, as the three approach definite limiting positions in the congruence.*

If a curve be minimal the osculating circle lies in a minimal plane.

Theorem 15] *A congruence of oriented circles will represent a minimal curve, not a line, when, and only when, the congruence is of the type described in* 14], *and the circles cutting twice at right angles two adjacent circles of the congruence do not tend to definite limits as the given circles approach coincidence.*

Theorem 16] *The points of a real or self-conjugate imaginary sphere will be represented by the totality of circles with which a given point is coplanar, and has a constant power.*

The system of circles representing a complex sphere is harder to describe.* We leave it aside, and pass on to our third method of representation.

* Cf. the author's *Treatise on the Circle*, cit. p. 544.

Let us take a point with the homogeneous rectangular Cartesian coordinates (x, y, z, t), subject to the restriction

$$x^2+y^2+z^2+t^2 \neq 0. \tag{3}$$

If this inequality were replaced by an equality, we should say that the point lay on the *fundamental sphere*, whose centre is the origin. This point will determine a unique linear transformation of the complex homogeneous variable (ζ)

$$\begin{aligned} \zeta_1' &= (it-z)\,\zeta_1-(x+iy)\,\zeta_2 \\ \zeta_2' &= (-x+iy)\,\zeta_1+(it+z)\,\zeta_2 \end{aligned} \;. \tag{4}$$

Conversely, every linear homogeneous transformation

$$\begin{aligned} \zeta_1' &= \alpha\zeta_1+\beta\zeta_2 \\ \zeta_2' &= \gamma\zeta_1+\delta\zeta_2 \end{aligned}, \qquad |\,\alpha\delta-\beta\gamma\,| \neq 0,$$

will give a complex point, not on the fundamental sphere, namely,

$$x=\beta+\gamma, \quad y=-i(\beta-\gamma), \quad z=\alpha-\delta, \quad t=i\,(\alpha+\delta). \tag{5}$$

We shall call this the representation of Stephanos.*

Theorem 17] *In the method of Stephanos, each point of a complex three-dimensional projective space which does not lie on the fundamental complex sphere is represented by a real directly conformal circular transformation of the Gauss plane, and every such transformation will represent a complex point not on the sphere.*

If we solve our equations (4), we find

$$\rho\zeta_1=(it+z)\,\zeta_1'+(x+iy)\,\zeta_1', \quad \rho\zeta_2=(x-iy)\,\zeta_1'+(it-z)\,\zeta_2'.$$

Theorem 18] *Mutually inverse transformations represent points which are the reflexions of one another in the origin. Involutory transformations represent points in the infinite plane. The origin represents the identical transformation.*

* This writer was not especially interested in the representation of complex points, but rather in the establishment of a correspondence between the point of a three-dimensional projective space and the linear transformations of the binary domain. See his article, 'Mémoire sur la représentation des homographies binaires par les points de l'espace', *Math. Annalen*, vol. vi, 1883.

Let us look for the product of the transformations corresponding to the point (x, y, z, t) and (x', y', z', t'). We shall find

$$\begin{aligned} x'' &= xt' - yz' + zy' + tx' \\ y'' &= xz' + yt' - zx' + ty', \\ z'' &= -xy' + yx' + zt' + tz', \\ t'' &= -xx' - yy' - zz' + tt'. \end{aligned}$$

The necessary and sufficient condition that the product of one of these and the inverse of the other should be involutory is

$$xx' + yy' + zz' + tt' = 0. \qquad (6)$$

We encountered this relation in II (14), and defined the transformations in this case as being 'orthogonal'.

Theorem 19] *The transformations orthogonal to a given transformation will be represented by the points of a plane not tangent to the fundamental sphere, and every such plane will be so represented.*

If equation (6) hold with the restriction

$$x'^2 + y'^2 + z'^2 + t'^2 = 0,$$

we may find $\zeta_1 : \zeta_2$ and $\zeta_1' : \zeta_2'$ so that

$$\zeta_1\zeta_1' - \zeta_2\zeta_2' = x', \quad \zeta_1\zeta_1' + \zeta_2\zeta_2' = iy', \quad \zeta_1\zeta_2' + \zeta_2\zeta_1' = -z',$$
$$\zeta_1\zeta_2' - \zeta_2\zeta_1' = -it',$$

which shows that the transformation carries (ζ) into (ζ').

Theorem 20] *The points of a plane tangent to the fundamental sphere will be represented by the totality of transformations which carry a given point into another given point. These two points will be identical when, and only when, the plane is a minimal one. Conversely, every set of transformations so transforming a given point, will be represented by points of a plane.*

Theorem 21] *The points of a line not tangent to the fundamental sphere will be represented by the transformations which carry two specified points into two specified points. A non-minimal line through the origin will be represented by the totality of transformations with two specified fixed points. When the line through the origin is minimal, the transformations will be parabolic, with specified invariant circles.*

A tangent to the fundamental sphere at a finite point may be determined by the intersection of a tangent plane with another plane whose pole is in the tangent plane.

Theorem 22] *The points of a tangent to the fundamental sphere at an infinite point will be represented by the transformations which carry a specified point into a specified point, and are orthogonal to a given transformation which does this.*

§ 2. Linear and Bilinear Systems.

We shall begin the present section with an enumeration of systems of points linearly dependent, in terms of real parameters, on a certain number of given points

$$x_i = X_1 y_i + X_2 z_i. \qquad \text{(A)}$$

Here we have clearly a chain, using the general definition of this term. In the same way we have the dual figure

$$u_i = X_1 v_i + X_2 w_i, \qquad \text{(A')}$$

which we define as a *chain of planes*, and whose nature needs no further discussion. We pass on to the next case

$$x_i = X_1 y_i + X_2 z_i + X_3 p_i. \qquad \text{(B)}$$

If (y), (z), and (p) be collinear, this represents their whole line, or merely a chain on it, otherwise a chain congruence in their plane, as defined in Ch. V.

$$x_i = X_1 y_i + X_2 z_i + X_3 p_i + X_4 q_i. \qquad \text{(C)}$$

If $$| xyzp | = 0$$

we have one of our previous systems, or else a chain of lines in a plane. In the opposite case we shall have what we define as a *chain space*.

Theorem 23] *Every chain space is projectively equivalent to the real domain. It meets every line in a single point or a chain, or not at all, and every plane in a line or a chain congruence.*

$$x_i = X_1 y_i + X_2 z_i + X_3 p_i + X_4 q_i + X_5 r_i. \qquad \text{(D)}$$

If the given points be coplanar we have the totality of points of their plane, or else one of our previous systems. When they are not coplanar, we may find a linear transformation to carry the system into

$$x_1 = X_1 + (R_1 + iR_1')X_5,$$
$$x_2 = X_2 + (R_2 + iR_2')X_5,$$
$$x_3 = X_3 + (R_3 + iR_3')X_5,$$
$$x_4 = X_4 + (R_4 + iR_4')X_5.$$

An arbitrary point on the line from (x) of the system to the point $(iR_i')x$ will be

$$x_1' = x_i + (P+Q)(iR_i')$$
$$= (X_i - PR_i - QR_i') + (R_i + iR_i')(X_5 + P).$$

This is also a point of the system. The system consists in the lines connecting the points of a chain congruence with a point outside its plane

$$x_i = X_1 y_i + Y_2 z_i + X_3 p_i + X_4 y_i + X_5 r_i + X_6 s_i. \qquad \text{(E)}$$

As before, we may reduce to the canonical form

$$x_i = X_i + X_5 \alpha_i + X_6 \beta_i.$$

If (U) be a real plane which satisfies the equations

$$(U\alpha) = (U\bar{\alpha}), \quad (U\beta) = (U\bar{\beta}),$$

then the conditions

$$(Ux) = 0, \quad (U\bar{x}) = 0$$

are equivalent, and all points of this plane belong to the system. We have a chain of planes such as was given in (A′).

The study of collineations and anti-collineations in three dimensions follows exactly the same lines in two dimensions and need not detain us. The study of antipolarities leads to the hyperquadrics, characterized by the equation

$$\sum a_{ij} x_i \bar{x}_j \equiv (ax)(\bar{a}\bar{x}).$$

When the discriminant is not zero, we can reduce to

$$\sum \pm A_i x_i \bar{x}_i = 0.$$

The number of positive and negative terms here is significant. If all the terms were positive, or all were negative, the equation would involve an absurdity. With three terms of one sign and one of the other, we may reduce to

$$B_1x_1\bar{x}_1+B_2x_2\bar{x}_2+B_3x_3\bar{x}_3-B_4x_4\bar{x}_4=0 \quad B_i>0.$$

This hyperquadric has ∞^3 points, but contains no line, for it has nothing in common with the plane

$$x_4=0.$$

The polar plane of a point on the hypersurface meets it there and nowhere else. On the other hand, with two terms of each sign we may write

$$B_1x_1\bar{x}_1+B_2x_2\bar{x}_2-B_3x_3\bar{x}_3-B_4x_4\bar{x}_4=0. \quad B_i>0.$$

The polar plane of $\left(0, \frac{1}{\sqrt{B_2}}, \frac{1}{\sqrt{B_3}}, 0\right)$, which we may assume to stand for any point on the hypersurface, meets the latter in a line chain. On the other hand, no plane is completely imbedded, as the hypersurface has no point on the line

$$x_3=x_4=0.$$

Let us see what happens to our hyperquadric when the discriminant vanishes. If the rank of the matrix be 3, there will be a single singular point whose polar plane is indeterminate, and the variety consists either in that point alone, or in all points on all lines from there to the points of a hyperquadric whose plane does not contain that point. When the rank of the matrix is 2 there is a line of singular points, and the variety consists either in that line alone, or in a chain of planes through it. When the rank is 1 we have a single plane.

The study of linear systems of hyperquadrics and their intersections may be carried on much as in two dimensions, but is naturally more complicated.* In the same way the study of the Hermitian metrics follows copy to a certain extent. At the same time there are places where the analogy breaks down, and a deeper investigation into this subject

* Segre, *Nuovo campo*, cit., vol. xxvi.

would seem to be desirable. For instance, take the elliptic case, where the fundamental form can be written

$$\sum a_{ij} x_i \bar{x}_j \equiv (x\bar{x}).$$

The distance and angle formulae of Ch. VI are applicable here as soon as we introduce an additional coordinate, and understand angles to refer to planes. There will be skew, parallel, or paratactic lines in space, like the well-known Clifford parallels or paratactics, which we reach as follows. We determine a line by its Plucker coordinates

$$p_{ij} = y_i z_j - y_j z_i.$$

The polar of (p) with regard to the fundamental form will be (p') where

$$p_{ij} = \rho \bar{p}_{kl}.$$

Two lines will intersect orthogonally if each intersect both the other and its polar in the fundamental form; (r) cuts (p) and (q) orthogonally if

$$\sum p_{ij} r_{kl} = \sum \bar{p}_{kl} r_{kl} = \sum q_{ij} r_{kl} = \sum \bar{q}_{kl} r_{kl} \equiv \sum r_{ij} r_{kl} = 0.$$

The summations in each case cover the six pairs of numbers (14, 24, 34, 23, 31, 12). These five equations have usually two solutions. There will, however, be an infinite number of solutions when the first four are linearly dependent. Let (q) go from (0, 0, 0, 1) to $(x_1, x_2, x_3, 0)$

$$q_{ij} = x_i, \quad q_{23} = q_{31} = q_{12} = 0.$$

Since the first four equations above are to be linearly dependent, we must have

$$x_i = \lambda p_{jk} + \mu \bar{p}_{ij}.$$

$$\bar{x}_i = \lambda p_{ij} + \mu \bar{p}_{jk}.$$

$$= \bar{\lambda} \bar{p}_{jk} + \bar{\mu} p_{ij}.$$

$$\lambda \bar{\lambda} = \mu \bar{\mu}.$$

It appears thus that (x) traces a chain on the line from $(p_{23}, p_{31}, p_{12}, 0)$ to $(\bar{p}_{14}, \bar{p}_{24}, \bar{p}_{34}, 0)$.

