HIGHER MATHEMATICS

CHAPTER I.

THE SOLUTION OF EQUATIONS.

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ART. 1. INTRODUCTION.

In this Chapter will be presented a brief outline of methods, not commonly found in text-books, for the solution of an equation containing one unknown quantity. Graphic, numeric, and algebraic solutions will be given by which the real roots of both algebraic and transcendental equations may be obtained, together with historical information and theoretic discussions.

An algebraic equation is one that involves only the operations of arithmetic. It is to be first freed from radicals so as to make the exponents of the unknown quantity all integers; the degree of the equation is then indicated by the highest exponent of the unknown quantity. The algebraic solution of an algebraic equation is the expression of its roots in terms of the literal coefficients; this is possible, in general, only for linear, quadratic, cubic, and quartic equations, that is, for equations of the first, second, third, and fourth degrees. A numerical equation is an algebraic equation having all its coefficients real numbers, either positive or negative. For the four degrees above mentioned the roots of numerical equations may be computed from the formulas for the algebraic solutions, unless they fall under the so-called irreducible case wherein real quantities are expressed in imaginary forms.

An algebraic equation of the n^{th} degree may be written with all its terms transposed to the first member, thus:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0,$$

and, for brevity, the first member will be called $f(x)$ and the equation be referred to as $f(x) = 0$. The roots of this equation are the values of x which satisfy it, that is, those values of x that reduce $f(x)$ to 0. When all the coefficients $a_1, a_2, \ldots a_n$ are real, as will always be supposed to be the case, Sturm's theorem gives the number of real roots, provided they are unequal, as also the number of real roots lying between two assumed values of x, while Horner's method furnishes a convenient process for obtaining the values of the roots to any required degree of precision.

A transcendental equation is one involving the operations of trigonometry or of logarithms, as, for example, $x + \cos x = 0$, or $a^{2x} + xb^x = 0$. No general method for the literal solution of these equations exists; but when all known quantities are expressed as real numbers, the real roots may be located and computed by tentative methods. Here also the equation may be designated as $f(x) = 0$, and the discussions in Arts. 2–5 will apply equally well to both algebraic and transcendental forms. The methods to be given are thus, in a sense, more valuable than Sturm's theorem and Horner's process, although for algebraic equations they may be somewhat longer. It should be remembered, however, that algebraic equations higher than the fourth degree do not often occur in physical problems, and that the value of a method of solution is to be measured not merely by the rapidity of computation, but also by the ease with which it can be kept in mind and applied.

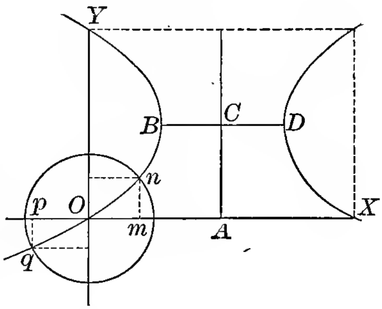
Prob. 1. Reduce the equation $(a + x)^{\frac{2}{5}} + (a - x)^{\frac{2}{5}} = 2b$ to an equation having the exponents of the unknown quantity all integers.

ART. 2. GRAPHIC SOLUTIONS.

Approximate values of the real roots of two simultaneous algebraic equations may be found by the methods of plane analytic geometry when the coefficients are numerically expressed. For example, let the given equations be

$$x^2 + y^2 = a^2, \qquad x^2 - bx = y^2 - cy,$$

the first representing a circle and the second a hyperbola. Drawing two rectangular axes OX and OY, the circle is described from O with the radius a. The coordinates of the center of the hyperbola are found to be $OA = \frac{1}{2}b$ and $AC = \frac{1}{2}c$, while its diameter $BD = \sqrt{b^2 - c^2}$, from which the two branches may be described. The intersections of the circle with the hyperbola give the real values of x and y. If $a = 1$, $b = 4$, and $c = 3$, there are but two real values for x and two real values for y, since the circle intersects but one branch of the hyperbola; here Om is the positive and Op the negative value of x, while mn is the positive and pq the negative value of y. When the radius a is so large that the circle intersects both branches of the hyperbola there are four real values of both x and y.

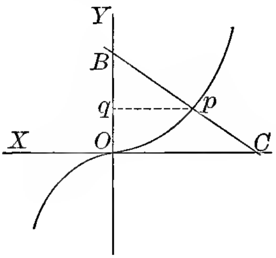


By a similar method approximate values of the real roots of an algebraic equation containing but one unknown quantity may be graphically found. For instance, let the cubic equation $x^3 + ax - b = 0$ be required to be solved.* This may be written as the two simultaneous equations

$$y = x^3, \qquad y = -ax + b,$$

* See Proceedings of the Engineers' Club of Philadelphia, 1884, Vol. IV, pp. 47–49

and the graph of each being plotted, the abscissas of their points of intersection give the real roots of the cubic. The curve $y = x^3$ should be plotted upon cross-section paper by the help of a table of cubes; then OB is laid off equal to b, and OC equal to a/b, taking care to observe the signs of a and b. The line joining B and C cuts the curve at p, and hence qp is the real root of $x^3 + ax - b = 0$. If the cubic equation have three real roots the straight line BC will intersect the curve in three points.



Some algebraic equations of higher degrees may be graphically solved in a similar manner. For the quartic equation $z^4 + Az^2 + Bz - C = 0$, it is best to put $z = A^{\frac{1}{2}}x$, and thus reduce it to the form $x^4 + x^2 + bx - c = 0$; then the two equations to be plotted are

$$y = x^4 + x^2, \qquad y = -bx + c,$$

the first of which may be drawn once for all upon cross-section paper, while the straight line represented by the second may be drawn for each particular case, as described above.*

This method is also applicable to many transcendental equations; thus for the equation $Ax - B\sin x = 0$ it is best to write $ax - \sin x = 0$; then $y = \sin x$ is readily plotted by help of a table of sines, while $y = ax$ is a straight line passing through the origin. In the same way $a^x - x^2 = 0$ gives the curve represented by $y = a^x$ and the parabola represented by $y = x^2$, the intersections of which determine the real roots of the given equation.

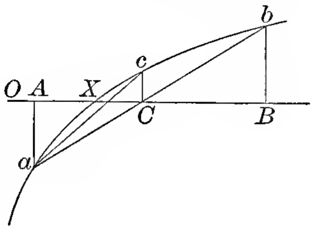
Prob. 2. Devise a graphic solution for finding approximate values of the real roots of the equation $x^5 + ax^3 + bx^2 + cx + d = 0$.

Prob. 3. Determine graphically the number and the approximate values of the real roots of the equation arc $x - 8\sin x = 0$. (Ans.—Six real roots, $x = \pm 159°$, $\pm 430°$, and $\pm 456°$.)

* For an extension of this method to the determination of imaginary roots, see Phillips and Beebe's Graphic Algebra, New York, 1882.

ART. 3. THE REGULA FALSI.

One of the oldest methods for computing the real root of an equation is the rule known as "regula falsi," often called the method of double position.* It depends upon the principle that if two numbers x_1 and x_2 be substituted in the expression $f(x)$, and if one of these renders $f(x)$ positive and the other renders it negative, then at least one real root of the equation $f(x)=0$ lies between x_1 and x_2. Let the figure represent a part of the real graph of the equation $y=f(x)$. The point X, where the curve crosses the axis of abscissas, gives a real root OX of the equation $f(x)=0$. Let OA and OB be inferior and superior limits of the root OX which are determined either by trial or by the method of Art. 5. Let Aa and Bb be the values of $f(x)$ corresponding to these limits. Join ab, then the intersection C of the straight line ab with the axis OB gives an approximate value OC for the root. Now compute Cc and join ac, then the intersection D gives a value OD which is closer still to the root OX.



Let x_1 and x_2 be the assumed values OA and OB, and let $f(x_1)$ and $f(x_2)$ be the corresponding values of $f(x)$ represented by Aa and Bb, these values being with contrary signs. Then from the similar triangle AaC and BbC the abscissa OC is

$$x_3=\frac{x_2f(x_1)-x_1f(x_2)}{f(x_1)-f(x_2)}=x_1+\frac{(x_2-x_1)f(x_1)}{f(x_1)-f(x_2)}=x_2+\frac{(x_2-x_1)f(x_2)}{f(x_1)-f(x_2)}.$$

By a second application of the rule to x_1 and x_3, another value x_4 is computed, and by continuing the process the value of x can be obtained to any required degree of precision.

As an example let $f(x)=x^5+5x^2+7=0$. Here it may be found by trial that a real root lies between -2 and -1.8.

* This originated in India, and its first publication in Europe was by Abraham ben Esra, in 1130. See Matthiesen, Grundzüge der antiken und modernen Algebra der litteralen Gleichungen, Leipzig, 1878.

For $x_1 = -2$, $f(x_1) = -5$, and for $x_2 = -1.8$, $f(x_2) = +4.304$; then by the regula falsi there is found $x_3 = -1.90$ nearly. Again, for $x_3 = -1.90$, $f(x_3) = +0.290$, and these combined with x_1 and $f(x_1)$ give $x_4 = -1.906$, which is correct to the third decimal.

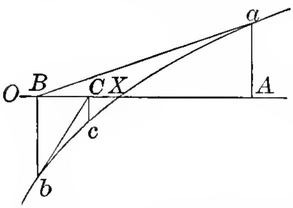
As a second example let $f(x) = \text{arc}\,x - \sin x - 0.5 = 0$. Here a graphic solution shows that there is but one real root, and that the value of it lies between 85° and 86° For $x_1 = 85°$, $f(x_1) = -0.01266$, and for $x_2 = 86°$, $f(x_2) = +0.00342$; then by the rule $x_3 = 85° 44'$, which gives $f(x_3) = -0.00090$. Again, combining the values for x_2 and x_3 there is found $x_4 = 85° 47'$, which gives $f(x_4) = -0.00009$. Lastly, combining the values for x_2 and x_4 there is found $x_5 = 85° 47'.4$, which is as close an approximation as can be made with five-place tables.

In the application of this method it is to be observed that the signs of the values of x and $f(x)$ are to be carefully regarded, and also that the values of $f(x)$ to be combined in one operation should have opposite signs. For the quickest approximation the values of $f(x)$ to be selected should be those having the smallest numerical values.

Prob. 4. Compute by the regula falsi the real roots of $x^5 - 0.25 = 0$. Also those of $x^2 + \sin 2x = 0$.

ART. 4. NEWTON'S APPROXIMATION RULE.

Another useful method for approximating to the value of the real root of an equation is that devised by Newton in 1666.*



If $y = f(x)$ be the equation of a curve, OX in the figure represents a real root of the equation $f(x) = 0$. Let OA be an approximate value of OX, and Aa the corresponding value of $f(x)$. At a let aB be drawn tangent to the curve; then OB is another approximate value of OX.

* See Analysis per equationes numero terminorum infinitas, p. 269, Vol. I of Horsely's edition of Newton's works (London, 1779), where the method is given in a somewhat different form.

Let Bb be the value of $f(x)$ corresponding to OB, and at b let the tangent bC be drawn; then OC is a closer approximation to OX, and thus the process may be continued.

Let $f'(x)$ be the first derivative of $f(x)$; or, $f'(x) = df(x)/dx$. For $x = x_1 = OA$ in the figure, the value of $f(x_1)$ is the ordinate Aa, and the value of $f'(x_1)$ is the tangent of the angle aBA; this tangent is also Aa/AB. Hence $AB = f(x_1)/f'(x_1)$, and accordingly OB and OC are found by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)},$$

which is Newton's approximation rule. By a third application to x_3 the closer value x_4 is found, and the process may be continued to any degree of precision required.

For example, let $f(x) = x^5 + 5x^2 + 7 = 0$. The first derivative is $f'(x) = 5x^4 + 10x$. Here it may be found by trial that -2 is an approximate value of the real root. For $x_1 = -2$ $f(x_1) = -5$, and $f'(x_1) = 60$, whence by the rule $x_2 = -1.92$. Now for $x_2 = -1.92$ are found $f(x_2) = -0.6599$ and $f'(x_2) = 29.052$, whence by the rule $x_3 = -1.906$, which is correct to the third decimal.

As a second example let $f(x) = x^2 + 4 \sin x = 0$. Here the first derivative is $f'(x) = 2x + 4 \cos x$. An approximate value of x found either by trial or by a graphic solution is $x = -1.94$, corresponding to about $-111° 09'$. For $x_1 = -1.94$, $f(x_1) = 0.03304$ and $f'(x_1) = -5.323$, whence by the rule $x_2 = -1.934$. By a second application $x_3 = -1.9328$, which corresponds to an angle of $-110° 54\frac{1}{2}'$.

In the application of Newton's rule it is best that the assumed value of x_1 should be such as to render $f(x_1)$ as small as possible, and also $f'(x_1)$ as large as possible. The method will fail if the curve has a maximum or minimum between a and b. It is seen that Newton's rule, like the regula falsi, applies equally well to both transcendental and algebraic equations, and moreover that the rule itself is readily kept in mind by help of the diagram.

Prob. 5. Compute by Newton's rule the real roots of the algebraic equation $x^4 - 7x + 6 = 0$. Also the real roots of the transcendental equation $\sin x + \text{arc}\, x - 2 = 0$.

ART. 5. SEPARATION OF THE ROOTS.

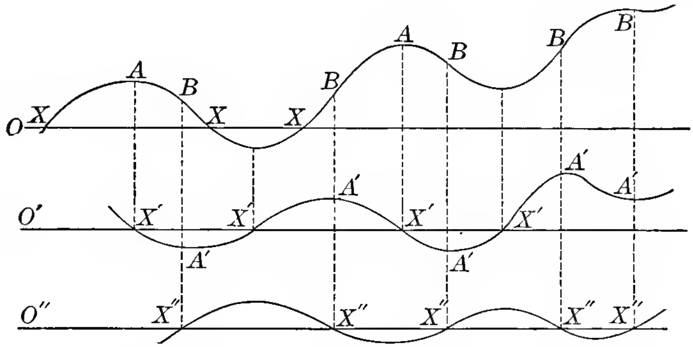
The roots of an equation are of two kinds, real roots and imaginary roots. Equal real roots may be regarded as a special class, which lie at the limit between the real and the imaginary. If an equation has p equal roots of one value and q equal roots of another value, then its first derivative equation has $p - 1$ roots of the first value and $q - 1$ roots of the second value, and thus all the equal roots are contained in a factor common to both primitive and derivative. Equal roots may hence always be readily detected and removed from the given equation. For instance, let $x^4 - 9x^2 + 4x + 12 = 0$, of which the derivative equation is $4x^3 - 18x + 4 = 0$; as $x - 2$ is a factor of these two equations, two of the roots of the primitive equation are $+ 2$.

The problem of determining the number of the real and imaginary roots of an algebraic equation is completely solved by Sturm's theorem. If, then, two values be assigned to x the number of real roots between those limits is found by the same theorem, and thus by a sufficient number of assumptions limits may be found for each real root. As Sturm's theorem is known to all who read these pages, no applications of it will be here given, but instead an older method due to Hudde will be presented which has the merit of giving a comprehensive view of the subject, and which moreover applies to transcendental as well as to algebraic equations.*

If any equation $y = f(x)$ be plotted with values of x as abscissas and values of y as ordinates, a real graph is obtained whose intersections with the axis OX give the real roots of the

* Devised by Hudde in 1659 and published by Rolle in 1690. See Œuvres de Lagrange, Vol. VIII, p. 190.

equation $f(x) = 0$. Thus in the figure the three points marked X give three values OX for three real roots. The curve which represents $y = f(x)$ has points of maxima and minima marked A, and inflection points marked B. Now let the first deriva-



tive equation $dy/dx = f'(x)$ be formed and be plotted in the same manner on the axis $O'X'$. The condition $f'(x) = 0$ gives the abscissas of the points A, and thus the real roots $O'X'$ give limits separating the real roots of $f(x) = 0$. To ascertain if a real root OX lies between two values of $O'X'$ these two values are to be substituted in $f(x)$: if the signs of $f(x)$ are unlike in the two cases, a real root of $f(x) = 0$ lies between the two limits; if the signs are the same, a real root does not lie between those limits.

In like manner if the second derivative equation, that is, $d^2y/dx^2 = f''(x)$, be plotted on $O''X''$, the intersections give limits which separate the real roots of $f'(x) = 0$. It is also seen that the roots of the second derivative equation are the abscissas of the points of inflection of the curve $y = f(x)$.