Theorem 24] *In the Hermitian metric of elliptic type there will pass through each point outside a given line a chain of lines having therewith an infinite number of common perpen-*

diculars. All of these lines are skew to the given line and the lengths cut off on all the common perpendiculars are the same.

This last fact is evident when we realize that if two skew lines have an infinite number of common perpendiculars, any two of these may be interchanged by a reflection in a properly chosen common perpendicular.

There is another sort of elliptic Hermitian metric in three dimensions, which we shall call pentaspherical and which deserves passing mention, as it has a certain connexion with the Laguerre representation discussed in the beginning of the present chapter. Let a point in space have the pentaspherical coordinates

$$\rho\xi_1 = 2\,x_1x_4$$
$$\rho\xi_2 = 2\,x_2x_4$$
$$\rho\xi_3 = 2\,x_3x_4$$
$$\rho\xi_4 = (xx) - 2\,x_4^{\,2}$$
$$\rho\xi_5 = i\,(xx)$$
$$(\xi\xi) \equiv 0.$$

We shall define the distance of two points (ξ) and (η) by the usual formula.

$$\cos d = \frac{\sqrt{(\xi\bar{\eta})}\,\sqrt{(\eta\bar{\xi})}}{\sqrt{(\xi\bar{\xi})}\,\sqrt{(\eta\bar{\eta})}}.$$

If we have given two pairs of conjugate imaginary points (ξ) $(\bar{\xi})$, (η) $(\bar{\eta})$, they determine two distances, d_1 and d_2, where

$$\cos d_2 = \frac{\sqrt{(\xi\eta)}\,\sqrt{(\bar{\xi}\bar{\eta})}}{\sqrt{(\xi\bar{\xi})}\sqrt{(\eta\bar{\eta})}}, \quad \cos d_1 = \frac{\sqrt{(\xi\bar{\eta})}\,\sqrt{(\eta\bar{\xi})}}{\sqrt{(\xi\bar{\xi})}\,\sqrt{(\eta\bar{\eta})}}.$$

Now (ξ) and $(\bar{\xi})$ are the foci of a real circle, and (η) and $(\bar{\eta})$ those of a second one. They have, as we saw earlier, two common perpendiculars, i.e. two circles cutting each twice orthogonally, exceptions occurring when they are cospherical, coplanar, or paratactic. Through each common perpendicular will pass a pair of spheres, one through each of the given circles, and the angle of a pair of this sort may be defined as one of the two angles of the given circles. If $\theta_1\theta_2$ be the angles of the two circles whose foci are given above, it may be shown that their values are obtained from the equations *

$$\cos^2\theta_1+\cos^2\theta_2=\frac{2\left[(\xi\eta)(\bar{\xi}\bar{\eta})+(\xi\bar{\eta})(\eta\bar{\xi})\right]}{(\eta\bar{\xi})(\eta\bar{\eta})},$$

$$\cos\theta_1\cos\theta_2=\frac{(\xi\eta)(\bar{\xi}\bar{\eta})-(\xi\bar{\eta})(\eta\bar{\xi})}{(\xi\bar{\xi})(\eta\bar{\eta})},$$

$$\cos\theta_1-\cos\theta_2=2\cos d_1,$$

$$\cos\theta_1+\cos\theta_2=2\cos d_2.$$

Theorem 25] *If the points of space be represented by oriented circles according to the Laguerre scheme, the cosine of the distance of two points in the elliptic pentaspherical Hermitian measurement is one-half the difference of the cosines of the angles of the representing circles, while the cosine of the distance of each from the conjugate of the other is one-half the sum of the cosines of these angles.*

§ 3. Geometry of the Minimal Plane.

The formulae and point of view of the Hermitian metrics form a natural introduction to our next topic, which also deals with distances and angles. Our starting point is the fact, often overlooked, that certain geometrical principles which are fundamental in the real domain, and form the basis of real geometry, suffer exception when the domain includes complex elements as well. What we do in actual fact is this. We start with a set of axioms for real geometry and develop the science as far as need be. Then we introduce complex coordinate values, complex parameters in our various functions and invariants, and extend our language by saying that the expressions which in the real case defined real distances, angles, &c., shall be said to define complex distances, angles, &c. Since the functions used are analytic, identities which hold in the real domain hold in the complex domain also, and we sail merrily ahead under the delusion that nothing has been changed by this extension of our domain. This is not the case. Some of the fundamental properties which we used

* See the author's *Treatise on the Circle and Sphere*, cit. p. 452.

to build up our real geometry are no longer necessarily true in the complex domain. For instance, the principle which some writers would like to set up as an axiom, that the sum of two sides of a triangle is greater than the third side, is certainly invalid when one of the three lies on a minimal line. In three-dimensional geometry we have an exceptional system of metrics, namely, that which obtains in a minimal plane, and our next task shall be to look into it.*

Distances in a minimal plane of Euclidean space should not be given a different definition from the usual one, for two points in general position, that is to say, two points whose line does not meet the circle at infinity, always lie in two minimal planes. On the other hand, if we remember Laguerre's theorem, whereby the angle of two intersecting lines is a constant multiple of the logarithm of the cross ratio which they form with the minimal lines of their pencil, then, in a minimal plane, where these minimal lines coalesce the angle of the given lines will appear as zero. There is no absolute invariant for two lines under the congruent transformations of the minimal plane. They have, however, a relative invariant, i.e. an expression which is multiplied by a constant factor. This invariant we shall now develop. Consider the plane

$$y = lx.$$

Let two lines have the direction coefficients $x:lx:z$ and $x':lx':z'$.

$$\sin\theta = \frac{\sqrt{l+l^2}\,(xz'-zx')}{\sqrt{(l+l^2)\,x^2+z^2}\;\sqrt{(l+l'^2)\,x'^2+z'^2}}$$

Change the angular unit, writing

$$\sqrt{l+l^2}\,\delta = \theta,$$

$$\frac{\sin\sqrt{l+l^2}\,\delta}{\sqrt{l+l^2}} = \frac{xz'-zx'}{\sqrt{(l+l^2)x^2+z^2}\;\sqrt{(l+l^2)x'^2+z'^2}}.$$

* See Beck, 'Zur Geometrie in der Minimalebene', *Sitzungsberichte d. Berliner Math. Gesellschaft*, vol. xii, 1912, and Moore, 'Geometry where the Element of Arc is an Exact Differential', *Proceedings American Academy of Arts and Sciences*, vol. l, 1914.

Now the limit of this, as $l+l^2$ approaches zero, is

$$\delta = \frac{x}{z} - \frac{x'}{z'},$$

and the limiting position of our plane is a minimal plane through the z axis. This expression shall be called the *divergence* of the two lines.* It comes from the angle by just the same sort of limiting process that can be used to derive the Euclidean metrical formulae from the non-Euclidean ones, as the absolute quadric approaches the degenerate form of a plane counted twice and a conic therein.

The best way to handle the minimal plane analytically is by means of a parametric representation. Let the plane be

$$x+iy = 0;$$

we may write

$$x = v, \quad y = iv, \quad z = u. \tag{7}$$

The distance of two points will be

$$d = (u_2 - u_1). \tag{8}$$

For the differential of arc, we have

$$ds = du. \tag{9}$$

If a line have the equation

$$ax+by+cz+e = 0,$$

the direction ratios are

$$X : Y : Z = c : ci : -(a+bi).$$

Writing the equation of the line

$$\phi u + \chi v + \psi = 0,$$

$$\phi = 0, \quad \chi = a+bi, \quad \psi = l.$$

The divergence formula for the lines (ϕ_1, χ_1, ψ_1) (ϕ_2, χ_2, ψ_2) is

$$\delta = \frac{\phi_2}{\chi_2} - \frac{\phi_1}{\chi_1}. \tag{10}$$

Theorem 26] *In any finite triangle in the minimal plane with no minimal side, the sum of the lengths of the sides, when*

* Beck, loc. cit., uses 'Sperrung', which seems to be a typographical term for 'spacing'; Moore uses 'angle', which seems objectionable.

properly oriented, and the sum of the divergences of the angle is equal to zero.

Theorem 27] *The locus of points at a given distance from a finite point is a minimal line, the envelope of lines having a given divergence from a given non-minimal line is an infinite point.*

Let the vertices of a triangle be

$$(u_1, v_1)\ (u_2, v_2)\ (u_3, v_3).$$

The divergence of two sides will be

$$\frac{v_1 - v_3}{u_1 - u_3} - \frac{v_1 - v_2}{u_1 - u_2}.$$

Half the product of this and the length of the third side will be

$$\tfrac{1}{2} \mid u_1\ v_2\ 1 \mid.$$

On the other hand, the product of two divergences and the included length will be

$$\frac{\mid u_1 v_2\ 1 \mid^2}{\prod (u_i - v_i)}.$$

Theorem 28] *In a finite triangle of the minimal plane, where no side is minimal, the product of the lengths of two sides and the divergence included, and the product of two divergences and the included length, is independent of the choice of sides.*

A collineation of the minimal plane shall be called *conformal* if it leave the infinite line, and its point on the circle at infinity invariant. Such a transformation may always be written in the form

$$u' = au + b,$$

$$v' = cu + dv + e.$$

It will keep distances invariant if

$$a = 1.$$

Theorem 29] *The general congruent collineation of the minimal plane will multiply divergences by a constant factor which is equal to unity only in the special case where the line at infinity is the only fixed line.*

Theorem 30] *In a congruent collineation of a minimal plane, either there is no finite fixed point, or all points of a finite line are fixed.*

The curvature of every curve in a minimal plane is zero, since all angles are zero. There is, however, an analogous function involving divergence which is worth notice. Let us seek the limit of the divergence of adjacent tangents divided by the connecting arc, a limit which we shall call the *deviation* of the curve at that point.* If the curve be expressed in the form

$$u = u(t), \quad v = v(t),$$

the equation of the tangent is

$$(U-u)v' - (V-v)u' = 0.$$

The divergence of adjacent tangents will be

$$\frac{-(u'v''-v'u'')}{u'^2}\,dt,$$

$$ds = v'dt\,;$$

hence the deviation is equal to

$$\frac{v'u''-u'v''}{u'^3}.$$

When the curve is written

$$v = v(u),$$

the deviation takes the simple form

$$-v''.$$

This will be constant only for the simple curves

$$v = au^2 + bu + c.$$

Theorem 31] *The only curves in the minimal plane which have a constant deviation are straight lines and parabolic circles.*

If the vertices of a triangle be (u_1, v_1) (u_2, v_2) (u_3, v_3) the equation of the circumscribed parabolic circle is

$$|\, u^2 u_1 v_2 \; 1 \,| = 0.$$

* Beck, loc. cit., uses 'Abweichung', Moore, 'Curvature'.