To illustrate this method let the given equation be the quintic $f(x) = x^5 - 5x^3 + 6x + 2 = 0$. The first derivative equation is $f'(x) = 5x^4 - 15x^2 + 6 = 0$, the roots of which are approximately -1.59, -0.69, $+0.69$, $+1.59$. Now let each of these values be substituted for x in the given quintic, as also the values $-\infty$, 0, and $+\infty$, and let the corresponding values of $f(x)$ be determined as follows:

$$x = -\infty,\quad -1.59,\quad -0.69,\quad 0,\quad +0.69,\quad +1.59,\quad +\infty;$$
$$f(x) = -\infty,\quad +2.4,\quad -0.6,\quad +2,\quad +4.7,\quad +1.6,\quad +\infty.$$

Since $f(x)$ changes sign between $x_0 = -\infty$ and $x_1 = -1.59$, one real root lies between these limits; since $f(x)$ changes sign between $x_1 = -1.59$ and $x_2 = -0.69$, one real root lies between these limits; since $f(x)$ changes sign between $x_2 = -0.69$ and $x_3 = 0$, one real root lies between these limits; since $f(x)$ does not change sign between $x_3 = 0$ and $x_4 = \infty$, a pair of imaginary roots is indicated, the sum of which lies between $+0.69$ and ∞.

As a second example let $f(x) = e^x - e^{2x} - 4 = 0$. The first derivative equation is $f'(x) = e^x - 2e^{2x} = 0$, which has two roots $e^x = \frac{1}{2}$ and $e^x = 0$, the latter corresponding to $x = -\infty$. For $x = -\infty$, $f(x)$ is negative; for $e^x = \frac{1}{2}$, $f(x)$ is negative; for $x = +\infty$, $f(x)$ is negative. The equation $e^x - e^{2x} - 4 = 0$ has, therefore, no real roots.

When the first derivative equation is not easily solved, the second, third, and following derivatives may be taken until an equation is found whose roots may be obtained. Then, by working backward, limits may be found in succession for the roots of the derivative equations until finally those of the primative are ascertained. In many cases, it is true, this process may prove lengthy and difficult, and in some it may fail entirely; nevertheless the method is one of great theoretical and practical value.

Prob. 6. Show that $e^x + e^{-3x} - 4 = 0$ has two real roots, one positive and one negative.

Prob. 7. Show that $x^6 + x + 1 = 0$ has no real roots; also that $x^6 - x - 1 = 0$ has two real roots, one positive and one negative.

ART. 6. NUMERICAL ALGEBRAIC EQUATIONS.

An algebraic equation of the n^{th} degree may be written with all its terms transposed to the first member, thus:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0;$$

and if all the coefficients and the absolute term are real numbers, this is commonly called a numerical equation. The first member may for brevity be denoted by $f(x)$ and the equation itself by $f(x) = 0$.

The following principles of the theory of algebraic equations with real coefficients, deduced in text-books on algebra, are here recapitulated for convenience of reference:

(1) If x_1 is a root of the equation, $f(x)$ is divisible by $x - x_1$; and conversely, if $f(x)$ is divisible by $x - x_1$, then x_1 is a root of the equation.

(2) An equation of the n^{th} degree has n roots and no more.

(3) If $x_1, x_2, \ldots x_n$ are the roots of the equation, then the product $(x - x_1)(x - x_2) \ldots (x - x_n)$ is equal to $f(x)$.

(4) The sum of the roots is equal to $-a_1$; the sum of the products of the roots, taken two in a set, is equal to $+a_2$; the sum of the products of the roots, taken three in a set, is equal to $-a_3$; and so on. The product of all the roots is equal to $-a_n$ when n is odd, and to $+a_n$ when n is even.

(5) The equation $f(x) = 0$ may be reduced to an equation lacking its second term by substituting $y - a_1/n$ for x.*

(6) If an equation has imaginary roots, they occur in pairs of the form $p \pm qi$ where i represents $\sqrt{-1}$.

(7) An equation of odd degree has at least one real root whose sign is opposite to that of a_n.

(8) An equation of even degree, having a_n negative, has at least two real roots, one being positive and the other negative.

(9) A complete equation cannot have more positive roots than variations in the signs of its terms, nor more negative roots than permanences in signs. If all roots be real, there are as many positive roots as variations, and as many negative roots as permanences.†

(10) In an incomplete equation, if an even number of terms, say $2m$, are lacking between two other terms, then it has at least $2m$

* By substituting $y^2 + py + q$ for x, the quantities p and q may be determined so as to remove the second and third terms by means of a quadratic equation, the second and fourth terms by means of a cubic equation, or the second and fifth terms by means of a quartic equation.

† The law deduced by Harriot in 1631 and by Descartes in 1639.

imaginary roots; if an odd number of terms, say $2m + 1$, are lacking between two other terms, then it has at least either $2m + 2$ or $2m$ imaginary roots, according as the two terms have like or unlike signs.*

(11) Sturm's theorem gives the number of real roots, provided that they are unequal, as also the number of real roots lying between two assumed values of x.

(12) If a_r is the greatest negative coefficient, and if a_s is the greatest negative coefficient after x is changed into $-x$, then all real roots lie between the limits $a_r + 1$ and $-(a_s + 1)$.

(13) If a_h is the first negative and a_r the greatest negative coefficient, then $a_r^{\frac{1}{n-h}} + 1$ is a superior limit of the positive roots. If a_k be the first negative and a_s the greatest negative coefficient after x is changed into $-x$, then $a_s^{\frac{1}{n-k}} + 1$ is a numerically superior limit of the negative roots.

(14) Inferior limits of the positive and negative roots may be found by placing $x = z^{-1}$ and thus obtaining an equation $f(z) = 0$ whose roots are the reciprocals of $f(x) = 0$.

(15) Horner's method, using the substitution $x = z - r$ where r is an approximate value of x_1, enables the real root x_1 to be computed to any required degree of precision.

The application of these principles and methods will be familiar to all who read these pages. Horner's method may be also modified so as to apply to the computation of imaginary roots after their approximate values have been found.† The older method of Hudde and Rolle, set forth in Art. 5, is however one of frequent convenient application, for such algebraic equations as actually arise in practice. By its use, together with principles (13) and (14) above, and the regula falsi of Art. 3, the real roots may be computed without any assumptions whatever regarding their values.

For example, let a sphere of diameter D and specific gravity

* Established by Du Gua; see Memoirs Paris Academy, 1741, pp. 435–494.

† Sheffler, Die Auflösung der algebraischen und transzendenten Gleichungen, Braunschweig, 1859; and Jelink, Die Auflösung der höheren numerischen Gleichungen, Leipzig, 1865.

g float in water, and let it be required to find the depth of immersion. The solution of the problem gives for the depth x the cubic equation

$$x^3 - \tfrac{3}{2}Dx^2 + \tfrac{1}{2}D^3g = 0.$$

As a particular case let $D = 2$ feet and $g = 0.65$; then the equation

$$x^3 - 3x^2 + 2.6 = 0$$

is to be solved. The first derivative equation is $3x^2 - 6x = 0$ whose roots are 0 and 2. Substituting these, there is found one negative root, one positive root less than 2, and one positive root greater than 2. The physical aspect of the question excludes the first and last root, and the second is to be computed. By (13) and (14) an inferior limit of this root is about 0.5, so that it lies between 0.5 and 2. For $x_1 = 0.5$, $f(x_1) = +1.975$, and for $x_2 = 2$, $f(x_2) = -1.4$; then by the regula falsi $x_3 = 1.35$. For $x_3 = 1.35$, $f(x_3) = -0.408$, and combining this with x_1 the regula falsi gives $x_4 = 1.204$ feet, which, except in the last decimal, is the correct depth of immersion of the sphere.

Prob. 8. The diameter of a water-pipe whose length is 200 feet and which is to discharge 100 cubic feet per second under a head of 10 feet is given by the real root of the quintic equation $x^5 - 38x - 101 = 0$. Find the value of x.

ART. 7. TRANSCENDENTAL EQUATIONS.

Rules (1) to (15) of the last article have no application to trigonometrical or exponential equations, but the general principles and methods of Arts. 2–5 may be always used in attempting their solution. Transcendental equations may have one, many, or no real roots, but those arising from problems in physical science must have at least one real root. Two examples of such equations will be presented.

A cylinder of specific gravity g floats in water, and it is required to find the immersed arc of the circumference. If this be expressed in circular measure it is given by the transcedental equation

$$f(x) = x - \sin x - 2\pi g = 0.$$

The first derivative equation is $1 - \cos x = 0$, whose root is any even multiple of 2π. Substituting such multiples in $f(x)$ it is found that the equation has but one real root, and that this lies between 0 and 2π; substituting $\frac{1}{2}\pi$, $\frac{3}{4}\pi$, and π for x, it is further found that this root lies between $\frac{3}{4}\pi$ and π.

As a particular case let $g = 0.424$, and for convenience in using the tables let x be expressed in degrees; then

$$f(x) = x - 57^\circ.2958 \sin x - 152^\circ.64.$$

Now proceeding by the regula falsi (Art. 3) let $x_1 = 180^\circ$ and $x_2 = 135^\circ$, giving $f(x_1) = +27^\circ.36$ and $f(x_2) = -58^\circ.16$, whence $x_3 = 166^\circ$. For $x_3 = 166^\circ$, $f(x_3) = -0^\circ.469$, and hence 166° is an approximate value of the root. Continuing the process, x is found to be $166^\circ.237$, or in circular measure $x = 2.9014$ radians.

As a second example let it be required to find the horizontal tension of a catenary cable whose length is 22 feet, span 20 feet, and weight 10 pounds per linear foot, the ends being suspended from two points on the same level. If l be the span, s the length of the cable, and z a length of the cable whose weight equals the horizontal tension, the solution of the problem leads to the transcendental equation $s = \left(e^{\frac{l}{2z}} - e^{-\frac{l}{2z}}\right) z$, or inserting the numerical values,

$$f(z) = 22 - \left(e^{\frac{10}{z}} - e^{-\frac{10}{z}}\right) z = 0$$

is the equation to be solved. The first derivative equation is

$$f'(z) = -\left(e^{\frac{10}{z}} - e^{-\frac{10}{z}}\right) + \frac{10}{z}\left(e^{\frac{10}{z}} + e^{-\frac{10}{z}}\right) = 0,$$

and this substituted in $f(z)$ shows that one real root is less than about 20. Assume $z_1 = 15$, then $f(z_1) = 0.486$ and $f'(z_1) = 0.206$, whence by Newton's rule (Art. 4) $z_2 = 13$ nearly. Next for $z_2 = 13$, $f(z_2) = -0.0298$ and $f'(z_2) = 0.322$, whence $z_3 = 13.1$. Lastly for $z_3 = 13.1$ $f(z_3) = 0.0012$ and $f'(z_3) = 0.3142$, whence $z_4 = 13.096$, which is a sufficiently close approximation. The horizontal tension in the given catenary is hence 130.96 pounds.*

* Since $e^{\theta} - e^{-\theta} = 2 \sinh \theta$, this equation may be written $11\theta - 10 \sinh \theta$, where $\theta = 10z^{-1}$, and the solution may be expedited by the help of tables of hyperbolic functions. See Chapter IV

Prob. 9. Show that the equation $3 \sin x - 2x - 5 = 0$ has but one real root, and compute its value.

Prob. 10. Find the number of real roots of the equation $2x + \log x - 10\,000 = 0$, and show that the value of one of them is $x = 4995.74$.

ART. 8. ALGEBRAIC SOLUTIONS.

Algebraic solutions of complete algebraic equations are only possible when the degree n is less than 5. It frequently happens, moreover, that the algebraic solution cannot be used to determine numerical values of the roots as the formulas expressing them are in irreducible imaginary form. Nevertheless the algebraic solutions of quadratic, cubic, and quartic equations are of great practical value, and the theory of the subject is of the highest importance, having given rise in fact to a large part of modern algebra.

The solution of the quadratic has been known from very early times, and solutions of the cubic and quartic equations were effected in the sixteenth century. A complete investigation of the fundamental principles of these solutions was, however, first given by Lagrange in 1770.* This discussion showed, if the general equation of the n^{th} degree, $f(x) = 0$, be deprived of its second term, thus giving the equation $f(y) = 0$, that the expression for the root y is given by

$$y = \omega s_1 + \omega^2 s_2 + \ldots + \omega^{n-1} s_{n-1},$$

in which n is the degree of the given equation, ω is, in succession, each of the n^{th} roots of unity, $1, \epsilon, \epsilon^2, \ldots \epsilon^{n-1}$, and $s_1, s_2, \ldots s_{n-1}$ are the so-called elements which in soluble cases are determined by an equation of the $n - 1^{\text{th}}$ degree. For instance, if $n = 3$ the equation is of the third degree or a cubic, the three values of ω are

$$\omega_1 = 1, \quad \omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3} = \epsilon, \quad \omega = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3} = \epsilon^2,$$

*Memoirs of Berlin Academy, 1769 and 1770; reprinted in Œuvres de Lagrange (Paris, 1868), Vol. II, pp. 539–562. See also Traité de la résolution des équations numeriques, Paris, 1798 and 1808.

and the three roots are expressed by

$$y_1 = s_1 + s_2, \qquad y_2 = \epsilon s_1 + \epsilon^2 s_2, \qquad y_3 = \epsilon^2 s_1 + \epsilon s_2,$$

in which s_1^3 and s_2^3 are found to be the roots of a quadratic equation (Art. 9).

The n values of ω are the n roots of the binomial equation $\omega^n - 1 = 0$. If n be odd, one of these is real and the others are imaginary; if n be even, two are real and $n - 2$ are imaginary.* Thus the roots of $\omega^2 - 1 = 0$ are $+1$ and -1; those of $\omega^3 - 1 = 0$ are given above; those of $\omega^4 - 1 = 0$ are $+1$, $+i$, -1, and $-i$ where i is $\sqrt{-1}$. For the equation $\omega^5 - 1 = 0$ the real root is $+1$, and the imaginary roots are denoted by ϵ, ϵ^2, ϵ^3, ϵ^4; to find these let $\omega^5 - 1 = 0$ be divided by $\omega - 1$, giving

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0,$$

which being a reciprocal equation can be reduced to a quadratic, and the solution of this furnishes the four values,

$$\epsilon = -\tfrac{1}{4}(1 - \sqrt{5} + \sqrt{-10 - 2\sqrt{5}}), \quad \epsilon^2 = -\tfrac{1}{4}(1 + \sqrt{5} + \sqrt{-10 + 2\sqrt{5}}),$$

$$\epsilon^4 = -\tfrac{1}{4}(1 - \sqrt{5} - \sqrt{-10 - 2\sqrt{5}}), \quad \epsilon^3 = -\tfrac{1}{4}(1 + \sqrt{5} - \sqrt{-10 + 2\sqrt{5}}),$$

where it will be seen that $\epsilon . \epsilon^4 = 1$ and $\epsilon^2 . \epsilon^3 = 1$, as should be the case, since $\epsilon^5 = 1$.

In order to solve a quadratic equation by this general method let it be of the form

$$x^2 + 2ax + b = 0,$$

and let x be replaced by $y - a$, thus reducing it to

$$y^2 - (a^2 - b) = 0.$$

Now the two roots of this are $y_1 = + s_1$ and $y_2 = - s_1$, whence the product of $(y - s_1)$ and $(y + s_1)$ is

$$y^2 - s^2 = 0.$$

Thus the value of s^2 is given by an equation of the first degree,

* The values of ω are, in short, those of the n "vectors" drawn from the center which divide a circle of radius unity into n equal parts, the first vector $\omega_1 = 1$ being measured on the axis of real quantities. See Chapter X.

$s^2 = a^2 - b$; and since $x = -a + y$, the roots of the given equation are

$$x_1 = -a + \sqrt{a^2 - b}, \qquad x_2 = -a - \sqrt{a^2 - b},$$

which is the algebraic solution of the quadratic.

The equation of the $n - 1^{\text{th}}$ degree upon which the solution of the equation of the n^{th} degree depends is called a resolvent. If such a resolvent exists, the given equation is algebraically solvable; but, as before remarked, this is only the case for quadratic, cubic, and quartic equations.

Prob. 11. Show that the six 6^{th} roots of unity are $+1$, $+\frac{1}{2}(1 + \sqrt{-3})$, $-\frac{1}{2}(1 - \sqrt{-3})$, -1, $-\frac{1}{2}(1 + \sqrt{-3})$, $-\frac{1}{2}(1 - \sqrt{-3})$.

ART. 9. THE CUBIC EQUATION.