Its deviation is

$$\frac{-2\,|\,u_1 v_2 1\,|}{(u_2-u_3)\,(u_3-u_1)\,(u_1-u_2)}.$$

Theorem 32] *The ratio of a side of a triangle to the opposite divergence is equal to twice the reciprocal of the deviation of the circumscribing circle.*

§ 4. Differential Geometry of Complex Space.

The totality of points of a complex three-dimensional space will depend upon six real parameters. We have, therefore, one-, two-, three-, four-, and five-parameter systems of points to consider. The differential geometry of the one-parameter system or thread presents very little interest. We shall begin with the congruence

$$x = x(u_1, u_2), \quad y = y(u_1, u_2), \quad z = z(u_1, u_2).$$

The following theorem is perfectly obvious:

Theorem 33] *The necessary and sufficient condition that a congruence should be a curve is that its projection on two non-parallel planes should be curves.*

The analytic condition for a curve will thus be

$$\frac{\partial(y, z)}{\partial(u_1, u_2)} = \frac{\partial(z, x)}{\partial(u_1, u_2)} = \frac{\partial(x, y)}{\partial(u_1, u_2)} \equiv 0. \tag{11}$$

If these equations be not fulfilled, we have, at an arbitrary point,

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2; \quad dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2;$$

$$dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2.$$

Theorem 34] *If a congruence be not a curve, the tangents at an arbitrary point generate a line-chain.*

We pass on to the three-parameter system

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3).$$

Suppose first that

$$\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \equiv 0. \tag{12}$$

If this equation hold identically for all real sets of parameter values u_1, u_2, u_3, it is an absolute identity for all values of this parameter, and

$$f(x, y, z) = 0,$$

so that our system lies on some surface. Conversely, when an equation like the last equation holds,

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u_1} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u_1} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u_1} = 0,$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u_2} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u_2} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u_2} = 0,$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u_3} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u_3} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u_3} = 0,$$

and (12) is satisfied. The rank of the matrix of the determinant could not sink to two, for then we should have a curve, and no three-parameter system at all. Assuming that (12) is satisfied, we see that dx, dy, and dz are linearly dependent, and conversely.

Theorem 35] *The necessary and sufficient condition that a three-parameter system of points should lie upon a surface is that the tangents at an arbitrary point should generate a pencil of lines.*

Theorem 36] *If a three-parameter system of points do not lie on a surface, the tangents at an arbitrary non-singular point will meet an arbitrary plane in a chain congruence.*

There is another way of expressing the condition that a three-parameter system should lie upon a surface, which leads to a curious result. Reverting to our equations (1), we have

$$\frac{\partial(X-iX',\ Y-iY',\ Z-iZ')}{\partial(u_1, u_2, u_3)} = 0.$$

Let us choose X, Y, and Z as our real parameters, so that

$$u_1 = X,\quad u_2 = Y,\quad u_3 = Z,$$

$$\begin{vmatrix} 1-i\dfrac{\partial X'}{\partial X} & -i\dfrac{\partial Y'}{\partial X} & -i\dfrac{\partial Z'}{\partial X} \\ -i\dfrac{\partial X'}{\partial Y} & 1-i\dfrac{\partial Y'}{\partial Y} & -i\dfrac{\partial Z'}{\partial Y} \\ -i\dfrac{\partial X'}{\partial Z} & -i\dfrac{\partial Y'}{\partial Z} & 1-i\dfrac{\partial Z'}{\partial Z} \end{vmatrix} = 0.$$

Dividing into real and imaginary parts, we find

$$\frac{\partial(X', Y', Z')}{\partial(X, Y, Z)} = \frac{\partial X'}{\partial X} + \frac{\partial Y'}{\partial Y} + \frac{\partial Z'}{\partial Z},$$

$$\frac{\partial(Y', Z')}{\partial(Y, Z)} + \frac{\partial(Z', X')}{\partial(Z, X)} + \frac{\partial(X', Y')}{\partial(X, Y)} = 0.$$

On the other hand, let us look for real directions at (X', Y', Z') which are parallel to the corresponding directions at (X, Y, Z)

$$dX' = \rho dX, \quad dY' = \rho dY, \quad dZ' = \rho dZ,$$

$$\begin{vmatrix} \dfrac{\partial X'}{\partial X}-\rho & \dfrac{\partial Y'}{\partial X} & \dfrac{\partial Z'}{\partial X} \\ \dfrac{\partial X'}{\partial Y} & \dfrac{\partial Y'}{\partial Y}-\rho & \dfrac{\partial Z'}{\partial Y} \\ \dfrac{\partial X'}{\partial Z} & \dfrac{\partial Y'}{\partial Z} & \dfrac{\partial Z'}{\partial Z}-\rho \end{vmatrix} = 0.$$

$$\rho^3 - \rho^2\left[\frac{\partial X'}{\partial X} + \frac{\partial Y'}{\partial Y} + \frac{\partial Z'}{\partial Z}\right]$$

$$+\rho\left[\frac{\partial(Y', Z')}{\partial(Y, Z)} + \frac{\partial(Z', X')}{\partial(Z, X)} + \frac{\partial(X', Y')}{\partial(X, Y)}\right] - \frac{\partial(X', Y', Z')}{\partial(X, Y, Z)} = 0. \tag{13}$$

In the present instance this becomes

$$\left(\rho - \frac{\partial(X', Y', Z')}{\partial(X, Y, Z)}\right)(\rho^2 + 1) = 0.$$

Our three-parameter system will lie upon a surface, when, and only when, (13) has the roots $\dfrac{\partial(X', Y', Z')}{\partial(X, Y, Z)}$, i, $-i$. What will be the geometric meaning of these facts? The real root will give us a single real fixed direction, and along this the ratio of stretching is $\frac{\partial(X', Y', Z')}{\partial(X, Y, Z)}$ which is the ratio of corresponding infinitesimal volumes. The complex roots will give us two conjugate imaginary fixed directions,

$$dX' = \sigma dX, \quad dY' = \sigma dY, \quad dZ' = \sigma dZ,$$
$$\bar{d}X' = \bar{\sigma}\bar{d}X, \quad \bar{d}Y' = \bar{\sigma}\bar{d}Y, \quad \bar{d}Z' = \bar{\sigma}\bar{d}Z.$$

There is a fixed elliptic involution of directions given by

$$\lambda dX + \bar{\lambda}\bar{d}X, \quad \lambda dY + \bar{\lambda}\bar{d}Y, \quad \lambda dZ + \bar{\lambda}\bar{d}Z,$$
$$\lambda dX - \bar{\lambda}\bar{d}X, \quad \lambda dY - \bar{\lambda}\bar{d}Y, \quad \lambda dZ - \bar{\lambda}\bar{d}Z.$$

The square of the ratio of stretching for corresponding directions is

$$\frac{\alpha\sigma^2\lambda^2 + 2\beta\sigma\bar{\sigma}\lambda\bar{\lambda} + \bar{\alpha}\bar{\sigma}^2\bar{\lambda}^2}{\alpha\lambda^2 + 2\beta\lambda\bar{\lambda} + \bar{\alpha}\bar{\lambda}^2},$$

$$\alpha = dX^2 + dY^2 + dZ^2, \quad \bar{\alpha} = \bar{d}X^2 + \bar{d}Y^2 + \bar{d}Z^2,$$
$$\beta = dX\bar{d}X + dX\bar{d}Y + dZ\bar{d}Z.$$

This is a maximum or mininum when, and only when,

$$\alpha\beta^2\lambda^2\sigma + \alpha\bar{\alpha}(\sigma + \bar{\sigma})\lambda\bar{\lambda} + \bar{\sigma}\beta\bar{\lambda}^2\bar{\sigma} = 0.$$

The roots will give a pair in the involution when

$$\sigma + \bar{\sigma} = 0,$$

and it is only under these circumstances that we shall have an orthogonal pair which belongs to the involution and whose members correspond to one another. The roots of the equation for maximum and minimum stretching are now

$$\frac{\lambda^2}{\bar{\lambda}^2} = \frac{\bar{\alpha}}{\alpha}.$$

Putting $\qquad \alpha = Re^{i\phi}, \quad \bar{\alpha} = Re^{-i\phi},$

$$\bar{\lambda} = \pm e^{i\phi}\lambda.$$

The squares of the two stretching ratios are now

$$\frac{\sigma^2(R-B)}{(R+B)}, \quad \frac{\sigma^2(R+B)}{(R-B)}.$$

The product of these is σ^4.

Our reasoning is reversible, so that we have

Theorem 37] *Necessary and sufficient conditions that a three-parameter system of points should lie on a surface are:*

a) that in the corresponding real transformation of space in the Marie system of representation there should be in each point in general position just one real invariant direction, and

b) that the stretching ratio for this direction should be the ratio of corresponding infinitesimal volumes at the corresponding points, and

c) that there should be an invariant real elliptic involution pencil of directions, and

d) that in this pencil there should be an orthogonal pair that correspond to one another, and

e) that the product of the stretching ratios for this pair, which are the maximum and minimum stretching ratios for the pencil, should be equal to unity.

We turn next to a four-parameter system, where x, y, and z are functions of u_1, u_2, u_3, and u_4. If this system constitute a surface, then from (12)

$$\frac{\partial(x, y, z)}{\partial(u_j, u_k, u_l)} \equiv 0, \quad i = 1, 2, 3, 4.$$

Conversely, suppose that these equations are identically satisfied. We may put

$$x = u_1 + iu_2, \quad y = u_3 + iu_4, \quad \frac{\partial z}{\partial u_1} + i\frac{\partial z}{\partial u_2} = \frac{\partial z}{\partial u_3} + i\frac{\partial z}{\partial u_4} = 0.$$

$$\text{Let} \quad z = Z_1 + iZ_2,$$

$$\frac{\partial Z_1}{\partial u_1} = \frac{\partial Z_2}{\partial u_2}, \quad \frac{\partial Z_1}{\partial u_2} = -\frac{\partial Z_2}{\partial u_1},$$

$$\frac{\partial Z_1}{\partial u_3} = \frac{\partial Z_2}{\partial u_2}, \quad \frac{\partial Z_1}{\partial u_2} = -\frac{\partial Z_2}{\partial u_3}.$$

This shows that z is an analytic function of x, when y is constant, and of y when x is constant, hence it is analytic in x and y together, and we have a surface.*

* Cf. Osgood, 'Zweite Note über analytische Funktionen', *Math. Annalen*, vol. liii, 1900, pp. 460 ff.