All methods for the solution of the cubic equation lead to the result commonly known as Cardan's formula.* Let the cubic be

$$x^3 + 3ax^2 + 3bx + 2c = 0, \tag{1}$$

and let the second term be removed by substituting $y - a$ for x, giving the form,

$$y^3 + 3By + 2C = 0, \tag{1'}$$

in which the values of B and C are

$$B = -a^2 + b, \qquad C = a^3 - \tfrac{3}{2}ab + c. \tag{2}$$

Now by the Lagrangian method of Art. 8 the values of y are

$$y_1 = s_1 + s_2, \qquad y_2 = \epsilon s_1 + \epsilon^2 s_2, \qquad y_3 = \epsilon^2 s_1 + \epsilon s_2,$$

in which ϵ and ϵ^2 are the imaginary cube roots of unity. Forming the products of the roots, and remembering that $\epsilon^3 = 1$ and $\epsilon^3 + \epsilon^2 + 1 = 0$, there are found

$$y_1y_2 + y_1y_3 + y_2y_3 = -3s_1s_2 = +3B,$$
$$y_1y_2y_3 = s_1^3 + s_2^3 = -2C.$$

For the determination of s_1 and s_2 there are hence two equations from which results the quadratic resolvent

$s^6 + 2Cs^3 - B^3 = 0$, and thus

$$s_1 = (-C + \sqrt{B^3 + C^2})^{\frac{1}{3}}, \quad s_2 = (-C - \sqrt{B^3 + C^2})^{\frac{1}{3}}. \tag{3}$$

*Deduced by Ferreo in 1515, and first published by Cardan in 1545.

One of the roots of the cubic in y therefore is

$$y_1 = (-C + \sqrt{B^3 + C^2})^{\frac{1}{3}} + (-C - \sqrt{B^3 + C^2})^{\frac{1}{3}},$$

and this is the well-known formula of Cardan.

The algebraic solution of the cubic equation (1) hence consists in finding B and C by (2) in terms of the given coefficients, and then by (3) the elements s_1 and s_2 are determined. Finally,

$$\begin{aligned} x_1 &= -a + (s_1 + s_2), \\ x_2 &= -a - \tfrac{1}{2}(s_1 + s_2) + \tfrac{1}{2}\sqrt{-3}(s_1 - s_2), \\ x_3 &= -a - \tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{2}\sqrt{-3}(s_1 - s_2), \end{aligned} \qquad (4)$$

which are the algebraic expressions of the three roots.

When $B^3 + C^2$ is negative the numerical solution of the cubic is not possible by these formulas, as then both s_1 and s_2 are in irreducible imaginary form. This, as is well known, is the case of three real roots, $s_1 + s_2$ being a real, while $s_1 - s_2$ is a pure imaginary.* When $B^3 + C^2$ is 0 the elements s_1 and s_2 are equal, and there are two equal roots, $x_2 = x_3 = -a + C^{\frac{1}{3}}$, while the other root is $x_1 = -a - 2C^{\frac{1}{3}}$.

When $B^3 + C^2$ is positive the equation has one real and two imaginary roots, and formulas (2), (3), and (4) furnish the numerical values of the roots of (1). For example, take the cubic

$$x^3 - 4.5x^2 + 12x - 5 = 0,$$

whence by comparison with (1) are found $a = -1.5$, $b = +4$, $c = -2.5$. Then from (2) are computed $B = 1.75$, $C = +3.125$. These values inserted in (3) give $s_1 = +0.9142$, $s_2 = -1.9142$; thus $s_1 + s_2 = -1.0$ and $s_1 - s_2 = +2.8284$. Finally, from (4)

$$\begin{aligned} x_1 &= 1.5 - 1.0 = +0.5, \\ x_2 &= 1.5 + 0.5 + 1.4142\sqrt{-3} = 2 + 2.4495i, \\ x_3 &= 1.5 + 0.5 - 1.4142\sqrt{-3} = 2 - 2.4495i, \end{aligned}$$

which are the three roots of the given cubic.

* The numerical solution of this case is possible whenever the angle whose cosine is $-C/\sqrt{-B^3}$ can be geometrically trisected.

Prob. 12. Compute the roots of $x^3 - 2x - 5 = 0$. Also the roots of $x^3 + 0.6x^2 - 5.76x + 4.32 = 0$.

Prob. 13. A cone has its altitude 6 inches and the diameter of its base 5 inches. It is placed with vertex downwards and one fifth of its volume is filled with water. If a sphere 4 inches in diameter be then put into the cone, what part of its radius is immersed in the water? (Ans. 0.5459 inches).

ART. 10. THE QUARTIC EQUATION.

The quartic equation was first solved in 1545 by Ferrari, who separated it into the difference of two squares. Lagrange in 1637 resolved it into the product of two quadratic factors. Tschirnhausen in 1683 removed the second and fourth terms. Euler in 1732 and Lagrange in 1767 effected solutions by assuming the form of the roots. All these methods lead to cubic resolvents, the roots of which are first to be found in order to determine those of the quartic.

The methods of Euler and Lagrange, which are closely similar, first reduce the quartic to one lacking the second term,

$$y^4 + 6By^2 + 4Cy + D = 0;$$

and the general form of the roots being taken as

$$y_1 = +\sqrt{s_1} + \sqrt{s_2} + \sqrt{s_3}, \qquad y_3 = -\sqrt{s_1} + \sqrt{s_2} - \sqrt{s_3},$$
$$y_2 = +\sqrt{s_1} - \sqrt{s_2} - \sqrt{s_3}, \qquad y_4 = -\sqrt{s_1} - \sqrt{s_2} + \sqrt{s_3},$$

the values s_1, s_2, s_3, are shown to be the roots of the resolvent,

$$s^3 + 3Bs^2 + \tfrac{1}{4}(9B^2 - D)s - \tfrac{1}{4}C^2 = 0.$$

Thus the roots of the quartic are algebraically expressed in terms of the coefficients of the quartic, since the resolvent is solvable by the process of Art. 9.

Whatever method of solution be followed, the following final formulas, deduced by the author in 1892, will result.* Let the complete quartic equation be written in the form

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0. \qquad (1)$$

* See American Journal Mathematics, 1892, Vol. XIV, pp. 237–245.

First, let g, h, and k be determined from

$$g = a^2 - b, \quad h = b^3 + c^2 - 2abc + dg, \quad k = \tfrac{4}{3}ac - b^2 - \tfrac{1}{3}d. \qquad (2)$$

Secondly, let l be obtained by

$$l = \tfrac{1}{2}(h + \sqrt{h^2 + k^3})^{\frac{1}{3}} + \tfrac{1}{2}(h - \sqrt{h^2 + k^3})^{\frac{1}{3}} \qquad (3)$$

Thirdly, let u, v, and w be found from

$$u = g + l, \quad v = 2g - l, \quad w = 4u^2 + 3k - 12gl. \qquad (4)$$

Then the four roots of the quartic equation are

$$\left.\begin{aligned} x_1 &= -a + \sqrt{u} + \sqrt{v + \sqrt{w}}, \\ x_2 &= -a + \sqrt{u} - \sqrt{v + \sqrt{w}}, \\ x_3 &= -a - \sqrt{u} + \sqrt{v - \sqrt{w}}, \\ x_4 &= -a - \sqrt{u} - \sqrt{v - \sqrt{w}}, \end{aligned}\right\} \qquad (5)$$

in which the signs are to be used as written provided that $2a^3 - 3ab + c$ is a negative number; but if this is positive all radicals except $\sqrt{w}$ are to be reversed in sign.

These formulas not only serve for the complete theoretic discussion of the quartic (1), but they enable numerical solutions to be made whenever (3) can be computed, that is, whenever $h^2 + k^3$ is positive. For this case the quartic has two real and two imaginary roots. If there be either four real roots or four imaginary roots $h^2 + k^3$ is negative, and the irreducible case arises where convenient numerical values cannot be obtained, although they are correctly represented by the formulas.

As an example let a given rectangle have the sides p and q, and let it be required to find the length of an inscribed rectangle whose width is m. If x be this length, this is a root of the quartic equation

$$x^4 - (p^2 + q^2 + 2m^2)x^2 + 4pqmx - (p^2 + q^2 - m^2)m^2 = 0,$$

and thus the problem is numerically solvable by the above formulas if two roots are real and two imaginary. As a special case let $p = 4$ feet, $q = 3$ feet, and $m = 1$ foot; then

$$x^4 - 27x^2 + 48x - 24 = 0.$$

By comparison with (1) are found $a = 0$, $b = -4\frac{1}{2}$, $c = +12$, and $d = -24$. Then from (2), $g = +4\frac{1}{2}$, $h = -\frac{441}{8}$, and $k = +\frac{49}{4}$. Thus $h^2 + k^3$ is positive, and from (3) the value of l is -3.6067. From (4) are now found, $u = +0.8933$, $v = 12.6067$, and $w = +161.20$. Then, since c is positive, the values of the four roots are, by (5),

$$x_1 = -0.945 - \sqrt{12.607 + 12.697} = -5.975 \text{ feet},$$
$$x_2 = -0.945 + \sqrt{12.607 + 12.697} = +4.085 \text{ feet},$$
$$x_3 = -0.945 + \sqrt{12.607 - 12.697} = +0.945 + 0.30i,$$
$$x_4 = -0.945 - \sqrt{12.607 - 12.697} = +0.945 - 0.30i,$$

the second of which is evidently the required length. Each of these roots closely satisfies the given equation, the slight discrepancy in each case being due to the rounding off at the third decimal.*

Prob. 14. Compute the roots of the equation $x^4 + 7x + 6 = 0$. (Ans. -1.388, -1.000, $1.194 \pm 1.701i$.)

ART. 11. QUINTIC EQUATIONS.

The complete equation of the fifth degree is not algebraically solvable, nor is it reducible to a solvable form. Let the equation be

$$x^5 + 5ax^4 + 5bx^3 + 5cx^2 + 5dx + 2e = 0,$$

and by substituting $y - a$ for x let it be reduced to

$$y^5 + 5By^3 + 5Cy^2 + 5Dy + 2E = 0.$$

The five roots of this are, according to Art. 8,

$$y_1 = s_1 + s_2 + s_3 + s_4,$$
$$y_2 = \epsilon s_1 + \epsilon^2 s_2 + \epsilon^3 s_3 + \epsilon^4 s_4,$$
$$y_3 = \epsilon^2 s_1 + \epsilon^4 s_2 + \epsilon s_3 + \epsilon^3 s_4,$$
$$y_4 = \epsilon^3 s_1 + \epsilon s_2 + \epsilon^4 s_3 + \epsilon^2 s_4,$$
$$y_5 = \epsilon^4 s_1 + \epsilon^3 s_2 + \epsilon^2 s_3 + \epsilon s_4,$$

in which ϵ, ϵ^2, ϵ^3 ϵ^4 are the imaginary fifth roots of unity. Now if the several products of these roots be taken there will be

* This example is known by civil engineers as the problem of finding the length of a strut in a panel of the Howe truss.

found, by (4) of Art. 6, four equations connecting the four elements s_1, s_2, s_3, and s_4, namely,

$$-\ B = s_1s_4 + s_2s_3,$$
$$-\ C = s_1^2s_3 + s_2^2s_1 + s_3^2s_4 + s_4^2s_2,$$
$$-\ D = s_1^3s_2 + s_2^3s_4 + s_3^3s_1 + s_4^3s_3 - s_1^2s_4^2 - s_2^2s_3^2 + s_1s_2s_3s_4,$$
$$-\ 2E = s_1^5 + s_2^5 + s_3^5 + s_4^5 + 5(s_1^2s_2^2s_4 + s_1^2s_3^2s_2 + s_2^2s_4^2s_3 + s_3^2s_4^2s_1)$$
$$- 5(s_1^3s_3s_4 + s_2^3s_1s_3 + s_3^3s_2s_4 + s_4^3s_1s_2);$$

but the solution of these leads to an equation of the 120th degree for s, or of the 24th degree for s^5. However, by taking $s_1s_4 - s_2s_3$ or $s_1^5 + s_2^5 + s_3^5 + s_4^5$ as the unknown quantity, a resolvent of the 6th degree is obtained, and all efforts to find a resolvent of the fourth degree have proved unavailing.

Another line of attack upon the quintic is in attempting to remove all the terms intermediate between the first and the last. By substituting $y^2 + py + q$ for x, the values of p and q may be determined so as to remove the second and third terms by a quadratic equation, or the second and third by a cubic equation, or the second and fourth by a quartic equation, as was first shown by Tschirnhausen in 1683. By substituting $y^3 + py^2 + qy + r$ for x, three terms may be removed, as was shown by Bring in 1786. By substituting $y^4 + py^3 + qy^2 + ry + t$ for x it was thought by Jerrard in 1833 that four terms might be removed, but Hamilton showed later that this leads to equations of a degree higher than the fourth.

In 1826 Abel gave a demonstration that the algebraic solution of the general quintic is impossible, and later Galois published a more extended investigation leading to the same conclusion.* Although these discussions are complex, and not devoid of points of doubt,† they have been generally accepted as conclusive. Moreover, the fact that the quintic is still unsolved in spite of the enormous amount of work done upon it during the past two centuries, is strong evidence that the problem is an impossible one.

* See Jordan's Traité des substitutions et des équations algébriques, 1870.

† See Kronecker, Verhandlungen der Berliner Akademie, 1853, p. 38; also Cockle, Philosophical Magazine, 1854, Vol. VII, p. 134.

There are, however, numerous special forms of the quintic whose algebraic solution is possible. The oldest of these is the quintic of De Moivre,

$$y^5 + 5By^3 + 5B^2y + 2E = 0,$$

which is solved at once by making $s_2 = s_3 = 0$ in the element equations; then $-B = s_1s_4$ and $-2E = s_1^5 + s_4^5$, from which s_1 and s_4 are found, and $y_1 = s_1 + s_4$, or

$$y_1 = (-E + \sqrt{B^5 + E^2})^{\frac{1}{5}} + (-E - \sqrt{B^5 + E^2})^{\frac{1}{5}},$$

while the other roots are $y_2 = \epsilon s_1 + \epsilon^4 s_4$, $y_3 = \epsilon^2 s_1 + \epsilon^3 s_4$, $y_4 = \epsilon^3 s_1 + \epsilon^2 s_4$, and $y_5 = \epsilon^4 s_1 + \epsilon s_4$. If $B^5 + E^2$ be negative, this quintic has five real roots; if positive, there are one real and four imaginary roots.

When any relation, other than those expressed by the four element equations, exists between s_1, s_2, s_3, s_4, the quintic is solvable algebraically. As an infinite number of such relations may be stated, it follows that there are an infinite number of solvable quintics. In each case of this kind, however, the coefficients of the quintic are also related to each other by a certain equation of condition.

The complete solution of the quintic in terms of one of the roots of its resolvent sextic was made by McClintock in 1884.* By this method s_1^5, s_2^5, s_3^5, and s_4^5 are expressed as the roots of a quartic in terms of a quantity t which is the root of a sextic whose coefficients are rational functions of those of the given quintic. Although this has great theoretic interest, it is, of course, of little practical value for the determination of numerical values of the roots.

By means of elliptic functions the complete quintic can, however, be solved, as was first shown by Hermite in 1858. For this purpose the quintic is reduced by Jerrard's transformation to the form $x^5 + 5dx + 2e = 0$, and to this form can also be reduced the elliptic modular equation of the sixth degree. Other solutions by elliptic functions were made by

* American Journal of Mathematics, 1886, Vol. VIII, pp. 49–83.

Kronecker in 1861 and by Klein in 1884.* These methods, though feasible by the help of tables, have not yet been systematized so as to be of practical advantage in the numerical computation of roots.

Prob. 15. If the relation $s_1 s_4 = s_2 s_3$ exists, between the elements show that $s_1^5 + s_2^5 + s_3^5 + s_4^5 = -2E$.

Prob. 16. Compute the roots of $y^5 + 10y^3 + 20y + 6 = 0$, and also those of $y^5 - 10y^3 + 20y + 6 = 0$.

ART. 12. TRIGONOMETRIC SOLUTIONS.

When a cubic equation has three real roots the most convenient practical method of solution is by the use of a table of sines and cosines. If the cubic be stated in the form (1) of Art. 9, let the second term be removed, giving

$$y^3 + 3By + 2C = 0.$$

Now suppose $y = 2r \sin \theta$, then this equation becomes

$$8 \sin^3 \theta + 6\frac{B}{r^2} \sin \theta + 2\frac{C}{r^3} = 0,$$

and by comparison with the known trigonometric formula

$$8 \sin^3 \theta - 6 \sin \theta + 2 \sin 3\theta = 0,$$

there are found for r and $\sin 3\theta$ the values

$$r = \sqrt{-B}, \qquad \sin 3\theta = C/\sqrt{-B^3},$$

in which B is always negative for the case of three real roots (Art. 9). Now $\sin 3\theta$ being computed, 3θ is found from a table of sines, and then θ is known. Thus,

$$y_1 = 2r \sin \theta, \quad y_2 = 2r \sin (120° + \theta), \quad y_3 = 2r \sin (240° + \theta),$$

are the real roots of the cubic in y.†

* For an outline of these transcendental methods, see Hagen's Synopsis der höheren Mathematik, Vol. I, pp. 339–344.