Theorem 38] *The necessary and sufficient condition that a four-parameter system of points should constitute a surface is that every three-parameter sub-set should lie on the same surface, or that all the tangents at an arbitrary point should be coplanar.*

The tangents to a surface will generate a plane, and two of them, usually distinct, will osculate the surface. When the four-parameter system of points is not a surface the tangents at an arbitrary point will not be coplanar; they will, by V. 12] either generate a chain of planes, or meet an arbitrary plane in chain congruence of points. The condition for the former is that they be linearly dependent on three of their number, i.e.

$$\sum L_i \frac{\partial x}{\partial u_i} = \sum L_i \frac{\partial y}{\partial u_i} = \sum L_i \frac{\partial z}{\partial u_i} \equiv 0,$$

$$\sum L_i \frac{\partial \bar{x}}{\partial u_i} = \sum L_i \frac{\partial \bar{y}}{\partial u_i} = \sum L_i \frac{\partial \bar{z}}{\partial u_i} \equiv 0.$$

$$\frac{\partial(x, y, z)}{\partial(u_j, u_k, u_l)} \equiv R \frac{\partial(\bar{x}, \bar{y}, \bar{z})}{\partial(u_j, u_k, u_l)} \qquad i = 1, 2, 3, 4. \qquad (14)$$

Leaving this equation hanging in the air for a moment, let us inquire under what circumstances our system will include three-parameter sub-systems which lie on surfaces, where it is necessary and sufficient that when

$$F(u_1, u_2, u_3, u_4) = 0,$$

$$\frac{\partial(x, y, z)}{\partial(u_j, u_k, u_l)} = 0.$$

The partial derivative of x to u_j in this expression is

$$\frac{\partial x}{\partial u_j} + \frac{\partial x}{\partial u_i}\frac{\partial u_i}{\partial u_j} = \frac{\partial x}{\partial u_j} - \frac{\partial x}{\partial u_i}\frac{\dfrac{\partial F}{\partial u_j}}{\dfrac{\partial F}{\partial u_i}}.$$

Substituting, we get

$$\sum \frac{\partial F}{\partial u_i}\frac{\partial(x, y, z)}{\partial(u_j, u_k, u_l)} = 0,$$

$$\sum \frac{\partial F}{\partial u_i}\frac{\partial(\bar{x}, \bar{y}, \bar{z})}{\partial(u_j, u_k, u_l)} = 0.$$

These equations, when independent, cannot have more than two independent solutions—they may have only one, or none at all. When they are equivalent, there are three independent systems.*

Theorem 39] *The necessary and sufficient condition that the tangents at an arbitrary point of a four-parameter system should meet an arbitrary plane in a chain congruence is that there should be three independent systems of surfaces, each of which contains a three-parameter sub-set of points of the given system.*

We pass finally to the five-parameter system, where x, y, and z are functions of u_1, u_2, u_3, u_4, and u_5. When will this constitute a one-parameter system of surfaces? Let these surfaces be given by

$$F(u_1, u_2, u_3, u_4, u_5) = \text{Const.}$$

Assuming that $\dfrac{\partial F}{\partial u_5} \neq 0$, we have for the partial derivatives of our coordinates

$$\frac{\partial x}{\partial u_i} = \frac{\dfrac{\partial(x, F)}{\partial(u_i, u_5)}}{\dfrac{\partial F}{\partial u_5}}, \qquad i = 1, 2, 3, 4.$$

The conditions for a surface are given by 38]

$$\frac{\partial(x, y, z)}{\partial(u_2, u_3, u_4)} = \frac{\partial(x, y, z)}{\partial(u_3, u_2, u_1)} = \frac{\partial(x, y, z)}{\partial(u_3, u_1, u_2)} = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = 0.$$

The first of these gives, when slightly expanded,

$$-\frac{1}{\dfrac{\partial F}{\partial u_5}} \begin{vmatrix} \dfrac{\partial F}{\partial u_2} & \dfrac{\partial F}{\partial u_3} & \dfrac{\partial F}{\partial u_4} & \dfrac{\partial F}{\partial u_5} \\ \dfrac{\partial(x, F)}{\partial(u_2, u_5)} & \dfrac{\partial(x, F)}{\partial(u_3, u_5)} & \dfrac{\partial(x, F)}{\partial(u_4, u_5)} & 0 \\ \dfrac{\partial(y, F)}{\partial(u_2, u_5)} & \dfrac{\partial(y, F)}{\partial(u_3, u_5)} & \dfrac{\partial(y, F)}{\partial(u_4, u_5)} & 0 \\ \dfrac{\partial(z, F)}{\partial(u_2, u_5)} & \dfrac{\partial(z, f)}{\partial(u_3, u_5)} & \dfrac{\partial(z, F)}{\partial(u_4, u_5)} & 0 \end{vmatrix} = 0,$$

* Cf. Goursat and Bourlet, *Leçons sur l'intégration des équations à dérivées partielles*, Paris, 1891, pp. 49 ff.

which can be written also

$$\frac{\partial(F, x, y, z)}{\partial(u_2, u_3, u_4, u_5)} = 0.$$

Our condition is, thus, that the rank of the matrix

$$\left\| \begin{array}{ccc} \frac{\partial F}{\partial u_1} & \cdots\cdots & \frac{\partial F}{\partial u_5} \\ \frac{\partial x}{\partial u_1} & \cdots\cdots & \frac{\partial x}{\partial u_5} \\ \frac{\partial y}{\partial u_1} & \cdots\cdots & \frac{\partial y}{\partial u_5} \\ \frac{\partial z}{\partial u_1} & \cdots\cdots & \frac{\partial z}{\partial u_5} \end{array} \right\|$$

should be 4 or less.

$$R\frac{\partial F}{\partial u_i} = \alpha \frac{\partial x}{\partial u_i} + \beta \frac{\partial y}{\partial u_i} + \gamma \frac{\partial z}{\partial u_i}.$$

Since the left-hand side is real,

$$\alpha \frac{\partial x}{\partial u_i} + \beta \frac{\partial y}{\partial u_i} + \gamma \frac{\partial z}{\partial u_i} - \bar{\alpha} \frac{\partial \bar{x}}{\partial u_i} - \bar{\beta} \frac{\partial \bar{y}}{\partial u_i} - \bar{\gamma} \frac{\partial \bar{z}}{\partial u_i} = 0.$$

The values of α, β, γ, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ must be such that the expression

$$\alpha \frac{\partial x}{\partial u_i} + \beta \frac{\partial y}{\partial u_i} + \gamma \frac{\partial z}{\partial u_i}$$

is proportional to the partial derivative of F, i.e.

$$\sum_{j,k=1}^{j,k=5} \frac{\partial(x, y, z)}{\partial(u_i, u_j, u_k)} \frac{\partial(\bar{x}, \bar{y}, \bar{z})}{\partial(u_i, u_l, u_m)} = R\frac{\partial F}{\partial u_i}.$$

We therefore write

$$\sum_{i,j,k=1}^{i,j,k=5} \frac{\partial(x, y, z)}{\partial(u_i, u_j, u_k)} \frac{\partial(\bar{x}, \bar{y}, \bar{z})}{\partial(u_i, u_l, u_m)} du_i = 0$$

$$(j-l)(j-m)(k-l)(k-m) \neq 0. \qquad (15)$$

Theorem 40] *If the coordinates of the points of a five-parameter system be analytic functions of the independent parameters* u_1, u_2, u_3, u_4, u_5, *the necessary and sufficient condition that the system should contain one, and, hence, an infinite number of surfaces is that the Pfaff equation* (15) *should be integrable.*

Let us now assume that our system is given in the other form, namely,

$$F(x, y, z, \bar{x}, \bar{y}, \bar{z}) = 0. \tag{16}$$

Let the surfaces be given by the equations

$$z = z(x, y, R), \quad \bar{z} = \bar{z}(\bar{x}, \bar{y}, R),$$

$$\phi(x, y, \bar{x}, \bar{y}, R) \equiv 0.$$

We may treat x, y, $\bar{x}$, $\bar{y}$, as independent variables, and write

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,$$

$$\frac{\partial F}{\partial \bar{x}} + \frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \bar{x}} = 0,$$

$$\frac{\partial^2 F}{\partial x\,\partial \bar{x}}\frac{\partial F}{\partial z}\frac{\partial F}{\partial \bar{z}} + \frac{\partial^2 F}{\partial z\,\partial \bar{z}}\frac{\partial F}{\partial x}\frac{\partial F}{\partial \bar{x}} - \frac{\partial F}{\partial x}\frac{\partial F}{\partial \bar{z}}\frac{\partial^2 F}{\partial z\,\partial \bar{x}} - \frac{\partial F}{\partial z}\frac{\partial F}{\partial \bar{x}}\frac{\partial^2 F}{\partial x\,\partial \bar{z}} = 0,$$

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial x\,\partial \bar{x}} & \dfrac{\partial^2 F}{\partial x\,\partial \bar{z}} & \dfrac{\partial F}{\partial x} \\ \dfrac{\partial^2 F}{\partial z\,\partial \bar{x}} & \dfrac{\partial^2 F}{\partial z\,\partial \bar{z}} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial F}{\partial \bar{x}} & \dfrac{\partial F}{\partial \bar{z}} & 0 \end{vmatrix} = 0.$$

Similarly,

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial y\,\partial \bar{y}} & \dfrac{\partial^2 F}{\partial y\,\partial \bar{z}} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial^2 F}{\partial z\,\partial \bar{y}} & \dfrac{\partial^2 F}{\partial z\,\partial \bar{z}} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial F}{\partial \bar{y}} & \dfrac{\partial F}{\partial \bar{z}} & 0 \end{vmatrix} = 0.$$

By cross-differentiation

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x \partial \bar{y}} & \frac{\partial^2 F}{\partial x \partial \bar{z}} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial z \partial \bar{y}} & \frac{\partial^2 F}{\partial z \partial \bar{z}} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial \bar{y}} & \frac{\partial F}{\partial \bar{z}} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial^2 F}{\partial y \partial \bar{x}} & \frac{\partial^2 F}{\partial y \partial \bar{z}} & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial z \partial \bar{x}} & \frac{\partial^2 F}{\partial z \partial \bar{z}} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial \bar{x}} & \frac{\partial F}{\partial \bar{y}} & 0 \end{vmatrix} = 0.$$

Hence it is a necessary condition that for all values where $F = 0$, the matrix

$$\left\| \begin{matrix} \frac{\partial^2 F}{\partial x \partial \bar{x}} & \frac{\partial^2 F}{\partial x \partial \bar{y}} & \frac{\partial^2 F}{\partial x \partial \bar{z}} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial \bar{x}} & \frac{\partial^2 F}{\partial y \partial \bar{y}} & \frac{\partial^2 F}{\partial y \partial \bar{z}} & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial z \partial \bar{x}} & \frac{\partial^2 F}{\partial z \partial \bar{y}} & \frac{\partial^2 F}{\partial z \partial \bar{z}} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial \bar{x}} & \frac{\partial F}{\partial \bar{y}} & \frac{\partial F}{\partial \bar{z}} & 0 \end{matrix} \right\| \qquad (17)$$

should be of rank 2. Conversely, when this condition is fulfilled for every point of the variety, every plane which meets it in ∞^3 points will meet it in ∞^1 curves, and these may be assembled into ∞^1 surfaces.

Theorem 41] *The necessary and sufficient condition that a five-parameter system of points, given by an equation such as* (16) *should contain a singly infinite system of surfaces is that the rank of the matrix* (17) *should be* 2.