† When B^3 is negative and numerically less than C^2, as also when B^3 is positive, this solution fails, as then one root is real and two are imaginary. In this case, however, a similar method of solution by means of hyperbolic sines is possible. See Grunert's Archiv für Mathematik und Physik, Vol. xxxviii, pp. 48–76.

For example, the depth of flotation of a sphere whose diameter is 2 feet and specific gravity 0.65, is given by the cubic equation $x^3 - 3x^2 + 2.6 = 0$ (Art. 6). Placing $x = y + 1$ this reduces to $y^3 - 3y + 0.6 = 0$, for which $B = -1$ and $C = +0.3$. Thus $r = 1$ and $\sin 3\theta = +0.3$. Next from a table of sines, $3\theta = 17° 27'$, and accordingly $\theta = 5° 49'$. Then

$$y_1 = 2 \sin \quad 5° 49' = +0.2027,$$
$$y_2 = 2 \sin 125° 49' = +1.6218,$$
$$y_3 = 2 \sin 245° 49' = -1.8245.$$

Adding 1 to each of these, the values of x are

$$x_1 = +1.203 \text{ feet}, \quad x_2 = +2.622 \text{ feet}, \quad x_3 = -0.825 \text{ feet};$$

and evidently, from the physical aspect of the question, the first of these is the required depth. It may be noted that the number 0.3 is also the sine of 162° 11′, but by using this the three roots have the same values in a different order.

When the quartic equation has four real roots its cubic resolvent has also three real roots. In this case the formulas of Art. 10 will furnish the solution if the three values of l be obtained from (3) by the help of a table of sines. The quartic being given, g, h, and k are found as before, and the value of k will always be negative for four real roots. Then

$$r = \sqrt{-k}, \qquad \sin 3\theta = -h/r^3,$$

and 3θ is taken from a table; thus θ is known, and the three values of l are

$$l_1 = r \sin \theta, \qquad l_2 = r \sin(120° + \theta), \qquad l_3 = r \sin(240° + \theta).$$

Next the three values of u, of v, and of w are computed, and those selected which give u, w, and $v - \sqrt{w}$ all positive quantities. Then (5) gives the required roots of the quartic.

As an example, take the case of the inscribed rectangle in Art. 10, and let $p = 4$ feet, $q = 3$ feet, $m = \sqrt{13}$ feet; then the quartic equation is

$$x^4 - 51x^2 + 48\sqrt{13}\, x - 156 = 0.$$

Here $a = 0$, $b = -8\frac{1}{2}$, $c = +12\sqrt{13}$, and $d = -156$. Next $g = +8\frac{1}{2}$, $h = -\frac{545}{8}$, and $k = -\frac{81}{4}$. The trigonometric work now begins; the value of r is found to be $+4\frac{1}{2}$, and that of $\sin 3\theta$ to be $+0.7476$; hence from the table $3\theta = 48° 23'$, and $\theta = 16° 07' 40''$. The three values of l are then computed by logarithmic tables, and found to be,

$$l_1 = +1.250, \qquad l_2 = +3.1187, \qquad l_4 = -4.3687.$$

Next the values of u, v, and w are obtained, and it is seen that only those corresponding to l_1 will render all quantities under the radicals positive; these quantities are $u = 9.75$, $v = 15.75$, and $w = 192.0$. Then the four roots of the quartic are

$$x_1 = -8.564, \quad x_2 = +2.319, \quad x_3 = +1.746, \quad x_4 = +4.499 \text{ feet},$$

of which only the second and third belong to inscribed rectangles, while the first and fourth belong to rectangles whose corners are on the sides of the given rectangle produced.

Trigonometric solutions of the quintic equation are not possible except for the binomial $x^5 \pm a$, and the quintic of De Moivre. The general trigonometric expression for the root of a quintic lacking its second term is $y = 2r_1 \cos\theta_1 + 2r_2 \cos\theta_2$, and to render a solution possible, r_1 and r_2, as well as $\cos\theta_1$ and $\cos\theta_2$, must be found; but these in general are roots of equations of the sixth or twelfth degree: in fact r_1^2 is the same as the function s_1s_4 of Art. 11, and r_2^2 is the same as s_2s_3. Here $\cos\theta_1$ and $\cos\theta_2$ may be either circular or hyperbolic cosines, depending upon the signs and values of the coefficients of the quintic.

Trigonometric solutions are possible for any binomial equation, and also for any equation which expresses the division of an angle into equal parts. Thus the roots of $x^6 + 1 = 0$ are $\cos m\, 30° \pm i \sin m\, 30°$, in which m has the values 1, 2, and 3. The roots of $x^5 - 5x^3 + 5x - 2\cos 5\theta = 0$ are $2\cos(m\, 72° + \theta)$ where m has the values 0, 1, 2, 3, and 4.

Prob. 17. Compute by a trigonometric solution the four roots of the quartic $x^4 + 4x^3 - 24x^2 - 76x - 29 = 0$. (Ans. -6.734, -1.550, $+0.262$, $+4.022$).

Prob. 18. Give a trigonometric solution of the quintic equation $x^5 - 5bx^3 + 5b^2x - 2e = 0$ for the case of five real roots. Compute the roots when $b = 1$ and $e = 0.752798$. (Ans. -1.7940, -1.3952, 0.2864, 0.9317, 1.9710.)

ART. 13. REAL ROOTS BY SERIES.

The value of x in any algebraic equation may be expressed as an infinite series. Let the equation be of any degree, and by dividing by the coefficient of the term containing the first power of x let it be placed in the form

$$a = x + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \ldots$$

Now let it be assumed that x can be expressed by the series

$$x = a + ma^2 + na^3 + pa^4 + qa^5 + \ldots$$

By inserting this value of x in the equation and equating the coefficients of like powers of a, the values of m, n, etc., are found, and then

$$x = a - ba^2 + (2b^2 - c)a^3 - (5b^3 - 5bc + d)a^4 + (14b^4 - 21b^2c + 6bd + 3c^2 - e)a^5$$
$$-(42b^5 - 84b^3c + 28b^2d + 28bc^2 - 7be - 7cd + f)a^6 + \ldots,$$

is an expression of one of the roots of the equation. In order that this series may converge rapidly it is necessary that a should be a small fraction.*

To apply this to a cubic equation the coefficients d, e, f, etc., are made equal to 0, For example, let $x^3 - 3x + 0.6 = 0$; this reduced to the given form is $0.2 = x - \frac{1}{3}x^3$, hence $a = 0.2$, $b = 0$, $c = -\frac{1}{3}$, and then

$$x = 0.2 + \tfrac{1}{3} \cdot 0.2^3 + \tfrac{1}{3} \cdot 0.2^5 + \text{etc.} = +0.20277,$$

which is the value of one of the roots correct to the fourth decimal place. This equation has three real roots, but the series gives only one of them; the others can, however, be found if their approximate values are known. Thus, one root is about $+1.6$, and by placing $x = y + 1.6$ there results an equation in y whose root by the series is found to be $+0.0218$, and hence $+1.6218$ is another root of $x^3 - 3x + 0.6 = 0$.

*This method is given by J. B. Mott in The Analyst, 1882, Vol. IX, p. 104.

Cardan's expression for the root of a cubic equation can be expressed as a series by developing each of the cube roots by the binomial formula and adding the results. Let the equation be $y^3 + 3By + 2C = 0$, whose root is, by Art. 9,

$$y = (-C + \sqrt{B^3 + C^2})^{\frac{1}{3}} + (-C - \sqrt{B^3 + C^2})^{\frac{1}{3}},$$

then this development gives the series,

$$y = 2(-C)^{\frac{1}{3}}\left(1 - \frac{2}{2}r - \frac{2 \cdot 5 \cdot 8}{2 \cdot 3 \cdot 4}r^2 - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}r^3 - \ldots\right),$$

in which r represents the quantity $(B^3 + C^2)/3C^2$. If $r = 0$ the equation has two equal roots and the third root is $2(-C)^{\frac{1}{3}}$. If r is numerically greater than unity the series is divergent, and the solution fails. If r is numerically less than unity and sufficiently small to make a quick convergence, the series will serve for the computation of one real root. For example, take the equation $x^3 - 6x + 6 = 0$, where $B = -2$ and $C = 3$; hence $r = 1/81$, and one root is

$$y = -2.8845(1 - 0.01235 - 0.00051 - 0.00032 -) = -2.846,$$

which is correct to the third decimal. In comparatively few cases, however, is this series of value for the solution of cubics.