It seems certain that some simple geometrical condition must be fulfilled when the rank of (17) is 3; unfortunately the present writer has been unable to find it. We find exactly as in VI (19) that the tangent plane has the equation

$$(x'-x)\frac{\partial F}{\partial x} + (y'-y)\frac{\partial F}{\partial y} + (z'-z)\frac{\partial F}{\partial z} = 0.$$

If the tangent plane at (x, y, z) be

$$z' - z = 0,$$

the three-parameter variety will meet the plane in a two-parameter variety given by

$$0 = (x'-x)\frac{\partial^2 F}{\partial x^2} + 2(x'-x)(y'-y)\frac{\partial^2 F}{\partial x \partial y} + (y'-y)^2\frac{\partial^2 F}{\partial y^2} + \dots$$

$$+ (\bar{x}'-\bar{x})\frac{\partial^2 F}{\partial \bar{x}^2} + 2(\bar{x}'-\bar{x})(\bar{y}'-\bar{y})\frac{\partial^2 F}{\partial \bar{x}\partial \bar{y}} + (\bar{y}'-\bar{y})^2\frac{\partial^2 F}{\partial \bar{y}^2} + \dots$$

$$+ (x'-x)(\bar{x}'-\bar{x})\frac{\partial^2 F}{\partial x \partial \bar{x}} + (x'-x)(\bar{y}'-\bar{y})\frac{\partial^2 F}{\partial x \partial \bar{y}}$$

$$+ (\bar{x}'-\bar{x})(y'-y)\frac{\partial^2 F}{\partial y \partial \bar{x}} + (y'-y)(\bar{y}'-\bar{y})\frac{\partial^2 F}{\partial y \partial \bar{y}} + \dots.$$

To find where a line through the point of contact meets this system, let us put

$$y'-y = \lambda(x'-x) = \lambda[X+iY];\ \bar{y}'-\bar{y} = \bar{\lambda}(\bar{x}'-\bar{x}) = \bar{\lambda}[X-iY],$$

$$0 = (X+iY)^2\left[\frac{\partial^2 F}{\partial x^2} + 2\lambda\frac{\partial^2 F}{\partial x \partial y} + \lambda^2\frac{\partial^2 F}{\partial y^2}\right]$$

$$+ (X-iY)^2\left[\frac{\partial^2 F}{\partial \bar{x}^2} + 2\bar{\lambda}\frac{\partial^2 F}{\partial \bar{x}\partial \bar{y}} + \bar{\lambda}^2\frac{\partial^2 F}{\partial \bar{y}^2}\right] + \dots$$

$$+ (X^2+Y^2)\left[\frac{\partial^2 F}{\partial x \partial \bar{x}} + \lambda\frac{\partial^2 F}{\partial y \partial \bar{x}} + \bar{\lambda}\frac{\partial^2 F}{\partial x \partial \bar{y}} + \lambda\bar{\lambda}\frac{\partial^2 F}{\partial y \partial \bar{y}}\right] + \dots.$$

If $$\frac{\partial^2 F}{\partial x \partial \bar{x}} + \lambda\frac{\partial^2 F}{\partial y \partial \bar{x}} + \bar{\lambda}\frac{\partial^2 F}{\partial x \partial \bar{y}} + \lambda\bar{\lambda}\frac{\partial^2 F}{\partial y \partial \bar{y}} = 0.$$

The two branches of the thread on this line have mutually orthogonal tangent chains. There may be a chain of such tangent lines, or none at all. There will be just one, by II. 25], if

$$\left\|\begin{matrix} \dfrac{\partial^2 F}{\partial x \partial \bar{x}} & \dfrac{\partial^2 F}{\partial x \partial \bar{y}} \\ \dfrac{\partial^2 F}{\partial y \partial \bar{x}} & \dfrac{\partial^2 F}{\partial y \partial \bar{y}} \end{matrix}\right\| = 0,$$

and this means that the matrix (17) has a rank (3) at this point.

Theorem 42] *The matrix* (17) *has the rank* 3 *at those points where there is just one tangent in the tangent plane meeting the variety in a thread with a double point at the point of contact of such a nature that tangent chains to the two branches are mutually orthogonal.*

CHAPTER VIII

THE VON STAUDT THEORY

§ 1. The Basis of Real Projective Geometry

The problem of complex geometry, as we have studied it so far, can be stated in the following terms:

'There exists a class of undefined objects called *points* which are in one to one correspondence with groups of real numbers called *coordinates*. Let the coordinates be allowed to take complex values; what real objects may then be found to correspond to them, and what geometrical relations will obtain?'

As a matter of mathematical history, geometry was a highly developed science before the birth of Descartes; many geometers before and since have considered coordinates and the whole analytical apparatus as one method, and by no means always the best one, to reach geometrical truth. In particular, since the time of Poncelet and Steiner, projective geometry, treated by rarely synthetic methods, has reached a very high state of perfection. But even here the student is pursued by complex elements. It is intolerably clumsy to say that a straight line coplanar with a conic may intersect it or may not. Consequently, geometers have tried one expedient after another to smuggle the imaginary back into their science.

One school of writers have used what they style 'the principle of continuity.' Here is a typical statement of this principle*:

'The principle of continuity enables us to combine the elegance of geometric methods with the generality of algebraic methods. For instance, if we wish to determine the points in which a line meets a circle, the neatest method is afforded by pure geometry. But in certain relative positions of the line and the circle, the line does not cut the circle in visible points.

* Cf. Russell, *Elementary Treatise on Pure Geometry*, second edition, Oxford, 1905, pp. 269, 270.

Here Algebraical Geometry comes to our aid. For if we solve the same problem by Algebraical Geometry, we shall ultimately have to solve a quadratic equation, and the quadratic equation will have two solutions, real, coincident or imaginary. Hence we conclude that a line always meets a circle in two points, real, coincident or imaginary.'

The fundamental difficulty is that we are given no hint whatever as to what the so-called *imaginary* points may be. They certainly are not points in the sense in which the substantive has been used so far. They seem to be something which is common to a line and a circle which have no common points. The only explanation which this writer gives is on the following page:

'The best way of defining a pair of imaginary points is as the double points of an elliptic involution. This statement would be unexceptionable if the word "defining" were replaced by "determining", but the phrase as it stands is meaningless, for the one certain thing about an elliptic involution is that it has no double points, in the sense in which the words have been used so far.'

It is perfectly clear that all such procedures, however defensible on purely didactic grounds, are repugnant to the spirit of modern rigorous mathematics. We must find a better way.

Whenever it is necessary to enlarge our universe of mathematical discourse by including new elements, there are just two ways of proceeding:

A) We may build a new mathematical structure from the ground up, where the new elements are included among the objects whose existence is postulated at the beginning, and which are defined only in so far as independent assumptions of a purely logical nature are made about them:

B) We may define the new elements in terms of the elements already recognized. Each of these methods is sound and legitimate, although it should be noticed that if we had recourse to A) every time we introduced a new concept, our progress would not be rapid. Still, as a logical method of proceeding it is unexceptionable. In the present case we are not obliged to define imaginary points *per se* any more than we are obliged to define real ones, provided that we start from a set of fundamental assumptions which will take account of them automatically. This method has been successfully followed by one or two recent writers who have laid down sets of independent postulates sufficient to build up the geometry of the complex domain.*

It is not our present intention to follow this method, or to show the correspondence between the objects so defined and complex sets of coordinate values. We shall follow the other lead, and define imaginary elements in pure geometry in terms of real elements which have been already admitted. This method was first worked out by that geometer whose name appears in the heading of the present chapter, and subsequent writers have done little in the field but refine and improve the details of his presentations. Yet all of the discussions which have so far been published are subject to one important criticism. The discoverer, and most of his successors, wrote before the age of modern abstract views of geometry, and the critique of geometrical assumptions which has been so active in recent years. Consequently, in all their treatments, there are certain assumptions taken directly from intuition, notably the idea of sense of description of a line. Such a procedure is to-day intolerable. A theory of imaginaries in pure geometry has no *raison d'être* unless it be built up logically from a definite set of axioms, with no intuitional element whatever. It is the object of this concluding chapter to show how the Von Staudt theory can be developed in this way.†

* Pieri, 'Nuovi principii di geometria proiettiva', *Memorie della R. Accademia delle Scienze di Torino*, Series 2, vol. lv, 1905. Also Veblen and Young, *Projective Geometry*, vol. i, Boston, 1910, Assumptions, A, E, P, H_2, K_2.

† The principal existing articles dealing with the Von Staudt theory are 'Von Staudt, Beiträge zur Geometrie der Lage', cit., Part ii; Pfaff, *Neuere Geometrie*, Erlangen, 1867 (which the present author has not been able to see); August, 'Untersuchungen über das Imaginäre in der Geometrie', Berlin, 1872, and a long article by Luroth, 'Das Imaginäre in der Geometrie und das Rechnen mit Würfen', *Math. Annalen*, vol. viii, 1874. Two writers have explained the theory in algebraic form. Stolz, 'Die geometrische Bedeutung der komplexen Elemente', *Math. Annalen*, vol. iv, 1871, and Stephanos, 'Sur la définition géométrique des points imaginaires,' *Bulletin des Sciences mathématiques*, Série 2, vol. vii, 1883.

We start with a system of assumptions for projective geometry in three dimensions.*

Axiom I] There exists a class of objects called *points* which contains at least two distinct elements.

Axiom II] Each two distinct points belong to a single sub-class called a *line*.

The line shall be said to *contain* the points, while the points are *on* the line. The line determined by points A and B shall be AB.

Axiom III] Two distinct points determine among the remaining points of their line two mutually exclusive sub-classes of points, neither of which is empty.

These classes are called *separation classes*. Points in different classes shall be said to be *separated* by the given points. Points in the same class are naturally *not separated*. If A and B separate P and Q, we shall write $AB \int PQ$, while for non-separation we shall write $AB \not\int PQ$.

Axiom IV] If $AB \int PQ$ then $PQ \int AB$.

Axiom V] If four collinear points be given, there is one and only one way in which they can be divided into two mutually separating pairs.

From these axioms we easily find:

Theorem 1] If $AB \int CD$, $AE \int CD$, $EB \not\int CD$.

Theorem 2] *If five collinear points be given, a chosen pair will separate two and only two of the pairs determined by the other three, or none at all.*

Theorem 3] If $AC \int BD$, $AE \int CD$, *then* $AE \int BD$.

This theorem is absolutely fundamental. We may give its significance in intuitional form by saying that if D be very

* The development of projective geometry from axioms here given is taken from the author's *Non-Euclidean Geometry*, cit. pp. 246 ff.

distant, and we start from A, and find that C is beyond B, while E is beyond C, then E is beyond B.

Theorem 4] $PA \int CD$, $PB \int CD$, $PQ \int AB$, *then* $PQ \int CD$.

In intuitional terms this is a betweenness theorem. It tells us that if C and D be between A and B, Q between C and D, then Q is between A and B. We have thus established axiomatically betweenness and beyondness; we also need continuity.

Axiom VI] If all points of the separation class bounded by A and B be so divided into two sub-classes that no point of the first sub-class is separated from A by B and a point of the second sub-class, then there will be a single point C in the class such that no point of the first sub-class is separated from A by B and C, and none of the second is separated from B by A and C.

The point C may be reckoned as belonging to the one or the other class, according to the definition of the dichotomy. It is the existence of this point that forces us to state our axioms in terms of non-separation rather than in the simpler terms of separation. The intuitional meaning of the axiom is this. If A be to the left of B, and all the points between A and B be divided into two sub-classes such that a point of the first is never to the right of one of the second, then there is such a point C, that a point of the first class is never to the right of it, and one of the second is never to the left of it.

Axiom VII] All points are not collinear.

Definition. The assemblage of all points collinear with a given point and with the points of a line that does not contain that point shall be called a *plane*.

Axiom VIII] If a line intersect in distinct points two of the lines determined by three non-collinear points, it intersects the third line.

Theorem 5] *A plane contains every line whereof it contains two points.*

Theorem 6] *If $A'B'C'$ be three non-collinear points of the plane determined by A and BC, then this latter is identical with the plane determined by A' and $B'C'$.*

Theorem 7] *Two lines in the same plane always intersect.*

Theorem 8] *Any three non-collinear points in a plane will determine it.*

Axiom IX] Not all points are in one plane.

Definition. The assemblage of all points collinear with a fixed point and with the points of a plane which does not contain the given point shall be called a *space*.

In the remainder of the present chapter all points are in one space.

It is easy to show by straightforward reasoning * that a space contains every line whereof it contains two points, and every plane whereof it contains three points which are not collinear. Moreover, we may show that a space is equally well determined by any four of its points which are not coplanar, so that two spaces sharing four non-coplanar points are identical. It will follow from this that a line and plane in the same space will always have one point in common, and two planes will share a line. A system of planes through a line is said to be *co-axal*, the line being their *axis*, or to form a *pencil*. This latter name is also applied to a set of concurrent coplanar lines.

Axiom X] If four co-axal planes meet one transversal in A, B, C, and D while they meet a second in A', B', C', and D' respectively, and if $AC \int BD$, then $A'C' \int B'D'$.

This axiom amounts to saying that the relation of separation is invariant for projection and intersection. We may also define separation in a pencil of lines by means of the separation relation cut on any transversal.

It is now possible to give the usual demonstration of Desargues two-triangle theorem, and to draw therefrom the

* Ibid., pp. 250 ff.

theorem that if two complete quadrangles be so situated that five sides of the one meet five of the other in points of a certain line, the same is true of the sixth pair of sides.

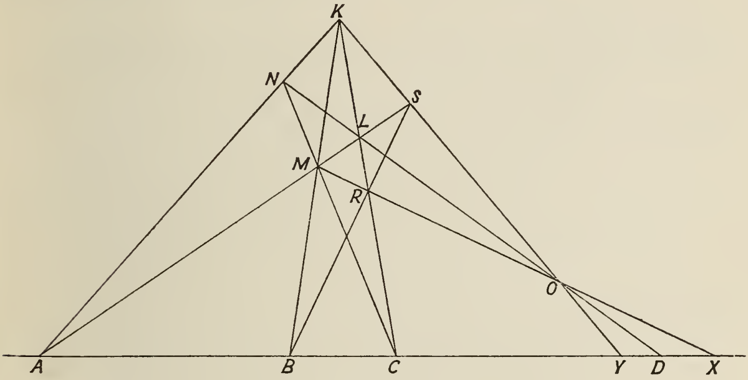

FIG. 2.

Theorem 9] *If the pairs of opposite vertices of a complete quadrilateral be AC, KM, and LM respectively, then KM and LN will meet AC in two distinct points B and D such that* $AC \int BD$.*

Let S be such a point of the line ML that $AL \int MS$.

Project S from B into R on KL; then $CL \int KR$.

In the triangles KNM, SLR, all of whose vertices we may suppose distinct, corresponding sides are concurrent in the collinear points A, C, B.

Hence KS, NL, and MR meet in a point O. Let KS meet AC in Y, while MR meets it in X, both points being distinct from A, B, C, D.

Now, since $AL \int MS$, projecting on AC from K and from O we have

$$AC \int BY, \qquad AD \int XY.$$

* Cf. Enriques, *Lezioni di geometria proiettiva*, Bologna, 1898, p. 56. In some systems of axioms for projective geometry, as Veblen and Young, loc. cit., it is necessary to introduce the specific axiom that the diagonals of a complete quadrilateral are not concurrent.

In the same way, since $CL \int RK$,

$$CD \int XY, \qquad AC \int BX.$$

Since $CY -\!\!\!\!\!\int DX$, $AY -\!\!\!\!\!\int DX$, then, by 2] $AC -\!\!\!\!\!\int DX$.

Hence $$AC -\!\!\!\!\!\int BD.$$

Definition. Given two pairs of points whereof the first is composed of opposite vertices of a complete quadrilateral, and the second of the intersection of this diagonal with the opposite two, the first pair are said to separate the other pair *harmonically.*

We are able next to give the usual proof of

Theorem 10] *If A and C separate B and D harmonically, then B and D do the same for A and C.*

We see in this way that a point has just one harmonic conjugate with regard to two given points collinear with it; the relation of harmonic separation is invariant for projection and intersection, and we may define harmonic separation in a pencil by means of any transversal. This enables us to introduce the idea of cross ratio, which is done in the following terms.*

Suppose that we have given three collinear points P_∞, P_0 and P_1, the harmonic conjugate of P_0 with regard to P_1 and P_∞ shall be called P_2. That of P_1 with regard to P_2 and P_∞, P_3. That of P_1 with regard to P_0 and P_∞ shall be P_{-1}, and in general P_{n+1} and P_{n-1} shall be harmonically separated by P_n and P_∞, where the subscripts are integral.

Next take a positive integer m and construct a new scale where $P_{\frac{1}{m}}$ replaces P_1 in such wise that $P_{\frac{m}{m}}$ is the same point as P_1. We shall then find that $P_{\frac{rm}{rn}}$ is identical with $P_{\frac{m}{n}}$. There will thus be a single definite point to correspond to each

* Cp. the author's *Non-Euclidean Geometry*, cit. pp. 255 ff.

rational subscript. Lastly, we shall find that every dichotomy in the rational number system will give a dichotomy in a separation class and a limiting point, exactly as is required for Axiom VI, and conversely every such dichotomy in the separation class will give a dichotomy in the number system, and so a rational or irrational number. The proof follows step by step the usual one where we show that any length can be expressed rationally or irrationally in terms of any other, our Axiom VI replacing that of Archimedes. If

$$x > 0, \quad P_\infty P_0 \not\!\int P_1 P_x, \qquad \text{if} \quad x < 0, \quad P_\infty P_0 \int P_1 P_x.$$

If A, B, C, and D be four distinct points, and if A, C, and B be made to play the rôles of P_∞, P_0, and P_1, then the subscript to be attached to D shall be called a *cross ratio* of the four given points, and written (AC, BD). Since this expression depends merely on a succession of harmonic constructions, it is invariant for projections and intersections, and may be used equally to define the cross ratio of four elements of a pencil. We find by a simple construction

$$(AC, BD) = (CA, DB) = (BD, AC) = (DB, CA).$$

The range of collinear points, the pencil of coplanar lines, and the pencil of co-axal planes shall be defined as *fundamental one-dimensional forms*. Two such forms shall be said to be *projective* if their elements are in one to one correspondence and corresponding cross ratios are equal. If the two be so related that they are connected by a sequence of projections and intersections, they are surely projective. Conversely, as we may easily find a sequence of projections and intersections to carry three chosen elements of line form into any three of the other, and as these may be taken as the basis of cross ratio measurement, we see that we have proved the fundamental theorem of projective geometry whereby the necessary and sufficient condition that two fundamental one-dimensional forms should be projective is that they should be connected by a succession of projections and intersections.

It is now necessary to examine a little more closely the relations of the various cross ratios of given point systems. We may easily show that *

$$(P_\infty P_\alpha, P_\beta P_\gamma) = \frac{\gamma-\alpha}{\beta-\alpha},$$

$$(P_0 P_\gamma, P_\beta P_\delta) = \frac{\beta}{\gamma} : \frac{\gamma-\delta}{\gamma-\beta},$$

$$(P_\alpha P_\gamma, P_\beta P_\delta) = \frac{\alpha-\beta}{\alpha-\delta} : \frac{\gamma-\delta}{\gamma-\beta}.$$

If we assign to the three points P_∞, P_0, and P_1 the pairs of homogeneous coordinate values (0, 1), (1, 0), (1, 1), then the point P_x will have coordinates (x_0, x_1), where $x = \frac{x_1}{x_0}$.

A cross ratio of the four points P_x, P_y, P_z, P_t will be expressed

$$\frac{\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} \cdot \begin{vmatrix} z_0 & z_1 \\ t_0 & t_1 \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 \\ t_0 & t_1 \end{vmatrix} \cdot \begin{vmatrix} z_0 & z_1 \\ y_0 & y_1 \end{vmatrix}}.$$

This will be absolutely unaltered on the linear transformation

$$\begin{aligned} x_0 &= a_{00}x_0' + a_{01}x_1', \\ x_1 &= a_{10}x_0' + a_{11}a_1'. \end{aligned} \qquad (1)$$

This is surely a projective transformation, and, as there are parameters available to carry any three points into any other three, it is the most general projective transformation. It will be involutory if

$$a_{00} + a_{11} = 0.$$

We distinguish between the *hyperbolic* involution, where each pair of corresponding points are harmonically separated by the self-corresponding ones, and the *elliptic* involution, where there are no self-corresponding points. In a hyperbolic involution two pairs do not separate one another, for their cross ratio is positive; in an elliptic they do. An involution is completely determined by two pairs, and a one-dimensional

* Ibid., pp. 261, 262.

projectivity is involutory if a single pair correspond interchangeably.

The whole of real projective geometry in three dimensions is now open to us without further axioms, so that we may assume any of the standard theorems in this science which we desire. In particular, let us assume that we have developed the usual theory of collineations and correlations, including polar systems, and have laid down the broad lines for the theory of conic sections. Let us recall, among other facts, that if an involution of points be given on a conic, the lines connecting corresponding points pass through a fixed point called the *centre* of the involution. If the involution be hyperbolic, the double points are the intersections of the conic with the polar of the centre.

A point shall be said to be *inside* a conic if its polar fails to meet the latter; it is *outside* if its polar meets the conic twice. Suppose that we have a self-conjugate triangle whose vertices are A_1, A_2, and A_3. Let P be a point on the conic, and let PA_i meet A_jA_k in A_i', while the tangent at P meets this latter line in A_i''. Then $A_i'A_i''$ and A_jA_k are pairs in the involution of conjugate points on A_jA_k.

Suppose that $$A_1A_1'' \not\!\!\int A_2A_3.$$

Then $$A_1A_3'' \not\!\!\int A_2A_3'.$$

If $$A_1A_3' \int A_2A_3'', \quad \text{then} \quad A_1A_3 \int A_2A_2'.$$

If $$A_1A_2 \int A_3'A_3'', \quad \text{then} \quad A_1A_2' \int A_3A_2''.$$

It will follow from this that two sides of that triangle will bear hyperbolic involutions, and the third side will bear an elliptic one:

Theorem 11] *If a point be inside a conic, every point conjugate to it is outside.*

Theorem 12] *Every line through a point inside a conic meets the latter in two distinct points.*

Theorem 13] *Given an elliptic involution and another involution on the same fundamental one-dimensional form, they have necessarily one common pair.*

This is proved by projecting them into two involutions on the same conic, and connecting the centres, one of which is inside. As a corollary we find a theorem of first importance in the constructions which we shall take up presently:

Theorem 14] *If any pair be taken in an elliptic involution, a second pair may always be found which divides the first one harmonically.*

Suppose that we have an involution of collinear points which are conjugate with regard to a conic. Let PP' be an arbitrary pair of this involution, Q the pole of their line, S any point of the conic. Let PS meet the curve again in S' while $S'Q$ meets it in T, and PT meets it in T'. The self-conjugate triangle PQP' is the diagonal triangle of the quadrangle $SS'TT'$, so that ST' goes through P', and S, which is an arbitrary point of the curve, projects our involution of conjugate points into the involution on the conic whose centre is the pole of the given line.

Suppose, conversely, that we have an elliptic involution of collinear points which is projected from two different points R and S on a conic into the same involution.