Many other series for the expression of the roots of equations, particularly for trinomial equations, have been devised. One of the oldest is that given by Lambert in 1758, whereby the root of $x^n + ax - b = 0$ is developed in terms of the ascending powers of b/a. Other solutions were published by Euler and Lagrange. These series usually give but one root, and this only when the values of the coefficients are such as to render convergence rapid.

Prob. 19. Consult Euler's Anleitung zur Algebra (St. Petersburg, 1771), pp. 143–150, and apply his method of series to the solution of a quartic equation.

ART. 14. COMPUTATION OF ALL ROOTS.

A comprehensive and valuable method for the solution of equations by series was developed by McClintock, in 1894, by means of his Calculus of Enlargement.* By this method all the roots, whether real or imaginary, may be computed from a single series. The following is a statement of the method as applied to trinomial equations:

Let $x^n = nAx^{n-k} + B^n$ be the given trinomial equation. Substitute $x = By$ and thus reduce the equation to the form $y^n = nay^{n-k} + 1$ where $a = A/B^k$. Then if B^n is positive, the roots are given by the series

$$y = \omega + \omega^{1-k} a + \omega^{1-2k}(1 - 2k + n)a^2/2!$$
$$+ \omega^{1-3k}(1-3k+n)(1-3k+2n)a^3/3!$$
$$+ \omega^{1-4k}(1-4k+n)(1-4k+2n)(1-4k+3n)a^4/4! + \ldots,$$

in which ω represents in succession each of the roots of unity. If, however, B^n is negative, the given equation reduces to $y^n = nay^{n-k} - 1$, and the same series gives the roots if ω be taken in succession as each of the roots of -1.

In order that this series may be convergent the value of a^n must be numerically less than $k^{-k}(n - k)^{k-n}$; thus for the quartic $y^4 = 4ax + 1$, where $n = 4$ and $k = 3$, the value of a must be less than $27^{-\frac{1}{4}}$.

To apply this method to the cubic equation $x^3 = 3Ax \pm B^3$, place $n = 3$ and $k = 2$, and put $y = Bx$. It then becomes $y^3 = 3ay \pm 1$ where $a = A/B^2$, and the series is

$$y = \omega + \omega^2 a - \tfrac{1}{3}\omega a^3 + \tfrac{1}{3}\omega^2 a^4 + \ldots,$$

in which the values to be taken for ω are the cube roots of 1 or -1, as the case may be. For example, let $x^3 - 2x - 5 = 0$. Placing $y = 5^{\frac{1}{3}}x$, this reduces to $y^3 = 0.684y + 1$. Here $a = 0.228$, and as this is less than $4^{-\frac{1}{3}}$ the series is convergent. Making $\omega = 1$, the first root is

$$y = 1 + 0.2280 - 0.0039 + 0.0009 = 1.2250.$$

* See Bulletin of American Mathematical Society, 1894, Vol. I, p. 3; also American Journal of Mathematics, 1895, Vol. XVII, pp. 89–110.

Next making $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, ω^2 is $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, and the corresponding root is found to be

$$y = -0.6125 + 0.3836\sqrt{-3}.$$

Again, making $\omega = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ the third root is found to be the conjugate imaginary of the second. Lastly, multiplying each value of y by $5^{\frac{1}{3}}$,

$$x = 2.095, \qquad x = -1.047 \pm 1.136\sqrt{-1},$$

which are very nearly the roots of $x^3 - 2x - 5 = 0$.

In a similar manner the cubic $x^3 + 2x + 5 = 0$ reduces to $y^3 = -0.684y - 1$, for which the series is convergent. Here the three values of ω are, in succession, -1, $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and the three roots are $y = -0.777$ and $y = 0.388 \pm 1.137i$.

When all the roots are real, the method as above stated fails because the series is divergent. The given equation can, however, be transformed so as to obtain $n - k$ roots by one application of the general series and k roots by another. As an example, let $x^3 - 243x + 330 = 0$. For the first application this is to be written in the form

$$x = \frac{x^3}{243} + \frac{330}{243},$$

for which $n = 1$ and $k = -2$. To make the last term unity place $x = \frac{330}{243}y$, and the equation becomes

$$y = \frac{330^3}{243^2}y^3 + 1,$$

whence $a = 330^3/3.243^2$. These values of n, k, and a are now inserted in the above general value of y, and ω made unity: thus $y = 0.9983$, whence $x_1 = 1.368$ is one of the roots. For the second application the equation is to be written

$$x^2 = -\frac{330}{243}x^{-1} + 243,$$

for which $n = 2$ and $k = 3$. Placing $x = 243^{\frac{1}{2}}y$, this becomes

$$y^2 = -\frac{340}{243^{\frac{3}{2}}}y^{-1} + 1,$$

whence $a = -110/243^{\frac{3}{2}}$, and the series is convergent. These values of n, k, and a are now inserted in the formula for y, and ω is made $+1$ and -1 in succession, thus giving two values for y, from which $x_2 = 14.86$ and $x_3 = -16.22$ are the other roots of the given cubic.

McClintock has also given a similar and more general method applicable to other algebraic equations than trinomials. The equation is reduced to the form $y^n = na \,.\, \phi y \pm 1$, where $na \,.\, \phi y$ denotes all the terms except the first and the last. Then the values of y are expressed by the series

$$y = \omega + \omega^{1-n}\phi\omega \,.\, a + \omega^{1-n}\frac{d}{d\omega}\omega^{1-n}(\phi\omega)^2 \,.\, \frac{a^2}{2!} + $$
$$+ \left(\omega^{1-n}\frac{d}{d\omega}\right)^2 \omega^{1-n}(\phi\omega)^3 \,.\, \frac{a^3}{3!} + \ldots,$$

in which the values of ω are to be taken as before. The method is one of great importance in the theory of equations, as it enables not only the number of real and imaginary roots to be determined, but also gives their values when the convergence of the series is secured.

Prob. 20. Compute by the above method all the roots of the quartic $x^4 + x + 10 = 0$.

ART. 15. CONCLUSION.

While this Chapter forms a supplement to the theory of equations as commonly given in college text-books, yet the brief space allotted to it has prevented the discussion and development of many interesting branches. Chief among these is the topic of complex or imaginary roots, particularly of their graphical representation and their numerical computation. Although such roots rarely, if ever, are required in the solution of problems in physical science, their determination is a matter of much theoretic interest. It may be mentioned, however, that both the regula falsi and Newton's approximation rule may, by a slight modification, be adapted to the computation of these imaginary roots, approximate values of them being first obtained by trial.

A method of solution of numerical algebraic equations, which may be called a logarithmic process, was published by Gräffe in 1837, and exemplified by Encke in 1841.* It consists in deriving from the given equation another equation whose roots are high powers of those of the given one, the coefficients of the latter then easily furnishing the real roots and the moduluses of the imaginary roots. The method, although little known, is without doubt one of high practical values, as logarithmic tables are used throughout; moreover, Encke states that the time required to completely solve an equation of the seventh degree with six imaginary roots, as accurately as can be done with seven-place tables, is less than three hours.

The algebraic solutions of the quadratic, cubic, and quartic equations are valid not only for real coefficients, but also for imaginary ones. In the latter case the imaginary roots do not necessarily occur in pairs. The method of McClintock has the great merit that it is applicable also to equations with imaginary coefficients; it constitutes indeed the only general method by which the roots in such cases can be computed.

Prob. 21. Compute by McClintock's series the roots of the equation $x^3 - ix - 1 = 0$.

Prob. 22. Solve the equation $\cos x \cosh x + 1 = 0$, and also the equation $x - e^x = 0$. (For answers see Crelle's Journal für Mathematik, 1841, Vol. XXII, pp. 1–62.)

* See Crelle's Journal für Mathematik, 1841, pp. 193–248.