Let L be the pole of the given line, C the centre of the involution on the conic, this latter point being necessarily within the conic. The line CL will meet the conic in two points P, P', which are projected back on the given line from R and S into pairs of points which belong to the given involution and are conjugate with regard to the conic. If they were distinct pairs, the given involution would consist in conjugate pairs with regard to the conic. Suppose that P, P' are projected from R and S into the same pair of points on the given line. L, the pole of the line, will then be the intersection of PP' and RS, and the tangents at R and S will intersect on the given line. But the mate of this point in the involution on the given line must lie on RC and SC, for this point could not be C itself as it is outside. Hence, once more, $C = L$.

Theorem 15] *The necessary and sufficient condition that an elliptic involution of collinear points should be projected from two points of a conic into the same involution upon the conic is that it should be an involution of conjugate points. The centre of the involution on the conic will be the pole of the given line.*

Definition. A projective transformation of a fundamental one-dimensional form shall be defined as *infinitesimal* when all cross ratios with three fixed points are altered infinitesimally in value. The necessary and sufficient condition for this is that the corresponding algebraic equations should differ infinitesimally from those which give rise to the identical projectivity.

Definition. Two ordered triads of points on the same line shall be said to have the *same sense* when the transformation that carries the one into the other can be altered continuously into the identical transformation. When this is not the case they are said to have *opposite senses.*

Theorem 16] *The necessary and sufficient condition that two ordered triads of collinear points should have the same sense is that the projective transformation* (1) *that carries the one into the other should have a positive discriminant.**

Theorem 17] *Two triads whose senses are the same as, or opposite to, the sense of a third triad, have the same senses as one another.*

Theorem 18] *If one of two triads have the same sense as a third, and the other have a sense opposite to that of the third, the two have opposite senses.*

Theorem 19] *If the members of a triad be permuted cyclically, the sense is not altered.*

We may confine our attention to the triad (0, 1) (1, 0) (1, 1). They are permuted cyclically by the projectivity with positive discriminant

$$x_0' = rx_1, \quad x_1' = -rx_0 + rx_1.$$

* There is a certain lapse from the purity of the Von Staudt method in using this theorem and the related algebra, but the gain in simplicity is great. Cf. Veblen and Young's *Projective Geometry*, cit. vol. ii, pp. 40 ff.

Theorem 20] *If two members of a triad be interchanged, the sense is reversed.*

We take the previous three points, and the transformation

$$x_0' = sx_1, \quad x_1' = sx_0.$$

We may define sameness and oppositeness of sense for triads of a pencil in analogous fashion, or by the senses on any transversal, for a continuous change in the intersections with one transversal will produce a continuous change on all. Incidentally, we may easily prove:

Theorem 21] *The necessary and sufficient condition that two triads on the same one-dimensional fundamental form should have opposite senses is that in the projectivity which carries the one into the other there should be two self-corresponding elements, which separate all pairs of corresponding elements.*

Theorem 22] *If l_1 and l_2 be two coplanar lines, V_1 and V_2 two distinct points of their plane, the necessary and sufficient condition that a triad of points on l_1 should be projected from V_1 and V_2 upon l_2 into two triads with the opposite senses is that $V_1 V_2$ should meet $l_1 l_2$ in two points which are separated by the first two points.*

It is clear that similar theorems will subsist when the triads are projected by co-axal planes. Finally, let us note that we are free to speak of a sense of description of a whole fundamental one-dimensional form if we mean thereby that all triads have the same sense as a given triad to which an order of elements has been assigned.

§ 2. Imaginary Elements in Pure Geometry.

'Ain't I glad to get out o' de Wilderness.' We have at least completed the long preliminary discussion necessary to take up the Von Staudt theory.

Definition. An elliptic involution on a line, together with a sense of description for that line, shall be called an *imaginary point*. The same involution coupled with the opposite sense shall be called the conjugate imaginary point.

The line shall be said to *contain* the point, which latter *lies* on the line.

Definition. An elliptic involution among the planes of a pencil, together with a sense of description, shall be defined as an *imaginary plane.* The same involution coupled with the opposite sense shall be defined as the conjugate imaginary plane.

The axis of the pencil shall be said to be *in* each of these planes, which latter are said to *contain all points* on the former.

Definition. An imaginary point shall be said to be *on an imaginary plane* when it is on the axis of the axial pencil, or when the involution determining the imaginary plane cuts the involution determining the imaginary point, and the desired senses in the two correspond.

Definition. The totality of points common to two imaginary planes shall be called an *imaginary line.*

There are three kinds of lines altogether.

(*a*) Real lines, each lying in ∞^1 real planes.

(*b*) Imaginary lines of the first sort lying in imaginary planes whose real lines intersect.

(*c*) Imaginary lines of the second sort, common to two planes whose real lines do not intersect.

It is to be noted that lines (*a*) contain ∞^1 real points, lines (*b*) one real point each, and lines (*c*) no real points.

A word or two as to the apparent unnaturalness of these definitions. We saw in Ch. III, p. 77, in a discussion which connected the Poncelet supplementaries with the work of Paulus and Marie, that there is a perfect one to one correspondence between the pairs of conjugate imaginary points of a plane and the elliptic involutions of collinear points therein, and the same will hold in space. The introduction of sense is a device, a 'Kunststück' as the Germans say, in order to attach to each involution two opposite marks to correspond with the separation of conjugate imaginaries. As for the use of an infinite number of points to define a single point, that is exactly analogous to what we do when we define an irrational number as a cut in the rational number system.

Theorem 23] *Two points in a real plane will lie on a line, all of whose points are in that plane, and two lines in a real plane will intersect in a point thereof.*

It will be sufficient to prove the first part; the second comes therefrom by polar reciprocation in a conic. If at least one of the given points be real the proof is immediate, for the real plane must contain the real line connecting the imaginary point and its conjugate. The imaginary point and a real line through the real point will determine an imaginary plane meeting the real plane in the line desired. Assuming, then, that both given points are imaginary, let the real lines through them meet in A, whose mates in the two involutions are A_1' and A_2' respectively. Let B_1 and B_1' be the pair of the first involution which by 14] divide AA_1' harmonically, while B_2B_2' perform the same function for the second involution. The complete quadrangle whose vertices are B_1B_1', B_2B_2' has, besides A, two diagonal points, HK, whence the one harmonic set is projected into the other. These, and these only, are the points from which one given elliptic involution is projected into the other, and since they are separated by the lines AB_1, AB_2 from one of them only, say from H, the given sense on AA_1' is projected into the given sense on AA_2'. The two imaginary points will thus determine, with a real line through H, an imaginary plane which meets the given plane in the imaginary line required.

The steps here taken are typical of those which one must frequently take. For instance, by polarizing the whole construction in a quadric we reach:

Theorem 24] *An imaginary line of the first sort lies with its conjugate in a real plane, and is the intersection of this plane and an imaginary plane. An imaginary line of the second sort lies in no real plane, and contains no real point.*

We must now turn to the arduous but vital task of proving

Theorem 25] *If two points of a line lie in a plane, the whole line is in that plane.*

The theorem follows from 23] when the plane is real, and is equally evident when the line is real. If the line be imaginary of the first sort, it will be determined by a pencil of lines in elliptic involution in the real plane connecting it with its conjugate, together, of course, with a sense of description. Since two points of a real plane can be connected by only one imaginary line of that plane, our imaginary line must be identical with the intersection of the real plane and the imaginary one, and the theorem is proved. There remains the case of an imaginary line of the second sort, the plane being necessarily imaginary. Let the plane be given by the involution $\alpha\alpha'$, $\beta\beta'$ about the line l, and let it contain the imaginary points given by the linear involutions $A_1 A_1'$, $B_1 B_1'$ and $A_2 A_2'$, $B_2 B_2'$, each having a prescribed sense. Lastly, let $A_3 A_3'$, $B_3 B_3'$, with a given sense, determine another point of our line. As a preliminary step, we shall show that we may assume that our line is given by the intersection of a plane through $A_1 B_1$, and one through $A_2 B_2$. Let the real lines of the planes giving the imaginary line be m and n. The involutions on $A_1 B_1$ and $A_3 B_3$ determine the same elliptic involutions about m and n. Consider the regulus of lines gliding along m, n, and $A_1 B_1$. Its members are paired in an elliptic involution by the involution on $A_1 B_1$. This regulus in involution will determine an involution on the conic cut by any plane through $A_3 B_3$. Since the involution on the conic is projected from m and n into the same involution $A_3 A_3'$, $B_3 B_3'$, it is by 15] projected from $A_1 B_1$ into this same involution, or the regulus in involution determines about $A_1 B_1$ an involution giving an imaginary plane which contains $A_3 A_3'$, $B_3 B_3'$, which is any imaginary point of the original line. It thus appears that we may replace m by $A_1 B_1$, and similarly n by $A_2 B_2$. Now consider the regulus of lines gliding on l, $A_1 B_1$, and $A_2 B_2$. About each of these lines there is an involution and the generators of the regulus are paired in such a way as to determine about l and $A_1\,B_1$ the involution cutting $A_2 A_2'$, $B_2 B_2'$, just as before we cut in a conic by a plane through $A_3 B_3$. The involution on the conic is projected from $A_1 B_1$, and from $A_2 B_2$, into the elliptic involution $A_3\,A_3'$, B_3', hence, by 15], it is projected from the line l into this same involution, so that if the senses are right this imaginary point lies on the imaginary plane through l.

With regard to senses, if we take a triad of points on a conic, they will project from two points on a conic into two triads on a line which have opposite senses when, and only when, the line meets the conic in two points which separate the given points, as we see from 21]. In the present case $A_3 A_3'$, $B_3 B_3'$ is an elliptic involution of conjugate points, so $A_3 B_3$ fails to meet the conic. A triad on the conic is projected from l, $A_1 B_1$, and $A_2 B_2$ into three triads having like sense, and our proof is complete.*

The reasoning used in 23] gives at once

Theorem 26] *Two conjugate imaginary elements are harmonically separated by each pair of the real elliptic involution that defines them.*

We must now undertake another long proof of a fundamental proposition.

Theorem 27] *Two lines in the same plane have always one common point.*

If the plane be real this follows from 23]. If the plane be imaginary, and both lines be imaginary of the first sort, we have merely to find the intersection of the imaginary plane with the real line common to the two real planes, one of which passes through each of the given lines. If one line be imaginary of the first sort, and one of the second, it is still a problem of finding the intersection of two imaginary lines in the same real plane, namely, that which connects the line of

* This theorem shows the simplification when it is not necessary to distinguish between conjugate imaginaries. For problems of this sort see Segre, 'Le coppie di elementi imaginarii,' &c., *Memorie della R. Accademia delle Scienze di Torino*, vol. xxxviii, 1886.

the first sort with its conjugate, and may be solved by the methods of 23].

Suppose that we have two imaginary lines of the second sort, one determined by axial pencils about the lines l_1 and l_2, the other by the same involution about l_1 and another about l_2. Draw a line to meet the three axes. Find the mates of its intersection with l_1 and l_2 in the involutions upon these lines, and the pair in each involution dividing this pair harmonically. We thus determine a regulus in involution whose lines cut the required involutions on l_1 and l_2 or determine the required involutions about these lines. This regulus can be determined in two ways according to the way that we join up the harmonic pairs criss-cross. We choose that way which determines among the planes about l_1 and l_2 the senses required by the given imaginary planes through these lines, as in 23]. A line of this regulus meets l_3 by the original construction, and we are looking essentially for a point common to an imaginary plane through this last line, one through l_3, and one through l_1. But the former planes intersect in an imaginary line of the first sort, since their real lines meet, and we are back on the previous case of a line of the first sort and one of the second, and the same plane.

We have now reached the point where we can say that all of the fundamental theorems of concurrence, collinearity, and coplanarity hold in the complex domain exactly as in the real one, and all theorems which do not involve either separation or continuity, as the quadrilateral and quadrangle ones, hold in one domain exactly as in the other. In particular, we may take over the definition of harmonic separation just as soon as we are certain that the diagonal of a complex complete quadrilateral are not concurrent. The proof given in the real domain is not applicable, as it depended upon separation. However, a moment's thought shows us that by means of two successive projections a complex complete quadrilateral may be carried into a real one, so that the theorem of non-concurrence holds in the complex domain as well as in the real. Our previous definition of harmonic reparation is universally applicable.

There are a good many problems in construction where some of the elements or data are imaginary which might be considered at this point, but as they are not vital to what follows we content ourselves with referring the reader to other sources.* It is time to see what we have in the complex domain to correspond to a projective transformation of the real one. This involves somewhat circuitous reasoning, beginning with a re-examination of the imaginary line of the second sort.

We take three points on such a line lying on three real lines l_1, l_2, and l_3. The lines meeting these three will form a regulus, whose members are paired in an elliptic involution, cutting a point involution on each line of the conjugate regulus. Moreover, there will be a definite sense attached to each line by the senses of the given involutions. We have thus a system of ∞^1 points of our given line, and this system will be defined, according to Von Staudt's original plan, as a *chain*.† The points of this chain are carried by the planes through a line of one regulus into real points on a line of the conjugate regulus whose real cross ratios may be defined as the cross ratios of the points of the chain. In any case, four harmonic points belong to a chain, as do all points reached from them by successive harmonic constructions. The elliptic involution of points determining a point of the imaginary line that does not belong to the chain will, by 15] be an involution of points which are conjugate with regard to the quadric of the regulus.

Let us now take a real plane containing the real line through one imaginary point of our given line. Every other imaginary point will be represented by the intersection of its real line with that plane.

* Beyel, 'Zur Geometrie des Imaginären', *Vierteljahresschrift der naturforschenden Gesellschaft in Zürich*, vol. xxxi, 1386; Servais, 'Sur les imaginaires en géométrie', *Mémoires couronnés et autres mémoires de l'Académie de Belgique*, vol. lii, 1889; Grünwald, 'Lösung der Aufgaben über Verbinden und Schneiden imaginärer Punkte, *Zeitschrift für Math. und Phys.*, vol. xlv, 1900, and Grünwald, 'Ueber die Konstruktion mit imaginären Punkten,' Ibid., vol. xlvi, 1901.

† loc. cit., p. 137.

Theorem 28] *The points of an imaginary line of the second sort may be put into one to one correspondence with a given real line and with the individual real points, not on this line, but in a real plane through this line. A chain of points which does not contain the point represented by the special real line will be represented by the points of a conic with regard to which the involution of points on the special line is an involution of conjugate points. A chain including the point represented by the special line will be represented by this line, and by the points of another real line. Harmonic points not including the point represented by the special line will be represented by harmonic points on a conic or line of the sort described.*

The neatest way to realize this is to assume a Euclidean measurement in our projective space, take the special line as the line at infinity, the involution thereon being that of orthogonal directions. A chain will then be represented by the points of a line or circle as in Ch. II.

Every transformation of the given imaginary line into itself may be expressed as a transformation of the lines of the real linear congruence of lines meeting the given imaginary line and its conjugate. The transformation will be infinitesimal if the cross ratios of each line with such fixed elements as are necessary to locate it are infinitesimally transformed. A transformation will be continuous if these cross ratios be continuously altered. Such is certainly the case when the transformation is a product of projections and intersections. Two transformations can be carried over continuously into one another, if such be the case with the parameter values which serve to determine them, and if for no intervening set of parameter values the transformation becomes improper.

Suppose, thus, that we have a continuous transformation of our complex line into itself which has the property that it carries harmonic points into harmonic points. This may be represented by a continuous transformation of the Gauss plane which carries points into points, and leaves invariant the relation of harmonic separation. But we saw in Ch. II. 18] that every such transformation might be represented in the form

$$z = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad z' = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta} \alpha\delta - \beta\gamma \neq 0.$$

The first of these is distinguished by the fact that it can be continuously changed into the identity.* Each of these transformations is completely determined by the fate of three points.

We next define as *projective* any transformation of our fundamental one-dimensional forms which is the result of a sequence of projections and intersections. Such a transformation is continuous, carries harmonic elements into harmonic elements, and can be continuously changed into the identity. Hence it is completely determined by the fate of three elements. This yields once more the fundamental theorem of projective geometry.

Theorem 29] *A projective transformation is completely determined by the fate of three elements. If three elements be invariant, the transformation is the identity.*

Theorem 30] *A one to one continuous transformation of the elements of a fundamental one-dimensional form which leaves harmonic separation invariant is either a projective transformation, or the product of that and an involutory transformation projectively equivalent to the interchange of conjugate imaginary elements.*

We find from II. 19]:

Theorem 31] *A one to one transformation of the points of an imaginary line of the second sort into itself which carries the points of a chain into the points of a chain, is either a projective transformation, or the product of that and a transformation projectively equivalent to the interchange of conjugate imaginary points.*

Theorem 32] *A projective transformation has always two*

* It is saddening to think how much the statement of this transformation might be simplified if the supposition in the note to II. 18] were proved correct, and we were enabled to dispense with the requirement of continuity.

distinct fixed elements, except in the case where there is a fixed element harmonically separated from a given element by those which correspond to it in the transformation and its inverse.

Theorem 33] *If in a projective transformation a single pair of elements correspond interchangeably, the same is true of every pair, and corresponding elements are harmonically separated by the two fixed elements.*

We may properly call this last transformation an *involution.*

Theorem 34] *If two projective pencils of lines be coplanar, and if the common line be self-corresponding, the locus of their points of intersection is a straight line.*

The locus of the intersections in the general case when the common line is not self-corresponding is called a *conic.* This is Reye's definition, and we can go on to prove by very familiar methods the truth of Pascal's theorem, as well as the theorem whereby any two points of the conic will serve equally well to determine it.* The polar theory will also follow easily.

Theorem 35] *A line not tangent to a conic will meet it twice.*

Theorem 36] *If an involution of points be given on a conic the lines connecting corresponding points are concurrent.*

It is not too much to say that we have now laid the basis for the complete development of projective geometry in the complex domain, and all theorems of the real domain which do not explicitly depend upon linear order hold equally well in the new universe of discourse. We are, in fact, able to take the last step, and apply our usual complex number system to this extended geometry. The procedure is as follows:

Definition. Four collinear or conconic points, co-axal planes, or lines of a pencil shall be called a *throw.*†

* Cf. Reye, *Geometrie der Lage,* third edition, Leipzig, 1886, chap. iii.

† German 'Wurf', cf. Von Staudt, loc. cit., p. 132.

Definition. Two throws which can be connected by a series of projections and intersections shall be defined as *equal*.

If X, Y, Z, and T be four elements of one of our fundamental forms or a conic, we shall use the notation $[XY, ZT]$ for one of their throws. Since we may easily find projective sequence to interchange the two pairs in a tetrad, we have

$$[AB, CD] = [CA, BD] = [BD, AC] = [DB, CA].$$

Definition.

$$[AC, BB] = 1, \quad [AC, BC] = 0, \quad [AC, BA] = \infty.$$

Definition. If AA, OZ, XY be three pairs of an involution,

$$[AO, IX] + [AO, IY] = [AO, XY].$$

Note that, by this definition, the addition of throws is commutative, and that the result of adding zero to a throw is to leave the latter unaltered. The associative law is proved by taking the points on a conic and applying Pascal's theorem.

Definition. If AO, IZ, and XY be pairs of an involution,

$$[AO, IX] \times [AO, IY] = [AO, IZ].$$

Multiplication is clearly commutative. We also find that a throw multiplied by zero gives a product equal to zero, and multiplied by one gives the multiplicant. The associative law may be proved by the use of Pascal's theorem, the pointer in question being supposed to be on a conic. The distributive law is proved as follows:

Theorem 37] $[AO, IX] \times [AO, XY] = [AO, XY]$.

Let the pairs AO, IY, XZ form an involution. Then

$$[AO, IZ] = [OA, YX] = [AO, XY].$$

But, by definition,

$$[AO, IX] \times [AO, IZ] = [AO, XY].$$

Theorem 38] $[AO, IX] \times [AO, XI] = 1$.

This follows immediately from the preceding, and from the definition of unity. Let us now try to multiply $[AO, XI]$ by the sum of $[AO, IY]$ and $[AO, IZ]$. This sum is $[AO, IT]$, where AA, YZ, and OT are pairs of an involution, and the product is $[AO, XT]$. But, by the definition of addition, this last throw is the sum of $[AO, XY]$ and $[AO, XN]$, which proves the theorem.*

It is well known that the only system of symbols which obey all the fundamental laws of multiplication and addition are the symbols for real, and those for the usual complex numbers, so that, strictly speaking, the algebra which we have now defined is the ordinary complex algebra. It is wise, however, to go just a little further and show how the actual connexion may be made.

Theorem 39] *If A and C be harmonically separated by B and D,* $[AC, BD] = -1$.

Its square is unity, but it is not equal to unity itself by the definition of that number. If now we compare our former construction of cross ratios with our present system of throws, in the case of real elements we find that

$$[AC, BD] = (AC, BD).$$

This identity holds by the succession of harmonic constructions as long as the value is rational. If irrational we may assume it by definition of the value of the irrational throw. If three points be given, there will thus be a single point making with them any assigned real throw.

Suppose now that we have four points, not projectively equal to those of a chain. We may carry them by projection and intersection into A, B, C, three real points of a real conic, and D, a complex point thereof, D' being the conjugate imaginary point. These two are the double points of a real elliptic involution on a line running clear of the conic. Find the point where the tangent at A meets this line, and connect it with C by a line meeting the conic again at E,

$$[AC, BD] + [AC, BD'] = [AC, BE],$$

and this is a real value. Again, since AC is not the same

* For other proofs of the fundamental laws see Sturm, 'Ueber die Von Staut'schen Würfen', *Math. Annalen*, vol. ix, 1875.

line as DD' these two will intersect in a point which we may connect with B, thus cutting the conic again in F,

$$[AC, BD] \times [AC, BD'] = [AC, BF].$$

It thus appears that $[AC, BD]$ is a root of a quadratic equation with real coefficients. It certainly could not be a real root, for then we could find such a real point H of the conic that

$$[AC, BD] = [AC, BF]$$

$$[AC, EB] \times [AC, BD] = [AC, HD] = 1,$$

and this will imply a contradiction with the definition of unity. The given throw is then a complex root of the ordinary sort.

It should be noticed in conclusion that, if we take five points in space, of which no four are coplanar, we may assign to them the coordinates (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), and (1, 1, 1, 1), we may assign to any point four homogeneous coordinates $x_0 : x_1 : x_2 : x_3$, not all zero, of such a nature that the ratio of two is the throw of two faces of the tetrahedron and the planes through the included edge to the unit point and the given point.* Our system of complex points and complex values is thus brought into accord with the usual analytic expressions.†

* See the author's *Non-Euclidean Geometry*, cit. pp. 263, 264.

† Cf. Servais, 'Sur la projection imaginaire', *Mémoires couronnés et autres mémoires de l'Académie de Belgique*, vol. lii, 1899, where another method of treating complex cross ratios is given.

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